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Comment on a Paper of Amrein, Martin, and Misra

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Abstract. We point out a gap in the proof of a proposition in a paper of Amrein, Martin and Misra and give a proof of their result.

The paper by Amrein, Martin and Misra [1] is an interesting and important contribution to the literature of time-dependent scattering theory. In this note we point out a gap in the proof of their Proposition 1 and furnish a proof of this result.

The scattering theory of AMM is based upon three conditions which they call (A1), (A2), and (A3). Using their notation we define the unitary groups $V_t = \exp(-iHt)$ and $U_t = \exp(-iH_0t)$ ($-\infty < t < \infty$) which respectively describe the total evolution and free evolution of the scattering system, and the von Neumann algebra \mathcal{A}_0 consisting of all bounded linear operators on the Hilbert space $\mathcal{H} = L^2(R^3)$ that commute with all spectral projections of the positive self-adjoint operator H_0 . We will be concerned with the following two conditions of AMM:

(A1) There exists a projection operator P on \mathcal{H} such that

a) $[P, V_t] = 0, t \in (-\infty, \infty),$

b) for every operator $A \in \mathcal{A}_0$ there exist two operators A_{\pm} such that

$$s\text{-}\lim_{t \rightarrow \pm\infty} V_t^* A V_t P = A_{\pm} = \mu_{\pm}(A), \quad (1)$$

c) $[P, A_{\pm}] = 0.$

(A3) For every vector $f \in \mathcal{H}$ there exists a vector $g \in P\mathcal{H}$ such that for all $A \in \mathcal{A}_0$

$$\lim_{t \rightarrow -\infty} (g, V_t^* A V_t g) = (f, Af).$$

In their Proposition 1 AMM want to prove that the images $\mu_{\pm}(\mathcal{A}_0)$ of the mappings defined by (1) are von Neumann algebras by assuming only their first condition (A1). After establishing that μ_{\pm} are *-homomorphisms they quote a theorem of Feldman and Fell [2] thereby arriving at the conclusion that μ_{\pm} are ultraweakly continuous on \mathcal{A}_0 . However, the theorem of Feldman and Fell does not apply in this case, because, as one easily proves, \mathcal{A}_0 is not properly infinite. This result was also discovered by AMM subsequent to the publication of their paper [3].

It is possible to prove that $\mu_-(\mathcal{A}_0)$ is a von Neumann algebra by making use of (A3) as well as (A1). W. O. Amrein has informed the author that such a proof has been discussed in detail by Mourre [4]. However, in order to use this method to prove that $\mu_+(\mathcal{A}_0)$ is a von Neumann algebra it is necessary to require the validity of (A3) also in the limit $t \rightarrow +\infty$. Call this new condition (A3)₊. Alternatively, (A3)₊ can be derived by assuming the validity of (A1), (A2) and (A3) and that the scattering system is time reversal invariant [1]. With the additional condition (A3)₊ the scattering operator S is necessarily unitary when it exists, as is noted in [1]. Thus, this method of proof seems to be too restrictive (at least to the author).

In view of the above remarks it would appear to be useful to have a proof of the proposition in question assuming only the validity of (A1). We will give such a proof.

Proposition. Assume the validity of (A1). Then μ_{\pm} are ultraweakly continuous on \mathcal{A}_0 .

Once this much has been proved, the results stated by AMM as their Proposition 1 can be obtained by, for example, following the procedure in the latter part of their proof. In the proof given below we will use freely the result of AMM that μ_{\pm} are *-homomorphisms on \mathcal{A}_0 without proving it again. We note that Lavine [5] proved that μ_{\pm} are *-isomorphisms on a certain operator algebra under more restrictive assumptions. In the present situation μ_{\pm} are not necessarily isomorphisms as in the cases discussed by Mourre [4] and Lavine [5].

Proof. We write \mathcal{A}_0 as the direct sum of finite and properly infinite von Neumann algebras $(\mathcal{A}_0)_{\mathcal{G}}$ and $(\mathcal{A}_0)_{I-\mathcal{G}}$ ([6], Proposition 8, p. 98), and consider the finite summand $(\mathcal{A}_0)_{\mathcal{G}}$ by assuming that \mathcal{A}_0 is finite.

Let A denote an arbitrary non-zero element of \mathcal{A}'_0 . Then A is a bounded normal operator and consequently has a polar decomposition

$$A = U|A|, \quad |A| \in \mathcal{A}'_0, \tag{2}$$

where U is unitary and $|A|$ is bounded, positive, and self-adjoint. It follows that $A \in \text{Ker } \mu_{\pm}$ if and only if $|A| \in \text{Ker } \mu_{\pm}$.

Since $|A|$ is bounded and self-adjoint, and the spectrum of H_0 is the non-negative real axis [1], we can write,

$$(\Psi, |A|\Psi) = \int_0^{\infty} f(\lambda) d(\Psi, E_0(\lambda)\Psi), \quad \Psi \in \mathcal{H}, \quad f \in \mathcal{B}, \tag{3}$$

where E_0 denotes the resolution of the identity of H_0 and \mathcal{B} the class of all real-valued bounded Borel measurable functions on $[0, \infty)$. It follows that

$$F(\delta) = E_0(f^{-1}(\delta)) \tag{4}$$

for every Borel set δ of the spectrum of $|A|$, where F denotes the resolution of the identity of $|A|$ ([7], Corollary X.2.10).

Now suppose $A \in \text{Ker } \mu_{\pm}$. From (1) and (2) one finds

$$\lim_{t \rightarrow \pm\infty} \|V_t^* |A| V_t g\| = 0, \quad \text{all } g \in P\mathcal{H}. \tag{5}$$

Since $V_t^* F V_t$ is the resolution of the identity of $V_t^* |A| V_t$, we find, using (4), (5) and ([7], Corollary X.7.3), that

$$\lim_{t \rightarrow \pm\infty} \|V_t^* E_0(f^{-1}(\delta)) V_t g\| = 0 \tag{6}$$

for all $g \in P\mathcal{H}$ and all Borel sets δ of the spectrum of $|A|$.

Take f to be the characteristic function of a Borel set Δ of $[0, \infty)$, $f = \chi_{\Delta}$. Then $f \in \mathcal{B}$ and we find from (3), $|A| = E_0(\Delta)$, which is non-zero by assumption. The set $\delta = \{0, 1\}$ is a Borel set of the spectrum of this operator so that

$$E_0(f^{-1}(\delta)) = E_0([0, \infty)) = I = \text{identity operator}$$

and we obtain a contradiction with (6). Consequently,

$$E_0(\Delta) \notin \text{Ker } \mu_{\pm} \tag{7}$$

for all Borel sets Δ of $[0, \infty)$ such that $E_0(\Delta) \neq 0$.

Let f_n denote a simple function,

$$f_n = \sum_{i=1}^n \alpha_i \chi_{\Delta_i}, \quad n < \infty, \tag{8}$$

where the α_i are non-zero real numbers and $\{\Delta_i\}$ is a sequence of disjoint Borel sets of $[0, \infty)$. Then $f_n \in \mathcal{B}$ and the corresponding operator is obtained from (3),

$$B_n = \sum_{i=1}^n \alpha_i E_0(\Delta_i).$$

Such operators will be called simple. Since μ_{\pm} are linear we find

$$\mu_{\pm}(B_n) = \sum_{i=1}^n \alpha_i \mu_{\pm}(E_0(\Delta_i)). \tag{9}$$

From the disjointness of the sequence $\{\Delta_i\}$ and the standard properties of a spectral measure one finds that $\{E_0(\Delta_i)\}$ is a sequence of pairwise orthogonal projections. The homomorphisms μ_{\pm} preserve this property so that $\{\mu_{\pm}(E_0(\Delta_i))\}$ are also sequences of this type. If $B_n \neq 0$ then there is at least one Borel set Δ_k occurring in the sequence $\{\Delta_i\}$ of (8) such that $E_0(\Delta_k) \neq 0$ so that (7) obtains. Take $\Psi_{\pm} \neq 0$ in the range of $\mu_{\pm}(E_0(\Delta_k)) \neq 0$ so that (9) yields

$$\mu_{\pm}(B_n) \Psi_{\pm} = \alpha_k \Psi_{\pm} \neq 0,$$

and thus $\mu_{\pm}(B_n) \neq 0$ or

$$B_n \notin \text{Ker } \mu_{\pm} \tag{10}$$

for all non-zero simple operators.

It follows from (3) that for each positive operator $|A| \neq 0$ there exists a Borel set Δ_0 of $[0, \infty)$ such that f assumes only positive values on Δ_0 and $E_0(\Delta_0) \neq 0$. We then find that, since $f\chi_{\Delta_0}$ belongs to \mathcal{B} and is non-negative, there exists an increasing sequence of non-negative simple functions $\{f_n\}$ which converges pointwise to $f\chi_{\Delta_0}$ [8]. Because $f\chi_{\Delta_0}$ majorizes f_n for each n we find that $f_n \in \mathcal{B}$ for all n and that the simple operators B_n corresponding to f_n by (3) are positive and bounded. Moreover, each f_n vanishes outside the Borel set Δ_0 and for a sufficiently large value of n (n_0 say) $f_{n_0} > 0$ on Δ_0 . It follows that $B_{n_0} \neq 0$ so that we obtain from (10),

$$\mu_{\pm}(B_{n_0}) > 0 \tag{11}$$

because μ_{\pm} preserve positivity. Since $f\chi_{\Delta_0} - f_{n_0} \in \mathcal{B}$ is non-negative we find that the operator $|A|E_0(\Delta_0) - B_{n_0}$ to which this function corresponds by (3) is positive. We

now again use the fact that μ_{\pm} preserve positivity coupled with (11) to show that

$$|A|E_0(\Delta_0) \notin \text{Ker } \mu_{\pm} \quad \text{for } |A| \neq 0,$$

from which $|A| \notin \text{Ker } \mu_{\pm}$ immediately follows.

Thus, we have shown that $\text{Ker } \mu_{\pm} \cap \mathcal{A}'_0 = \{0\}$. Then, since \mathcal{A}_0 has been assumed finite, one finds $\text{Ker } \mu_{\pm} = \{0\}$ ([6], Corollaire 1 of Proposition 2, p. 256). Hence, μ_{\pm} are injective on the finite summand $(\mathcal{A}_0)_G$. It follows that the restrictions of μ_{\pm} to $(\mathcal{A}_0)_G$ are direct summand representations in the sense of Fell [9] and are consequently ultraweakly continuous because $(\mathcal{A}_0)_G$ is of type I [2, 9].

We can now follow AMM and invoke the theorem of Feldman and Fell [2] to deduce that the restrictions of μ_{\pm} to the properly infinite summand $(\mathcal{A}_0)_{I-G}$ are ultraweakly continuous. Finally, we use the linearity of μ_{\pm} , the ultraweak continuity of their restrictions to the two summands, and the characterization of ultraweakly continuous homomorphisms in [2] to prove that μ_{\pm} are ultraweakly continuous on \mathcal{A}_0 .

W. O. Amrein has recently informed the author that the result of this note has also been proved by V. Georgescu (unpublished) by a different method.

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