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A New Method for the General Solution of Partial-Wave Dispersion Relations

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Abstract. We describe a new method for obtaining explicit solutions, of a very general type, of a partial-wave dispersion relation for arbitrary l in which the left-hand singularities are replaced by a finite set of poles. The solutions are given by a representation theorem as the product of a Herglotz function and the quotient of two polynomials. Various conditions are obtained for solutions to exist. The special cases of $l = 0$ and one pole and of $l = 1$ and two poles are worked out fully.

1. Introduction

In this paper we take up again the problem considered by Rasche and Woolcock [1]. The problem is that of finding the solutions of a partial-wave dispersion relation in which the left-hand singularities are replaced by a finite set of poles. The physical origins of the problem are discussed fully in Section I of [1]. We modify the notation of [1] a little and use a variable $x = s/s_0$, so that the physical region extends from 1 to ∞ along the real axis. The N poles are now taken to be at the points $x_1, \ldots, x_N$, where

$$x_1 < x_2 < \ldots < x_N < 1,$$

with residues $\Gamma_1, \ldots, \Gamma_N$. The pair of equations corresponding to (2) and (3) of [1] is

$$f_1(x) - \frac{1}{\pi} \int_1^\infty \frac{f_2(t)}{t-x}\, dt = \sum_{i=1}^{N} \frac{\Gamma_i}{x - x_i}, \tag{1.1}$$

$$f_2(x) = \rho(x)[(f_1(x))^2 + (f_2(x))^2], \tag{1.2}$$

and we look for pairs of real-valued functions $f_1(x), f_2(x)$ which satisfy these equations for all $x > 1$. There is a principal value integral on the left side of (1.1). The function $\rho(x)$ is just the product $q(x)\ R(x)$, where

$$q(x) = \tfrac{1}{2}(m_1 + m_2)\, x^{-\frac{1}{2}}(x-1)^{\frac{1}{2}} \left[x - \left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2 \right]^{\frac{1}{2}}, \quad x \geqslant 1.$$

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The function $R(x)$ is the ratio of the total cross-section to the total elastic cross-section for the partial wave in question, so that $R(x) \geqslant 1$ for $x \geqslant 1$. Other conditions will be imposed on $R(x)$ when the mathematical need for them arises later in the paper. If the partial wave being considered belongs to orbital angular momentum l, then we impose the third condition

$$\lim_{x \downarrow 1} (x-1)^{-l} f_1(x) \text{ exists (in } \mathbb{R}). \tag{1.3}$$

Until recently, studies of the solutions of partial-wave dispersion relations have been made within the general framework of the N/D method. In the case where the connection between $f_1(x)$ and $f_2(x)$ is made using the inelasticity parameter $\eta(x)$, there is the work of Hamilton and Tromborg [2], who consider both the case of a continuous left-hand cut extending to $-\infty$ and the case with a finite set of poles. For the case where the function $R(x)$ is given as in (1.2), with very general assumptions about the form of the contribution from the left-hand singularities, Lyth [3] has discussed the N/D method in considerable detail. Wanders and Reuse [4] have presented an improved N/D method for the case of a continuous left-hand cut extending to $-\infty$, with the function $R(x)$ prescribed. For the case of a finite set of poles, the work of reference [1] also uses the framework of the N/D method. It has, however, been realized that for this special case of a finite set of poles there are alternative methods available. For the case where the function $\eta(x)$ is prescribed, Nenciu [5] has shown that the Schur–Pick–Nevanlinna interpolation theory is a natural framework for the study of partial-wave dispersion relations. In this paper we are going to show that, when the function $R(x)$ is prescribed, it is possible to characterize a very general class of solutions of the system (1.1)–(1.3) as the boundary values from above of analytic functions which may be represented as the product of a Herglotz function and the quotient of two polynomials.

The paper proceeds as follows. At the end of this section we list the precise properties of the analytic functions we wish to find, and then transform the problem to a form where the representation theorem of Section 2 applies. This theorem is an interesting mathematical result in its own right. The theorem is applied to our problem in Section 3, and various conditions are obtained for solutions to exist. The special cases $l=0$, $N=1$ and $l=1$, $N=2$ are worked out fully in Section 4.

Our solutions of the system (1.1)–(1.3) will be obtained in the usual way by constructing complex-valued functions $f(z)$ satisfying the following conditions:

1) $f(z)$ is analytic on $\mathbb{C} - ([1,\infty) \cup \{x_i\})$, where $x_1 < x_2 < \ldots < x_N < 1$;
2) $f(\bar{z}) = \overline{f(z)}$;
3) $f(z)$ has simple poles at the isolated singularities $x_1, \ldots, x_N$, with real residues $\Gamma_1, \ldots, \Gamma_N$ respectively;
4) there exists a function $f(x)$, defined on $[1,\infty)$, such that, given $a \geqslant 1$ and $\epsilon > 0$, there exists $\delta > 0$ (depending on a and ϵ) for which
$|f(z) - f(a)| < \epsilon$ when $z \in \{|z-a| < \delta, \operatorname{Im} z > 0\}$;
5) if $f(x) = f_1(x) + i f_2(x)$, where f_1 and f_2 are real-valued functions defined on $[1,\infty)$, then (1.2) holds for $x \geqslant 1$;
6) there exists a non-negative integer k such that $z^{-k} f(z) \to 0$ as $|z| \to \infty$, uniformly in $0 \leqslant \operatorname{Arg} z \leqslant \pi$;
7) for some fixed non-negative integer l, $\lim_{z \to 1} (z-1)^{-l} f(z)$ exists, uniformly in $0 \leqslant \operatorname{Arg} z \leqslant \pi$.

We denote by $\mathscr{P}_l(\Gamma_i; x_i; \rho(x))$ the class of functions f satisfying the conditions (1)–(7). The limit in (7) is the scattering length for the partial wave in question, multiplied by $(m_1 m_2)^l$; it is clearly real.

We shall sometimes describe the property (4) more briefly by saying that $f(z)$ is continuous onto $[1, \infty)$ from above. Note that $f(z)$ is also continuous onto $[1, \infty)$ from below, the limit function being $\overline{f(x)}$ by property (2). We have used the *same* symbol f to denote the original analytic function defined on $\mathbb{C} - [1, \infty)$ (in (1)–(3)), the limit function defined on $[1, \infty)$ (in (4) and (5)) and, in (6) and (7), the obvious function which is defined and continuous on $\operatorname{Im} z \geqslant 0$ (in the sense of the relative topology). This deliberate confusion is convenient and causes no difficulty. Note that for the consistency of the conditions $R(x)$ must be assumed to be continuous at each point of $[1, \infty)$ where $f(x) \neq 0$.

The next step is to define a new function $h(z)$ to be

$$h(z) = \frac{f(z)\,Q(z)}{(z-1)^l\,p(z)}, \tag{1.4}$$

where

$$Q(z) = \prod_{i=1}^{N} (z - x_i)$$

and

$$p(z) = \prod_{j=1}^{m} (z - z_j),$$

the z_j being all the zeros of $f(z)$ away from $[1,\infty)$. For zeros of order greater than 1, the same factor is repeated the appropriate number of times. The points z_j either belong to $(-\infty, 1) - \{x_i\}$ or else occur in complex conjugate pairs; thus $p(x) > 0$ for $x \geqslant 1$. If $f(z)$ does not vanish away from $[1, \infty)$, we set $p(z) = 1$. That the number of factors in the definition of $p(z)$ is finite can be seen by using an argument in Section II of Jin and Martin [6]. Indeed, if k is the smallest integer for which property (6) holds, and if $(k + N - l) > 0$, then we must have

$$m \leqslant (k + N - l).$$

For, if $m - (k + N - l) = m' > 0$, then $z^{m'} h(z) \to 0$ as $|z| \to \infty$, uniformly in $0 \leqslant \operatorname{Arg} z \leqslant \pi$, and we may write the usual dispersion relation

$$z^{m'}\,h(z) = \frac{1}{\pi} \int_1^\infty \frac{t^{m'} \operatorname{Im} h(t)}{t - z}\, dt.$$

The left side vanishes at $z = 0$, but the right does not unless $\operatorname{Im} h(x)$ (and therefore $f(x)$ itself) is identically zero on $[1, \infty)$.

Having taken care of the poles of $f(z)$ and of its zeros away from $[1, \infty)$, we now consider the function $h(z)$. Rather than list all its properties immediately, we proceed in two stages. In the next section we prove a representation theorem for functions $h(z)$ having most (but not all) of the properties of the function $h(z)$ above. Then in Section 3 we add the extra properties required for a complete specification of $f(z)$ and proceed to determine the conditions on $\Gamma_i, x_i, \rho(x)$ for which the class $\mathscr{P}_l(\Gamma_i; x_i; \rho(x))$ is non-void and to describe the class $\mathscr{P}_l$.

2. Representation Theorem

Let the complex-valued function h of the complex variable z have the following properties:

1) $h(z)$ is analytic on $\mathbb{C}-[1,\infty)$;
2) $h(\bar{z})=\overline{h(z)}$;
3) $h(z)\neq 0$;
4) there exists a function $h(x)$, defined on $[1,\infty)$, such that, given $a\geqslant 1$ and $\epsilon>0$, there exists $\delta>0$ (depending on a and ϵ) for which $|h(z)-h(a)|<\epsilon$ when $z\in\{|z-a|<\delta,\operatorname{Im}z>0\}$;
5) $\operatorname{Im}h(x)\geqslant 0$ when $x\geqslant 1$;
6) there exists a non-negative integer n such that $z^{-n}h(z)\to 0$ as $|z|\to\infty$, uniformly in $0\leqslant\operatorname{Arg}z\leqslant\pi$. Let F be the set of points in $[1,\infty)$ for which $h(x)=0$; then F is closed and has Lebesgue measure zero (see, for example, part of Theorem 19.2.4 of [7]). In fact we impose the stronger condition
7) F is a closed set consisting only of isolated points. For the definition of an isolated point, see for example, Section (6.61) of [8]. The set F has no cluster points; it therefore has a finite number of points in every closed subinterval of $[1,\infty)$ and so is countable.

We shall prove in the rest of this section that each h with properties (1)–(7) can be represented in the form

$$h(z)=\phi(z)\,\psi(z), \tag{2.1}$$

where ψ is a Herglotz function which is analytic and positive in $(-\infty,1)$ and ϕ is a polynomial with zeros only in $[1,\infty)$, satisfying $\phi(x)\geqslant 0$ when $x\geqslant 1$.

From (4) and (5) it is clear that $\operatorname{Arg}h(x)$ is defined and continuous on $[1,\infty)-F$ and that

$$0\leqslant\operatorname{Arg}h(x)\leqslant\pi,\quad x\in[1,\infty)-F. \tag{2.2}$$

Therefore

$$\int_1^\infty\frac{\operatorname{Arg}h(t)}{t(t-z)}\,dt$$

exists in the Riemann sense when $z\in\mathbb{C}-[1,\infty)$ and

$$\psi(z)=\exp\left(\frac{z}{\pi}\int_1^\infty\frac{\operatorname{Arg}h(t)}{t(t-z)}\,dt\right) \tag{2.3}$$

is analytic on $\mathbb{C}-[1,\infty)$ and satisfies $\psi(\bar{z})=\overline{\psi(z)}$. We now show that ψ has the properties stated above.

Consider the analytic logarithm of ψ defined by

$$\Psi(z)=\frac{z}{\pi}\int_1^\infty\frac{\operatorname{Arg}h(t)}{t(t-z)}\,dt. \tag{2.4}$$

Then $\operatorname{Im} \Psi(z)$ is an argument of $\psi(z)$ and

$$\operatorname{Im} \Psi(z) = \frac{\operatorname{Im} z}{\pi} \int_1^\infty \frac{\operatorname{Arg} h(t)}{|t-z|^2}\, dt. \tag{2.5}$$

It follows from (2.2) and (2.5) that, when $\operatorname{Im} z > 0$,

$$0 < \operatorname{Im} \Psi(z) < \operatorname{Im} z \int_1^\infty \frac{1}{|t-z|^2}\, dt = \arc\tan \frac{t - \operatorname{Re} z}{\operatorname{Im} z}\Bigg|_1^\infty < \pi.$$

Thus $\operatorname{Im} \psi(z) > 0$ when $\operatorname{Im} z > 0$, so that ψ defined by (2.3) is a Herglotz function which is clearly analytic and positive in $(-\infty, 1)$.

We note here two simple results which will be used later. First, it is clear from (2.2) and (2.3) that $\psi(x) > 1$ when $0 < x < 1$, and so

$$\liminf_{x \uparrow 1} \psi(x) \geqslant 1. \tag{2.6}$$

Second, it is shown in Appendix A that

$$\lim_{n\to\infty} \operatorname{Im} \Psi(z_n) = \operatorname{Arg} h(x_0) \tag{2.7}$$

for every sequence $\{z_n\}$ in the upper half-plane which converges to a point x_0 in $[1, \infty) - F$.

We now construct the function

$$\phi(z) = h(z)/\psi(z); \tag{2.8}$$

ϕ is analytic on $\mathbb{C} - [1, \infty)$ and has the properties

$$\phi(\bar{z}) = \overline{\phi(z)}, \quad \phi(z) \neq 0. \tag{2.9}$$

To complete the proof of our theorem we need to show that ϕ has the properties stated above. To do this, we show first that ϕ can be extended to a function analytic on $\mathbb{C} - F$. We then show that ϕ has removable singularities at the points of F. Finally we prove that ϕ has only a finite number of zeros in $[1, \infty)$ and that ϕ must be a polynomial.

To extend ϕ to a function analytic on $\mathbb{C} - F$, we work with the analytic logarithms of h and ϕ, defined as follows. Let H be the analytic logarithm of h with $\operatorname{Im} H(x) = 0$ (resp. π) for $x < 1$ if $h(x) > 0$ (resp. <0) for $x < 1$. The analytic logarithm Φ of ϕ we then define to be

$$\Phi(z) = H(z) - \Psi(z), \tag{2.10}$$

where Ψ is given by (2.4). Since $h(z) \neq 0$, $H(z)$ is analytic on $\mathbb{C} - [1, \infty)$ and so is $\Phi(z)$. We now consider the limit of $\operatorname{Im} \Phi(z_n)$ for a sequence $\{z_n\}$ in the upper half-plane which converges to $x_0 \in [1, \infty) - F$. From (2.7) and (2.10),

$$\operatorname{Im} \Phi(z_n) \to \lim_{n\to\infty} \operatorname{Im} H(z_n) - \operatorname{Arg} h(x_0).$$

From the continuity of h (property (4)), $\lim_{n\to\infty} \operatorname{Im} H(z_n)$ is independent of the sequence $\{z_n\}$ and is an argument of $h(x_0)$, which can differ from the principal argument only by an integral multiple of 2π. Thus

$$\lim_{n\to\infty} \operatorname{Im} \Phi(z_n) = 2n_0\pi, \tag{2.11}$$

where the integer n_0 is constant between two consecutive points of F, again because of the continuity of h. We can therefore apply the Schwarz principle (see, for example, Section 11.17 of [9]) to continue Φ analytically through the real axis into the lower half-plane between two consecutive points of F. The values of each of the functions thus obtained may not coincide with the values of the original function Φ in the lower half-plane, but from (2.9) and (2.11) the difference is $4\pi n_0 i$ (resp. $4\pi n_0 i - 2\pi i$) in the case when $h(x) > 0$ (resp. $h(x) < 0$) for $x < 1$. From this it follows that ϕ can also be continued analytically through the real axis into the lower half-plane between two consecutive points of F and that each function thus obtained coincides with the original function ϕ in the lower half-plane. Thus ϕ can be extended to a function which is analytic and does not vanish on $\mathbb{C} - F$. From (2.11) it is clear that $\phi(x) > 0$ when $x \in [1, \infty) - F$.

Let x_s be a point of F. The next step is to show that each x_s is a removable singularity of ϕ. We note first that the change in the argument of ϕ in going around x_s is just the change in $\operatorname{Im}\Phi$ in going around x_s. From (2.9) and (2.11),

$\operatorname{Im} \Phi$ changes by an even multiple of 2π if $x_s \neq 1$; (2.12)

$\operatorname{Im} \Phi$ changes by an even multiple of 2π if $x_s = 1$ and $h(x) > 0$ when $x < 1$; (2.13a)

$\operatorname{Im} \Phi$ changes by an odd multiple of 2π if $x_s = 1$ and $h(x) < 0$ when $x < 1$. (2.13b)

Now we estimate $|\phi(z)| = \exp(\operatorname{Re}\Phi(z))$ in the neighbourhood of x_s.

Let A be the set of points z such that

$$a < |z - x_s| < b,$$

where $0 < a < \min\{1, b\}$. The value of b is fixed so that no other point of F belongs to $[x_s - b, x_s + b]$. Now $\operatorname{Re}\Phi(z)$ is harmonic in A and

$$\operatorname{Re} \Phi(z) = \int_{|z'-x_s|=a} G'_A(z', z) \operatorname{Re} \Phi(z')\, ds' + \int_{|z'-x_s|=b} G'_A(z', z) \operatorname{Re} \Phi(z')\, ds' \tag{2.14}$$

when $z \in A$, where $G_A(z', z)$ is the Green's function for A and $G'_A(z', z)$ is the derivative of G_A with respect to z' in the direction of the *inner* normal. Let

$$M = \max_{|z-x_s|=b} \operatorname{Re} \Phi(z), \quad M' = \max_{\substack{|z-x_s|\leqslant b \\ 0\leqslant \operatorname{Arg} z \leqslant \pi}} \operatorname{Re} H(z). \tag{2.15}$$

Note that $\operatorname{Re}H(z) = \ln|h(z)|$ and that $|h|$ is continuous on $\{|z - x_s| \leqslant b, 0 \leqslant \operatorname{Arg} z \leqslant \pi\}$. Now using (2.14), (2.10), (2.15), (B.2) and (C.8) we have, for $z \in A$,

$$\begin{aligned}\operatorname{Re} \Phi(z) \leqslant M \int_{|z'-x_s|=b} G'_A(z', z)\, ds' + (M' + \pi + 2x_s) \int_{|z'-x_s|=a} G'_A(z', z)\, ds' \\ - \int_{|z'-x_s|=a} G'_A(z', z) \ln(a \sin \alpha)\, ds',\end{aligned} \tag{2.16}$$

where $z' - x_s = ae^{i\alpha}$ in the last integral.

Next we write

$$z - x_s = ce^{i\gamma}, \quad a < c < b;$$

then from (2.16), (C.6), (C.7), (C.2), (C.3) and (C.9),

$$\operatorname{Re} \Phi(x_s + ce^{i\gamma}) \leqslant (M + M' + \pi + 2x_s) - \frac{1}{2\pi}\frac{c+a}{c-a}\int_0^{2\pi} \ln(a \sin \alpha)\, d\alpha.$$

Now take $0 < c < \min\{5, b\}$ and $a = \frac{1}{5}c$; then

$$\operatorname{Re} \Phi(x_s + c^{i\gamma}) \leqslant \Lambda - \tfrac{3}{2} \ln c,$$

where

$$\Lambda = (M + M' + \pi + 2x_s) - \frac{3}{4\pi}\int_0^{2\pi} \ln(\sin \alpha)\, d\alpha + \tfrac{3}{2} \ln 5.$$

Thus

$$|\phi(x_s + ce^{i\gamma})| = \exp(\operatorname{Re} \Phi(x_s + ce^{i\gamma})) \leqslant \lambda c^{-\frac{3}{2}}, \tag{2.17}$$

where $\lambda = \exp \Lambda$.

Equation (2.17) shows that ϕ cannot have an essential singularity at a point of F and that, if it has a pole at such a point, this pole must be of first order. However, from (2.12) we see that the pole must be of even order if $x_s \neq 1$; thus all points of F except $x_s = 1$ are removable singularities. The same is true for $z = 1$; equation (2.8) shows that a pole of ϕ at $z = 1$ would imply that $\lim_{x \uparrow 1} \psi(x) = 0$, in contradiction to (2.6). We conclude that ϕ can be extended to an entire function, with possible zeros only at the points of F. From (2.12) it follows that these zeros must be of even order for $x_s \neq 1$, while (2.13) shows that the possible zero at $z = 1$ may be of any order.

To complete the proof of the theorem we have to show that the number of zeros of ϕ is finite and that ϕ cannot be a transcendental entire function. To prove this we write ϕ in the form

$$\phi(z) = \phi_X(z)/\psi_\infty(z),$$

where

$$\phi_X(z) = \frac{h(z)}{\exp\left(\dfrac{z}{\pi}\displaystyle\int_1^X \frac{\operatorname{Arg} h(t)}{t(t-z)}\, dt\right)},$$

$$\psi_\infty(z) = \exp\left(\frac{z}{\pi}\int_X^\infty \frac{\operatorname{Arg} h(t)}{t(t-z)}\, dt\right).$$

Clearly $\phi_X(z)$ is analytic on $\mathbb{C} - [X, \infty)$, has possible zeros only in $[1, X)$ and satisfies property (6) of the function h. Thus, by the same argument as we used in the last paragraph but one of Section 1, the number of zeros of ϕ_X (and therefore of ϕ) in $[1, X)$ is less than or equal to n, where n is the smallest non-negative integer for which property (6) holds. The number of zeros of ϕ in $[1, \infty)$ is therefore less than or equal to n and we have

$$\phi(z) = \pi(z)\, q(z),$$

where π is a polynomial of degree $\leqslant n$ which satisfies $\pi(x) \geqslant 0$ for $x \geqslant 1$ and q is an entire function which never vanishes. The argument of Section 2 of Jin and MacDowell [10] now shows that q is bounded by a power of $|z|$, uniformly for all directions, so that q is a constant (which must be positive, since we saw earlier that $\phi(x) > 0$ when $x \in [1, \infty) - F$). This completes the proof of the theorem.

3. Application of the Representation Theorem

There appears in the theorem of Section 2 a function ψ which is Herglotz and, in addition, is analytic on $\mathbb{C} - [1, \infty)$ and non-negative on $(-\infty, 1)$. We now appeal to results from [11] (corollary to Theorem P.3 and Theorems P.4 and P.5) to conclude that the function $-1/(z-1)\psi(z)$ is also a Herglotz function with the same additional properties as $\psi(z)$ and that this function has the representation

$$-\frac{1}{(z-1)\,\psi(z)} = \gamma + \int_1^\infty \frac{d\sigma(t)}{t-z}, \tag{3.1}$$

where $\gamma \geqslant 0$ and σ is a real-valued monotone non-decreasing function on $[1, \infty)$ with the property that

$$\int_1^\infty \frac{d\sigma(t)}{t} < \infty. \tag{3.2}$$

Combining now (1.4), (2.1) and (3.1), we have arrived at the following representation for the function $f(z)$:

$$(z-1)^{-l} f(z) = \frac{-P(z)}{(z-1)\,Q(z)} \left(\gamma + \int_1^\infty \frac{d\sigma(t)}{t-z} \right)^{-1}, \tag{3.3}$$

where

$$P(z) = p(z)\, \phi(z).$$

It follows from the properties of p and ϕ that

$$P(x) \geqslant 0, \quad x \geqslant 1. \tag{3.4}$$

The property (3.4) in fact provides a complete characterization of the polynomial P.

We now write the function σ as

$$\sigma = \sigma_c + \sigma_d, \tag{3.5}$$

where σ_c is continuous on $[1, \infty)$ and σ_d is a saltus function; it has a jump $c_j(>0)$ at each of the points of an at most countable set $\{\xi_j\}(1 \leqslant \xi_1 < \xi_2 < \ldots)$ and is constant on each of the open intervals $(\xi_1, \xi_{j+1})(j = 1, 2, \ldots)$ and on $[1, \xi_1)$ if $\xi_1 > 1$. For the decomposition (3.5) of σ, see Sections (19.54)–(19.58) of [8]. Notice that if $P(1) > 0$, then $\xi_1 = 1$ in

order that property (7) of f (Section 1) be satisfied. Note too that, when $\xi_j > 1$, $f(\xi_j)$ must be defined to be zero in order to have continuity of f at ξ_j. Thus σ can be discontinuous only at $x = 1$ and at points of the set F. This means in particular that the set $\{\xi_j\}$ must not have a cluster point. The points ξ_j with $\xi_j > 1$ are usually called the CDD zeros of f, since zeros of this type were first discovered by Castillejo, Dalitz and Dyson [12] in the solutions of an s-wave dispersion relation arising from a static model.
From (3.3) and (3.5) we have

$$(z-1)^{-l}f(z) = \frac{-P(z)}{(z-1)Q(z)}\left(\chi(z) + \int_1^\infty \frac{d\sigma_c(t)}{t-z}\right)^{-1}, \tag{3.6}$$

where

$$\chi(z) = \gamma + \sum_j \frac{c_j}{\xi_j - z}, \quad \gamma \geqslant 0, \quad c_j > 0 \; (j = 1, 2, \ldots), \quad 1 \leqslant \xi_1 < \xi_2 < \ldots. \tag{3.7}$$

It is to be understood that $\chi(z) = \gamma$, $\gamma \geqslant 0$, is possible; there may be no point ξ_j for which $c_j > 0$.

From (3.2) follow the two conditions

$$\sum_j \frac{c_j}{\xi_j} < \infty, \tag{3.8}$$

$$\int_1^\infty \frac{d\sigma_c(t)}{t} < \infty. \tag{3.9}$$

In the discussion leading to (3.6) we have not taken into account fully the properties (3) and (5) of $f(z)$. We need now to impose the condition of equation (1.2), and later to fix the residues at the poles. Equation (1.2) may be written

$$\operatorname{Im}\,(-1/f(x)) = \rho(x), \quad x \in [1, \infty) - F.$$

Thus, using (3.6), we have

$$\operatorname{Im}\left[\lim_{y \downarrow 0} \int_1^\infty \frac{d\sigma_c(t)}{t - x - iy}\right] = \frac{\rho(x)(x-1)^{l-1}P(x)}{Q(x)}, \quad x \in [1, \infty) - F.$$

Now if $[a, b]$ is a closed interval contained in $[1, \infty) - F$, we may use Theorem 7a, Chapter VIII of [13], to show that

$$\sigma_c(b) - \sigma_c(a) = \frac{1}{\pi}\int_a^b \frac{\rho(t)(t-1)^{l-1}P(t)}{Q(t)}\,dt.$$

Since $R(x)$ (and therefore $\rho(x)$) has already been assumed to be continuous on $[1, \infty) - F$, it follows that σ_c is differentiable on $[1, \infty) - F$ and that

$$\sigma_c'(x) = \frac{1}{\pi}\frac{\rho(x)(x-1)^{l-1}P(x)}{Q(x)}. \tag{3.10}$$

This fixes the absolutely continuous part of σ_c; the singular part cannot appear, since F consists only of isolated points. From (3.6) and (3.10),

$$(z-1)^{-l} f(z) = \frac{-P(z)}{(z-1)\,Q(z)} \left(\chi(z) + \frac{1}{\pi} \int_1^\infty \frac{\rho(t)(t-1)^{l-1} P(t)}{Q(t)(t-z)}\, dt \right)^{-1}, \tag{3.11}$$

where $\chi(z)$ is given by (3.7). The condition (3.9) becomes

$$\frac{1}{\pi} \int_1^\infty \frac{\rho(t)(t-1)^{l-1} P(t)}{t Q(t)} < \infty. \tag{3.12}$$

Equation (3.12) leads to a necessary condition for the set $\mathscr{P}_l(\Gamma_i; x_i; \rho(x))$ to be non-void. We must clearly assume that there exists a non-negative integer K (which we take to be the smallest possible) for which

$$\int_1^\infty \frac{\rho(t)}{t^{K+2}}\, dt < \infty. \tag{3.13}$$

We then have the following lemma.

Lemma 1: If $\mathscr{P}_l(\Gamma_i; x_i; \rho(x))$ is non-void, then $N \geqslant (l + p + K)$, where p is the degree of the polynomial P.

Notice that if m is the number of points x_i for which Γ_i and Γ_{i+1} have the same sign, then $p \geqslant m$.

The next step is to fix the residues at the poles of f. We put this in the form of another lemma.

Lemma 2: A necessary and sufficient condition for the class $\mathscr{P}_l(\Gamma_i; x_i; \rho(x))$ to be non-void is that there should exist a polynomial P with the property (3.4) and a function χ of the form (3.7) such that

$$\chi(x_i) = -\frac{1}{\pi} \int_1^\infty \frac{\rho(t)(t-1)^{l-1} P(t)}{Q(t)(t - x_i)}\, dt - \frac{P(x_i)(x_i - 1)^{l-1}}{\Gamma_i \prod_{j \neq i} (x_i - x_j)}, \quad i = 1, \ldots, N, \tag{3.14}$$

and also such that (3.8) and (3.12) are satisfied and $\xi_1 = 1$ if $P(1) > 0$.

The sufficiency asserted in Lemma 2 must be checked by proving that all the properties (1)–(7) of Section 1 are satisfied by the function f of (3.11) when the conditions of the lemma hold. To ensure that the continuity property (4) holds, the only condition known to us is a Hölder continuity condition on ρ. We have assumed throughout that the first inelastic threshold is at $x_0 > 1$ and that $\rho(x) = q(x)$ for $1 \leqslant x < x_0$. We now add the assumption that, for each $x \geqslant x_0$, there exist $\delta > 0$, $A > 0$, $\mu \in (0,1)$ such that

$$|\rho(t_2) - \rho(t_1)| < A\,|t_2 - t_1|^\mu, \quad t_1, t_2 \in [x - \delta, x + \delta]. \tag{3.15}$$

The numbers δ, A, μ may depend on x. The continuity of $D(z)$ then follows for $x > 1$, where

$$D(z) = \chi(z) + \frac{1}{\pi}\int_1^\infty \frac{\rho(t)(t-1)^{l-1}P(t)}{Q(t)(t-z)}\,dt, \quad z \notin [1,\infty), \tag{3.16}$$

$$D(x) = \chi(x) + \frac{1}{\pi}\int_1^\infty \frac{\rho(t)(t-1)^{l-1}P(t)}{Q(t)(t-x)}\,dt + i\,\frac{\rho(x)(x-1)^{l-1}P(x)}{Q(x)}, \tag{3.17}$$

$x > 1$, $x \notin \{\xi_j\}$. A full statement of the continuity result is given in Theorem B, Section II of [14]. With $f(\xi_j)$ defined to be zero if $\xi_j > 1$, the continuity of f for $x > 1$ then follows, except perhaps for points where $D(x) = 0$. From (3.17), $D(x)$ can vanish on $(1,\infty)$ only where $P(x) = 0$. But if $\operatorname{Re} D(x)$ vanishes at a zero $x = x_s$ of $P(x)$, then the derivative of $\operatorname{Re} D(x)$ exists at $x = x_s$; indeed

$$\operatorname{Re} D'(x_s) = \frac{1}{\pi}\int_1^\infty \frac{\rho(t)(t-1)^{l-1}}{Q(t)}\frac{P(t)}{(t-x_s)^2}\,dt + \sum_j \frac{c_j}{(\xi_j - x_s)^2} < 0.$$

Thus if $f(x_s)$ is defined to be zero, f will be continuous at x_s. Finally it is clear from (3.11) that, since $\rho(x) = q(x)$ on $[1, x_0)$, property (7) holds provided that $\xi_1 = 1$ if $P(1) > 0$.

The only remaining property which is not obviously fulfilled is (6). We show in Appendix D that in fact this property holds with $k = 0$. To summarize, then, we started out by looking for functions f satisfying properties (1)–(7). In the course of constructing these functions we have imposed on the function $\rho(x)$ which appears in property (5) (equation (1.2)) two restrictions beside the original one, namely $\rho(x) = q(x)\,R(x)$ with $R(x) \geqslant 1$. These further restrictions are the integrability property (3.13) and the Hölder continuity property (3.15). We found (Lemma 1) that for any functions f to exist, it is necessary to assume that $N \geqslant l + K$. We proved that every function f satisfying properties (1)–(7) must be of the form (3.11), where χ is given by (3.7), with the subsidiary condition (3.8), and P is a polynomial of degree $p \leqslant N - l - K$ having the property (3.4). We then established that, if the functions χ and P satisfy the N conditions of (3.14) and the further condition that $\xi_1 = 1$ if $P(1) > 0$, then the function f does indeed have all the properties (1)–(7). In particular, property (6) holds with $k = 0$.

Our next remark concerns the sign of the scattering length for functions f in the class $\mathscr{P}_l(\Gamma_i; x_i; \rho(x))$. From (3.11), if $P(1) > 0$ and consequently $\xi_1 = 1$, then the limit in property (7) is just $P(1)/Q(1)c_1 > 0$ and the solution has positive scattering length. However, if P has a zero of order r at $z = 1$, then the scattering length is negative if $r = 1$ and $\xi_1 > 1$, zero if $r = 1$ and $\xi_1 = 1$ or if $r > 1$. To obtain a function with a non-positive scattering length, it is necessary that $P(1) = 0$. For given pole positions and prescribed $\rho(x)$, therefore, the class $\mathscr{P}_l(\Gamma_i; x_i; \rho(x))$ will contain functions with non-positive scattering length only for a proper subset of the set $\{(\Gamma_1, \dots, \Gamma_N)\}$ of N-tuples of residues for which it is non-void.

This introduces the interesting question of characterizing the set $\{(\Gamma_1, \dots, \Gamma_N)\}$ for which the class $\mathscr{P}_l(\Gamma_i; x_i; \rho(x))$ is non-void, the pole positions $x_1, \dots, x_N$ and function $\rho(x)$ being already given. By referring to (3.14) this problem may be put in the following form. The set of all points in $\mathbb{R}^N$ of the form

$$(\chi(x_1), \dots, \chi(x_N)),$$

where χ ranges over all functions of the form (3.7) with a finite or countably infinite set $\{\xi_j\}$ and satisfying (3.8), is a convex cone in $\mathbb{R}^N$, with the origin corresponding to $\chi(z) = 0$. In a similar way, if

$$F_i(P) = -\frac{1}{\pi}\int_1^\infty \frac{\rho(t)(t-1)^{l-1}P(t)}{Q(t)(t-x_i)}\,dt - \frac{P(x_i)(x_i-1)^{l-1}}{\Gamma_i \prod_{j\neq i}(x_i-x_j)}, \quad i=1,\ldots,N,$$

then the set of all points in $\mathbb{R}^N$ of the form

$$(F_1(P), \ldots, F_N(P))$$

where P varies over all non-trivial polynomials of degree $\leqslant N-l-K$ which satisfy (3.4), is a second convex cone in $\mathbb{R}^N$, again with the origin corresponding to $\chi(z) = 0$[3]. By (3.14), the class $\mathscr{P}_l(\Gamma_i; x_i; \rho(x))$ is non-void if and only if the intersection of these two cones, minus any points for which $P(1) > 0$ and $\chi(1)$ is finite, is non-void. The second convex cone depends on the N-tuple $(\Gamma_1, \ldots, \Gamma_N)$ of residues; determination of the set of N-tuples for which $\mathscr{P}_l(\Gamma_i; x_i; \rho(x))$ is non-void becomes a problem of convex programming. This way of approaching the problem would probably be useful in making a computer search for such N-tuples, in cases for which analytical methods become too difficult.

There are three further comments of a general nature to be made. The first is that the set of N-tuples $(\Gamma_1, \ldots, \Gamma_N)$ for which $\mathscr{P}_l(\Gamma_i; x_i; \rho(x))$ contains functions with $\leqslant m$ CDD zeros increases as m increases. The second is that the full set of such N-tuples (for a finite or countably infinite set of CDD zeros) is a bounded set in $\mathbb{R}^N$; this result is proved in Appendix E. The third is that, if the intersection of the two cones in $\mathbb{R}^N$ which are defined in the previous paragraph has a non-void interior, then there are solutions with an infinite number of CDD zeros (see [11], Chapter V, Sections 3–5).

We conclude this section with another lemma.

Lemma 3: Let $R_1(x) \geqslant R_2(x)$, $x \geqslant 1$. Then, if $\mathscr{P}_l(\Gamma_i; x_i; \rho_1(x))$ is non-void, so is $\mathscr{P}_l(\Gamma_i; x_i; \rho_2(x))$.

The proof is straightforward. By Lemma 2, corresponding to $R_1(x)$ there is a polynomial P_1 and a function χ_1 such that

$$\chi_1(x_i) = -\frac{1}{\pi}\int_1^\infty \frac{q(t)\,R_1(t)(t-1)^{l-1}P_1(t)}{Q(t)(t-x_i)} - \frac{P_1(x_i)(x_i-1)^{l-1}}{\Gamma_i \prod_{j\neq i}(x_i-x_j)}, \quad i=1,\ldots,N,$$

and $\xi_1 = 1$ if $P_1(1) > 0$. Then

$$-\frac{1}{\pi}\int_1^\infty \frac{q(t)\,R_2(t)(t-1)^{l-1}P_1(t)}{Q(t)(t-x_i)}\,dt - \frac{P_1(x_i)(x_i-1)^{l-1}}{\Gamma_i \prod_{j\neq i}(x_i-x_j)} = \chi_1(x_i) + k(x_i),$$

where

$$k(z) = \frac{1}{\pi}\int_1^\infty \frac{q(t)(R_1(t)-R_2(t))(t-1)^{l-1}P_1(t)}{Q(t)(t-z)}\,dt.$$

[3]) If there is a non-trivial polynomial P of degree $\leqslant N-l-K$ which satisfies (3.4), $P(1) = 0$ and $F_i(P) = 0$, $i = 1, \ldots, N$, then (3.11) with $\chi(z) = 0$ gives a solution which is free of CDD zeros.

Now there is an uncountably infinite set of functions $\chi'(z)$ of the form (3.7) such that

$$\chi'(x_i) = k(x_i), \quad i = 1, \ldots, N.$$

Indeed, from Theorem 7.2, Chapter V of [15], χ' can be chosen to have as few CDD poles as $N/2$ if N is even or $(N+1)/2$ if N is odd. Since $\chi_2 = \chi_1 + \chi'$ is also of the form (3.7) and χ_2 has a pole at $z = 1$ if χ_1 has, we can find for the function $R_2(x)$ functions with the same polynomial P_1 and an infinity of functions χ_2, which satisfy the conditions of Lemma 2.

4. Special Cases

We turn now to some simple special cases which are nevertheless interesting from the point of view of practical calculations with partial wave dispersion relations. We give explicitly the sets of singles $\{\Gamma_1\}$ and of pairs $\{(\Gamma_1, \Gamma_2)\}$ for which the classes $\mathscr{P}_0(\Gamma_1; x_1; \rho(x))$ and $\mathscr{P}_1(\Gamma_1, \Gamma_2; x_1, x_2; \rho(x))$ respectively are non-void, and discuss the sign of the scattering length for the functions in each class. In working through these cases, we use simple analytical methods, which are easy enough for cases with $N \leqslant 2$ and $p \leqslant 1$ but become very difficult if $N > 2$ or if $p > 1$. Since it follows from Lemma 1 of Section 3 that, for fixed l and N, p becomes more restricted as K increases, we take $K = 0$ in what follows. We then have

$$l \geqslant 0, \quad N \geqslant \max\{1, l\}, \quad 0 \leqslant p \leqslant N - l. \tag{4.1}$$

In Subsection 4.1 we consider the simplest case, namely $l = 0$, $N = 1$; in Subsection 4.2 we consider the case $l = 1$, $N = 2$. Before proceeding, note from (3.7) that

$$\chi(x) \geqslant 0, \quad x < 1, \tag{4.2}$$

equality occurring in (4.2) if and only if $\chi(z) = 0$.

4.1. The case $l = 0$, $N = 1$

From (4.1) we have $p = 0, 1$. When $p = 0$ it follows from (3.11) that

$$f(z) = \frac{-1}{(z-1)(z-x_1)} \left(\chi(z) + \frac{1}{\pi} \int_1^\infty \frac{\rho(t)}{(t-1)(t-x_1)(t-z)}\, dt \right)^{-1}, \tag{4.3}$$

where $\chi(z)$ is given by (3.7). Since $P(z) = 1$, we have $\xi_1 = 1$ from Lemma 2 of Section 3, and the scattering length is always positive in this subcase. From (3.14) or (4.3) we have

$$\Gamma_1 = \frac{1}{1-x_1} \left(\chi(x_1) + \frac{1}{\pi} \int_1^\infty \frac{\rho(t)}{(t-1)(t-x_1)^2}\, dt \right)^{-1},$$

where $\chi(x_1) > 0$ from (4.2), since χ is not identically zero. Thus $\mathscr{P}_0(\Gamma_1; x_1; \rho(x))$ is non-void if and only if

$$0 < \Gamma_1 < \frac{1}{1-x_1} \left(\frac{1}{\pi} \int_1^\infty \frac{\rho(t)}{(t-1)(t-x_1)^2}\, dt \right)^{-1}. \tag{4.4}$$

Note particularly that the inequalities in (4.4) are strict.

When $p = 1$ we have

$$P(z) = (z - a),$$

where, from (3.4),

$$a \leqslant 1.$$

From (3.11) we have

$$f(z) = \frac{-(z-a)}{(z-1)(z-x_1)} \left(\chi(z) + \frac{1}{\pi} \int_1^\infty \frac{\rho(t)(t-a)}{(t-x_1)(t-1)(t-z)}\, dt \right)^{-1}, \tag{4.5}$$

with $\chi(z)$ again given by (3.7). From Lemma 2, $\xi_1 = 1$ if $a < 1$ and the scattering length is then positive. However, if $a = 1$ we may have $\xi_1 > 1$; in this case it is clear from (4.5) that

$$f(1) < 0 \quad \text{if } a = 1 \text{ and } \chi(1) \text{ is finite,}$$

$$f(1) = 0 \quad \text{if } a = 1 \text{ and } \xi_1 = 1.$$

From (4.5) or (3.14) we have

$$\Gamma_1 = -\frac{x_1 - a}{x_1 - 1} \left(\chi(x_1) + \frac{1}{\pi} \int_1^\infty \frac{\rho(t)(t-a)}{(t-1)(t-x_1)^2}\, dt \right)^{-1}, \tag{4.6}$$

so that Γ_1 and $(x_1 - a)$ have the same sign. We now fix x_1 and observe that the set of Γ_1 for which $\mathscr{P}_0(\Gamma_1; x_1; \rho(x))$ is non-void is the image of the set

$$\{(a,r) \,|\, a \leqslant 1,\ 0 < r\} \cup \{(1,0)\}$$

under the mapping

$$(a,r) \mapsto \Gamma_1(a,r), \tag{4.7}$$

where, from (4.6),

$$\Gamma_1(a,r) = \frac{(x_1 - a)}{(1 - x_1)\{r + I_1 + (x_1 - a) I_2\}},$$

with

$$I_1 = \frac{1}{\pi} \int_1^\infty \frac{\rho(t)}{(t-1)(t-x_1)}\, dt, \quad I_2 = \frac{1}{\pi} \int_1^\infty \frac{\rho(t)}{(t-1)(t-x_1)^2}\, dt.$$

For fixed $r > 0$, it is clear that $\Gamma_1(a,r)$ strictly increases as a decreases from 1 to $-\infty$; thus the subset of values of Γ_1 for which r is fixed is the interval

$$-\{r + I_1 - (1 - x_1) I_2\}^{-1} \leqslant \Gamma_1 < \{(1 - x_1) I_2\}^{-1}.$$

The lower limit of this interval is strictly increasing as r increases from 0 to ∞; since the image of (1,0) is to be included, we see that the set of Γ_1 for which $\mathscr{P}_0(\Gamma_1; x_1; \rho(x))$

is non-void is just the interval

$$-\left(\frac{1}{\pi}\int_1^\infty \frac{\rho(t)}{(t-x_1)^2}\,dt\right)^{-1} \leqslant \Gamma_1 < \left(\frac{(1-x_1)}{\pi}\int_1^\infty \frac{\rho(t)}{(t-1)(t-x_1)^2}\,dt\right)^{-1}.$$

This is in fact the range of Γ_1 obtained in Sections II and III of [1] for functions to exist of the restricted type considered there. Though in this paper we have obtained more general functions, the restriction on Γ_1 remains the same. Note finally in this subsection that the class $\mathscr{P}_0(\Gamma_1; x_1; \rho(x))$ contains functions with non-positive scattering length only for Γ_1 in the image of the set $\{(1,r)\,|\,0 \leqslant r\}$ under the mapping (4.7); this image is just the interval

$$-\left(\frac{1}{\pi}\int_1^\infty \frac{\rho(t)}{(t-x_1)^2}\,dt\right)^{-1} \leqslant \Gamma_1 < 0.$$

4.2. The case $l = 1$, $N = 2$

From (4.1) we again have $p = 0, 1$. The positions of the poles satisfy

$$x_1 < x_2 < 1. \tag{4.8}$$

From (3.7) and (4.8) we have

$$\chi(x_1) \leqslant \chi(x_2),$$

$$(1-x_2)\,\chi(x_2) \leqslant (1-x_1)\,\chi(x_1),$$

or, combining these two inequalities,

$$\chi(x_1) \leqslant \chi(x_2) \leqslant \frac{(1-x_1)}{(1-x_2)}\,\chi(x_1). \tag{4.9}$$

The left-hand equality sign in (4.9) holds if and only if $\chi(z) = \gamma$ and the right-hand equality sign if and only if $\chi(z) = c_1/(1-z)$.

When $p = 0$ we have, from (3.11),

$$f(z) = \frac{-1}{(z-x_1)(z-x_2)}\left(\chi(z) + \frac{1}{\pi}\int_1^\infty \frac{\rho(t)}{(t-x_1)(t-x_2)(t-z)}\,dt\right)^{-1}, \tag{4.10}$$

with $\chi(z)$ given by (3.7). We require $\xi_1 = 1$ and the scattering length is always positive in this subcase. From (4.10) or (3.14),

$$\Gamma_1 = \frac{-1}{x_1-x_2}\left(\chi(x_1) + \frac{1}{\pi}\int_1^\infty \frac{\rho(t)}{(t-x_1)^2(t-x_2)}\,dt\right)^{-1}, \tag{4.11}$$

$$\Gamma_2 = \frac{-1}{x_2-x_1}\left(\chi(x_2) + \frac{1}{\pi}\int_1^\infty \frac{\rho(t)}{(t-x_1)(t-x_2)^2}\,dt\right)^{-1}. \tag{4.12}$$

Defining

$$\Gamma_1^0 = \frac{\pi}{x_2 - x_1}\left(\int_1^\infty \frac{\rho(t)}{(t-x_1)^2\,(t-x_2)}\,dt\right)^{-1} > 0, \tag{4.13}$$

$$\Gamma_2^0 = \frac{\pi}{x_1 - x_2}\left(\int_1^\infty \frac{\rho(t)}{(t-x_1)(t-x_2)^2}\,dt\right)^{-1} < 0, \tag{4.14}$$

the conditions (4.11) and (4.12) become

$$\Gamma_1 = \frac{\Gamma_1^0}{1 + (x_2 - x_1)\chi(x_1)\,\Gamma_1^0}, \tag{4.15}$$

$$\Gamma_2 = \frac{\Gamma_2^0}{1 + (x_2 - x_1)\chi(x_2)(-\Gamma_2^0)}. \tag{4.16}$$

As $\chi(x_1)$ increases from 0 to ∞, Γ_1 strictly decreases from Γ_1^0 to 0; thus Γ_1 is confined to the interval

$$0 < \Gamma_1 < \Gamma_1^0, \tag{4.17}$$

the inequalities being strict. For each Γ_1 satisfying (4.17), there is a unique value of $\chi(x_1)$ given by (4.15) and a range of values of $\chi(x_2)$ given by (4.9). There is thus a range of values of Γ_2 given by (4.16), namely,

$$\frac{\Gamma_2^0}{1 + (-\Gamma_2^0)(1/\Gamma_1 - 1/\Gamma_1^0)} < \Gamma_2 \leqslant \frac{\Gamma_2^0}{1 + (-\Gamma_2^0)\left(\dfrac{1-x_1}{1-x_2}\right)(1/\Gamma_1 - 1/\Gamma_1^0)}. \tag{4.18}$$

The left-hand inequality in (4.18) is strict, since we require $\xi_1 = 1$ and therefore $\chi(x_1) = \chi(x_2)$ is not possible. However, the upper limit in (4.18) is attained, and there is a unique $\chi(z)$ in this case, namely,

$$\chi(z) = \frac{(1-x_1)}{(x_2-x_1)}\left(\frac{1}{\Gamma_1} - \frac{1}{\Gamma_1^0}\right)\frac{1}{1-z}.$$

It should be noted that Γ_1^0 and Γ_2^0 depend on the values of x_1, x_2 and so the region given by (4.17) and (4.18) depends on x_1, x_2 also via Γ_1^0 and Γ_2^0. Note that the upper and lower limit curves in the (Γ_1, Γ_2) plane given by (4.17) and (4.18) are both arcs of rectangular hyperbolas with asymptotes parallel to the axes, which intersect at (0,0) and at (Γ_1^0, Γ_2^0). The upper limit curve is concave downwards, the lower concave upwards. The upper limit curve in (4.18) will appear again; we therefore define the function

$$\Lambda_+(\Gamma_1) = \frac{\Gamma_1^0\Gamma_2^0\Gamma_1}{\Gamma_1^0\Gamma_1 + \left(\dfrac{1-x_1}{1-x_2}\right)(-\Gamma_2^0)(\Gamma_1^0 - \Gamma_1)}, \tag{4.19}$$

which is in fact defined for all real Γ_1 except

$$\Gamma_1 = \frac{(1-x_1)\,\Gamma_1^0(-\Gamma_2^0)}{(1-x_1)(-\Gamma_2^0)-(1-x_2)\,\Gamma_1^0}(>\Gamma_1^0).$$

We turn now to the subcase $p=1$, for which $P(z)=(z-a)$, with $a \leqslant 1$ as before. From (3.11) we have

$$f(z) = \frac{-(z-a)}{(z-x_1)(z-x_2)}\left(\chi(z)+\frac{1}{\pi}\int_1^\infty \frac{(t-a)\,\rho(t)}{(t-x_1)(t-x_2)(t-z)}\,dt\right)^{-1}. \tag{4.20}$$

Exactly as in the case $N=1$, $l=0$, $\xi_1=1$ and the scattering length is positive if $a<1$, while the scattering length is negative if $a=1$ and $\chi(1)$ is finite, zero if $a=1$ and $\xi_1=1$. From (4.20) or (3.14) we have

$$\Gamma_1 = -\frac{x_1-a}{x_1-x_2}\left(\chi(x_1)+\frac{1}{\pi}\int_1^\infty \frac{(t-a)\,\rho(t)}{(t-x_1)^2\,(t-x_2)}\,dt\right)^{-1}, \tag{4.21}$$

$$\Gamma_2 = -\frac{x_2-a}{x_2-x_1}\left(\chi(x_2)+\frac{1}{\pi}\int_1^\infty \frac{(t-a)\,\rho(t)}{(t-x_1)(t-x_2)^2}\,dt\right)^{-1}. \tag{4.22}$$

Using the definitions (4.13) and (4.14) and the further definition

$$\lambda = \frac{1}{\pi}\int_1^\infty \frac{\rho(t)}{(t-x_1)(t-x_2)}\,dt>0, \tag{4.23}$$

we can write (4.21) and (4.22) in the form

$$\Gamma_1 = \frac{\Gamma_1^0(x_1-a)}{(x_1-a)+(x_2-x_1)(\lambda+\chi(x_1))\,\Gamma_1^0}, \tag{4.24}$$

$$\Gamma_2 = \frac{\Gamma_2^0(x_2-a)}{(x_2-a)+(x_2-x_1)(\lambda+\chi(x_2))(-\Gamma_2^0)}. \tag{4.25}$$

From (4.2). (4.9), (4.24) and (4.25), together with $a \leqslant 1$, we see that the set of pairs (Γ_1,Γ_2) for which $\mathscr{P}_1(\Gamma_1,\Gamma_2;x_1,x_2;\rho(x))$ is non-void is the image of the set

$$\left\{(a,r,\alpha)\,|\,a\leqslant 1,\ 0<r,\ 1<\alpha\leqslant\frac{1-x_1}{1-x_2}\right\}\cup\{(1,r,1)\,|\,0\leqslant r\}$$

under the mapping

$$(a,r,\alpha)\mapsto(\Gamma_1(a,r),\ \Gamma_2(a,\alpha r)), \tag{4.26}$$

where

$$\Gamma_1(a,r) = \frac{\Gamma_1^0(x_1-a)}{(x_1-a)+(x_2-x_1)(\lambda+r)\,\Gamma_1^0}, \tag{4.27}$$

$$\Gamma_2(a, \alpha r) = \frac{\Gamma_2^0(x_2 - a)}{(x_2 - a) + (x_2 - x_1)(\lambda + \alpha r)(-\Gamma_2^0)}. \tag{4.28}$$

Note that the denominators in (4.27) and (4.28) never vanish, since

$$(1 - x_1) < (x_2 - x_1)\,\lambda\Gamma_1^0, \quad (1 - x_2) < (x_2 - x_1)\,\lambda(-\Gamma_2^0).$$

Thus

$$\Gamma_1(1,0) = \Gamma_1^{00} < 0, \quad \Gamma_2(1,0) = \Gamma_2^{00} > 0. \tag{4.29}$$

We determine first the subset of pairs (Γ_1, Γ_2) for which α is fixed and not equal to unity. This subset may be covered by either of two one-parameter families of curves. If we fix a and vary r from 0 to ∞, we obtain a curve which moves away from $(\Gamma_1(a,0), \Gamma_2(a,0))$, whose slope always has the sign of $-(x_2 - a)/(x_1 - a)$ and which approaches $(0,0)$ with slope $-(x_2 - a)/\alpha(x_1 - a)$. For $x_2 < a \leqslant 1$ the curve lies in the second quadrant, for $a = x_2$ it is a segment of the negative Γ_1-axis, for $x_1 < a < x_2$ it lies in the third quadrant, for $a = x_1$ it is a segment of the negative Γ_2-axis and for $a < x_1$ it lies in the fourth quadrant. Alternatively, if we fix r and vary a from 1 to $-\infty$, then $\Gamma_1(a,r)$ strictly increases with decreasing a and $\Gamma_2(a, \alpha r)$ strictly decreases. Thus we obtain a one-parameter set of curves, each of which has negative slope at every point and approaches (Γ_1^0, Γ_2^0) as $a \to -\infty$. It is not difficult to see that no pair of curves from either of these one-parameter families intersect, though a detailed proof is rather tedious to write out. Thus the subset of pairs (Γ_1, Γ_2) for which α is fixed at a value greater than unity is bounded below by the curve

$$\Gamma_2 = \Lambda_-(\Gamma_1), \quad \Gamma_1^{00} < \Gamma_1 < \Gamma_1^0,$$

where $\Lambda_-(\Gamma_1)$ is obtained from (4.27) and (4.28) by putting $r = 0$ and eliminating a. This gives

$$\Lambda_-(\Gamma_1) = \frac{(\lambda\Gamma_1^0\Gamma_1 + \Gamma_1^0 - \Gamma_1)\,\Gamma_2^0}{\lambda\Gamma_1^0\Gamma_1 + (\Gamma_1^0 - \Gamma_1)(1 - \lambda\Gamma_2^0)}, \quad \Gamma_1^{00} < \Gamma_1 < \Gamma_1^0. \tag{4.30}$$

The function $\Lambda_-(\Gamma_1)$ is strictly decreasing on its interval of definition and $\Lambda_-(\Gamma_1^{00}) = \Gamma_2^{00}$, $\Lambda_-(\Gamma_1^0) = \Gamma_2^0$. Furthermore, the subset of pairs (Γ_1, Γ_2) for which α is fixed is bounded above by the curve

$$\Gamma_2 = \Lambda_+(\alpha, \Gamma_1), \quad \Gamma_1^{00} < \Gamma_1 < \Gamma_1^0,$$

which is obtained as follows. For $\Gamma_1^{00} < \Gamma_1 < 0$ it is given by putting $a = 1$ in (4.27) and (4.28) and eliminating r. We have $\Lambda_+(\alpha, 0) = 0$ for all α, while $\Lambda_+(\alpha, \Gamma_1)$ for $0 < \Gamma_1 < \Gamma_1^0$ is obtained by eliminating a from (4.27) and (4.28) and letting $r \to \infty$ or, alternatively, by eliminating r from (4.27) and (4.28) and letting $a \to -\infty$. Either way we obtain

$$\Lambda_+(\alpha, \Gamma_1) = \frac{\Gamma_1^0\Gamma_2^0\Gamma_1}{\Gamma_1^0\Gamma_1 - (\alpha - 1)\left(\dfrac{x_2 - x_1}{1 - x_2}\right)\lambda\Gamma_1^0\Gamma_2^0\Gamma_1 - \left(\dfrac{1 - x_1}{1 - x_2}\right)\alpha(\Gamma_1^0 - \Gamma_1)\,\Gamma_2^0},$$

$$\Gamma_1^{00} < \Gamma_1 \leqslant 0, \tag{4.31}$$

$$\Lambda_+(\alpha, \Gamma_1) = \frac{\Gamma_1^0 \Gamma_2^0 \Gamma_1}{\Gamma_1^0 \Gamma_1 - \alpha(\Gamma_1^0 - \Gamma_1)\Gamma_2^0}, \quad 0 \leqslant \Gamma_1 < \Gamma_1^0. \tag{4.32}$$

Note that $\Lambda_+(\alpha, \Gamma_1^{00}) = \Gamma_2^{00}$, $\Lambda_+(\alpha, \Gamma_1^0) = \Gamma_2^0$ for each α. The only part of the boundary for which $\mathscr{P}_1(\Gamma_1, \Gamma_2; x_1, x_2; \rho(x))$ has functions with the fixed value of α chosen is the curve $\Gamma_2 = \Lambda_+(\alpha, \Gamma_1)$ for $\Gamma_1^{00} < \Gamma_1 < 0$. Moreover, for this part of the boundary the functions have non-positive scattering length. However, for the interior of the subset under consideration all functions have positive scattering length.

The subset of pairs (Γ_1, Γ_2) for which $\alpha = 1$ is just the image of $\{(1, r, 1) \mid 0 \leqslant r\}$ under the mapping (4.26). This is just the arc of the curve $\Gamma_2 = \Lambda_+(1, \Gamma_1)$ for $\Gamma_1^{00} \leqslant \Gamma_1 < 0$. For each (Γ_1, Γ_2) on this curve, $\mathscr{P}_1(\Gamma_1, \Gamma_2; x_1, x_2; \rho(x))$ consists of a single function, with $\chi(z) = r$ and thus with negative scattering length.

Now as α varies from 1 to $(1 - x_1)/(1 - x_2)$, we see from (4.30) that the lower boundary $\Gamma_2 = \Lambda_-(\Gamma_1)$, with $\Gamma_1^{00} < \Gamma_1 < \Gamma_1^0$, is fixed. From (4.31) and (4.32) however, the upper boundary changes. By inspection of (4.31) it is seen that $\Lambda_+(\alpha, \Gamma_1)$, for fixed Γ_1 in $(\Gamma_1^{00}, 0)$, strictly decreases with increasing α in $[1, (1 - x_1)/(1 - x_2)]$. Similarly, from (4.32) it follows that $\Lambda_+(\alpha, \Gamma_1)$, for fixed Γ_1 in $(0, \Gamma_1^0)$, strictly increases with increasing α in $[1, (1 - x_1)/(1 - x_2)]$. Further, with $\Lambda_+(\Gamma_1)$ defined by (4.19), we see that

$$\Lambda_+(1, \Gamma_1) = \Lambda_+(\Gamma_1), \quad \Gamma_1^{00} \leqslant \Gamma_1 \leqslant 0,$$

$$\Lambda_+((1 - x_1)/(1 - x_2), \Gamma_1) = \Lambda_+(\Gamma_1), \quad 0 \leqslant \Gamma_1 \leqslant \Gamma_1^0.$$

It is now clear that the set of pairs (Γ_1, Γ_2) for which $\mathscr{P}_1(\Gamma_1, \Gamma_2; x_1, x_2; \rho(x))$ contains at least one function with $p = 1$ is just

$$\{(\Gamma_1, \Gamma_2) \mid \Gamma_1^{00} < \Gamma_1 < \Gamma_1^0, \Lambda_-(\Gamma_1) < \Gamma_2 < \Lambda_+(\Gamma_1)\} \cup$$
$$\{(\Gamma_1, \Lambda_+(\Gamma_1)) \mid \Gamma_1^{00} \leqslant \Gamma_1 < 0\}. \tag{4.33}$$

To obtain the full set of pairs (Γ_1, Γ_2) for which $\mathscr{P}_1(\Gamma_1, \Gamma_2; x_1, x_2; \rho(x))$ is non-void we need to take the union of the sets (4.18) and (4.33). This union is just the set

$$\{(\Gamma_1, \Gamma_2) \mid \Gamma_1^{00} \leqslant \Gamma_1 < \Gamma_1^0, \quad \Lambda_-(\Gamma_1) < \Gamma_2 \leqslant \Lambda_+(\Gamma_1)\} - \{(0, 0)\}. \tag{4.34}$$

All the constants and functions appearing in (4.34) are defined in (4.13), (4.14), (4.29), (4.30) and (4.19). Further, it is clear from our earlier discussion that the subset of (4.34) for which $\mathscr{P}_1(\Gamma_1, \Gamma_2; x_1, x_2; \rho(x))$ contains functions with non-positive scattering length is

$$\{(\Gamma_1, \Gamma_2) \mid \Gamma_1^{00} \leqslant \Gamma_1 < 0, \quad \Lambda_+((1 - x_1)/(1 - x_2), \Gamma_1) \leqslant \Gamma_2 \leqslant \Lambda_+(\Gamma_1)\}, \tag{4.35}$$

with $\Lambda_+((1 - x_1)/(1 - x_2), \Gamma_1)$ given by (4.31). Note finally that the lower limit curve $\Gamma_2 = \Lambda_-(\Gamma_1)$ in (4.34) is the arc of a rectangular hyperbola which is concave upwards, has asymptotes parallel to the axes and has endpoints $(\Gamma_1^{00}, \Gamma_2^{00})$ and (Γ_1^0, Γ_2^0). The lower limit curve $\Gamma_2 = \Lambda_+((1 - x_1)/(1 - x_2), \Gamma_1)$ in (4.35) is also the arc of a rectangular hyperbola which is concave upwards, has asymptotes parallel to the axes and has endpoints $(\Gamma_1^{00}, \Gamma_2^{00})$ and $(0, 0)$.

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APPENDIX A

Lemma: Let $g:\mathbb{R}\to\mathbb{C}$ be continuous at $x = x_0$ and bounded on $\mathbb{R}(|g(x)| \leqslant M$ for all $x \in \mathbb{R})$. Then

$$\lim_{n\to\infty} \frac{y_n}{\pi} \int_{-\infty}^{\infty} \frac{g(t)}{(t-x_n)^2 + y_n^2}\, dt = g(x_0) \tag{A.1}$$

for every sequence $\{x_n + iy_n\}$ in the upper half-plane which converges to x_0.

Proof: Put $u = (t - x_n)/y_n$ and let

$$g_n(u) = g(x_n + y_n u), \quad u \in \mathbb{R}.$$

Then

$$\frac{y_n}{\pi} \int_{-\infty}^{\infty} \frac{g(t)}{(t-x_n)^2 + y_n^2}\, dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g_n(u)}{u^2+1}\, du.$$

But

$$\lim_{n\to\infty} g_n(u) = g(x_0), \quad u \in \mathbb{R},$$

since g is continuous at $x = x_0$, and

$$|g_n(u)| \leqslant M, \quad n \in \mathbb{N}, u \in \mathbb{R}.$$

The result follows from Lebesgue's dominated convergence theorem (see for example, Section 1.34 of [9]).

Corollary: Let $\delta: [1, \infty) \to \mathbb{R}$ be continuous at $x = x_0 (>1)$ and bounded on $[1, \infty)$. Then

$$\lim_{n\to\infty} \operatorname{Im}\left[\frac{z_n}{\pi}\int_1^\infty \frac{\delta(t)}{t(t-z_n)}\,dt\right] = \delta(x_0) \tag{A.2}$$

for every sequence $\{z_n\}$ in the upper half-plane which converges to x_0.

APPENDIX B

Lemma: Let

$$\Psi(z) = \frac{z}{\pi}\int_1^\infty \frac{\delta(t)}{t(t-z)}\,dt, \tag{B.1}$$

where $0 \leqslant \delta(x) \leqslant \pi$ and $\delta \in L([1, \infty))$. Then

$$\operatorname{Re} \Psi(x + re^{i\theta}) > -\pi - 2x + \ln(r \sin\theta) \tag{B.2}$$

when $x \in [1, \infty)$, $0 < r < 1$ and $0 < \theta < \pi$.

Proof: We have

$$\operatorname{Re} \Psi(x + re^{i\theta}) = \chi_1 + \chi_2, \tag{B.3}$$

where

$$\chi_1 = \frac{x + r\cos\theta}{\pi}\int_1^\infty \frac{\delta(t)(t - x - r\cos\theta)}{t|t - x - re^{i\theta}|^2}\,dt,$$

$$\chi_2 = -\frac{r^2\sin^2\theta}{\pi}\int_1^\infty \frac{\delta(t)}{t|t - x - re^{i\theta}|^2}\,dt.$$

We estimate χ_1 and χ_2 separately. First,

$$\chi_1 > 0 \quad \text{when} \quad x + r\cos\theta \leqslant 1. \tag{B.4}$$

When $x + r\cos\theta > 1$,

$$(x + r\cos\theta)/t \leqslant 1 + x + r\cos\theta - t, \quad 1 \leqslant t \leqslant x + r\cos\theta. \tag{B.5}$$

Thus, separating χ_1 into integrals over $[1, x + r\cos\theta]$ and $[x + r\cos\theta, \infty)$, we have, when $x + r\cos\theta > 1$,

$$\chi_1 \geqslant (x + r\cos\theta) \int_1^{x+r\cos\theta} \frac{t - x - r\cos\theta}{t|t - x - re^{i\theta}|^2} dt$$

$$\geqslant - \int_1^{x+r\cos\theta} \frac{(t - x - r\cos\theta)^2}{(t - x - r\cos\theta)^2 + r^2\sin^2\theta} dt$$

$$+ \int_1^{x+r\cos\theta} \frac{t - x - r\cos\theta}{(t - x - r\cos\theta)^2 + r^2\sin^2\theta} dt, \quad \text{using (B.5)}$$

$$> 1 - x - r\cos\theta + \tfrac{1}{2}\ln\frac{r^2\sin^2\theta}{(1 - x - r\cos\theta)^2 + r^2\sin^2\theta}$$

$$> -x + \ln(r\sin\theta) - \ln[x^2 - 2(x-1)(1 - r\cos\theta)]^{\frac{1}{2}}$$

$$> -2x + \ln(r\sin\theta).$$

Taken together with (B.4), this shows that

$$\chi_1 > -2x + \ln(r\sin\theta), \quad x \geqslant 1, 0 < r < 1, 0 < \theta < \pi. \tag{B.6}$$

For χ_2 we have

$$\chi_2 > -r^2\sin^2\theta \int_1^\infty \frac{1}{(t - x - r\cos\theta)^2 + r^2\sin^2\theta} dt > -\pi r\sin\theta > -\pi. \tag{B.7}$$

The inequality (B.2) follows from (B.3), (B.6) and (B.7).

APPENDIX C

We state in this appendix some properties of Green's functions and Poisson kernels. We shall use the letter z (resp. z') in the argument of a Green's function to denote the ordered pair (x,y) (resp. (x',y')) of real arguments. Denote by $G_b(z',z)$ the Green's function for the region $\{|z - z_0| < b\}$, by $G_{(a)}(z',z)$ the Green's function for the region $\{|z - z_0| > a\}$ and by $G_A(z',z)$ the Green's function for the region $A = \{a < |z - z_0| < b\}$, where $z_0 \in \mathbb{C}$ and $0 < a < b$. The derivatives of the Green's functions with respect to z', evaluated on the boundaries of the respective regions in the direction of the *inner* normal, we denote by $G'_b(z',z)$, $G'_{(a)}(z',z)$ and $G'_A(z',z)$.

We now show that

$$G'_b(z',z) - G'_A(z',z) \geqslant 0 \quad \text{when} \quad |z' - z_0| = b,\ z \in A, \tag{C.1}$$

$$G'_{(a)}(z',z) - G'_A(z',z) \geqslant 0 \quad \text{when} \quad |z' - z_0| = a,\ z \in A. \tag{C.2}$$

To prove (C.1) we observe that $G_b(z',z) - G_A(z',z)$ is harmonic in z' in A and continuous in z' on the closure of A, that $G_b(z',z) - G_A(z',z) = 0$ when $|z' - z_0| = b, z \in A$ and that

$G_b(z',z) - G_A(z',z) = G_b(z',z) > 0$ when $|z' - z_0| = a, z \in A$. Thus

$$G_b(z',z) - G_A(z',z) > 0 \quad \text{when } z \in A,\ z' \in A$$

and (C.1) follows. The inequality (C.2) is proved in a similar way.

Let P_b (resp. $P_{(a)}$) be the Poisson kernel for the interior (resp. exterior) of a circle with centre z_0 and radius b (resp. a). Then

$$P_b(z',z) = 2\pi b G_b'(z',z), \quad |z' - z_0| = b$$

and

$$P_{(a)}(z',z) = 2\pi a G_{(a)}'(z',z), \quad |z' - z_0| = a. \tag{C.3}$$

From the corresponding relations for the Poisson kernels we have

$$\int_{|z'-z_0|=b} G_b'(z',z)\, ds' = 1, \quad |z - z_0| < b, \tag{C.4}$$

$$\int_{|z'-z_0|=a} G_{(a)}'(z',z)\, ds' = 1, \quad |z - z_0| > a. \tag{C.5}$$

From (C.1) and (C.4)

$$\int_{|z'-z_0|=b} G_A'(z',z)\, ds' \leqslant 1, \quad z \in A, \tag{C.6}$$

and from (C.2) and (C.5),

$$\int_{|z'-z_0|=a} G_A'(z',z)\, ds' \leqslant 1, \quad z \in A. \tag{C.7}$$

Note also that, when $z \in A$ and $|z' - z_0| = a$ or $|z' - z_0| = b$,

$$G_A'(z',z) > 0. \tag{C.8}$$

The explicit form of $P_{(a)}$ is needed in Section 2; it is

$$P_{(a)}(z_0 + ae^{i\alpha}, z_0 + |z - z_0|e^{i\beta}) = \frac{|z - z_0|^2 - a^2}{|z - z_0|^2 - 2|z - z_0|a\cos(\alpha - \beta) + a^2},$$

$$|z - z_0| > a. \tag{C.9}$$

APPENDIX D

We show here that property (6) of Section 1 holds for the function f of (3.11), with $k = 0$. For this we need the definition of $f(x)$ for $x > 1$. With $D(x)$ given by (3.17), this is simply

$$f(x) = -\frac{P(x)(x-1)^{l-1}}{Q(x)\,D(x)} \quad \text{when} \quad x > 1,\ D(x) \neq 0 \text{ and } x \notin \{\xi_j\},$$

$f(x) = 0$ when $x = \xi_j(>1)$ and when $D(x) = 0$. The method of proof follows that of Appendix I of [1]. With $D(z)$ defined by (3.16) and

$$g(x) = \frac{q(x)(x-1)^{l-1} P(x)}{Q(x)}, \quad x > 1,$$

we have, for $x > 1, y > 0$,

$$|D(z)| \geqslant \left| \operatorname{Im} \chi(z) + \frac{y}{\pi} \int_1^\infty \frac{\rho(t)(t-1)^{l-1} P(t)}{Q(t)[(t-x)^2 + y^2]} dt \right|$$

$$\geqslant \frac{y}{\pi} \int_1^\infty \frac{g(t)}{(t-x)^2 + y^2} dt \geqslant \frac{1}{\pi} \int_0^\infty \frac{g(x+yu)}{1+u^2} du.$$

This inequality also applies for $y = 0$.

Now it is easy to verify by direct differentiation that there is a number $X > 1$ such that $g(x)$ is monotone decreasing on $[X, \infty)$. Then, with $0 < \beta < \pi/2$, we have

$$|D(z)| \geqslant \frac{1}{\pi} \int_0^\infty \frac{g(x + x \tan \beta u)}{1+u^2} du, \quad x \geqslant X, 0 \leqslant y \leqslant x \tan \beta.$$

Since

$$g(x) \underset{x\to\infty}{\sim} c/x^{N-p-l+\frac{1}{2}}, \quad c > 0,$$

and $N - p - l \geqslant K \geqslant 0$, we can find a constant $c_1 > 0$ such that, for $0 \leqslant \operatorname{Arg} z \leqslant \beta$ and for all sufficiently large $|z|$,

$$|z|^{N-p-l+\frac{1}{2}} |D(z)| \geqslant c_1.$$

For the continuum $\{\operatorname{Re} z \geqslant 1\} \cap \{\operatorname{Arg} z \geqslant \beta\}$, the argument of Appendix I of [1] shows that, when $N = (p + l)$, we can find $c_2 > 0$ such that $|z|^{\frac{1}{2}} |D(z)| \geqslant c_2$ for all sufficiently large $|z|$. If, however, $N > p + l$, it is easy enough to show that the inequality becomes $|z|\ |D(z)| \geqslant c_3(>0)$.

For the continuum $\{\operatorname{Re} z \leqslant 1, \operatorname{Im} z \geqslant 0\}$ we have $D(z) \to \gamma$ as $|z| \to \infty$, uniformly in $\operatorname{Arg} z$, when $\gamma > 0$. When $\gamma = 0$, on the other hand, $|z|^{\frac{1}{2}} |D(z)| \geqslant c_4(>0)$ if $N = p + l$ and $|z| |D(z)| \geqslant c_5(>0)$ if $N > p + l$, in each case for all sufficiently large $|z|$ and uniformly in $\operatorname{Arg} z$. All these results are straightforward to prove.

Our conclusion is that in every case we can find $c > 0$ such that

$$|z|^{N-p-l+\frac{1}{2}} |D(z)| \geqslant c$$

for all sufficiently large z, uniformly in $0 \leqslant \operatorname{Arg} z \leqslant \pi$. From (3.11) it follows that

$$|z|^{\frac{1}{2}} |f(z)| \leqslant c',$$

again for all sufficiently large z, uniformly in $0 \leqslant \operatorname{Arg} z \leqslant \pi$. Thus property (6) holds with $k = 0$.

APPENDIX E

We prove in this appendix that, for given pole positions $x_1, \ldots, x_N$ and given function $R(x)$, the set of N-tuples $(\Gamma_1, \ldots, \Gamma_N)$ of pole residues for which the class $\mathscr{P}_l(\Gamma_i; x_i; \rho(x))$ is non-void is a bounded set in $\mathbb{R}^N$.

To prove this, it is convenient to map the complex plane cut along $[1, \infty)$ into the interior of the unit circle by the usual transformation

$$w(z) = \frac{1 + i(z-1)^{\frac{1}{2}}}{1 - i(z-1)^{\frac{1}{2}}},$$

where

$$(z-1)^{\frac{1}{2}} = |z-1|^{\frac{1}{2}} e^{i\theta/2},$$

θ being the argument of $(z-1)$ which lies in the interval $[0, 2\pi]$. The upper (resp. lower) side of the cut $[1, \infty)$ is mapped into the upper (resp. lower) half of the unit circle, with $1 \mapsto 1$, $\infty \mapsto -1$.

We continue to use the same symbols for functions with transformed arguments. Thus $f(w)$ is analytic in $\{|w| < 1\}$, except for poles at $w(x_i)$ $(i = 1, \ldots, N)$, and is continuous onto the unit circle from the interior. This is true even at $w = -1$, if we define $f(-1) = 0$, by property (6) of Section 1 with $k = 0$. Further,

$$|f(e^{i\theta})| \leqslant 1/q(\theta), \quad -\pi < \theta < \pi,$$

where $q(\theta)$ is given in terms of $q(x)$ by the transformation

$$x = 2/(1 + \cos\theta).$$

Now define the function $g(w)$ by

$$g(w) = B(w) f(w)$$

where $B(w)$ is the Blaschke product

$$B(w) = \prod_{i=1}^{N} \epsilon(w(x_i)) \left(\frac{w(x_i) - w}{1 - w(x_i)\, w} \right),$$

with $\epsilon(0) = -1$ and $\epsilon(a) = \bar{a}/|a|$, $0 < |a| < 1$. Then

$$|g(e^{i\theta})| = |f(e^{i\theta})|$$

and g is analytic in $\{|w| < 1\}$ and continuous on $\{|w| \leqslant 1\}$.

Next define the function

$$K(w) = \exp\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\phi} + w}{e^{i\phi} - w} \ln \frac{1}{q(\phi)}\, d\phi \right].$$

Then $K(w)$ is analytic in $\{|w| < 1\}$ and does not vanish there. From the theory of the Poisson kernel, $|K(w_n)| \to 1/q(\theta)$ for any sequence $\{w_n\}$ which converges to $e^{i\theta}$ from inside the unit circle. This result is fully covered by Theorem 59 of [16], even the awkward points $\theta = 0, \pi$. Thus g/K is analytic in $\{|w| < 1\}$ and $|g(w_n)/K(w_n)|$ approaches a limit $k(\theta)$ for any sequence $\{w_n\}$ which converges to $e^{i\theta}$ from the interior of the unit circle; moreover,

$$k(\theta) \leqslant 1.$$

Thus, from Theorem 101 of [16],

$$|g(w)| < |K(w)|, \quad |w| < 1.$$

Now denote by B_i the Blaschke product B *without* the ith factor. Then

$$g(w(x_i)) = \frac{-\epsilon(w(x_i))\, B_i(w(x_i))(1 + w(x_i))^2}{(1 - w(x_i))^2} \Gamma_i$$

and $|g(w(x_i))| < |K(w(x_i))|$ thus gives an upper bound on $|\Gamma_i|$, which proves the result.