Zeitschrift: Helvetica Physica Acta
Band: 46 (1973)
Heft: 3

Artikel: Relativistic dynamics
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DOI: https://doi.org/10.5169/seals-114486

Helvetica Physica Acta
Vol. 46, 1973.
Birkhäuser Verlag Basel

Relativistic Dynamics

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Abstract. A canonical formalism for the relativistic classical mechanics of many particles is constructed. The correct equations for the motion of a charged particle in an electromagnetic field are obtained in this formalism, and the relativistic two-body problem with an invariant interaction is solved by showing that there is a special frame in which the equations of motion are essentially the same as those for the non-relativistic theory. The classical canonical formalism is then used as a basis for the construction, by means of the correspondence principle, of a consistent relativistic quantum theory. A simple interpretation is provided for the Newton–Wigner position operator by showing that it is just the observable $\vec{q} - \frac{1}{2}[(\vec{p}/E)t + t(\vec{p}/E)]$ in a representation, called the mass representation, which diagonalizes the momentum and the free particle 'Schrödinger operator'.

1. Introduction

Most attempts to construct a relativistic quantum mechanics have led only to the case of the free particle. It was initially claimed that the reason for this impasse is the difficulty of mastering all representations of the Poincaré group. However, when the same group theoretical arguments were applied to the Galilean case, exactly the same result was found. The reason for the impasse is clear in this case. It is that the Galilean group has been interpreted as the group of motion from the active point of view, and one is therefore naturally led to a representation for a free particle which is given in the Heisenberg picture [1]. An effort has therefore recently been made to develop a group theoretical argument for the Galilean case in a way which avoids this difficulty. An effective procedure was found to be the following: First construct the set of observables which characterizes the system, and build the dynamics only afterwards. With this method, one naturally obtains the observables in the Schrödinger picture, which is the representation independent of the dynamics. This result was found by studying the action of the Galilean group on the measuring apparatus from the passive point of view; each observable was defined by an imprimitivity system according to Mackey's theory [2, 3].

Since it was possible to carry out this program completely for the Galilean case, and the general solution was found [4], our objective was to apply the same idea in the

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relativistic case. The first problem was to exhibit the set of observables without reference to the dynamics. There is no indication in the literature as to how to do this; reference is always made, at least, to the mass of the particle. For example, the representation of the Newton–Wigner position operator depends explicitly on the mass [5]. Such an observable, which is defined by the measuring of apparatus, must not depend on dynamical characteristics of the system such as the mass. We may prepare a state of the neutral K meson system which is not a mass eigenstate; the mass of the system will then not be well-defined, and it is not clear how to calculate the expectation value of the position of the K meson with the Newton–Wigner position operator.

In the Galilean case, the group which acts on the apparatus, the passive Galilean group, is not isomorphic to the active Galilean group [6]. There is therefore no reason why the 'passive Poincaré' group should be isomorphic to the usual Poincaré group. Since we do not yet know the group which acts on the apparatus in the relativistic case, we must turn to the classical case to see what are the observables there. We found that no more is really known about the classical than about the quantum problem. It is therefore the plan of this article to study first the classical then the quantum case. Our theory does not constitute a complete solution for the quantum case, since we have not yet defined the observables by means of the imprimitivity systems. As a first try we have postulated that we can apply the correspondence principle, which can be justified in the Galilean case.

2. Relativistic Analytical Mechanics

We wish to construct a canonical formalism for the relativistic mechanics of many particles. Let us first consider one particle, and identify the set of independent observables which describe the state of the particle. Consider the eight variables $\vec{p}$, E, $\vec{q}$, t. One must realize that the energy E is independent of $\vec{p}$ because the mass

$$m \equiv [(E^2/c^2) - \vec{p}^2]^{\frac{1}{2}}/c$$

depends on the state of interaction of the particle. It is well known, for example, that the bound proton in the nucleus has a smaller mass than that of free proton, and that this mass defect correponds to the energy of binding. In order to describe a system in interaction, it is necessary to consider states which are not restricted to the 'mass shell'. We will therefore postulate that the states of a particle are described by eight independent variables $p^\mu = (p, E/c^2)$ and $q^\mu = (q, t)$. For the N particles case, we take the $8N$ variables (p^α, q^α). We have adopted the metric $g_{\mu\nu} = (1, 1, 1, -c^2)$, then m is given by $(-g_{\mu\nu}p^\mu p^\nu)^{\frac{1}{2}}/c$.

The usual point of view is that the motion is described by a relation between all of these variables which defines a trajectory on the space of one-particle states (p^μ, q^μ). This is however a 'static' point of view; nothing really changes in this picture, which is completely reversible. We shall adopt another point of view. We introduce a parameter τ to describe the evolution of the system. We call this parameter the historical time because it corresponds to the ordering relation determined by successive measuring processes in quantum theory or given by the laws of thermodynamics. One must be careful not to confuse the historical time τ [7], which is an order relation, with the geometrical time t, one of the physical observables defining the state of a particle.

We are now in a position to formulate our dynamical principle, which must be a generalization of the Hamilton principle. Consider the differential one-form

$$p_\alpha dq^\alpha - K(p^\alpha, q^\alpha)\, d\tau, \tag{2.1}$$

where $K(p^\alpha, q^\alpha)$, characterizing the given system, is a function of all the $8N$ variables of the space of states. Our dynamical principle is the following [8]:

Given a closed curve C in $\Gamma = (p^\alpha, q^\alpha, \tau)$, the integral

$$\oint_C (p_\alpha dq^\alpha - K(p^\alpha, q^\alpha)\, d\tau) \tag{2.2}$$

is invariant for a deformation of C obtained by arbitrary displacements of the point of C along the trajectories corresponding to the evolution of the system. As in the classical case, this principle is equivalent to canonical equations:

$$\begin{aligned} \dot{p}_\alpha &= -\partial_{q^\alpha} K \\ \dot{q}^\alpha &= +\partial_{p_\alpha} K. \end{aligned} \tag{2.3}$$

For the one-particle case, these reduce to

$$\begin{aligned} \frac{dp^i}{d\tau} &= -\partial_{q^i} K \qquad & \frac{dE}{d\tau} &= +\partial_t K \\ \frac{dq^i}{d\tau} &= +\partial_{p^i} K \qquad & \frac{dt}{d\tau} &= -\partial_E K. \end{aligned} \tag{2.4}$$

In order that these equations satisfy Lorentz invariance, it is sufficient that K be a scalar for the Lorentz group, τ being invariant. In this case, K, which is conserved during the evolution, turns out to be related to the total mass of the system.

Independently of the Lorentz invariance, this principle is, in fact, a generalization of the Galilean dynamical principle, since if we impose $dt/d\tau = 1$, then

$$K(\vec{p}, \vec{q}, E, t) = H(\vec{p}, \vec{q}, t) - E \tag{2.5}$$

and the equations (2.4) reduce to the Galilean form:

$$\frac{dp^i}{dt} = -\partial_{q^i} H, \quad \frac{dq^i}{dt} = +\partial_{p^i} H, \quad \frac{dE}{dt} = \partial_t H. \tag{2.6}$$

Hence in the limit where K is of the form (2.5), the relativistic theory leads to the same results as the classical theory.

To interpret these equations we consider first the case of one free particle. Let

$$K = g_{\mu\nu} \frac{p^\mu p^\nu}{2M} = \frac{\vec{p}^2 - E^2/c^2}{2M}. \tag{2.7}$$

Then

$$\begin{aligned} \frac{d\vec{p}}{d\tau} &= 0 \qquad & \frac{dE}{d\tau} &= 0 \\ \frac{d\vec{q}}{d\tau} &= \frac{\vec{p}}{M} \qquad & \frac{dt}{d\tau} &= \frac{E}{Mc^2}. \end{aligned} \tag{2.8}$$

By eliminating the parameter τ in (2.8) we find the familar equations

$$\frac{d\vec{p}}{dt}=0 \qquad \frac{dE}{dt}=0$$

$$\frac{d\vec{q}}{dt}=c^2\frac{\vec{p}}{E}.$$

But, in the usual case one imposes the relation:

$$-\vec{p}^2+E^2/c^2\equiv m^2c^2=M^2c^2. \tag{2.9}$$

This is equivalent to the initial condition $K=-\frac{1}{2}Mc^2$. In this case

$$\frac{dt}{d\tau}=\frac{1}{c}\left(\left(\frac{d\vec{q}}{d\tau}\right)^2-\frac{2K}{M}\right)^{\frac{1}{2}}=\frac{1}{c}\left(\left(\frac{d\vec{q}}{d\tau}\right)^2+c^2\right)^{\frac{1}{2}} \tag{2.10}$$

and τ is the proper time of the system. If the relation (2.9) is satisfied, we shall call $\vec{p}$ and E the proper momentum and the proper energy. These are, in general, the variables used in describing a free particle. The identification of the historical time with the geometrical time for the free particle at rest is always possible because the group of change of scale of τ is, in this case, a symmetry group for the evolution. This symmetry is broken only when the system is really in interaction with another system.

In the following we consider a few examples (and put $c=1$ henceforth).

1. Particle in an electromagnetic field

$$K=g_{\mu\nu}\frac{(p^\mu-A^\mu)(p^\nu-A^\nu)}{2M}. \tag{2.11}$$

It is well known that this gives the correct equations in the absence of radiation [9]. In fact we find

$$\frac{d\vec{p}}{d\tau}=g_{\mu\nu}\frac{p^\mu-A^\mu}{M}\vec{\nabla}A^\nu$$

$$\frac{dE}{d\tau}=-g_{\mu\nu}\frac{p^\mu-A^\mu}{M}\partial_t A^\nu$$

$$\frac{d\vec{q}}{d\tau}=\frac{\vec{p}-\vec{A}}{M} \qquad \frac{dt}{d\tau}=\frac{E-A^4}{M}. \tag{2.12}$$

It follows from (2.12) that:

$$\left(\frac{dt}{d\tau}\right)^2-\left(\frac{d\vec{q}}{d\tau}\right)^2=-\frac{2K}{M} \tag{2.13}$$

is a constant. As for the free particle, choosing $K=-M/2$ the historical time τ corresponds to the proper time. Whatever the value of K chosen, it follows from (2.12) that:

$$\frac{dq^i}{dt}=\partial_{p^i}E \quad \frac{dp^i}{dt}=-\partial_{q^i}E,$$

where E is a function of $\vec{p}$, $\vec{q}$ and t is obtained by solving (2.11) for K fixed.

2. The two-body problem without interaction

$$K = K_1 + K_2 = g_{\mu\nu} \frac{p_1^\mu p_1^\nu}{2M_1} + g_{\mu\nu} \frac{p_2^\mu p_2^\nu}{2M_2}. \tag{2.14}$$

The corresponding equations of motion are simply (2.8) for each particle. If we impose $K_n = -\frac{1}{2}M_n$ for $n = 1$ and 2, τ is the proper time for each of the particles; it cannot however also be the proper time for the center of mass unless $\vec{p}_1/M_1 = \vec{p}_2/M_2$. Let us transform to the center of mass variables

$$\begin{aligned} P^\mu &= p_1^\mu + p_2^\mu & p^\mu &= \frac{M_1 p_2^\mu - M_2 p_1^\mu}{M_1 + M_2} \\ Q^\mu &= \frac{M_1 q_1^\mu + M_2 q_2^\mu}{M_1 + M_2} & q^\mu &= q_2^\mu - q_1^\mu. \end{aligned} \tag{2.15}$$

This is a canonical transformation which leaves K invariant, since

$$g_{\mu\nu}(p_1^\mu dq_1^\nu + p_2^\mu dq_2^\nu - P^\mu dQ^\nu - p^\mu dp^\nu) = 0 \tag{2.16}$$

and

$$K = g_{\mu\nu} \frac{P^\mu P^\nu}{2M} + g_{\mu\nu} \frac{p^\mu p^\nu}{2\mu} \tag{2.17}$$

where

$$M = M_1 + M_2, \quad \mu = \frac{M_1 M_2}{M_1 + M_2}. \tag{2.18}$$

If we choose the initial value of

$$K_0 \equiv g_{\mu\nu} \frac{P^\mu P^\nu}{2M} = -\tfrac{1}{2}M \tag{2.19}$$

then, as we have seen above (2.10), τ is the proper time of the center of mass

$$\frac{dT}{d\tau} = \left(\left| \frac{d\vec{Q}}{d\tau} \right|^2 + 1 \right)^{\frac{1}{2}}. \tag{2.20}$$

If, on the other hand, we take $K_n = -\frac{1}{2}M_n$ then (2.19) cannot be satisfied unless the second term of (2.17) vanishes, which is the condition $\vec{p}_1/M_1 = \vec{p}_2/M_2$, i.e. $\vec{p} = 0$.

3. The two-body problem with interaction

$$K = g_{\mu\nu} \frac{p_1^\mu p_1^\nu}{2M_1} + g_{\mu\nu} \frac{p_2^\mu p_2^\nu}{2M_2} + V(|q_1^\mu - q_2^\mu|), \tag{2.21}$$

where

$$|q^\mu| \equiv (g_{\mu\nu} q^\mu q^\nu)^{\frac{1}{2}}, \tag{2.22}$$

is real and positive for space like q^μ. We may transform to the center of mass variables (2.15):

$$K = g_{\mu\nu} \frac{P^\mu P^\nu}{2M} + g_{\mu\nu} \frac{p^\mu p^\nu}{2\mu} + V(|q^\mu|) = K_0 + K_{\text{int.}}. \tag{2.23}$$

As in the case without interaction, the motion of the center of mass is free, i.e. $P^\mu = dQ^\mu/d\tau$ is conserved. For the other variables, we have (2.4)

$$\begin{aligned} \frac{d\vec{p}}{d\tau} &= -\vec{\nabla} V(|q^\mu|) & \frac{de}{d\tau} &= +\partial_t V(|q^\mu|) \\ \frac{d\vec{q}}{d\tau} &= \frac{\vec{p}}{\mu} & \frac{dt}{d\tau} &= \frac{e}{\mu}. \end{aligned} \tag{2.24}$$

We have the two invariants K_0 and $K_{\text{int.}}$, but we also have others corresponding to angular momentum:

$$M_{\mu\nu} = p_\mu q_\nu - q_\mu p_\nu. \tag{2.25}$$

It can be verified that

$$\overrightarrow{\mu\nu\rho}\, M_{\mu\nu} q_\rho = 0 \quad \overrightarrow{\mu\nu\rho}\, M_{\mu\nu} p_\rho = 0 \tag{2.26}$$

and that

$$M_{\mu\nu} M_{\rho\lambda} \epsilon^{\mu\nu\rho\lambda} = 0, \tag{2.27}$$

where $\epsilon^{\mu\nu\rho\lambda}$ is the canonical antisymmetric tensor which takes the values ± 1. As for the electromagnetic field, the antisymmetric tensor $M_{\mu\nu}$ defines two vectors:

$$\vec{a} = (M_{23}, M_{31}, M_{12}) \quad \text{and} \quad \vec{b} = (M_{41}, M_{42}, M_{43})$$

so that (2.27)

$$\vec{a} \cdot \vec{b} = 0. \tag{2.28}$$

Taking into account (2.27), it is simple to check that the couple of four equations (2.26) correspond to only two linearly independent equations. They therefore define a plane. Since $M_{\mu\nu}$ is conserved, this plane is constant.

A good choice of initial conditions is

$$g_{\mu\nu} p_1^\mu p_1^\nu = -M_1^2, \quad g_{\mu\nu} p_2^\mu p_2^\nu = -M_2^2. \tag{2.29}$$

In this case p^μ is space-like at the intial historical time, since from (2.15)

$$\frac{1}{2}\left(\frac{M_1 + M_2}{M_1 M_2}\right)^2 g_{\mu\nu} p^\mu p^\nu = -\left(1 + g_{\mu\nu} \frac{p_1^\mu p_2^\nu}{M_1 M_2}\right) \geqslant 0$$

and

$$E_1 E_2 \geqslant \vec{p}_1 \cdot \vec{p}_2 + M_1 M_2.$$

On the other hand, the initial value of q^μ must be space-like so that the potential is well defined. These two conditions define a plane which in general is space-like and it is the plane of the motion (2.29). It is then possible, by Lorentz transformation, to choose a frame such that at every τ, $t=0$, $e=0$ and $\vec{b}=0$. The equations of motion (2.24) then become

$$\frac{d\vec{p}}{d\tau}=-\vec{\nabla}V(|\vec{q}|) \qquad \frac{d\vec{q}}{d\tau}=\frac{\vec{p}}{\mu} \tag{2.30}$$

and, since E is constant,

$$T=t_2=t_1=\frac{E}{M}\tau=\frac{E_1+E_2}{M_1+M_2}\tau. \tag{2.31}$$

Then the geometrical times associated with each particle are the same as the geometrical time for the center of mass of the system. In this particular frame equations (2.30) are the usual equations, valid also in the classical theory.

3. Quantum Dynamics

Let us first consider the case of one particle with zero spin. We will assume that the quantum observables are the same as for the classical case, i.e. $\vec{q}$, t, $\vec{p}$, E, and we add the parameter τ to describe the evolution. As a first approach to the problem, we choose to take a 'naive' point of view and directly apply the correspondence principle based on our relativistic analytical mechanics.

For a given τ, the states of the system are described in the Hilbert space

$$L^2(R^4, d^3x\,dt),$$

the space of square integrable functions of four variables with the Lorentz invariant scalar product

$$\langle\phi,\psi\rangle=\int d^3x\,dt\,\phi^*(\vec{x},t)\,\psi(\vec{x},t). \tag{3.1}$$

The observables are given by the following operators (we take $\hbar=c=1$):

$$q^j=x^j, \quad t=t, \quad p^j=-i\partial_{x^j}, \quad E=i\partial_t \tag{3.2}$$

which can be written in a four-vector form as (recall the metric $+++-$)

$$q^\mu=x^\mu, \quad p^\mu=-ig^{\mu\nu}\,\partial_{x^\nu}. \tag{3.3}$$

These operators satisfy the commutation relations

$$i[p^\mu,q^\nu]=g^{\mu\nu}I \tag{3.4}$$

corresponding to the generalized Poisson brackets for the relativistic analytical mechanics developed in the previous section.

The evolution is described by the Schrödinger equation

$$i\partial_\tau\psi_\tau=K\psi_\tau, \tag{3.5}$$

where K is a self-adjoint operator called the Schrödinger operator. For the case of the free particle, by the correspondence principle we have

$$K=g_{\mu\nu}\frac{p^\mu p^\nu}{2M}=\frac{1}{2M}\Box \tag{3.6}$$

where

$$\Box \equiv -g^{\mu\nu}\partial_{x^\mu}\partial_{x^\nu} = \partial_t^2 - \vec{\nabla}^2.$$

It is easy to verify that this Schrödinger operator provides the good commutation rules (see (2.8))

$$\begin{aligned} \frac{dq^\mu}{d\tau} &\equiv i[K, q^\mu] = \frac{p^\mu}{M} \\ \frac{dp^\mu}{d\tau} &\equiv i[K, p^\mu] = 0. \end{aligned} \tag{3.7}$$

Let us recall the momentum representation. This representation is defined by the four-dimensional Fourier transform

$$\phi(\vec{p}, E) = \left(\frac{1}{2\pi}\right)^2 \int d^3x\, dt\, e^{-i(\vec{p}\vec{x}-Et)}\,\phi(\vec{x}, t). \tag{3.8}$$

In the momentum space the states of the system are described in the Hilbert space $L^2(R^4, d^3p\,dE)$ corresponding to the scalar product

$$\langle\phi, \psi\rangle = \int d^3p\,dE\,\phi^*(\vec{p}, E)\,\psi(\vec{p}, E) \tag{3.9}$$

and the observables q^μ, p^μ are given by

$$q^\mu = ig^{\mu\nu}\partial_{p^\nu}, \quad p^\mu = p^\mu. \tag{3.10}$$

It is important to realize that the Schrödinger operator K is conserved during the evolution. We can therefore restrict ourselves to the physical free particle states corresponding to the condition $K > 0$, i.e. $m^2 \equiv -g_{\mu\nu}p^\mu p^\nu > 0$. The corresponding subspace of $L^2(R^4, d^3p\,dE)$ is isomorphic to the space $L^2(R^4, d^3p\,dm)$ which is defined by the scalar product:

$$\langle g, f\rangle = \int d^3p\,dm\,g^*(\vec{p}, m)\,f(\vec{p}, m). \tag{3.11}$$

The isomorphism is explicitly given by

$$\phi(\vec{p}, E) \to f(\vec{p}, m) = \frac{m^{\frac{1}{2}}}{(\vec{p}^2 + m^2)^{\frac{1}{4}}}\,\phi(\vec{p}, (\vec{p}^2 + m^2)^{\frac{1}{2}}) \tag{3.12}$$

for E and m positive and

$$\phi(\vec{p}, E) \to f(\vec{p}, m) = \frac{(-m)^{\frac{1}{2}}}{(\vec{p}^2 + m^2)^{\frac{1}{4}}}\,\phi(\vec{p}, -(\vec{p}^2 + m^2)^{\frac{1}{2}}) \tag{3.13}$$

for E and m negative. We shall call this new representation the mass representation, since it diagonalizes the momentum and the Schrödinger operator for the free particle. It is to be noted that the corresponding 'eigenstate' equation:

$$(g_{\mu\nu}p^\mu p^\nu + m^2)\,\phi = 0 \tag{3.14}$$

is equivalent to the Klein–Gordon equation. This proves that the usual spin zero equation is not an equation of evolution, but just an 'eigenstate' equation corresponding to a 'stationary' state.

In this context, it is easy to interpret the Newton–Wigner position operator. Let us consider the observable

$$\vec{q}_r = \vec{q} - \tfrac{1}{2}\left(\frac{\vec{p}}{E}t + t\frac{\vec{p}}{E}\right). \tag{3.15}$$

It is easy to verify the following commutation rules (see (3.4))

$$\begin{aligned} i[q_r^i, q_r^j] &= 0 \\ i[p^i, q_r^j] &= \delta^{ij} I \\ i[E, \vec{q}_r] &= -i[E, t]\frac{\vec{p}}{E} = \frac{\vec{p}}{E}. \end{aligned} \tag{3.16}$$

Furthermore $\vec{q}_r$ is a constant of motion since (see (3.7))

$$i[K, \vec{q}_r] = \frac{\vec{p}}{M} - \frac{\vec{p}}{E}\frac{E}{M} = 0. \tag{3.17}$$

The interpretation of $\vec{q}_r$ is clear, this is the position extrapolated for the time $t = 0$. We claim that $\vec{q}_r$ is nothing else but the Newton–Wigner position operator. In fact, in the momentum representation we have (see (3.10))

$$q_r^k \phi(\vec{p}, E) = i\left(\partial_{p^k} + \frac{p^k}{E}\partial_E - \frac{p^k}{2E}\right)\phi(\vec{p}, E)$$

and, since

$$i\left(\partial_{p^k} + \frac{p^k}{E}\partial_E - \frac{p^k}{2E}\right)\phi(\vec{p}, E)\big|_{E=\pm(\vec{p}^2+m^2)^{\frac{1}{2}}} = i\left(\partial_{p^k} - \frac{p^k}{2(\vec{p}^2 + m^2)}\right)\phi(\vec{p}, \pm(\vec{p}^2 + m^2)^{\frac{1}{2}})$$

$$= (\vec{p}^2 + m^2)^{\frac{1}{4}}\, i\partial_{p^k}(\vec{p}^2 + m^2)^{-\frac{1}{4}}\phi(\vec{p}, \pm(\vec{p}^2 + m^2)^{\frac{1}{2}})$$

in the mass representation we find with the help of (3.12) and (3.13)

$$q_r^k f(\vec{p}, m) = i\partial_{p^k} f(\vec{p}, m) \tag{3.18}$$

which is just what we want [10].

Furthermore, according to (3.16) it is easy to see that

$$e^{iE\tau}\vec{q}_r e^{-iE\tau} = \vec{q}_r + \frac{\vec{p}}{E}\tau = \vec{q} - \tfrac{1}{2}\left(\frac{\vec{p}}{E}(t-\tau) + (t-\tau)\frac{\vec{p}}{E}\right). \tag{3.19}$$

But the so-called 'Schrödinger' equation

$$i\partial_\tau f_\tau(\vec{p}, m) = \pm(\vec{p} + m^2)^{\frac{1}{2}} f_\tau(\vec{p}, m) \tag{3.20}$$

has nothing to do with the real evolution given by our equation (3.5).

Finally let us remark that in the Galilean case the Newton–Wigner position operator $\vec{q}_r$ (3.15) corresponds exactly to the 'position' operator

$$e^{-i(p^2/2M)t}\vec{q}\, e^{+i(p^2/2M)t} = \vec{q} - \frac{\vec{p}}{M}t. \tag{3.21}$$

Let us now consider the spin zero particle in an electromagnetic field. The states and the observables of the system are the same as in the free case. But the Schrödinger operator corresponding to (3.5) is given by

$$K = g_{\mu\nu} \frac{(p^\mu - A^\mu)(p^\nu - A^\nu)}{2M} \tag{3.22}$$

the analog of (2.11). The stationary states are given by 'eigenstate' equation

$$(g_{\mu\nu}(p^\mu - A^\mu)(p^\nu - A^\nu) + m^2)\,\phi = 0. \tag{3.23}$$

All results given by 'eigenstate' equations are the same in our theory as in the usual. On the other hand, the explicit dynamics described by (3.5) and (3.22) gives the good classical approximation by virtue of the correspondence principle.

Furthermore, the correspondence principle, applied to the third example from the previous section, furnishes a model for the two-body problem with interaction. In this case the states of the system are described in the tensor product of the Hilbert space $L^2(R^4, d^3x\,dt)$ by itself.

4. On the Parameter τ

We introduced the parameter τ in Section 2 as an order relation, and used it in the construction of a dynamical principle which is equivalent to a canonical system of equations. A parameter of this kind has been introduced by other authors, but in each case, as a mathematical convenience without physical interpretation [11]. Schwinger [12], referring to earlier work by Fock [13] and Nambu [14], discusses an operator algebra of the form (3.4), but does not specify precisely the one 'particle' Hilbert space on which these operators are defined. The 'proper time' parameter τ enters his formulation in a parametric integral for a Green's function. Feynman [15] stressed the purely formal nature of the derivation of a Klein–Gordon equation as a stationary condition on the wave functions arising from an application of the path integral method over four-dimensional paths specified by four functions of a parameter τ.

On the contrary, we interpret τ as the parameter describing the true evolution of the system, playing a role analogous to that of t in the usual interpretation of the Galilean theory. According to this interpretation, the Schrödinger wave function $\psi_t(\vec{x})$ (belonging to an $\mathscr{H}_t \approx L^2(R^3)$) describes the state of the system at a given time t. At another time, t', there is another Hilbert space $\mathscr{H}_{t'}$ in which the function $\psi_{t'}(\vec{x})$ describes the state of the systems. These states are connected dynamically by Schrödinger's equation. In fact, the scalar product between these functions is not defined since they belong to different Hilbert spaces. It is only by taking advantage of the isomorphism between these spaces $\mathscr{H}_t$ and $\mathscr{H}_{t'}$ that we may imbed $\mathscr{H}_{t'}$ into $\mathscr{H}_t$ and carry out a scalar product between the image of $\psi_{t'}$ and ψ_t, defining in this way a transition probability amplitude.

In the same way, our wave function $\psi_\tau(x,t)$ belongs to an $\mathscr{H}_\tau = L^2(R^4)$, the state of the system is deaned over the geometrical space-time. The evolution of the system is then described by our Schrödinger equation (3.5), providing the dynamical relation between $\mathscr{H}_\tau$ and $\mathscr{H}_{\tau'}$. Transition probability amplitudes may then be defined by imbedding $\mathscr{H}_{\tau'}$ into $\mathscr{H}_\tau$, as for the Galilean case.

The existence and meaning of τ therefore rests on a postulate on the same level as that defining t in the Galilean world.

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