

**Zeitschrift:** Helvetica Physica Acta  
**Band:** 46 (1973)  
**Heft:** 3  
  
**Artikel:** Comment on a paper of Amrein, Martin, and Misra  
**Autor:** Zachary, W.W.  
**DOI:** <https://doi.org/10.5169/seals-114487>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 07.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## Comment on a Paper of Amrein, Martin, and Misra

by W. W. Zachary

Naval Research Laboratory, Washington, D.C., USA

(12. II. 73)

*Abstract.* We point out a gap in the proof of a proposition in a paper of Amrein, Martin and Misra and give a proof of their result.

The paper by Amrein, Martin and Misra [1] is an interesting and important contribution to the literature of time-dependent scattering theory. In this note we point out a gap in the proof of their Proposition 1 and furnish a proof of this result.

The scattering theory of AMM is based upon three conditions which they call (A1), (A2), and (A3). Using their notation we define the unitary groups  $V_t = \exp(-iHt)$  and  $U_t = \exp(-iH_0t)$  ( $-\infty < t < \infty$ ) which respectively describe the total evolution and free evolution of the scattering system, and the von Neumann algebra  $\mathcal{A}_0$  consisting of all bounded linear operators on the Hilbert space  $\mathcal{H} = L^2(R^3)$  that commute with all spectral projections of the positive self-adjoint operator  $H_0$ . We will be concerned with the following two conditions of AMM:

(A1) There exists a projection operator  $P$  on  $\mathcal{H}$  such that

a)  $[P, V_t] = 0, t \in (-\infty, \infty),$

b) for every operator  $A \in \mathcal{A}_0$  there exist two operators  $A_{\pm}$  such that

$$s\text{-}\lim_{t \rightarrow \pm\infty} V_t^* A V_t P = A_{\pm} = \mu_{\pm}(A), \quad (1)$$

c)  $[P, A_{\pm}] = 0.$

(A3) For every vector  $f \in \mathcal{H}$  there exists a vector  $g \in P\mathcal{H}$  such that for all  $A \in \mathcal{A}_0$

$$\lim_{t \rightarrow -\infty} (g, V_t^* A V_t g) = (f, Af).$$

In their Proposition 1 AMM want to prove that the images  $\mu_{\pm}(\mathcal{A}_0)$  of the mappings defined by (1) are von Neumann algebras by assuming only their first condition (A1). After establishing that  $\mu_{\pm}$  are  $*$ -homomorphisms they quote a theorem of Feldman and Fell [2] thereby arriving at the conclusion that  $\mu_{\pm}$  are ultraweakly continuous on  $\mathcal{A}_0$ . However, the theorem of Feldman and Fell does not apply in this case, because, as one easily proves,  $\mathcal{A}_0$  is not properly infinite. This result was also discovered by AMM subsequent to the publication of their paper [3].

It is possible to prove that  $\mu_-(\mathcal{A}_0)$  is a von Neumann algebra by making use of (A3) as well as (A1). W. O. Amrein has informed the author that such a proof has been discussed in detail by Mourre [4]. However, in order to use this method to prove that  $\mu_+(\mathcal{A}_0)$  is a von Neumann algebra it is necessary to require the validity of (A3) also in the limit  $t \rightarrow +\infty$ . Call this new condition (A3)<sub>+</sub>. Alternatively, (A3)<sub>+</sub> can be derived by assuming the validity of (A1), (A2) and (A3) and that the scattering system is time reversal invariant [1]. With the additional condition (A3)<sub>+</sub> the scattering operator  $S$  is necessarily unitary when it exists, as is noted in [1]. Thus, this method of proof seems to be too restrictive (at least to the author).

In view of the above remarks it would appear to be useful to have a proof of the proposition in question assuming only the validity of (A1). We will give such a proof.

*Proposition.* Assume the validity of (A1). Then  $\mu_{\pm}$  are ultraweakly continuous on  $\mathcal{A}_0$ .

Once this much has been proved, the results stated by AMM as their Proposition 1 can be obtained by, for example, following the procedure in the latter part of their proof. In the proof given below we will use freely the result of AMM that  $\mu_{\pm}$  are \*-homomorphisms on  $\mathcal{A}_0$  without proving it again. We note that Lavine [5] proved that  $\mu_{\pm}$  are \*-isomorphisms on a certain operator algebra under more restrictive assumptions. In the present situation  $\mu_{\pm}$  are not necessarily isomorphisms as in the cases discussed by Mourre [4] and Lavine [5].

*Proof.* We write  $\mathcal{A}_0$  as the direct sum of finite and properly infinite von Neumann algebras  $(\mathcal{A}_0)_G$  and  $(\mathcal{A}_0)_{I-G}$  ([6], Proposition 8, p. 98), and consider the finite summand  $(\mathcal{A}_0)_G$  by assuming that  $\mathcal{A}_0$  is finite.

Let  $A$  denote an arbitrary non-zero element of  $\mathcal{A}'_0$ . Then  $A$  is a bounded normal operator and consequently has a polar decomposition

$$A = U|A|, \quad |A| \in \mathcal{A}'_0, \quad (2)$$

where  $U$  is unitary and  $|A|$  is bounded, positive, and self-adjoint. It follows that  $A \in \text{Ker } \mu_{\pm}$  if and only if  $|A| \in \text{Ker } \mu_{\pm}$ .

Since  $|A|$  is bounded and self-adjoint, and the spectrum of  $H_0$  is the non-negative real axis [1], we can write,

$$(\Psi, |A|\Psi) = \int_0^{\infty} f(\lambda) d(\Psi, E_0(\lambda)\Psi), \quad \Psi \in \mathcal{H}, \quad f \in \mathcal{B}, \quad (3)$$

where  $E_0$  denotes the resolution of the identity of  $H_0$  and  $\mathcal{B}$  the class of all real-valued bounded Borel measurable functions on  $[0, \infty)$ . It follows that

$$F(\delta) = E_0(f^{-1}(\delta)) \quad (4)$$

for every Borel set  $\delta$  of the spectrum of  $|A|$ , where  $F$  denotes the resolution of the identity of  $|A|$  ([7], Corollary X.2.10).

Now suppose  $A \in \text{Ker } \mu_{\pm}$ . From (1) and (2) one finds

$$\lim_{t \rightarrow \pm\infty} \|V_t^* |A| V_t g\| = 0, \quad \text{all } g \in P\mathcal{H}. \quad (5)$$

Since  $V_t^* F V_t$  is the resolution of the identity of  $V_t^* |A| V_t$ , we find, using (4), (5) and ([7], Corollary X.7.3), that

$$\lim_{t \rightarrow \pm\infty} \|V_t^* E_0(f^{-1}(\delta)) V_t g\| = 0 \quad (6)$$

for all  $g \in P\mathcal{H}$  and all Borel sets  $\delta$  of the spectrum of  $|A|$ .

Take  $f$  to be the characteristic function of a Borel set  $\Delta$  of  $[0, \infty)$ ,  $f = \chi_\Delta$ . Then  $f \in \mathcal{B}$  and we find from (3),  $|A| = E_0(\Delta)$ , which is non-zero by assumption. The set  $\delta = \{0, 1\}$  is a Borel set of the spectrum of this operator so that

$$E_0(f^{-1}(\delta)) = E_0([0, \infty)) = I = \text{identity operator}$$

and we obtain a contradiction with (6). Consequently,

$$E_0(\Delta) \notin \text{Ker } \mu_\pm \quad (7)$$

for all Borel sets  $\Delta$  of  $[0, \infty)$  such that  $E_0(\Delta) \neq 0$ .

Let  $f_n$  denote a simple function,

$$f_n = \sum_{i=1}^n \alpha_i \chi_{\Delta_i}, \quad n < \infty, \quad (8)$$

where the  $\alpha_i$  are non-zero real numbers and  $\{\Delta_i\}$  is a sequence of disjoint Borel sets of  $[0, \infty)$ . Then  $f_n \in \mathcal{B}$  and the corresponding operator is obtained from (3),

$$B_n = \sum_{i=1}^n \alpha_i E_0(\Delta_i).$$

Such operators will be called simple. Since  $\mu_\pm$  are linear we find

$$\mu_\pm(B_n) = \sum_{i=1}^n \alpha_i \mu_\pm(E_0(\Delta_i)). \quad (9)$$

From the disjointness of the sequence  $\{\Delta_i\}$  and the standard properties of a spectral measure one finds that  $\{E_0(\Delta_i)\}$  is a sequence of pairwise orthogonal projections. The homomorphisms  $\mu_\pm$  preserve this property so that  $\{\mu_\pm(E_0(\Delta_i))\}$  are also sequences of this type. If  $B_n \neq 0$  then there is at least one Borel set  $\Delta_k$  occurring in the sequence  $\{\Delta_i\}$  of (8) such that  $E_0(\Delta_k) \neq 0$  so that (7) obtains. Take  $\Psi_\pm \neq 0$  in the range of  $\mu_\pm(E_0(\Delta_k)) \neq 0$  so that (9) yields

$$\mu_\pm(B_n) \Psi_\pm = \alpha_k \Psi_\pm \neq 0,$$

and thus  $\mu_\pm(B_n) \neq 0$  or

$$B_n \notin \text{Ker } \mu_\pm \quad (10)$$

for all non-zero simple operators.

It follows from (3) that for each positive operator  $|A| \neq 0$  there exists a Borel set  $\Delta_0$  of  $[0, \infty)$  such that  $f$  assumes only positive values on  $\Delta_0$  and  $E_0(\Delta_0) \neq 0$ . We then find that, since  $f\chi_{\Delta_0}$  belongs to  $\mathcal{B}$  and is non-negative, there exists an increasing sequence of non-negative simple functions  $\{f_n\}$  which converges pointwise to  $f\chi_{\Delta_0}$  [8]. Because  $f\chi_{\Delta_0}$  majorizes  $f_n$  for each  $n$  we find that  $f_n \in \mathcal{B}$  for all  $n$  and that the simple operators  $B_n$  corresponding to  $f_n$  by (3) are positive and bounded. Moreover, each  $f_n$  vanishes outside the Borel set  $\Delta_0$  and for a sufficiently large value of  $n$  ( $n_0$  say)  $f_{n_0} > 0$  on  $\Delta_0$ . It follows that  $B_{n_0} \neq 0$  so that we obtain from (10),

$$\mu_\pm(B_{n_0}) > 0 \quad (11)$$

because  $\mu_\pm$  preserve positivity. Since  $f\chi_{\Delta_0} - f_{n_0} \in \mathcal{B}$  is non-negative we find that the operator  $|A|E_0(\Delta_0) - B_{n_0}$  to which this function corresponds by (3) is positive. We

now again use the fact that  $\mu_{\pm}$  preserve positivity coupled with (11) to show that

$$|A|E_0(\Delta_0) \notin \text{Ker } \mu_{\pm} \quad \text{for } |A| \neq 0,$$

from which  $|A| \notin \text{Ker } \mu_{\pm}$  immediately follows.

Thus, we have shown that  $\text{Ker } \mu_{\pm} \cap \mathcal{A}'_0 = \{0\}$ . Then, since  $\mathcal{A}_0$  has been assumed finite, one finds  $\text{Ker } \mu_{\pm} = \{0\}$  ([6], Corollaire 1 of Proposition 2, p. 256). Hence,  $\mu_{\pm}$  are injective on the finite summand  $(\mathcal{A}_0)_G$ . It follows that the restrictions of  $\mu_{\pm}$  to  $(\mathcal{A}_0)_G$  are direct summand representations in the sense of Fell [9] and are consequently ultraweakly continuous because  $(\mathcal{A}_0)_G$  is of type I [2, 9].

We can now follow AMM and invoke the theorem of Feldman and Fell [2] to deduce that the restrictions of  $\mu_{\pm}$  to the properly infinite summand  $(\mathcal{A}_0)_{I-G}$  are ultraweakly continuous. Finally, we use the linearity of  $\mu_{\pm}$ , the ultraweak continuity of their restrictions to the two summands, and the characterization of ultraweakly continuous homomorphisms in [2] to prove that  $\mu_{\pm}$  are ultraweakly continuous on  $\mathcal{A}_0$ .

W. O. Amrein has recently informed the author that the result of this note has also been proved by V. Georgescu (unpublished) by a different method.

### Acknowledgment

The author wishes to thank Dr. W. O. Amrein for illuminating correspondence concerning his results subsequent to the publication of [1] and especially for information regarding the paper [4].

### REFERENCES

- [1] W. O. AMREIN, PH. A. MARTIN and B. MISRA, *Helv. Phys. Acta* **43**, 313 (1970). This paper will hereafter be referred to as AMM.
- [2] J. FELDMAN and J. M. G. FELL, *Ann. Math.* **65**, 241 (1957), Theorem 1. See also, H. TAKEMOTO, *Tôhoku Math. J.* **21**, 152 (1969); **22**, 210 (1970).
- [3] W. O. AMREIN, private communication.
- [4] E. MOURRE, *Quelques résultats dans la théorie algébrique de la diffusion*, preprint (Centre de Physique Théorique, C.N.R.S., Marseille 1971).
- [5] R. B. LAVINE, *J. Func. Anal.* **5**, 368 (1970).
- [6] J. DIXMIER, *Les Algèbres d'opérateurs dans l'espace Hilbertien (algèbres de von Neumann)*, 2nd edition (Gauthier-Villars, Paris 1969).
- [7] N. DUNFORD and J. T. SCHWARTZ, *Linear Operators Part II: Spectral Theory* (Interscience, New York-London 1963).
- [8] P. R. HALMOS, *Measure Theory* (Van Nostrand, Princeton, N.J. 1950), Theorem 20B, p. 85.
- [9] J. M. G. FELL, *Math. Ann.* **133**, 118 (1957).