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Euclidean Fermi Fields and a Feynman–Kac Formula for Boson–Fermion Models

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Abstract. We define free, covariant Euclidean Bose and Fermi fields and establish their relation with the corresponding relativistic free fields. Using this correspondence, we prove a Feynman–Kac formula for boson–fermion models.

1. Introduction

Euclidean scalar boson fields play a fundamental role in the construction of Lorentz covariant fields. Their importance was stressed by Schwinger [Sc 1, 2], Nakano [N 1] and Symanzik [Sy 1, 2], whose ideas led to an abstract formulation by Nelson [Ne 1]. His work has been very stimulating and has provided new results in the constructive quantum field theory of boson models [G 1], [GRS 1]. Our paper resulted in an attempt to find a formulation which also incorporated Fermi fields. We introduce free Euclidean (spin $\frac{1}{2}$) Fermi fields. Starting from a Hamiltonian H in relativistic Fock space, which describes a Yukawa interaction plus a polynomial boson selfinteraction, we prove a path space formula. This is the Feynman–Kac formula and it relates $\exp[-tH]$ to $\exp[-V]$, where V is a Euclidean action and is expressed in terms of Euclidean boson and fermion fields. $\exp[-V]$ is the generating expression for the Feynman diagrams in the Euclidean region. Hence the Euclidean fields may be viewed as creating and annihilating virtual particles.

Euclidean fermion fields involve complications absent for bosons:

I. The Euclidean scalar boson field $\Phi(\mathbf{x})$ agrees at time zero with the relativistic boson field: $\Phi(0, \boldsymbol{x}) = \varphi(\boldsymbol{x})$. The Euclidean boson Fock space $\mathscr{E}_b$ then contains the relativistic boson Fock space $\mathscr{F}_b$ ([Ne 1], [Fe 1]). This does not hold for fermions, i.e. $\mathscr{F}_f \not\subset \mathscr{E}_f$ and sharp time Euclidean Fermi fields create non-renormalizeable wave-functions.

II. Euclidean boson and fermion fields transform under the analytic continuation of the representation of the inhomogeneous Lorentz group to the inhomogeneous rotation group iSO_4 (Euclidean group). However, it is necessary to introduce two independent anti-commuting Euclidean fields $\Psi^{(1)}$ and $\Psi^{(2)}$ corresponding to the

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relativistic fields ψ and $\bar{\psi}$. These extra degrees of freedom avoid a contradiction between Euclidean covariance of the fields $\Psi^{(1)}$ and $\Psi^{(2)}$, the canonical anti-commutation relations and the form of the two-point function which has to be equal to the relativistic Feynman propagator at imaginary times [N 1].

III. In contradistinction to the Euclidean boson action, the Euclidean action V involving fermions is non-hermitian. The adjoint transformation $V \to V^*$ is related to Euclidean time inversion (see below). This non-hermitian property causes no difficulty in the physical interpretation.

In spite of these differences:

IV. As for Euclidean boson fields, the action density for charge conserving theories is abelian. Thus our Feynman–Kac formula for boson–fermion systems gives a mathematically precise history integral for both fermions and bosons.

The Feynman–Kac formula can be used to define (cut off) Euclidean Green's functions for the type of interactions considered here. Removing the cutoffs we should obtain a set of Euclidean Green's functions, satisfying the axioms introduced in Ref. [OS 2] and thus defining a relativistic field theory model (in the sense of Wightman).

Our paper is organized as follows: We will construct Euclidean fields by starting with creation and annihilation operators satisfying the usual canonical (anti)-commutation relations and then defining the fields in terms of these operators. This Fock space construction will be carried out in Section 3. For this purpose we recall the definition of a relativistic free Bose field and a free spin $\frac{1}{2}$ Fermi field of mass $m_b > 0$ and $m_f > 0$ in Section 2. In Section 4 we establish the transition from Euclidean Fock space to the relativistic Fock space. In Section 5 we prove the Feynman–Kac formula for a system of scalar Bosons and spin $\frac{1}{2}$ Fermions coupled by a Yukawa interaction and a polynomial boson self-interaction. In Section 6 we define approximate Euclidean Green's functions for a theory with the interaction discussed in Section 5. We verify two of the axioms introduced in [OS 2], namely positivity and symmetry. To give an explicit example we consider the well-known $\lambda P(\varphi)_2$ model for small coupling constant λ. We show that the contour expansion estimate, recently established by Dimock, Glimm and Spencer [DG 1], [GS 1], combined with our results, is sufficient to prove the existence of limiting Euclidean Green's functions, satisfying all the axioms of [OS 2] and hence defining a Wightman field theory.

For notational conventions in the relativistic case, we mostly follow Bjorken and Drell [BD 1]. We write $\mathbf{x}$ for a real four-vector $(x_0, x_1, x_2, x_3) = (x_0, \boldsymbol{x})$ and we set $\mathbf{xy} = \sum_{i=0}^{3} x_i y_i$ for the Euclidean scalar product. This paper details the results announced in [OS 1].

2. The Relativistic Free Fields for Scalar Bosons and Spin $\frac{1}{2}$ Fermions

2.1. *The scalar Bose field*

The one particle boson space $\mathscr{F}_b^{(1)}$ is represented as $\mathscr{L}^2(\mathbb{R}^3)$ and the boson Fock space is the Hilbert space completion of the symmetric tensor algebra over $\mathscr{F}_b^{(1)}$,

$$\mathscr{F}_b = \Lambda_s(\mathscr{F}_b^{(1)}) = \mathbb{C} \oplus \mathscr{F}_b^{(1)} \oplus (\mathscr{F}_b^{(1)} \otimes_s \mathscr{F}_b^{(1)}) \oplus \cdots.$$

The vacuum is denoted by Ω_b. In the standard fashion we introduce boson annihilation and creation operators $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$ with the commutation relations $[a(\boldsymbol{k}), a^*(\boldsymbol{k}')] = \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}')$, (for mathematical details see e.g. [GJ 1]). By $H_{b,0}$ we denote the positive self-adjoint free Hamiltonian, defined by

$$H_{b,0} = \int a^*(\boldsymbol{k})\, a(\boldsymbol{k})\, \mu(\boldsymbol{k})\, d^3k, \tag{2.1}$$

where $\mu(\boldsymbol{k}) = (m_b^2 + \boldsymbol{k}^2)^{1/2}$ and $m_b > 0$ is the bare boson mass. The time zero boson field is given by the formula

$$\varphi(\boldsymbol{x}) = (2\pi)^{-3/2} \int e^{-i\boldsymbol{k}\boldsymbol{x}} (2\mu(\boldsymbol{k}))^{-1/2} (a^*(\boldsymbol{k}) + a(-\boldsymbol{k}))\, d^3k.$$

We will also need 'free fields at imaginary times'. For $x_0 > 0$ we define

$$\begin{aligned}\hat{\varphi}(\mathbf{x}) &= \hat{\varphi}(x_0, \boldsymbol{x}) = e^{-x_0 H_{b,0}}\, \varphi(\boldsymbol{x})\, e^{x_0 H_{b,0}} \\ &= (2\pi)^{-3/2} \int e^{-i\boldsymbol{k}\boldsymbol{x}} (2\mu(\boldsymbol{k})^{-1/2} (e^{-x_0\mu(\boldsymbol{k})}\, a^*(\boldsymbol{k}) + e^{x_0\mu(\boldsymbol{k})}\, a(-\boldsymbol{k}))\, d^3k.\end{aligned} \tag{2.2}$$

We note that

$$\hat{\varphi}(x_0, f) = \int \hat{\varphi}(x_0, \boldsymbol{x})\, f(\boldsymbol{x})\, d^3x \tag{2.3}$$

is defined on Range $(e^{-sH_{b,0}})$, for all $s > x_0, f \in \mathscr{H}^{-1/2}(\mathbb{R}^3)$, where

$$\mathscr{H}^{-s/2}(\mathbb{R}^d) = \{f \in \mathscr{S}'(\mathbb{R}^d), f \text{ real}, f(k_1, \dots k_d) \left(\sum_{i=1}^{d} k_i^2 + 1\right)^{-s/4} \in \mathscr{L}^2(\mathbb{R}^d)\}.$$

The anti-time-ordered two-point function for the fields $\hat{\varphi}$ is defined by

$$\langle \bar{T} \hat{\varphi}(\mathbf{x})\, \hat{\varphi}(\mathbf{y}) \rangle_0 = \begin{cases} \langle \hat{\varphi}(\mathbf{x})\, \hat{\varphi}(\mathbf{y}) \rangle_0, \text{ if } x_0 \leqslant y_0, \\ \langle \hat{\varphi}(\mathbf{y})\, \hat{\varphi}(\mathbf{x}) \rangle_0, \text{ if } x_0 \geqslant y_0, \end{cases} \tag{2.4}$$

where $\langle\ \rangle_0$ denotes the vacuum expectation value. Note that $\hat{\varphi}(\mathbf{x})$ and $\hat{\varphi}(\mathbf{y})$ commute at equal times, i.e. for $x_0 = y_0$. We now use the relation

$$\frac{\mu(\boldsymbol{p})^n}{2\mu(\boldsymbol{p})}\, e^{-\mu(\boldsymbol{p})x_0} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{(ip_0)^n\, e^{-ip_0 x_0}}{\mathbf{p}^2 + m_b^2}\, dp_0, \tag{2.5}$$

which is valid for all $x_0 > 0$ and all $n = 0, 1, 2 \dots$. Relation (2.5) is easily verified using contour integration. Thus we obtain

$$\langle \bar{T} \hat{\varphi}(\mathbf{x})\, \hat{\varphi}(\mathbf{y}) \rangle_0 = (2\pi)^{-4} \int e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} (\mathbf{k}^2 + m_b^2)^{-1}\, d^4k. \tag{2.6}$$

Cutoff fields are defined by

$$\varphi_\kappa(\boldsymbol{x}) = \int \chi_\kappa(\boldsymbol{x} - \boldsymbol{y})\, \varphi(\boldsymbol{y})\, d^3y$$

and

$$\hat{\varphi}_\kappa(\mathbf{x}) = e^{-x_0 H_{b,0}}\, \varphi_\kappa(\boldsymbol{x})\, e^{x_0 H_{b,0}},$$

where $\chi_\kappa(\boldsymbol{x})$ is a cutoff function, e.g.

$$\chi_\kappa(\boldsymbol{x}) = \frac{1}{(2\pi)^3} \int_{|\boldsymbol{k}| \leq \kappa} e^{i\boldsymbol{k}\boldsymbol{x}}\, d^3k.$$

Obviously we have

$$\langle \bar{T}\hat{\varphi}_\kappa(\mathbf{x})\, \hat{\varphi}_\kappa(\mathbf{y})\rangle_0 = \int \langle \bar{T}\hat{\varphi}(x_0, \boldsymbol{x}')\, \hat{\varphi}(y_0, \boldsymbol{y}')\rangle_0 \chi_\kappa(\boldsymbol{x}-\boldsymbol{x}')\, \chi_\kappa(\boldsymbol{y}-\boldsymbol{y}')\, d^3x'\, d^3y'. \tag{2.7}$$

2.2. The spin $\frac{1}{2}$ Fermi field

In this section we summarize the theory of a free spin $\frac{1}{2}$ Fermi field of mass $m_f > 0$. Our notation closely follows that of Bjorken and Drell [BD 1].

The one-particle space is $\mathscr{F}_f^{(1)} = \mathbb{C}^4 \otimes \mathscr{L}^2(\mathbb{R}^3)$ and the fermion Fock space is the Hilbert space completion of the alternating tensor algebra over $\mathscr{F}_f^{(1)}$,

$$\mathscr{F}_f = \Lambda_a(\mathscr{F}_f^{(1)}) = \mathbb{C} \oplus \mathscr{F}_f^{(1)} \oplus (\mathscr{F}_f^{(1)} \otimes_a \mathscr{F}_f^{(1)}) \oplus \dots.$$

The vacuum is denoted by Ω_f. We introduce fermion annihilation and creation operators $b(\boldsymbol{p},s)$ and $b^*(\boldsymbol{p},s)$ and anti-fermion annihilation and creation operators $d(\boldsymbol{p},s)$ and $d^*(\boldsymbol{p},s)$ for the momentum $\boldsymbol{p}$ and spin $\frac{1}{2}s = \pm\frac{1}{2}$. They satisfy the anticommutation relations

$$\{b(\boldsymbol{p},s), b^*(\boldsymbol{p}',s')\} = \{d(\boldsymbol{p},s), d^*(\boldsymbol{p}',s')\} = \delta_{ss'}\, \delta^3(\boldsymbol{p}-\boldsymbol{p}'),$$

all other anticommutators vanishing. The free Hamiltonian operator is defined to be

$$H_{f,0} = \sum_{s=\pm 1} \int (b^*(\boldsymbol{p},s)\, b(\boldsymbol{p},s) + d^*(\boldsymbol{p},s)\, d(\boldsymbol{p},s))\, \omega(\boldsymbol{p})\, d^3p, \tag{2.8}$$

where $\omega(\boldsymbol{p}) = (m_f^2 + \boldsymbol{p}^2)^{1/2}$; $m_f > 0$ is the bare fermion mass. $H_{f,0}$ is positive and self-adjoint. The free field $\psi(t, \boldsymbol{x})$ is given by

$$\psi(t,\boldsymbol{x}) = \left(\frac{1}{2\pi}\right)^{3/2} \sum_{s=\pm 1} \int \left(\frac{m_f}{\omega(\boldsymbol{p})}\right)^{1/2} e^{-i\boldsymbol{p}\boldsymbol{x}} [d^*(\boldsymbol{p},s)\, v(\boldsymbol{p},s)\, e^{i\omega(\boldsymbol{p})t}$$

$$+ b(-\boldsymbol{p},s)\, u(-\boldsymbol{p},s)\, e^{-i\omega(\boldsymbol{p})t}]\, d^3p. \tag{2.9}$$

There is a unitary representation $U(.)$ of the universal covering group of the inhomogeneous, orthochronous proper Lorentz group $iL_+^\uparrow$, which is the semidirect product of $SL(2,\mathbb{C})$ with $\mathbb{R}^4$, such that

$$U(\mathbf{a},A)\, \psi_\alpha(\mathbf{x})\, U(\mathbf{a},A)^{-1} = \sum_\beta S_{\alpha\beta}^{-1}(A,\bar{A})\, \psi_\beta(\Lambda(A,\bar{A})\,\mathbf{x}+\mathbf{a}),$$

$$U(\mathbf{a},A)\, \bar{\psi}_\alpha(\mathbf{x})\, U(\mathbf{a},A)^{-1} = \sum_\beta \bar{\psi}_\beta(\Lambda(A,\bar{A})\,\mathbf{x}+\mathbf{a})\, S_{\beta\alpha}(A,\bar{A}). \tag{2.10}$$

Here $A \in SL(2,\mathbb{C})$ and a bar over a 2×2 matrix A denotes its complex conjugate. Also for any A, $B \in SL(2,\mathbb{C})$, $\mathbf{z} \in \mathbb{C}^4$, $\Lambda(A,B)\,\mathbf{z}$ is given by

$$\widetilde{\Lambda(A,B)\,\mathbf{z}} = A\tilde{\mathbf{z}}B^T \tag{2.11}$$

with

$$\tilde{\mathbf{z}} = \sum z_i\, \sigma_i, \quad (\sigma_i\text{: Pauli matrices}).$$

Furthermore

$$(A,B) \to S(A,B) \tag{2.12}$$

is a 4-dimensional analytic representation of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ such that

$$S(A, A^{-1T}) = \begin{pmatrix} a_0 + \boldsymbol{a\sigma} & 0 \\ 0 & a_0 + \boldsymbol{a\sigma} \end{pmatrix}, \tag{2.12a}$$

$$S(A, A^{T}) = \begin{pmatrix} a_0 & \boldsymbol{a\sigma} \\ \boldsymbol{a\sigma} & a_0 \end{pmatrix}, \tag{2.12b}$$

where $A = a_0 + \boldsymbol{a\sigma}$ and $\det A = a_0^2 - \boldsymbol{a}^2 = 1$. If we now define

$$S(\boldsymbol{p}) = S\left(\left(\frac{1}{m_f}(\omega(\boldsymbol{p}) + \boldsymbol{p\sigma})\right)^{1/2}, \left(\frac{1}{m_f}(\omega(\boldsymbol{p}) + \boldsymbol{p\sigma})\right)^{1/2T}\right), \tag{2.13}$$

for any real 3-vector $\boldsymbol{p}$, then $S(\boldsymbol{p})$ is exactly the matrix given in equation (3.7) of Chapter 3 in [BD 1]. Hence $w^r(\boldsymbol{p})$, the 4-component spinors given by the rth column of $S(\boldsymbol{p})$, are just the spinors for momentum $\boldsymbol{p}$ obtained from the spinors in the rest-frame by a 'rotation-free' transformation. The spinors

$$u(\boldsymbol{p}, 1) = w^1(\boldsymbol{p}), \quad u(\boldsymbol{p}, -1) = w^2(\boldsymbol{p}),$$

$$v(\boldsymbol{p}, 1) = w^4(\boldsymbol{p}), \quad v(\boldsymbol{p}, -1) = w^3(\boldsymbol{p}),$$

$$\bar{u}_\alpha(\boldsymbol{p}, s) = \Sigma_\beta u_\beta^*(\boldsymbol{p}, s)(\gamma^0)_{\beta\alpha},$$

$$\bar{v}_\alpha(\boldsymbol{p}, s) = \Sigma_\beta v_\beta^*(\boldsymbol{p}, s)(\gamma^0)_{\beta\alpha},$$

satisfy the following completeness relations

$$\begin{aligned} \sum_{s=\pm 1} u_\alpha(\boldsymbol{p}, s)\,\bar{u}_\beta(\boldsymbol{p}, s) &= \frac{1}{2m}(\not{p} + m)_{\alpha\beta}, \\ \sum_{s=\pm 1} v_\alpha(\boldsymbol{p}, s)\,\bar{v}_\beta(\boldsymbol{p}, s) &= \frac{1}{2m}(\not{p} - m)_{\alpha\beta}, \end{aligned} \tag{2.14}$$

with $p_0 = \omega(\boldsymbol{p})$.

Now we define 'free fields at imaginary times' as in the Bose case. For $x_0 \geqslant 0$ we set

$$\begin{aligned} \hat{\psi}(\mathbf{x}) &= e^{-x_0 H_{f,0}}\, \psi(0, \boldsymbol{x})\, e^{x_0 H_{f,0}} \\ &= \left(\frac{1}{2\pi}\right)^{3/2} \sum_{s=\pm 1} \int \left(\frac{m_f}{\omega(\boldsymbol{p})}\right)^{1/2} e^{-i\boldsymbol{px}} [d^*(\boldsymbol{p}, s)\, v(\boldsymbol{p}, s)\, e^{-\omega(\boldsymbol{p})x_0} \\ &\quad + b(-\boldsymbol{p}, s)\, u(-\boldsymbol{p}, s)\, e^{\omega(\boldsymbol{p})x_0}]\, d^3 p, \end{aligned} \tag{2.15}$$

$$\begin{aligned} \bar{\psi}(\mathbf{x}) &= e^{-x_0 H_{f,0}}\, \bar{\psi}(0, \boldsymbol{x})\, e^{x_0 H_{f,0}} = e^{-x_0 H_{f,0}}\, \psi^*(0, \boldsymbol{x})\, \gamma_0\, e^{x_0 H_{f,0}} \\ &= \left(\frac{1}{2\pi}\right)^{3/2} \sum_{s=\pm 1} \int \left(\frac{m_f}{\omega(\boldsymbol{p})}\right)^{1/2} e^{-i\boldsymbol{px}} [b^*(\boldsymbol{p}, s)\, \bar{u}(\boldsymbol{p}, s)\, e^{-\omega(\boldsymbol{p})x_0} \\ &\quad + d(-\boldsymbol{p}, s)\, \bar{v}(-\boldsymbol{p}, s)\, e^{\omega(\boldsymbol{p})x_0}]\, d^3 p. \end{aligned}$$

Again the operators

$$\hat{\psi}(x_0, f) = \int \hat{\psi}(x_0, \boldsymbol{x}) f(\boldsymbol{x})\, d^3x$$

and

$$\hat{\bar{\psi}}(x_0, f) = \int \hat{\bar{\psi}}(x_0, \boldsymbol{x}) f(\boldsymbol{x})\, d^3x$$

are defined on Range $(e^{-sH_{f,0}})$, for all $s > x_0$, $f \in \mathscr{L}^2(\mathbb{R}^3)$. We now define

$$\langle \bar{T}\hat{\psi}_\alpha(\mathbf{x})\, \hat{\bar{\psi}}_\beta(\mathbf{y})\rangle_0 = \begin{cases} \langle \hat{\psi}_\alpha(\mathbf{x})\, \hat{\bar{\psi}}_\beta(\mathbf{y})\rangle_0, & \text{if} \quad x_0 \leqslant y_0, \\ -\langle \hat{\bar{\psi}}_\beta(\mathbf{y})\, \hat{\psi}_\alpha(\mathbf{x})\rangle_0, & \text{if} \quad x_0 \geqslant y_0. \end{cases}$$

Now we set

$$\gamma_0^E = \gamma_0;\ \gamma_j^E = +i\gamma_j\ (j = 1, 2, 3) \quad \text{and} \quad \not{p}^E = \sum_{j=0}^{3} p_j \gamma_j^E.$$

Then formula (2.5) and the completeness relations (2.14) give

$$\langle \bar{T}\hat{\psi}_\alpha(\mathbf{x})\, \hat{\bar{\psi}}_\beta(\mathbf{y})\rangle_0 = \frac{1}{(2\pi)^4} \int \frac{(m - i\not{p}^E)_{\alpha\beta}}{\mathbf{p}^2 + m_f^2}\, e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})}\, d^4p.$$

In analogy to the boson field we introduce cutoff fields by

$$\psi_{\kappa,\alpha}(0, \boldsymbol{x}) = \int \chi_\kappa(\boldsymbol{x} - \boldsymbol{y})\, \psi_\alpha(0, \boldsymbol{y})\, d^3y,$$

$$\bar{\psi}_{\kappa,\alpha}(0, \boldsymbol{x}) = \int \chi_\kappa(\boldsymbol{x} - \boldsymbol{y})\, \bar{\psi}_\alpha(0, \boldsymbol{y})\, d^3y,$$

and

$$\hat{\psi}_{\kappa,\alpha}(\mathbf{x}) = e^{-x_0 H_{f,0}}\, \psi_{\kappa,\alpha}(0, \boldsymbol{x})\, e^{x_0 H_{f,0}},$$

$$\hat{\bar{\psi}}_{\kappa,\alpha}(\mathbf{x}) = e^{-x_0 H_{f,0}}\, \bar{\psi}_{\kappa,\alpha}(0, \boldsymbol{x})\, e^{x_0 H_{f,0}},$$

such that

$$\langle \bar{T}\hat{\psi}_{\kappa,\alpha}(\mathbf{x})\, \hat{\bar{\psi}}_{\kappa,\beta}(\mathbf{y})\rangle_0 = \int \langle \bar{T}\hat{\psi}_\alpha(x_0, \boldsymbol{x}')\, \hat{\bar{\psi}}_\beta(y_0, \boldsymbol{y}')\rangle_0\, \chi_\kappa(\boldsymbol{x} - \boldsymbol{x}')\, \chi_\kappa(\boldsymbol{y} - \boldsymbol{y}')\, d^3x'\, d^3y'. \tag{2.16}$$

The total Fock space is the tensor product of $\mathscr{F}_b$ and $\mathscr{F}_f$,

$$\mathscr{F} = \mathscr{F}_b \otimes \mathscr{F}_f.$$

The vacuum is $\Omega = \Omega_b \otimes \Omega_f$. All operators introduced previously are considered as operators in $\mathscr{F}$. The total free Hamiltonian is $H_0 = H_{b,0} + H_{f,0}$. We may replace $H_{b,0}$ and $H_{f,0}$ by H_0 in the definitions of $\hat{\varphi}(\mathbf{x})$, $\hat{\psi}(\mathbf{x})$ and $\hat{\bar{\psi}}(\mathbf{x})$. N will denote the total number operator in $\mathscr{F}$:

$$N = \int a^*(\boldsymbol{k})\, a(\boldsymbol{k})\, d^3k + \sum_{s=\pm 1} (b^*(\boldsymbol{p}, s)\, b(\boldsymbol{p}, s) + d^*(\boldsymbol{p}, s)\, d(\boldsymbol{p}, s))\, d^3p.$$

3. Euclidean Fields

In this section we introduce free Euclidean boson and fermion fields, which will be related to the relativistic boson and fermion fields constructed in the preceding section. These Euclidean fields act in a space $\mathscr{E}$ which again has Fock space structure (see Section 1). The fields transform covariantly under a unitary representation of $\overline{\mathrm{iSO}_4}$.

3.1. Euclidean boson fields

The Euclidian Fock space $\mathscr{E}_b$ for bosons is defined to be the Hilbert space completion of the symmetric tensor algebra over $\mathscr{E}_b^{(1)} = \mathscr{L}_2(\mathbb{R}^4)$ (the Euclidean one-particle space),

$$\mathscr{E}_b = \Lambda_s(\mathscr{E}_b^{(1)}).$$

The vacuum is denoted by $\Omega_{\mathscr{E},b}$. Again in the standard fashion we introduce annihilation and creation operators $A(\mathbf{k})$ and $A^*(\mathbf{k})$:

$$(A(\mathbf{k})\, X)^{n-1}(\mathbf{k}_1, \dots \mathbf{k}_{n-1}) = n^{1/2} X^{(n)}(\mathbf{k}, \mathbf{k}_1, \dots \mathbf{k}_{n-1}),$$

where $X^{(n)}$ is the n-particle component of an element $X \in \mathscr{E}_b$. Furthermore

$$[A(\mathbf{k}), A^*(\mathbf{k}')] = \delta^{(4)}(\mathbf{k} - \mathbf{k}').$$

The Euclidean boson field is then defined by

$$\Phi(\mathbf{x}) = (2\pi)^{-2} \int (\mathbf{k}^2 + m_b^2)^{-1/2} e^{-i\mathbf{kx}} (A^*(\mathbf{k}) + A(-\mathbf{k}))\, d^4k.$$

Obviously Euclidean boson fields are commutative, i.e.

$$[\Phi(\mathbf{x}), \Phi(\mathbf{y})] = 0, \quad \text{for all} \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^4.$$

This is in contrast to relativistic free boson fields, which commute only for space-like separated points.

As usual these Euclidean boson fields become well-defined (unbounded) operators only after smearing with some test function f:

$$\Phi(f) = (2\pi)^{-2} \int \tilde{f}(\mathbf{k}) (\mathbf{k}^2 + m_b^2)^{-1/2} (A^*(\mathbf{k}) + A(-\mathbf{k}))\, d^4k$$

is a self-adjoint operator in $\mathscr{E}_b$ if $f \in \mathscr{H}^{-1}(\mathbb{R}^4)$. In fact, let $\mathscr{D}_{\mathscr{E},0}$ be the dense set of vectors having at most a finite number of particles. Then $\Phi(f)$ is a well-defined symmetric operator on $\mathscr{D}_{\mathscr{E},0}$, it leaves $\mathscr{D}_{\mathscr{E},0}$ invariant and any element in $\mathscr{D}_{\mathscr{E},0}$ is an analytic vector for $\Phi(f)$. Thus $\Phi(f)$, which by definition is the closure of $\Phi(f) \upharpoonright \mathscr{D}_{\mathscr{E},0}$, is self-adjoint. We note that the test function space $\mathscr{H}^{-1}(\mathbb{R}^4)$ contains elements of the form

$$f(x_0, \boldsymbol{x}) = \delta(x_0 - t) f_1(\boldsymbol{x}),$$

with $f_1 \in \mathscr{H}^{-1/2}(\mathbb{R}^3)$ and t arbitrary. Thus $\Phi(f) = \Phi(t, f_1)$ is a well-defined self-adjoint operator. We call $\Phi(t, \cdot)$ the Euclidean boson field at time t. This existence of sharp time Euclidean boson fields was essential in Nelson's construction, see [Ne 1], [Fe 1]. In our approach it will still be a useful (but not necessary) tool for the construction of a relativistic field theory from Euclidean field theory.

The two-point function is given by

$$\langle \Phi(\mathbf{x})\, \Phi(\mathbf{y})\rangle_0 = (2\pi)^{-4} \int e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}(\mathbf{k}^2 + m_b^2)^{-1}\, d^4 k$$

$$= \langle \bar{T}\hat{\varphi}(\mathbf{x})\, \hat{\varphi}(\mathbf{y})\rangle_0, \tag{3.1}$$

(see equation (2.6)). This relation will become crucial in Section 4. If we define

$$\Phi_\kappa(\mathbf{x}) = \int \Phi(x_0, \boldsymbol{x}')\, \chi_\kappa(\boldsymbol{x} - \boldsymbol{x}')\, d^3 x',$$

then relation (3.1) is replaced by

$$\langle \Phi_\kappa(\boldsymbol{x})\, \Phi_\kappa(\mathbf{y})\rangle_0 = \langle \bar{T}\hat{\varphi}_\kappa(\mathbf{x})\, \hat{\varphi}_\kappa(\mathbf{y})\rangle_0, \tag{3.2}$$

(see rel. (2.7)).

We have a unitary representation of iSO_4 on $\mathscr{E}_b$ defined by

$$U_b(\mathbf{a}, R)\, \Omega_{\mathscr{E},b} = \Omega_{\mathscr{E},b},$$

$$U_b(\mathbf{a}, R)\, A^*(\mathbf{k})\, U_b(\mathbf{a}, R)^{-1} = e^{-i(R\mathbf{k})\cdot \mathbf{a}} A^*(R\mathbf{k}), \quad \mathbf{a} \in \mathbb{R}^4, \quad R \in SO_4.$$

Therefore the field transforms as

$$U_b(\mathbf{a}, R)\, \Phi(\mathbf{x})\, U_b(\mathbf{a}, R)^{-1} = \Phi(R\mathbf{x} + \mathbf{a}).$$

For notational convenience we shall write U_b^t for $U_b((t,0),\mathbb{1})$.

3.2. Euclidean Fermi fields

Having constructed a Euclidean Bose field, our aim is to find a Euclidean Fermi field by a similar procedure. The first ansatz would be to look for a field $\Psi_\alpha(\mathbf{x})$ such that

$$\langle \Psi_\alpha(\mathbf{x})\, \Psi_\beta^*(\mathbf{y})\rangle_0 = \sum_\delta \langle \bar{T}\hat{\psi}_\alpha(\mathbf{x})\, \hat{\bar{\psi}}_\delta(\mathbf{y})\rangle_0\, \gamma_{\delta\beta}^0.$$

This, however, is not possible, since the right-hand side is non-hermitian, hence *a fortiori* not positive definite. A way out is to double the number of fields. More precisely, we construct two anti-commuting 4-component free fields $\Psi^{(1)}(\mathbf{x})$ and $\Psi^{(2)}(\mathbf{y})$ such that

$$\langle \Psi_\alpha^{(1)}(\mathbf{x})\, \Psi_\beta^{(2)}(\mathbf{y})\rangle_0 = \langle \bar{T}\hat{\psi}_\alpha(\mathbf{x})\, \hat{\bar{\psi}}_\beta(\mathbf{y})\rangle_0$$

$$\langle \Psi_\alpha^{(i)}(\mathbf{x})\, \Psi_\beta^{(j)*}(\mathbf{y})\rangle_0 = \delta_{ij}\, \delta_{\alpha\beta}\, \delta_{m_f}(\mathbf{x} - \mathbf{y}),$$

$$\langle \Psi_\alpha^{(i)}(\mathbf{x})\, \Psi_\beta^{(i)}(\mathbf{y})\rangle_0 = 0. \tag{3.3}$$

Here $\langle\ \rangle_0$ on the left-hand side denotes the vacuum expectation value in the Euclidean Fock space $\mathscr{E}_f$ and

$$\delta_{m_f}(\mathbf{x} - \mathbf{y}) = (2\pi)^{-4} \int \frac{e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})}}{(\mathbf{p}^2 + m_f^2)^{1/2}}\, d^4 p.$$

Also

$$\{\Psi_\alpha^{(i)}(\mathbf{x}), \Psi_\beta^{(j)}(\mathbf{y})\} = 0,$$

$$\{\Psi_\alpha^{(i)}(\mathbf{x}), \Psi_\beta^{(j)*}(\mathbf{y})\} = 2\delta_{\alpha\beta}\, \delta_{ij}\, \delta_{m_f}(\mathbf{x} - \mathbf{y}). \tag{3.4}$$

We choose the one-particle Hilbert space $\mathscr{E}_f^{(1)}$ to be

$$\mathscr{E}_f^{(1)} = \bigoplus_{j=1}^{8} \mathscr{H}_j,$$

where each $\mathscr{H}_j$ is isomorphic to $\mathscr{L}^2(\mathbb{R}^4)$. Alternatively

$$\mathscr{E}_f^{(1)} \cong \mathbb{C}^8 \otimes \mathscr{L}^2(\mathbb{R}^4).$$

Then the Euclidean Fock space $\mathscr{E}_f$ for fermions is defined to be the Hilbert space completion of the alternating tensor algebra over $\mathscr{E}_f^{(1)}$,

$$\mathscr{E}_f = \Lambda_a(\mathscr{E}_f^{(1)}).$$

The vacuum is denoted $\Omega_{\mathscr{E},f}$. We introduce fermion annihilation and creation operators $B(\mathbf{p},j)$ and $B^*(\mathbf{p},j)$ ($\mathbf{p} \in \mathbb{R}^4, j = 1,\ldots 4$) and anti-fermion annihilation and creation operators $D(\mathbf{p},j)$ and $D^*(\mathbf{p},j)$ ($\mathbf{p} \in \mathbb{R}^4, j = 1,\ldots 4$), satisfying the anticommutation relations

$$\{B(\mathbf{p},j), B^*(\mathbf{p}',j')\} = \{D(\mathbf{p},j), D^*(\mathbf{p}',j')\}$$
$$= \delta_{jj'}\delta^{(4)}(\mathbf{p} - \mathbf{p}'),$$

all other anti-commutators vanishing. $B^*(\mathbf{p},j)$ is the creation operator related to the one-particle space $\mathscr{H}_j$ ($j = 1,\ldots 4$) and $D^*(\mathbf{p},j)$ is the creation operator related to the one-particle space $\mathscr{H}_{j+4}$ ($j = 1,\ldots 4$). In order to construct Euclidean spinors, which will give us the fields, we define a matrix $S^E(\mathbf{p})$ in analogy to the matrix $S(\mathbf{p})$ (2.13). Namely for $p_0 + |\mathbf{p}| \neq 0$ we set

$$S^E(\mathbf{p}) = S\left(\left(\frac{1}{|\mathbf{p}|}(p_0 + i\boldsymbol{p\sigma})\right)^{1/2}, \left(\frac{1}{|\mathbf{p}|}(p_0 + i\boldsymbol{p\sigma})\right)^{1/2T}\right) \tag{3.5}$$

(see (2.12b)). We note that the sphere $|\mathbf{p}| = \text{const.}$ in $\mathbb{R}^4$ and the mass shell

$$V_+^m = \{\mathbf{p} \in \mathbb{R}^4 | p_0^2 - \boldsymbol{p}^2 = m^2 > 0;\ p_0 > 0\}$$

are dual symmetric spaces [H 1]. An easy calculation gives

$$S^E(\mathbf{p}) = \left(\frac{p_0 + |\mathbf{p}|}{2|\mathbf{p}|}\right)^{1/2} \begin{pmatrix} \mathbb{1} & \dfrac{i}{|\mathbf{p}| + p_0}\boldsymbol{p\sigma} \\ \dfrac{i}{|\mathbf{p}| + p_0}\boldsymbol{p\sigma} & \mathbb{1} \end{pmatrix}. \tag{3.6}$$

Note that $S^E(\mathbf{p})$ is homogeneous of order 0 in $\mathbf{p}$. We have the important relations

$$S^E(\mathbf{p})^* = S^E(\mathbf{p})^{-1} = S^E(p_0, -\boldsymbol{p}), \tag{3.7}$$

$$\gamma_0^E S^E(\mathbf{p})\, \gamma_0^E = S^E(\mathbf{p})^{-1}, \tag{3.8}$$

$$S^E(\mathbf{p})\,|\mathbf{p}|\,\gamma_0^E S^E(\mathbf{p})^{-1} = \not{p}^E. \tag{3.9}$$

In particular $S^E(\mathbf{p})$ is a unitary matrix, hence all matrix elements are uniformly bounded by 1. Now for $a > 0$ define the four-component spinors $w_\pm^j(a)$ $(j = 1, \ldots 4)$ by

$$w_\pm^1(a) = \begin{pmatrix} (-ia \pm m_f)^{1/2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad w_\pm^2(a) = \begin{pmatrix} 0 \\ (-ia \pm m_f)^{1/2} \\ 0 \\ 0 \end{pmatrix},$$

$$w_\pm^3(a) = \begin{pmatrix} 0 \\ 0 \\ (ia \pm m_f)^{1/2} \\ 0 \end{pmatrix}, \quad w_\pm^4(a) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (ia \pm m_f)^{1/2} \end{pmatrix},$$

and set

$$U^j(\mathbf{p}) = S^E(-\mathbf{p})\, w_+^j(|\mathbf{p}|),$$

$$V^j(\mathbf{p}) = S^E(-\mathbf{p})\, w_-^j(|\mathbf{p}|),$$

$$\hat{U}^j(\mathbf{p}) = w_+^j(|\mathbf{p}|)^T S^E(\mathbf{p})^{-1},$$

$$\hat{V}^j(\mathbf{p}) = w_-^j(|\mathbf{p}|)^T S^E(\mathbf{p})^{-1},$$

i.e.

$$U_\alpha^j(\mathbf{p}) = \sum S_{\alpha\beta}^E(-\mathbf{p})\, w_{+\beta}^j(|\mathbf{p}|), \tag{3.10}$$

etc. Then ($*$ denotes the complex conjugate)

$$\sum_{j=1}^4 U_\alpha^j(\mathbf{p})\, U_\beta^j(\mathbf{p})^* = \sum_{j=1}^4 V_\alpha^j(\mathbf{p})\, V_\beta^j(\mathbf{p})^* = \sum_{j=1}^4 \hat{U}_\alpha^j(\mathbf{p})\, \hat{U}_\beta^j(\mathbf{p})^*$$

$$= \sum_{j=1}^4 \hat{V}_\alpha^j(\mathbf{p})\, \hat{V}_\beta^j(\mathbf{p})^* = \delta_{\alpha\beta}(\mathbf{p}^2 + m_f^2)^{1/2}. \tag{3.11}$$

Also

$$\sum_{j=1}^4 U_\alpha^j(-\mathbf{p})\, \hat{U}_\beta^j(\mathbf{p})$$

$$= \sum_{j,\gamma,\delta} S_{\alpha\gamma}^E(\mathbf{p})\, w_{+\gamma}^j(|\mathbf{p}|)\, w_{+\delta}^j(|\mathbf{p}|)\, S_{\delta\beta}^E(\mathbf{p})^{-1}$$

$$= \sum_{\gamma,\delta} S_{\alpha\gamma}^E(\mathbf{p})(-i|\mathbf{p}|\,\gamma_0^E + m_f)_{\gamma\delta} S_{\delta\beta}^E(\mathbf{p})^{-1} = (-i\not{p}^E + m_f)_{\alpha\beta}. \tag{3.12a}$$

Similarly

$$\sum_{j=1}^4 V_\alpha^j(\mathbf{p})\, \hat{V}_\beta^j(-\mathbf{p}) = -(-i\not{p}^E + m_f)_{\alpha\beta}. \tag{3.12b}$$

We are now prepared to define the Euclidean Fermi fields $\Psi^{(1)}$ and $\Psi^{(2)}$:

$$\Psi_\alpha^{(1)}(\mathbf{x}) = (2\pi)^{-2} \sum_{j=1}^{4} \int \frac{e^{-i\mathbf{p}\mathbf{x}}}{(\mathbf{p}^2 + m_f^2)^{1/2}} \{D^*(\mathbf{p},j)\, V_\alpha^j(\mathbf{p}) + B(-\mathbf{p},j)\, U_\alpha^j(\mathbf{p})\}\, d^4 p,$$

$$\Psi_\alpha^{(2)}(\mathbf{x}) = (2\pi)^{-2} \sum_{j=1}^{4} \int \frac{e^{-i\mathbf{p}\mathbf{x}}}{(\mathbf{p}^2 + m_f^2)^{1/2}} \{B^*(\mathbf{p},j)\, \hat{U}_\alpha^j(\mathbf{p}) + D(-\mathbf{p},j)\, \hat{V}_\alpha^j(\mathbf{p})\}\, d^4 p. \quad (3.13)$$

Using relations (3.11–3.12) it is easily checked that these fields satisfy relations (3.3).[3]) If we define

$$\Psi_{\kappa,\alpha}^{(i)}(\mathbf{x}) = \int \Psi_\alpha^{(i)}(x_0, y)\, \chi_\kappa(x - y)\, d^3 y,$$

we have (see (2.16)):

$$\langle \Psi_{\kappa,\alpha}^{(1)}(\mathbf{x})\, \Psi_{\kappa,\beta}^{(2)}(\mathbf{y})\rangle_0 = \langle \bar{T} \hat{\psi}_{\kappa,\alpha}(\mathbf{x})\, \hat{\bar{\psi}}_{\kappa,\beta}(\mathbf{y})\rangle_0.$$

Using the relations in (3.4) it is easy to see that $\Psi_\alpha^{(i)}(f)$ is a bounded operator for all $f \in \mathscr{H}_{\mathbf{C}}^{-1/2}(\mathbb{R}^4)$

$$\mathscr{H}_{\mathbf{C}}^{-1/2}(\mathbb{R}^4) = \{f \in \mathscr{S}'(\mathbb{R}^4), f(\mathbf{p})(\mathbf{p}^2+1)^{-1/4} \in \mathscr{L}^2(\mathbb{R}^4)\},$$

with

$$\|\Psi_\alpha^{(i)}(f)\| \leqslant 2(2\pi)^{-4} \int (\mathbf{p}^2 + m_f^2)^{-1/2} |\tilde{f}(\mathbf{p})|^2\, d^4 p.$$

Now let $f \in \mathscr{L}^2(\mathbb{R}^4)$ and $j = 1, \ldots 4$ be arbitrary. Then $B^*(f,j)\,\Omega_{\mathscr{E},f}$ and $D^*(f,j)\,\Omega_{\mathscr{E},f}$ span the one-particle space. For given f, j define f_α and g_α by

$$\tilde{f}_\alpha(\mathbf{p}) = S^E(-\mathbf{p})_{j\alpha}^{-1}\, w_{-j}^{j}(|\mathbf{p}|)^{-1} (\mathbf{p}^2 + m_f^2)^{1/2} \tilde{f}(\mathbf{p}),$$

$$\tilde{g}_\alpha(\mathbf{p}) = S^E(\mathbf{p})_{\alpha j}\, w_{+j}^{j}(|\mathbf{p}|)^{-1} (\mathbf{p}^2 + m_f^2)^{1/2} \tilde{f}(\mathbf{p}).$$

Then $f_\alpha, g_\alpha \in \mathscr{H}_{\mathbf{C}}^{-1/2}(\mathbb{R}^4)$ if $f \in \mathscr{L}^2(\mathbb{R}^4)$ and

$$D^*(f,j)\,\Omega_{\mathscr{E},f} = \sum_\alpha \Psi_\alpha^{(1)}(f_\alpha)\,\Omega_{\mathscr{E},f},$$

$$B^*(f,j)\,\Omega_{\mathscr{E},f} = \sum_\alpha \Psi_\alpha^{(2)}(g_\alpha)\,\Omega_{\mathscr{E},f}.$$

This proves the following lemma.

Lemma 3.1. Vectors of the form

$$:\prod_{k=1}^{n} \Psi_{\alpha_k}^{(i_k)}(f_k):\Omega_{\mathscr{E},f}, \qquad f_k \in \mathscr{H}_{\mathbf{C}}^{-1/2}(\mathbb{R}^4),$$

span the n-particle space.

As usual : : denotes Wick ordering.

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[3]) When we presented this material at the New York meeting on constructive quantum field theory (Sept. 1972) F. Guerra informed us that he has constructed similar fields.

We conclude this section with a discussion of the covariance properties of the fields. For this purpose we will have to deal with the universal covering group $\overline{iSO_4}$ of iSO_4 which is the semi-direct product of $\mathbb{R}^4$, considered as an additive group, and $SU(2) \times SU(2)$:

$$(\mathbf{a}, (u_1, u_2)) \in \overline{iSO_4} \Leftrightarrow \mathbf{a} \in \mathbb{R}^4,\ u_1, u_2 \in SU(2).$$

The unit element is $(0, (\mathbb{1}, \mathbb{1}))$. The group multiplication is defined by

$$(\mathbf{a}, (u_1, u_2))(\mathbf{a}', (u_1', u_2')) = (\mathbf{a} + R(u_1, u_2)\,\mathbf{a}', (u_1 u_1', u_2 u_2')).$$

Here $R(u_1, u_2) \in SO_4$ is given by $R(u_1, u_2)\mathbf{x} = \mathbf{x}'$ with

$$(-ix_0' + \boldsymbol{x}'\boldsymbol{\sigma}) = u_1(-ix_0 + \boldsymbol{x\sigma})\,u_2^T, \tag{3.14}$$

(compare with 2.11). We will show there exists a unitary representation $U_f(\cdot)$ of $\overline{iSO_4}$, such that $\Omega_{\mathscr{E},f}$ is an invariant vector and

$$U_f(\mathbf{a}, (u_1, u_2))\,\Psi_\alpha^{(1)}(\mathbf{x})\,U_f(\mathbf{a}, (u_1, u_2))^{-1}$$
$$= \sum_\beta S_{\alpha\beta}^{-1}(u_1, u_2)\,\Psi_\beta^{(1)}(R(u_1, u_2)\,\mathbf{x} + \mathbf{a}),$$

$$U_f(\mathbf{a}, (u_1, u_2))\,\Psi_\alpha^{(2)}(\mathbf{x})\,U_f(\mathbf{a}, (u_1, u_2))^{-1}$$
$$= \sum_\beta \Psi_\beta^{(2)}(R(u_1, u_2)\,\mathbf{x} + \mathbf{a})\,S_{\beta\alpha}(u_1, u_2). \tag{3.15}$$

S is given by (2.12).

Comparing this with relation (2.10), we see that these transformation properties are in agreement with the axiomatic formulation for the Euclidean Green's functions ([OS 2], Section 6). Of course (3.15) is consistent with the value of the two-point functions. Note that $S(u_1, u_2)$ is unitary for all $u_1,\ u_2 \in SU(2)$. The relations

$$\gamma_0^E S(u, u^T)\,\gamma_0^E = S(u^{-1}, \bar{u})$$

$$\gamma_0^E S(u, \bar{u})\,\gamma_0^E = S(u, \bar{u})$$

$$u \in SU(2);\ \bar{u} = u^{-1T}$$

(see (3.8) and (2.12a)) easily show that $\Psi^{(2)}$ transforms like $\Psi^{(1)*}\gamma_0^E$. To construct this representation, we first note that the translation group is unitarily implemented by defining $U_f(\mathbf{a})$ through

$$U_f(\mathbf{a})\,\Omega_{\mathscr{E},f} = \Omega_{\mathscr{E},f},$$

$$U_f(\mathbf{a})\,B^*(\mathbf{p}, j)\,U_f(\mathbf{a})^{-1} = e^{-i\mathbf{pa}}\,B^*(\mathbf{p}, j),$$

$$U_f(\mathbf{a})\,D^*(\mathbf{p}, j)\,U_f(\mathbf{a})^{-1} = e^{-i\mathbf{pa}}\,D^*(\mathbf{p}, j), \tag{3.16}$$

making (3.15) true for translations. In order to define $U_f(u_1, u_2)$ for $u_1, u_2 \in SU(2)$, we have to introduce a 'Wigner rotation' [W 1]. First we note that the subgroup

$$\mathscr{G} = \{(u, \bar{u})\,|\,u \in SU(2)\}$$

of $SU(2) \times SU(2)$ is isomorphic to $SU(2)$ and is the subgroup of $SU(2) \times SU(2)$ which leaves the point (1,0,0,0) invariant under the transformation given by (3.14). For

$u_1, u_2 \in SU(2)$, $\mathbf{p} \in \mathbb{R}^4$ define the element $(u(u_1, u_2, \mathbf{p}), \bar{u}(u_1, u_2, \mathbf{p})) \in \mathscr{G}$ by

$$\begin{aligned}(u(u_1, u_2, \mathbf{p}), \bar{u}(u_1, u_2, \mathbf{p}))\\ = (|\mathbf{p}'|^{1/2}(p_0' + i\mathbf{p}'\boldsymbol{\sigma})^{-1/2}, |\mathbf{p}'|^{1/2}(p_0' + i\mathbf{p}'\boldsymbol{\sigma})^{-1/2T}) \cdot (u_1, u_2)\\ \cdot (|\mathbf{p}|^{-1/2}(p_0 + i\mathbf{p}\boldsymbol{\sigma})^{1/2}, |\mathbf{p}|^{-1/2}(p_0 + i\mathbf{p}\boldsymbol{\sigma})^{1/2T}),\end{aligned} \tag{3.17}$$

where $\mathbf{p}' = R(u_1, u_2)\,\mathbf{p}$ (in particular $|\mathbf{p}'| = |\mathbf{p}|$). Setting

$$T(u_1, u_2, \mathbf{p}) = S(u(u_1, u_2, \mathbf{p}), \bar{u}(u_1, u_2, \mathbf{p}))$$

the fact that $S(\cdot,\cdot)$ is a representation of $SU(2) \times SU(2)$ gives

$$\begin{aligned}T(u_1, u_2, \mathbf{p}) &= S^E(R(u_1, u_2)\,\mathbf{p})^{-1} S(u_1, u_2)\, S^E(\mathbf{p})\\ &= S^E(-R(u_1, u_2)\,\mathbf{p})^{-1} S(u_1, u_2)\, S^E(-\mathbf{p}).\end{aligned} \tag{3.18}$$

We are now in a position to define $U_f(u_1, u_2)$. We set

$$\begin{aligned}&U_f(u_1, u_2)\,\Omega_{\mathscr{E},f} = \Omega_{\mathscr{E},f},\\ &U_f(u_1, u_2)\,B^*(\mathbf{p}, j)\,U_f(u_1, u_2)^{-1} = \sum_{j'} B^*(R(u_1, u_2)\,\mathbf{p}, j')\,T_{j'j}(u_1, u_2)\,\mathbf{p}),\\ &U_f(u_1, u_2)\,D^*(\mathbf{p}, j)\,U_f(u_1, u_2)^{-1} = \sum_{j'} T_{jj'}(u_1, u_2, \mathbf{p})^{-1} D^*(R(u_1, u_2)\,\mathbf{p}, j').\end{aligned} \tag{3.19}$$

Relation (3.18) immediately shows that this is a representation. Noting that

$$w^{j'}_{\pm j'}(|\mathbf{p}|)\,T_{j'j}(u_1, u_2; \mathbf{p}) = T_{j'j}(u_1, u_2; \mathbf{p})\,w^{j}_{\pm j}(|\mathbf{p}|)$$

and again using relation (3.18), the covariance properties (3.15) of the fields are easily verified. We may incidentally note that relations (3.16) and (3.19) exhibit $U_f(\mathbf{a})\,U_f(u_1, u_2) = U_f(\mathbf{a}, (u_1, u_2))$ as an induced representation on $\mathscr{E}_f^{(1)}$ [M 1]. (The representation is highly reducible.)

3.3. *The total Euclidean Fock space*

The total Euclidean Fock space is the tensor product of $\mathscr{E}_b$ and $\mathscr{E}_f$:

$$\mathscr{E} = \mathscr{E}_b \otimes \mathscr{E}_f.$$

The vacuum is $\Omega_{\mathscr{E}} = \Omega_{\mathscr{E},b} \otimes \Omega_{\mathscr{E},f}$. All the operators introduced previously are now considered as operators in $\mathscr{E}$. The unitary representation of $\overline{iSO_4}$ is defined by

$$U(\mathbf{a}, (u_1, u_2)) = U_b(\mathbf{a}, R(u_1, u_2)) \otimes U_f(\mathbf{a}, u_1, u_2).$$

We shall write U^t for $U((t, 0), (\mathbb{1}, \mathbb{1}))$. Also we introduce the total number operator in $\mathscr{E}$:

$$N_{\mathscr{E}} = \int A^*(\mathbf{k})\,A(\mathbf{k})\,d^4k + \sum_{j=1}^{4} \int B^*(\mathbf{p}, j)\,B(\mathbf{p}, j)\,d^4p + \sum_{j=1}^{4} \int D^*(\mathbf{p}, j)\,D(\mathbf{p}, j)\,d^4p.$$

4. Relativistic and Euclidean Fields

In this section we show how to reconstruct $\mathscr{F}$, φ, ψ_α and $\bar{\psi}_\alpha$ from the corresponding Euclidean quantities $\mathscr{E}$, Φ, $\Psi_\alpha^{(1)}$ and $\Psi_\alpha^{(2)}$. In a theory involving only scalar Bosons it is possible to identify $\mathscr{F}$ with a subspace of $\mathscr{E}$. The Euclidean field Φ at time zero (see Section 3.1) restricted to that subspace becomes the free field φ at time zero, see [Ne 1] and [Fe 1]. This procedure, however, does not work for fermions, mainly because the fermion operators are not well-defined at sharp times. Our procedure generalizes the methods used in [Ne 1] and [Fe 1].

We introduce the 'subspace of positive times' $\mathscr{E}_+$ of $\mathscr{E}$. On $\mathscr{E}_+ \times \mathscr{E}_+$ we define a sesquilinear form $\langle X, Y\rangle = (\Theta X, Y)$. Here Θ is a unitary involution on $\mathscr{E}$, which may be interpreted as a time inversion followed by charge conjugation. The form $\langle\ ,\ \rangle$ turns out to be positive semi-definite. The set $\mathscr{N}$ of vectors X with $\langle X, X\rangle = 0$ is a closed linear subspace of $\mathscr{E}_+$, thus $\{\mathscr{E}_+\backslash\mathscr{N}, \langle\ ,\ \rangle\}$ defines a pre-Hilbert space, whose closure can be identified in a natural way with the Fock space $\mathscr{F}$. Using this identification, we can reconstruct anti-time ordered products of operators $\hat{\varphi}$, $\hat{\psi}$ and $\hat{\bar{\psi}}$ in $\mathscr{F}$ from the corresponding ordinary products of Φ, $\Psi^{(1)}$ and $\Psi^{(2)}$ in $\mathscr{E}_+$. Using the time translation group U^t in $\mathscr{E}$ we construct a contraction semigroup on $\mathscr{F}$ which turns out to be $\exp[-tH_0]$, for $t \geqslant 0$.

This transition from $\mathscr{E}_+$ to $\mathscr{F}$ will be used in Section 5 to prove a relativistic Feynman–Kac formula for systems with boson-fermion interactions.

4.1. *Reconstruction of the Fock space* $\mathscr{F}$

We start our construction by defining the unitary involution Θ. Set $\vartheta\mathbf{x} = (-x_0\, \boldsymbol{x})$. Our aim is to find a Θ such that

$$\begin{aligned}
\Theta\Omega_{\mathscr{E}} &= \Omega_{\mathscr{E}},\\
\Theta\Phi(\mathbf{x})\,\Theta^{-1} &= \Phi(\vartheta\mathbf{x}),\\
\Theta\Psi_\alpha^{(i)}(\mathbf{x})\,\Theta^{-1} &= \sum_\beta \Psi_\beta^{(3-i)}(\vartheta\mathbf{x})^*(\gamma_0)_{\beta\alpha}\\
&= (\Psi^{(3-i)}\gamma_0)_\alpha^*(\vartheta\mathbf{x}).
\end{aligned} \tag{4.1}$$

We define the linear operator Θ by:

$$\begin{aligned}
\Theta\Omega_{\mathscr{E}} &= \Omega_{\mathscr{E}},\\
\Theta A(\mathbf{k})\,\Theta^{-1} &= A^*(\vartheta\mathbf{k}),\\
\Theta B^*(\mathbf{p}, j)\,\Theta^{-1} &= \sum_k B^*(\vartheta\mathbf{p}, k)\, C_{kj}(-\boldsymbol{p}),\\
\Theta D^*(\mathbf{p}, j)\,\Theta^{-1} &= \sum_k C_{jk}(\boldsymbol{p})\, D^*(\vartheta\mathbf{p}, k),
\end{aligned} \tag{4.2}$$

where $C(\boldsymbol{p})$ is defined by

$$C(\boldsymbol{p}) = \frac{1}{|\boldsymbol{p}|}\begin{pmatrix} 0 & -i\boldsymbol{p\sigma} \\ i\boldsymbol{p\sigma} & 0 \end{pmatrix}. \tag{4.3}$$

We have $C(p)^* = C(p)$; $C(p)^2 = 1$. Thus $C(p)$ is a unitary involution and therefore Θ has the same property. The relations in (4.1) now follow from the definition of the fields, from the relation (3.7) and the fact that

$$w^{j}_{\pm j}(|\mathbf{p}|)\, C_{jk}(p) = C_{jk}(p)\, w^{k}_{\pm k}(|\mathbf{p}|)^*,$$

for all j and k. Also

$$S^E(-\mathbf{p})^{-1}\gamma_0 S^E(-\vartheta\mathbf{p}) = C(p).$$

The following relation is easy to verify

$$\Theta : \prod_{k=1}^{n} \Psi^{(i_k)}_{\alpha_k}(f_k) \prod_{l=1}^{m} \Phi(h_l) : \Omega_{\mathscr{E}} = \left(: \prod_{k=n}^{1} (\Psi^{(3-i_k)}\gamma_0)_{\alpha_k}(\vartheta f_k) \prod_{l=1}^{m} \Phi(\vartheta h_l) : \right)^* \Omega_{\mathscr{E}},$$

where

$$f_k \in \mathscr{H}^{-1/2}(\mathbb{R}^4);\quad h_l \in \mathscr{H}^{-1}(\mathbb{R}^4),\quad n, m \in \mathbb{Z}^+,\quad (\vartheta f)(\mathbf{x}) = f(\vartheta\mathbf{x}).$$

For given $r, s (r < s)$ we define subspaces $\mathscr{E}_{r,s}$ of $\mathscr{E}$ by $\mathscr{E}_{r,s}$ = closed linear hull of

$$\left\{ : \prod_{k=1}^{n} \Psi^{(i_k)}_{\alpha_k}(f_k) \prod_{l=1}^{m} \Phi(h_l) : \Omega_{\mathscr{E}};\qquad f_k \in \mathscr{H}^{-1/2}_{rs};\, h_l \in \mathscr{H}^{-1}_{rs} \right\},$$

where

$$\mathscr{H}^{\alpha}_{r,s} = \{h \in \mathscr{H}_{\alpha}(\mathbb{R}^4);\ \operatorname{supp} h \subset [r, s] \times \mathbb{R}^3\}.$$

We shall write $\mathscr{E}_+$ for $\mathscr{E}_{0,\infty}$ and $\mathscr{E}_-$ for $\mathscr{E}_{-\infty,0}$. With this definition we have

$$U^t\,\mathscr{E}_{r,s} = \mathscr{E}_{r+t,s+t};\quad \Theta\mathscr{E}_{r,s} = \mathscr{E}_{-s,-r}.$$

Note also that $\Theta U^t = U^{-t}\Theta$. Next we introduce the notation

$$\Psi^{(i)}(\boldsymbol{f}, \boldsymbol{\alpha}) = \prod_{j=1}^{K} \Psi^{(i)}_{\alpha_j}(f_j),$$

$$\Psi^{(i)}\gamma_0(\boldsymbol{f}, \boldsymbol{\alpha}) = \sum_{\beta} \Psi^{(i)}(\boldsymbol{f}, \boldsymbol{\beta}) \prod_{j=1}^{K} (\gamma_0)_{\beta_j,\alpha_j},$$

$$\Phi(\boldsymbol{h}) = \prod_{l=1}^{M} \Phi(h_l),$$

for

$$\boldsymbol{\alpha} = (\alpha_1, \dots \alpha_K),\quad \boldsymbol{\beta} = (\beta_1, \dots \beta_K)$$

$$\boldsymbol{f} = (f_1, \dots f_K),\ \boldsymbol{h} = (h_1, \dots h_M);\ K, M \in \mathbb{Z}^+.$$

$$f_j \in \mathscr{H}^{-1/2}(\mathbb{R}^4),$$

$$h_l \in \mathscr{H}^{-1}(\mathbb{R}^4).$$

Then we define $\mathscr{E}^0_+$ to be the linear hull of the set of all vectors of the form

$$: \Psi^{(1)}(\boldsymbol{f}, \boldsymbol{\alpha})\, \Psi^{(2)}(\boldsymbol{g}, \boldsymbol{\beta})\, \Phi(\boldsymbol{h}) : \Omega_{\mathscr{E}} \tag{4.4}$$

$$\boldsymbol{\alpha} = (\alpha_1, \dots \alpha_K),\quad \boldsymbol{\beta} = (\beta_1, \dots \beta_L);\ K, L, M \in \mathbb{Z}^+,$$

with

$$f_i, g_j \in \mathscr{H}_{0,\infty}^{-1/2}; \quad h_l \in \mathscr{H}_{0,\infty}^{-1}.$$

$\mathscr{E}_+^0$ is dense in $\mathscr{E}_+$ by definition. Using a similar notation in Fock space, the set of vectors

$$:\hat{\psi}(\boldsymbol{f}, \boldsymbol{\alpha})\, \hat{\bar{\psi}}(\boldsymbol{g}, \boldsymbol{\beta})\, \hat{\varphi}(\boldsymbol{h}):\Omega, \tag{4.5}$$

with $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$ as above, is total in $\mathscr{F}$. Its linear hull will be denoted by $\mathscr{F}^0$.

Remark. It is instructive to note that the vector in (4.5) may be written as

$$:\psi(\boldsymbol{\epsilon f}, \boldsymbol{\alpha})\, \bar{\psi}(\boldsymbol{\epsilon g}, \boldsymbol{\beta})\, \varphi(\boldsymbol{\epsilon h}):\Omega,$$

where $\boldsymbol{\epsilon f} = (\epsilon f_1, \epsilon f_2, \ldots)$ and ϵf is defined by

$$\widetilde{\epsilon f}(\boldsymbol{p}) = \tilde{f}(-i\omega(\boldsymbol{p}), \boldsymbol{p})$$

for fermions and

$$\widetilde{\epsilon h}(\boldsymbol{p}) = \tilde{h}(-i\mu(\boldsymbol{p}), \boldsymbol{p})$$

for bosons. The existence of ϵf is guaranteed by the support properties of f. An easy calculation shows for example

$$\widetilde{\epsilon f}(\boldsymbol{p}) = \frac{\omega(\boldsymbol{p})}{\pi} \int_{-\infty}^{+\infty} \frac{\tilde{f}(p_0, \boldsymbol{p})}{p_0^2 + \omega(\boldsymbol{p})^2}\, dp_0.$$

Thus for $f \in \mathscr{H}_{0,\infty}^{-1/2}$, ϵf is in $\mathscr{L}^2(\mathbb{R}^3)$, which is the correct test function space for the time zero fermion fields and similarly if $h \in \mathscr{H}_{0,\infty}^{-1}$, then ϵh is in $\mathscr{H}^{-1/2}(\mathbb{R}^3)$, which is the correct space of test functions for free time-zero boson fields.

We now define an operator W_0 mapping $\mathscr{E}_+^0$ into $\mathscr{F}^0$. We set

$$W_0 : \Psi^{(1)}(\boldsymbol{f}, \boldsymbol{\alpha})\, \Psi^{(2)}(\boldsymbol{g}, \boldsymbol{\beta})\, \Phi(\boldsymbol{h}):\Omega_{\mathscr{E}} = :\hat{\psi}(\boldsymbol{f}, \boldsymbol{\alpha})\, \hat{\bar{\psi}}(\boldsymbol{g}, \boldsymbol{\beta})\, \hat{\varphi}(\boldsymbol{h}):\Omega. \tag{4.6}$$

By linearity, equation (4.6) defines W_0 on $\mathscr{E}_+^0$. The following lemma justifies the construction of Θ.

Lemma 4.1. Let X, $Y \in \mathscr{E}_+^0$. Then

$$(W_0 X, W_0 Y) = (\Theta X, Y). \tag{4.7}$$

Note that the scalar product on the left-hand side is taken in $\mathscr{F}$, where as the scalar product on the right-hand side is taken in $\mathscr{E}$.

Proof. It is sufficient to prove relation (4.7) for vectors X and Y of the form (4.4), i.e.

$$X = :\Psi^{(1)}(\boldsymbol{f}, \boldsymbol{\alpha})\, \Psi^{(2)}(\boldsymbol{g}, \boldsymbol{\beta})\, \Phi(\boldsymbol{h}):\Omega_{\mathscr{E}},$$

$$Y = :\Psi^{(1)}(\boldsymbol{f}', \boldsymbol{\alpha}')\, \Psi^{(2)}(\boldsymbol{g}', \boldsymbol{\beta}')\, \Phi(\boldsymbol{h}'):\Omega_{\mathscr{E}}.$$

We use relation (4.1) to compute the right-hand side of relation (4.7).

$$(\Theta X, Y) = (\Omega_{\mathscr{E}}, :\Psi^{(1)}\gamma_0(\boldsymbol{\vartheta g}, \boldsymbol{\vartheta\beta})\,\Psi^{(2)}\gamma_0(\boldsymbol{\vartheta f}, \boldsymbol{\vartheta\alpha})\,\Phi(\boldsymbol{\vartheta h}): \\ \times :\Psi^{(1)}(\boldsymbol{f}', \boldsymbol{\alpha}')\,\Psi^{(2)}(\boldsymbol{g}', \boldsymbol{\beta}')\,\Phi(\boldsymbol{h}')\,\Omega_{\mathscr{E}}), \tag{4.8}$$

with

$$\boldsymbol{\vartheta f} = (\vartheta f_K, \ldots \vartheta f_1),$$

$$\boldsymbol{\vartheta\alpha} = (\alpha_K, \ldots \alpha_1),$$

for

$$\boldsymbol{f} = (f_1, \ldots f_K),$$

$$\boldsymbol{\alpha} = (\alpha_1, \ldots \alpha_K).$$

Now we use Wick's theorem (which of course remains true for Euclidean fields) to write equation (4.8) as a sum of products of the form

$$\langle(\Psi^{(1)}\gamma_0)_{\beta_k}(\vartheta g_k)\,\Psi^{(2)}_{\beta'_r}(g'_r)\rangle_0, \quad \langle(\Psi^{(2)}\gamma_0)_{\alpha_k}(\vartheta f_k)\,\Psi^{(1)}_{\alpha'_r}(f'_r)\rangle_0,$$

$$\langle\Phi(\vartheta h_k)\,\Phi(h'_r)\rangle_0.$$

Using relation (3.3) we may replace

$$\langle(\Psi^{(1)}\gamma_0)_{\beta_k}(\vartheta g_k)\,\Psi^{(2)}_{\beta'_r}(g'_r)\rangle_0$$

by

$$\langle\overline{T}(\hat{\psi}\gamma_0)_{\beta_k}(\vartheta g_k)\,\hat{\bar{\psi}}_{\beta'_r}(g'_r)\rangle_0$$

and hence by

$$\langle(\hat{\psi}\gamma_0)_{\beta_k}(\vartheta g_k)\,\hat{\bar{\psi}}_{\beta'_r}(g'_r)\rangle_0,$$

because $\vartheta g_k(x_0, \boldsymbol{x})$ is non-zero only if $x_0 \leqslant 0$ while $g'_r(y_0, \boldsymbol{y})$ is zero unless $y_0 \geqslant 0$, and therefore the anti-time ordering operator $\overline{T}$ is unnecessary. Similarly it follows that we can replace

$$\langle(\Psi^{(2)}\gamma_0)_{\alpha_k}(\vartheta f_k)\,\Psi^{(1)}_{\alpha'_r}(f'_r)\rangle_0$$

by

$$\langle(\hat{\bar{\psi}}\gamma_0)_{\alpha_k}(\vartheta f_k)\,\hat{\psi}_{\alpha'_r}(f'_r)\rangle_0,$$

and

$$\langle\Phi(\vartheta h_k)\,\Phi(h'_r)\rangle_0$$

by

$$\langle\hat{\varphi}(\vartheta h_k)\,\hat{\varphi}(h'_l)\rangle_0.$$

Now we use Wick's theorem in the other direction to conclude that

$$(\Theta X, Y) = (\Omega, :\hat{\psi}\gamma_0(\boldsymbol{\vartheta g}, \boldsymbol{\vartheta\beta})\,\hat{\bar{\psi}}\gamma_0(\boldsymbol{\vartheta f}, \boldsymbol{\vartheta\alpha})\,\hat{\varphi}(\boldsymbol{\vartheta h}): \\ \times :\hat{\psi}(\boldsymbol{f}', \boldsymbol{\alpha}')\,\hat{\bar{\psi}}(\boldsymbol{g}', \boldsymbol{\beta}')\,\hat{\varphi}(\boldsymbol{h}'):\Omega). \tag{4.9}$$

Finally we use the definitions of ψ, $\bar{\psi}$ and φ to verify

$$(\hat{\psi}\gamma_0)_\beta(\vartheta g)^* = \hat{\bar{\psi}}_\beta(g), \quad (\hat{\bar{\psi}}\gamma_0)_\alpha(\vartheta f)^* = \hat{\psi}_\alpha(f), \quad \hat{\varphi}(\vartheta h)^* = \hat{\varphi}(h). \tag{4.10}$$

Inserting relations (4.10) into (4.9) we obtain

$$(\Theta X, Y) = (W_0 X, W_0 Y),$$

which proves lemma 4.1.

Now let $X \in \mathscr{E}_+^0$, then $\|W_0 X\|^2 = (\Theta X, X) \leqslant \|\Theta X\| \|X\| \leqslant \|X\|^2$, using the unitarity of Θ. Thus $\|W_0\| \leqslant 1$ (a simple argument shows that in fact $\|W_0\| = 1$), and W_0 can be extended in a unique fashion to an operator W from all of $\mathscr{E}_+$ into $\mathscr{F}$ with $\|W\| \leqslant 1$. The range of W, denoted by $\mathscr{F}_+$ is dense in $\mathscr{F}$. Let $\mathscr{N}$ be the non-trivial kernel of W. Then

$$\mathscr{N} = \{X : X \in \mathscr{E}_+, (\Theta X, X) = 0\}$$

is a closed linear subspace of $\mathscr{E}_+$. The quotient space $\mathscr{E}_+ \backslash \mathscr{N}$ equipped with the positive definite sesquilinear form

$$(\{X\}, \{Y\})_{\mathscr{N}} = (\Theta X, Y)$$

is a pre-Hilbert space, whose closure we denote by $\tilde{\mathscr{F}}$. For X an element in $\mathscr{E}_+$ we write $\{X\}$ to denote the equivalence class of $X \bmod \mathscr{N}$ in $\mathscr{E}_+ \backslash \mathscr{N}$. Defining P as the mapping from $\mathscr{E}_+$ onto $\mathscr{E}_+ \backslash \mathscr{N} \subset \tilde{\mathscr{F}}$ which maps $X \in \mathscr{E}_+$ onto $\{X\} \in \mathscr{E}_+ \backslash \mathscr{N}$, we can write W as $\tilde{W}P$, where $\tilde{W}$ is an isometry from $\tilde{\mathscr{F}}$ onto $\mathscr{F}$. In order to relate our construction to the formalism introduced by Nelson [Ne 1], we remark that, in case only bosons are present, each equivalence class $\{X\}$ contains exactly one time zero vector (which does not exist for fermions). Since $(\Theta X, Y) = (X, Y)$ for time zero boson vectors, $\mathscr{F}_b$ can be identified with $\mathscr{N}^\perp$, the 'time zero subspace' of $\mathscr{E}_b$, and the operator P can be interpreted as projection onto $\mathscr{N}^\perp$. In our more general set up a natural embedding of $\mathscr{F}$ in $\mathscr{E}$ does not seem to be possible.

4.2. Reconstruction of Wick ordered polynomials. The free Hamiltonian.

In this section we show how to reconstruct field operators and Wick ordered polynomials in the field operators in $\mathscr{F}$ from corresponding objects in $\mathscr{E}$.

Let $\Xi(\mathbf{x})$ be a Wick ordered polynomial in the cutoff fields $\Psi^{(i)}_{\alpha,\kappa}$ and Φ_κ and let $\hat{\xi}(\mathbf{x})$ be the same expression as $\Xi(\mathbf{x})$ but with $\Psi^{(1)}_{\alpha,\kappa}(\mathbf{x})$, $\Psi^{(2)}_{\alpha,\kappa}(\mathbf{x})$, $\Phi_\kappa(\mathbf{x})$ replaced by $\hat{\psi}_{\alpha,\kappa}(\mathbf{x})$, $\hat{\bar{\psi}}_{\alpha,\kappa}(\mathbf{x})$, $\hat{\varphi}_\kappa(\mathbf{x})$ respectively. Set $\xi(\boldsymbol{x}) = \hat{\xi}(0, \boldsymbol{x})$. Then $\hat{\xi}(\mathbf{x}) = e^{-x_0 H_0} \xi(\boldsymbol{x}) e^{x_0 H_0}$. When smeared out with sufficiently smooth test functions $h(\mathbf{x})$, $\Xi(h)$ and $\hat{\xi}(h)$ are operators defined on the dense domains

$$\mathscr{D}_{\mathscr{E}} = \bigcap_{n \geqslant 0} \mathscr{D}(N_{\mathscr{E}}^n) \subset \mathscr{E}, \quad \mathscr{D} = \bigcap_{n \geqslant 0} \mathscr{D}(N^n) \subset \mathscr{F},$$

respectively. If $h(x_0, \boldsymbol{x})$ has the support in $[0, \infty) \times \mathbb{R}^3$, then $\Xi(h)\Omega_{\mathscr{E}}$ is a vector in $\mathscr{E}_+$.

Lemma 4.2. Let $\Xi_i(\mathbf{x})$ and $\xi_i(\mathbf{x})$ be defined as above and suppose all h_i have support in $[0, \infty) \times \mathbb{R}^3$ and are smooth enough such that $\Xi_i(h_i)$ and $\hat{\xi}_i(h_i)$ are operators defined on the dense domains $\mathscr{D}_{\mathscr{E}} \subset \mathscr{E}$ and $\mathscr{D} \subset \mathscr{F}$ respectively ($i = 1, \ldots N$). Then

$$W \prod_{i=1}^{N} \Xi_i(h_i) \Omega_{\mathscr{E}} = \bar{T}\left(\prod_{i=1}^{N} \hat{\xi}_i(h_i)\right) \Omega. \tag{4.11}$$

Remark. $\bar{T}$ again denotes the anti-time ordering, i.e.

$$\bar{T}\left(\prod_{i=1}^{N}\hat{\xi}_i(h_i)\right)=\sum_{\pi}\pm\int_0^{\infty}dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{N-1}}dt_{N-1}\prod_{i=1}^{N}h_{\pi(i)}(t_i,\boldsymbol{x}_i)\,d\boldsymbol{x}_i$$
$$\times\,\hat{\xi}_{\pi(N)}(t_N,\boldsymbol{x}_N)\ldots\hat{\xi}_{\pi(1)}(t_1,\boldsymbol{x}_1),$$

where $\sum_\pi$ means summation over all permutations π of N elements. Also the sign depends on the number of transpositions of Fermi fields. The proof of the lemma follows from arguments used in the proof of lemma 4.1. In order to construct the time zero field operators in $\mathscr{F}$, we use a limiting procedure. Let $X\in\mathscr{E}_{\tau,\infty}\cap\mathscr{E}_+$ for some $\tau>0$ and let $h(x_0,\boldsymbol{x})=\chi_{[0,t]}(x_0)g(\boldsymbol{x})$ where $\chi_{[s,t]}$ is the characteristic function of the interval $[s,t]$. Then using lemma 4.2 we obtain

$$\xi(g)\,WX=\lim_{t\to 0}\frac{1}{t}\hat{\xi}(\chi_{[0,t]}g)\,WX=\lim_{t\to 0}\frac{1}{t}\,W\Xi(\chi_{[0,t]}g)\,X,\tag{4.12}$$

for $\xi(g)$, $\hat{\xi}(\chi_{[0,t]}g)$, $\Xi(\chi_{[0,t]}g)$ defined as above. Equation (4.12) defines $\xi(g)$ on the dense set $W(\bigcup_{\tau>0}\mathscr{E}_{\tau,\infty}\cap\mathscr{E}_+^0)$. In particular we can define the cutoff fields. The next lemma shows how to reconstruct the free Hamiltonian from the Euclidean time translation:

Lemma 4.3. For $t\geqslant 0$ and $X\in\mathscr{E}_+$, we have

$$WU^tX=e^{-tH_0}WX.$$

The proof follows from equation (4.11) and the linearity and boundedness of W.

5. The Feynman–Kac formula

In this section we prove a relativistic Feynman–Kac formula for a system where scalar bosons and spin $\frac{1}{2}$ fermions are coupled through a Yukawa interaction and a polynomial boson self-interaction. We start with some technical preparations. In what follows $P(x)$ will denote a real polynomial of degree $2n$, the leading coefficient being equal to one. We define

$$\mathbb{P}_\kappa(t,g)=\lambda\int:P(\Phi_\kappa(t,\boldsymbol{x})):g(\boldsymbol{x})\,d^3x\restriction\mathscr{D}(N_{\mathscr{E}}^n)^-,$$

$$\mathbb{P}_\kappa^\Delta(t,g)=\int_t^{t+\Delta}\mathbb{P}_\kappa(\tau,g)\,d\tau,$$

$$P_\kappa(g)=\lambda\int:P(\varphi_\kappa(\boldsymbol{x})):g(\boldsymbol{x})\,d^3x\restriction\mathscr{D}(N^n)^-$$

$$0\leqslant g\leqslant 1,\quad g\in\mathscr{L}^1(\mathbb{R}^3)\cap\mathscr{L}^2(\mathbb{R}^3),\quad\lambda\leqslant 0.$$

Remember that

$$\Phi_\kappa(t,\boldsymbol{x})=\int\Phi(t,\boldsymbol{y})\,\chi_\kappa(\boldsymbol{y}-\boldsymbol{x})\,d^3y\restriction\mathscr{D}(N_{\mathscr{E}}^{1/2})^-$$

is a self-adjoint operator in $\mathscr{E}$, if the cutoff function is in $\mathscr{H}^{-1/2}(\mathbb{R}^3)$. For example this is true for

$$\chi_\kappa(\boldsymbol{x})=(2\pi)^{-3}\int_{|k|\leqslant\kappa}e^{i\mathbf{k}\cdot\mathbf{x}}d^3k.$$

Standard arguments (see e.g. [GJ 1] show that $\mathbb{P}_\kappa(t,g)$ and $\mathbb{P}_\kappa^\Delta(t,g)$ (respectively $P_\kappa(g)$) are self-adjoint operators, bounded below by a constant C_κ depending on κ, and essentially self-adjoint (e.s.a.) on $\mathscr{D}(N_{\mathscr{E}}^n) \subset \mathscr{E}$ (respectively $\mathscr{D}(N^n) \subset \tilde{\mathscr{F}}$). Thus for $s \geqslant 0$, $\exp[-s\mathbb{P}_\kappa(t,g)]$ and $\exp[-sP_\kappa(g)]$ are quasibounded semi-groups of self-adjoint operators. Furthermore

$$s-\lim_{m\to\infty} \exp\left[-\frac{t}{m}\sum_{k=1}^{m} \mathbb{P}_\kappa\left(\frac{kt}{m},g\right)\right] = s-\lim_{m\to\infty}\prod_{k=1}^{m}\exp\left[-\frac{t}{m}\mathbb{P}_\kappa\left(\frac{kt}{m},g\right)\right]$$

$$= \exp\left[-\int_0^t \mathbb{P}_\kappa(\tau,g)\,d\tau\right]. \tag{5.1}$$

Relation (5.1) follows using the Duhamel formula and the fact that

$$\lim_{m\to\infty}\frac{t}{m}\sum_{k=1}^{m}\mathbb{P}_\kappa\left(\frac{kt}{m},g\right) = \int_0^t \mathbb{P}_\kappa(\tau,g)\,d\tau$$

in $\mathscr{D}(N_{\mathscr{E}}^n)$. Since $\mathbb{P}_\kappa(0,g)$ leaves $\mathscr{E}_+$ invariant, lemma 4.2 implies

$$W\mathbb{P}_\kappa(0,g)\,X = P_\kappa(g)\,WX,$$

for

$$X \in \mathscr{E}_+ \cap \mathscr{D}(\mathbb{P}_\kappa(0,g)). \tag{5.2}$$

Let z be in the resolvent set of $\mathbb{P}_\kappa(0,g)$. Again since $\mathbb{P}_\kappa(0,g)$ leaves $\mathscr{E}_+$ invariant, the same is true for $(z+\mathbb{P}_\kappa(0,g))^{-1}$. Set $X = (z+\mathbb{P}_\kappa(0,g))^{-1}\,Y$ for $Y \in \mathscr{E}_+$. Then (5.2) gives

$$W\mathbb{P}_\kappa(0,g)\,X + zWX = P_\kappa(g)\,WX + zWX,$$

and hence

$$W(z+\mathbb{P}_\kappa(0,g))^{-1}\,Y = (z+P_\kappa(g))^{-1}\,WY.$$

But then we also have for any $s \geqslant 0$

$$W\,e^{-s\mathbb{P}_\kappa(0,g)}\,Y = e^{-sP_\kappa(g)}\,WY. \tag{5.3}$$

The discussion of the Yukawa part of the interaction is a bit more complicated because the Euclidean action turns out to be a non-Hermitian operator. We define the following operator in $\mathscr{D}(N_{\mathscr{E}})$:

$$\mathbb{Q}(h,\kappa) = \lambda'\int d^4x\,h(\mathbf{x})\sum_\alpha :\Psi_{\alpha,\kappa}^{(2)}(\mathbf{x})\,\Psi_{\alpha,\kappa}^{(1)}(\mathbf{x}):\Phi(\mathbf{x})$$

and set

$$\mathbb{Q}_\kappa^\Delta(t,g) = \mathbb{Q}(h,\kappa)$$

if

$$h(\mathbf{x}) = \chi_{[t,t+\Delta]}(x_0)\,g(\boldsymbol{x}),$$

where $g(\mathbf{x})$ is as above and $\chi_{[s,t]}$ is the characteristic function of the interval $[s,t]$. Using the relation $\Theta N_{\mathscr{E}}\Theta^{-1} = N_{\mathscr{E}}$, it is easily verified that in $D(N_{\mathscr{E}})$

$$\mathbb{Q}(h,\kappa) = \Theta\mathbb{Q}(\vartheta h,\kappa)^*\,\Theta^{-1}. \tag{5.4}$$

On $\mathscr{D}(Ne^{tH_0})$, $t \geqslant 0$ we define

$$\hat{Q}_\kappa(t,g) = \lambda' \int g(\mathbf{x})\, d^3 x \sum_\alpha : \hat{\bar{\psi}}_{\alpha,\kappa}(t, \boldsymbol{x})\, \hat{\psi}_{\alpha,\kappa}(t, \boldsymbol{x}) : \hat{\varphi}_\kappa(t, \boldsymbol{x})$$
$$= e^{-tH_0} Q_\kappa(g)\, e^{tH_0},$$

where $Q_\kappa(g) = \hat{Q}_\kappa(0,g)$ is e.s.a. in $\mathscr{D}(N)$. We also define the doubly cutoff Euclidean Yukawa action $\mathbb{Q}^\Delta_{\kappa_0,\kappa}(t,g)$ to be the same as $\mathbb{Q}^\Delta_\kappa(t,g)$ but with $\Psi^{(i)}_{\alpha,\kappa}(\mathbf{x})$ replaced by $\Psi^{(i)}_{\alpha,\kappa_0,\kappa}(\mathbf{x})$ where κ_0 denotes a cutoff in the x_0-direction. The following estimates are standard, see e.g. [GJ 1], proposition 1.2.3.

$$\|\mathbb{Q}^\Delta_\kappa(t,g)(N_\mathscr{E} + 1)^{-1}\| \leqslant C_{1,\kappa}, \tag{5.5}$$

$$\|\mathbb{Q}^\Delta_{\kappa_0,\kappa}(t,g)(N_\mathscr{E} + 1)^{-1/2}\| \leqslant C_{2,\kappa_0,\kappa}, \tag{5.6}$$

$$\|Q_\kappa(g)(H_0 + 1)^{-1/2}\| \leqslant C_{3,\kappa}, \tag{5.7}$$

$$\|[\mathbb{Q}^\Delta_\kappa(t,g) - \mathbb{Q}^\Delta_{\kappa_0,\kappa}(t,g)]\,(N_\mathscr{E} + 1)^{-1}\| \leqslant \rho(\kappa_0) C_{4,\kappa},$$

with

$$\lim_{\kappa_0 \to \infty} \rho(\kappa_0) = 0. \tag{5.8}$$

For fixed g and κ, these estimates are uniform when t and Δ vary in a compact set. They are strong enough to permit the construction of an exponential $\exp(-\mathbb{Q}^\Delta_\kappa(t,g))$ and to relate it to $\exp(-t(H_0 + Q(g)))$. Write $\mathbb{Q}, \mathbb{Q}_{\kappa_0}$ for $\mathbb{Q}^\Delta_\kappa(t,g)$ and $\mathbb{Q}^\Delta_{\kappa_0,\kappa}(t,g)$ respectively and set $\mathbb{Q}'_{\kappa_0} = \mathbb{Q} - \mathbb{Q}_{\kappa_0}$. Then for any $n \in \mathbb{Z}^+$

$$\begin{aligned}
\|\mathbb{Q}^n \Omega_\mathscr{E}\| = \|(\mathbb{Q}_{\kappa_0} + \mathbb{Q}'_{\kappa_0})^n \Omega_\mathscr{E}\| &\leqslant \sum_{k=0}^{n} \binom{n}{k} \|(\mathbb{Q}'_{\kappa_0})^k (\mathbb{Q}_{\kappa_0})^{n-k} \Omega_\mathscr{E}\| \\
&\leqslant \sum_{k=0}^{n} \binom{n}{k} \|\mathbb{Q}'_{\kappa_0}(N_\mathscr{E} + 1)^{-1}\|^k \|\mathbb{Q}_{\kappa_0}(N_\mathscr{E} + 1)^{-1/2}\|^{n-k} \\
&\quad \times (3(n) \ldots 3(n-k+1))(3.(n-k)\ldots 3.2)^{1/2} \\
&\leqslant \sum_{k=0}^{n} (\rho(\kappa_0)\, C_4)^k\, C_2^{n-k} \binom{n}{k} 3^n \frac{n!}{((n-k)!)^{1/2}} \\
&\leqslant (6(1 + C_2)\, C_4\, \rho(\kappa_0))^n\, n! \sum_{k=0}^{n} \frac{(\rho(\kappa_0)\, C_4)^{-k}}{(k!)^{1/2}} \leqslant C^n_{5\kappa_0}\, n!\, C_{6\kappa_0},
\end{aligned} \tag{5.9}$$

where

$$C_{5\kappa_0} = 3(1 + C_2)\, C_4\, \rho(\kappa_0), \qquad C_{6\kappa_0} = \sum_{k=0}^{\infty} \frac{(\rho(\kappa_0)\, C_4)^{-k}}{(k!)^{1/2}}.$$

Note that $C_{5\kappa_0} < 1$ for κ_0 large enough. Thus

$$s - \lim_{N \to \infty} \sum_{n=0}^{N} \frac{(-\mathbb{Q})^n}{n!}\, \Omega_\mathscr{E}$$

exists and defines $\exp - \mathbb{Q}_\kappa^\Delta(t,g)\,\Omega_\mathscr{E}$. Replacing $\Omega_\mathscr{E}$ by a vector $X \in \mathscr{D}_{\mathscr{E},0}$ ($\mathscr{D}_{\mathscr{E},0}$ = vectors with only a finite number of particles) or by a vector in the domain of $e^{\alpha N_\mathscr{E}}$, $\alpha > 0$ sufficiently large, and repeating the above argument, we also find that

$$\exp[-\mathbb{Q}_\kappa^\Delta(t,g)]\,X = s - \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(-\mathbb{Q}_\kappa^\Delta(t,g))^n}{n!} X$$

exists. By similar arguments, we may show that in $\mathscr{D}_{\mathscr{E},0}$ for any $m \in \mathbb{Z}^+$, $t_1, \ldots t_m \in \mathbb{R}^1$,

$$\prod_{i=1}^{m} \exp[-\mathbb{Q}_\kappa^\Delta(t_i,g)]\,X = \exp\left[-\sum_{i=1}^{m} \mathbb{Q}_\kappa^\Delta(t_i,g)\right] X = \prod_{i=1}^{m} (U^{t_i} \exp[-\mathbb{Q}_\kappa^\Delta(0,g)]\,U^{-t_i})\,X. \tag{5.10}$$

Now let A be $\exp[-t\mathbb{P}_\kappa(\tau,g)]$ or $\exp[-\int_{t_1}^{t_2} \mathbb{P}_\kappa(t,g)\,dt]$. Then A is a bounded operator and commutes with $\mathbb{Q}_\kappa(t,g)$. This allows us to extend the domain of the operator $\exp[-\mathbb{Q}_\kappa(t,g)]$ from $\mathscr{D}_{\mathscr{E},0}$ to vectors of the form AX ($X \in \mathscr{D}_{\mathscr{E},0}$). If $X \in \mathscr{D}_{\mathscr{E},0}$, then

$$\begin{aligned}\exp[-\mathbb{Q}_\kappa^\Delta(t,g)]\,AX &= s - \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(-\mathbb{Q}_\kappa^\Delta(t,g))^n}{n!} AX \\ &= s - \lim_{N\to\infty} A \sum_{n=0}^{N} \frac{(-\mathbb{Q}_\kappa^\Delta(t,g))^n}{n!} X \\ &= A \exp[-\mathbb{Q}_\kappa^\Delta(t,g)]\,X.\end{aligned} \tag{5.11}$$

Therefore

$$A \exp[-\mathbb{Q}_\kappa^\Delta(t,g)] \subset \exp[-\mathbb{Q}_\kappa^\Delta(t,g)]\,A. \tag{5.12}$$

Thus $\exp[-\mathbb{Q}_\kappa^\Delta(t,g)]$ is defined on vectors of the form $\prod_{i=1}^N A_i \prod_{j=1}^M B_j X$, where $X \in \mathscr{D}_{\mathscr{E},0}$, A_i is as above, $B_j = \exp[-\mathbb{Q}_\kappa^\Delta\ (t_j,g)]$. Call this set $\mathscr{D}'$. For

$$\exp[-\mathbb{Q}_\kappa^\Delta(t,g)]\exp[-\mathbb{P}_\kappa^{\Delta'}(t',g)]$$

we write symbolically $\exp[-\mathbb{Q}_\kappa^\Delta(t,g) - \mathbb{P}_\kappa^{\Delta'}(t',g)]$. We now want to prove that for $X \in \mathscr{E}_+ \cap \mathscr{D}'$ and $\Delta > 0$

$$W \exp[-\mathbb{Q}_\kappa^\Delta(0,g)]\,U^\Delta X = \exp[-\Delta(H_0 + Q_\kappa(g))]\,WX. \tag{5.13}$$

Using the boundedness of W we have

$$\begin{aligned}&W \exp[-\mathbb{Q}_\kappa^\Delta(0,g)]\,U^\Delta X \\ &= s - \lim_{N\to\infty} \sum_{n=0}^{N} W \frac{(-\mathbb{Q}_\kappa^\Delta(0,g))^n}{n!} U^\Delta X \\ &= s - \lim_{N\to\infty} \sum_{n=0}^{N} (-1)^n \int_0^\Delta dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n\, \hat{Q}_\kappa(t_n,g) \ldots \hat{Q}_\kappa(t_1,g)\, e^{-\Delta H_0} WX \\ &= s - \lim_{N\to\infty} \sum_{n=0}^{N} (-1)^n \int_0^\Delta dt_1 \ldots \int_0^{t_{n-1}} dt_n\, e^{-t_n H_0} Q_\kappa(g)\, e^{-(t_{n-1}-t_n)H_0} \\ &\qquad \ldots Q_\kappa(g)\, e^{-(\Delta - t_1)H_0}\, WX \\ &= e^{-\Delta(H_0 + Q_\kappa(g))}\, WX,\end{aligned} \tag{5.14}$$

which is (5.13). Note that for the second equality in (5.14), we have used lemma 4.2; the third equality follows from the definition of $Q_\kappa(g)$. The last equality follows from the estimate (5.7) and a standard theorem on perturbations of semigroups of type (C_0), see e.g. [HP 1], page 400, corollary 1. With these preliminaries, the proof of the Feynman–Kac formula is straightforward:

Theorem. Let the notation be as above and let $X \in \mathscr{E}_+$, $0 \leqslant t < \infty$, $0 < \kappa < \infty$. *Then*

$$W \exp[-\mathbb{Q}_\kappa^t(0,g) - \mathbb{P}_\kappa^t(0,g)]\, U^t X = \exp[-t(H_0 + Q_\kappa(g) + P_\kappa(g))]\, WX.$$

Proof. Using (5.1), (5.10) and (5.12) we have with $\Delta = t/m$

$$W \exp[-\mathbb{Q}_\kappa^t(0,g) - \mathbb{P}_\kappa^t(0,g)]\, U^t X$$

$$= s - \lim_{m\to\infty} W \left\{ \prod_{k=0}^{m-1} U^{k\Delta} \exp[-\Delta \mathbb{P}_\kappa(0,g)] \exp[-\mathbb{Q}_\kappa^\Delta(0,g)]\, U^{-k\Delta} \right\} U^{m\Delta} X$$

$$= s - \lim_{m\to\infty} W \{\exp[-\Delta \mathbb{P}_\kappa(0,g)] \exp[-\mathbb{Q}_\kappa^\Delta(0,g)]\, U^\Delta\}^m X$$

$$= s - \lim_{m\to\infty} \{\exp[-\Delta P_\kappa(g)] \exp[-\Delta(H_0 + Q_\kappa(g))]\}^m WX$$

(using equations (5.3) and (5.13))

$$= \exp[-t(H_0 + Q_\kappa(g) + P_\kappa(g))]\, WX, \tag{5.15}$$

which proves the theorem. To obtain the last equality in (5.15) we used the Trotter product formula (see e.g. [Ne 2]) which is applicable because $H_0 + Q_\kappa(g)$, $P_\kappa(g)$ and $H_0 + Q_\kappa(g) + P_\kappa(g)$ are generators of (C_0) contraction semi-groups [HP 1].

6. Application: Approximate Euclidean Green's functions

In another publication [OS 2] we established a set of conditions under which the Euclidean Green's functions may be used to define a Wightman field theory. In Chapter 7 of [OS 2] we also indicated how these conditions could possibly be verified in constructive quantum field theories. In this section we will show how approximate (cutoff) Euclidean Green's functions can be defined in terms of Euclidean fields. They satisfy the axiom of symmetry and for a particular choice of the cutoffs also the positivity axiom. Hence, if the limit 'no cutoffs' exists, the limiting Euclidean Green's functions will still have the properties of positivity and symmetry.

As an explicit example we consider the well-known $\lambda P(\varphi)_2$ model for small λ. We show that the above results combined with the contour expansion estimate, recently established by Dimock, Glimm and Spencer ([DG 1], [GS 1], see especially Theorem 2.1 in [GS 1]), are sufficient to verify all the axioms of [OS 2] for the limiting Euclidean Green's functions.

Let h be a measurable function on $\mathbb{R}^4$ with compact support, such that $0 \leqslant h \leqslant 1$. We define

$$V(h,\kappa) = \int \{\lambda : P(\Phi_\kappa(\mathbf{x})): + \lambda' \sum_\alpha : \Psi_{\alpha,\kappa}^{(2)}(\mathbf{x})\, \Psi_{\alpha,\kappa}^{(1)}(\mathbf{x})\, \Phi_\kappa(\mathbf{x})\}\, h(\mathbf{x})\, d^4x \tag{6.1}$$

and

$$V(t,g,\kappa) = V(h,\kappa) \tag{6.2}$$

when $h(\mathbf{x}) = \chi_{[0,t]}(x_0)g(\boldsymbol{x})$. Here $\chi_{[s,t]}$ denotes the characteristic function of the interval $[s,t]$. We note that relation (5.4) may be extended to

$$V(h,\kappa) = \Theta V(\vartheta h,\kappa)^* \Theta^{-1}. \tag{6.3}$$

We define the approximate Euclidean Green's functions for a model of Yukawa plus boson self-interaction in 4 dimensions to be

$$\mathfrak{S}^{(k)}_{n,k,\kappa}(\mathbf{x}_1,\ldots \mathbf{x}_n) = \langle \Sigma^{(k_1)}(\mathbf{x}_1)\ldots \Sigma^{(k_n)}(\mathbf{x}_n)\, e^{-V(h,\kappa)}\rangle_0 / \langle e^{-V(h,\kappa)}\rangle_0 \tag{6.4}$$

where $(k) = (k_1,\ldots k_n)$ and each $\sum^{(k_j)}(\mathbf{x}_j)$ is either $\Phi(\mathbf{x}_j)$ or $\Psi^{(i)}_\alpha(\mathbf{x}_j)$ $(i = 1,2;\ \alpha = 1,\ldots 4)$. h and κ parametrize the approximation. We note that for $h = \chi_{[-t,t]}g$, we have

$$\langle e^{-V(h,\kappa)}\rangle_0 = (\Theta\, e^{-V(t,g,\kappa)}\,\Omega_{\mathscr{E}}, e^{-V(t,g,\kappa)}\,\Omega_{\mathscr{E}}\rangle_0 \geqslant 0. \tag{6.5}$$

Now this expression is not zero since the Feynman–Kac formula shows it is equal to

$$\|e^{-t(H_0+P_\kappa(g)+Q_\kappa(g))}\,\Omega\|^2.$$

Next for fixed t, g, κ let $X \in \mathscr{E}_+$ be any finite linear combination of vectors $Y \in \mathscr{E}_+$ of the form

$$Y = \int \Sigma^{(k_1)}(\mathbf{x}_1)\ldots \Sigma^{(k_n)}(\mathbf{x}_n)\, e^{-V(t,g,\kappa)}\,\Omega_{\mathscr{E}} f(\mathbf{x}_1\ldots \mathbf{x}_n)\, d^4x_1\ldots d^4x_n$$

n, k_1, ... k_n arbitrary, $f \in \mathscr{S}(\mathbb{R}^{4n})$, supp $f \subset \{O < x_{1_0} \ldots < x_{n_0}\}$.

Then with the particular choice $h = \chi_{[-t,t]}g$ we obtain the positivity condition (E2) of [OS 2] if we write the inequality

$$(\Theta X, X) \geqslant 0$$

in lemma (4.1) in terms of the $\mathfrak{S}_{n,h,\kappa}$, using relations (4.1) and (6.3). Also, since the $\sum$'s commute or anti-commute, the symmetry condition (E3) of [OS 2] holds for the approximate Euclidean Green's functions $\mathfrak{S}_{n,h,\kappa}$.

With an appropriate redefinition of the Euclidean fields all the results of this paper can be immediately obtained for space dimensions different from 3.

Consider now the example of the $\lambda P(\varphi)_2$ model. The cutoff Euclidean Green's functions are given by

$$\mathfrak{S}_{n,h}(\mathbf{x}_1\ldots \mathbf{x}_n) = \langle \Phi(\mathbf{x}_1)\ldots \Phi(\mathbf{x}_n)\, e^{-V(h)}\rangle_0 / \langle e^{-V(h)}\rangle_0$$

$(x_i \in \mathbb{R}^2)$ and define multilinear forms in $C_0^\infty(\mathbb{R}^2)\times\ldots\times C_0^\infty(\mathbb{R}^2)$. No ultraviolet cutoff κ is needed. The Euclidean action is given by

$$V(h) = \lambda \int :P(\Phi(\mathbf{x})):h(\mathbf{x})\, d^2x. \tag{6.6}$$

We note that the definition (6.6) coincides with (1.4) of ref. [GS 1]. The following estimates are simple consequences of the contour expansion estimate (2.9–2.10) in [GS 1] (λ sufficiently small)

$$|\mathfrak{S}_{n,h}(f_1,\ldots f_n)| < C_1 \prod_{i=1}^n |f_i|_s, \tag{6.7}$$

$$|\mathfrak{S}_{n,h_1}(f_1,\ldots f_n) - \mathfrak{S}_{n,h_2}(f_1,\ldots f_n)| < C_2\, e^{-m_1 d}, \tag{6.8}$$

$$|\mathfrak{S}_{m+n,h}(f_1,\ldots f_n, f^a_{n+1},\ldots f^a_{n+m}) - \mathfrak{S}_{n,h}(f_1,\ldots f_n)\,\mathfrak{S}_{m,h}(f_{n+1},\ldots f_{n+m})| < C_3\, e^{-m_1 a}. \tag{6.9}$$

Here $f_i \in C_0^\infty(\mathbb{R}^2)$, C_1 is a constant depending only on n. $|\cdot|_s$ denotes some Schwartz's norm, C_2 and C_3 are constants depending on n and the f_i, but not on h, h_1, h_2. m_1 is some positive constant independent of h, h_1, h_2, the f_i and n. d is the distance from the support of the f_i to the set where $h_1 \neq h_2$ and $f^a(\mathbf{x}) = f(x_0, x_1 - a)$, $(\mathbf{x} = (x_0, x_1))$.

In ref. [GS 1] the estimates (6.8) and (6.9) are stated as theorems (1.1) and (1.3), respectively. Now estimate (6.8) implies that

$$\lim_{h \to 1} \mathfrak{S}_{n,h}(f_1, \ldots f_n) \equiv \mathfrak{S}_n(f_1, \ldots f_n) \tag{6.10}$$

exists and defines a multi-linear function in $C_0^\infty(\mathbb{R}^2) \times \ldots \times C_0^\infty(\mathbb{R}^2)$. Since estimate (6.7) is uniform in h, we conclude that it also holds if we replace $\mathfrak{S}_{n,h}$ by $\mathfrak{S}_n$. Hence by the nuclear theorem, $\mathfrak{S}_n$ can be uniquely extended to a distribution in $\mathscr{S}'(\mathbb{R}^{2n})$, which we again denote by $\mathfrak{S}_n$. Defining $\mathfrak{S}_0 \equiv 1$, we now verify axioms (E0–E4) of [OS 2] for $\{\mathfrak{S}_n\}_{n=0}^\infty$. (EO) follows from $\mathfrak{S}_n \in \mathscr{S}'(\mathbb{R}^{2n})$. Euclidean invariance (E1) follows from the uniqueness of the limit in (6.10). Again by the uniqueness of the limit, positivity (E2) follows from our previous discussion which showed that already the $\mathfrak{S}_{n,h}$ satisfy (E2) if h is the characteristic function of any cube centred at the origin. The symmetry (E3) is also true since it is true for $\mathfrak{S}_{n,h}$. Finally the cluster property (E4) follows from (6.9), which in addition implies the mass gap as shown in ref. [GS 1].

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