Zeitschrift: Helvetica Physica Acta
Band: 46 (1973)
Heft: 6

Artikel: Fluctuations in some mean-field models in quantum statistics
Autor: Wreszinski, Walter F.
DOI: https://doi.org/10.5169/seals-114514

Helvetica Physica Acta
Vol. 46, 1974.

Birkhäuser Verlag Basel

Fluctuations in Some Mean-Field Models in Quantum Statistics

by **Walter F. Wreszinski**[1])

Seminar fuer Theoretische Physik der E.T.H., 8049 Zuerich

(21. IX. 73)

Abstract. Following the work on the normality of fluctuations in the Dicke maser model in Ref. [1], we study equilibrium fluctuations and some of their properties in a few soluble ('mean-field-like') models in quantum statistical mechanics which exhibit a phase transition.

1. Introduction

In this article we shall study normality and a few other properties of fluctuations in some soluble models in quantum statistics. The models are, with one exception, the imperfect Bose gas-*mean-field* models. The method of proof is, however, almost identical in all cases, and certainly *not* extensible to non-mean-field models as, for instance, the Heisenberg model. Therefore, we regard this one exception as 'mean-field-like' with respect to fluctuations, and this is the justification for the terminology in the title and abstract.

The motivation for such a study is three-fold: firstly, intrinsically, normality of fluctuations plays a role in the axiomatic foundations of statistical mechanics, and therefore deserves to be studied in its own right (see, e.g., Ref. [13]). Secondly, and more technically, it involves a much finer type of limiting procedure than the one associated with intensive and local quantities. This is, in particular, reflected in the non-equivalence of two definitions of normality introduced (see Remark 3.2) and in the fact that the 'asymptotically exact' [1] Hamiltonians (leading to linear Heisenberg equations of motion) for local and intensive observables, on the one hand, and fluctuation observables, on the other, are, in general, different [1]. Thirdly, fluctuations have been studied rigorously to a rather small extent: in particular, Ruelle's book [12] does not tackle this subject.

2. Notations and Definitions

We consider throughout models described, in a unified notation, by a Hamiltonian H_L for a finite region, labelled by 'L' (which we take, for simplicity, to be the volume of a cubical box containing the system, ranging over a set which we identify with Z_+), which is a self-adjoint operator on a Hilbert space $\mathscr{H}_L$. The trace on $\mathscr{H}_L$ will be denoted by tr_L and the number operator for the region L, suitably defined, by N_L. Each model

[1]) Supported financially by the Fundação de Amparo à Pesquisa do Estado de São Paulo, São Paulo, Brazil.

exhibits a phase transition, either at a temperature T_c, for fixed density, or at a density ρ_c, for fixed temperature. The grand-partition function, grand-canonical density operator, pressure and finite volume Gibbs state over $\mathscr{A}_L = B(\mathscr{H}_L)$ are given, respectively, by

$$Z^L_{\beta,\mu} = \mathrm{tr}_L \exp[-\beta(H_L - \mu N_L)] \tag{2.1}$$

$$\sigma^L_{\beta,\mu} = \frac{1}{Z^L_{\beta,\mu}} \exp[-\beta(H_L - \mu N_L)] \tag{2.2}$$

$$p^L_{\beta,\mu} = \frac{\beta^{-1}}{L} \log Z^L_{\beta,\mu} \tag{2.3}$$

$$\rho^L_{\beta,\mu}(\cdot) = \mathrm{tr}_L(\sigma^L_{\rho,\mu}\cdot). \tag{2.4}$$

$\rho^L_{\beta,\mu}(X)$ will also be defined for some unbounded operators X on $\mathscr{H}_L$. Let $\mathscr{A}_\Gamma$ be the normed *-algebra generated by $\bigcup_{L \in Z_+} \mathscr{A}_L$, and $\mathscr{A}$ = norm closure of $\mathscr{A}_\Gamma$ be the algebra of quasi-local observables. In all models we shall analyse (with the possible exception of the ideal Bose gas), the following equality is seen to hold:

$$\exists \lim_{L\to\infty} \rho^L_{\beta,\mu}(A) \equiv \rho_{\beta,\mu}(A) \forall A \in \mathscr{A}_\Gamma. \tag{2.5}$$

Hence, $\rho_{\beta,\mu}$ extends to a state on $\mathscr{A}$, with the aid of which one may, under certain conditions, given in Ref. [6], define the dynamics of the infinite system. These conditions are met in the case of the strong-coupling B.C.S. model [3, 4] and in the Dicke maser model [1] if the conjecture in Ref. [1], p. 393, holds.

The canonical ensemble corresponds to putting $\mu = 0$ in (2.1)–(2.4) and the corresponding notation will be Z^L_β, σ^L_β, $-f^L_\beta$ (where f^L_β is the free energy per unit volume) and $\rho^L_\beta(\cdot)$. We shall call an operator on $\mathscr{H}_L$ *extensive* if it is an integral of local operators, suitably defined in case the local operators are unbounded (as the number operator N_L) or, if the region is a lattice and the operators are bounded in norm, a sum of operators localized at the sites of L, as

$$S^{(i)}_L = \tfrac{1}{2} \sum_{p=1}^{L} \sigma^{(i)}_p, \quad \sigma^{(i)}_p, i \in \{1, 2, 3\},$$

Pauli matrices on

$$\mathbb{C}^2(p) \tag{2.6}$$

on

$$\mathscr{H}^C_L \equiv \bigotimes_{i=1}^{L} \mathbb{C}^2(i). \tag{2.7}$$

Let J be a subset of Z_+. For each model and each L, β, μ we shall consider a set of operators

$$S_L(\beta,\mu) = \{A^{(i)}_L, i \in J, \text{ all } A^{(i)}_L \text{ extensive}\} \tag{2.8}$$

and denote with the corresponding small *Greek* letters the corresponding *intensive* operators

$$\alpha_L^{(i)} = A_L^{(i)}/L, \quad i \in J. \tag{2.9}$$

For all $\beta \in (0, \infty)$ and in all mean-field models we shall analyse the restriction of the Gibbs state to the quasi-local algebra $\mathscr{A}^s$ generated by the spin operators, denoted by ρ_β, is translation-invariant. Hence, if l is the dimension of the Lie algebra generated by the extensive spin operators $A_L^{(i)}$, it follows from Refs. [15] and [16] that ρ_β is *classical*, i.e., for all polynomials $P_L = P(\alpha_L^{(1)}, \ldots, \alpha_L^{(l)})$ in the intensive operators, the

$$\lim_{L\to\infty} \rho_\beta(P_L)$$

exists. Since the intensive operators are uniformly bounded, it also follows that, for all polynomials $P_L = P(\alpha_L^{(1)}, \ldots, \alpha_L^{(l)})$,

$$\lim_{L\to\infty} \rho_\beta^L(P_L) = \lim_{L\to\infty} \rho_\beta(P_L). \tag{2.10}$$

As described in Ref. [2], it follows from [15] that there exists a probability measure μ_{ρ_β} on the 'phase-space' $\mathbb{R}^l$ with support in $|\alpha^{(i)}| \leqslant \|A^{(i)}\|$, $1 \leqslant i \leqslant l$, such that, for all monomials,

$$\lim_{L\to\infty} \rho_\beta((\alpha_L^{(1)})^{m(1)} \ldots (\alpha_L^{(l)})^{m(l)}) = \int \mu_{\rho_\beta}(d\boldsymbol{\alpha})\, (\alpha^{(1)})^{m(1)} \ldots (\alpha^{(l)})^{m(l)}.$$

A state ρ on $\mathscr{A}^s$ is pure if μ_ρ is concentrated on one point $\boldsymbol{\alpha} \equiv (\alpha^{(1)}, \ldots, \alpha^{(l)}) \in \mathbb{R}^l$, and it is known [15, 16] that ρ is pure if it is an extremal invariant state of all translation-invariant states of $\mathscr{A}^s$. Examples of pure states will appear in Sections 3 and 5.

Given $S_L(\beta, \mu)$ by (2.8), we let $J = [1, k]$ and denote

$$\lim_{L\to\infty} \rho_{\beta,\mu}(\alpha_L^{(i)}) = \alpha_{\beta,\mu}^{(i)}, \quad \forall i \in J. \tag{2.11}$$

For each $A_L^{(i)} \in S_L(\beta, \mu)$, $i \in J$, we define the corresponding $((\beta, \mu)$-dependent) *fluctuation operator* for volume L, denoted by the corresponding small *roman* letter, by

$$a_L(\alpha_{\beta,\mu}^{(i)}) = (A_L^{(i)} - L\alpha_{\beta,\mu}^{(i)})/\sqrt{L} = \sqrt{L}(\alpha_L^{(i)} - \alpha_{\beta,\mu}^{(i)}). \tag{2.12}$$

Corresponding to each $\{r_i, i \in [1, k]\} \subset Z_+$ we define a fluctuation operator for the set $S_L(\beta, \mu)$ by

$$F_{\beta,\mu}^L(\{r_i, i \in [1, k]\}) = \prod_{i=1}^k (a_L(\alpha_{\beta,\mu}^{(i)}))^{r_i}. \tag{2.13}$$

Since we shall be computing limits of Gibbs states on operators of the form (2.13), which are not uniformly bounded in L, rather than products of operators $(\alpha_L^{(i)} - \alpha_{\beta,\mu}^{(i)})$, the argument for the validity of the analogue of (2.10) when the P_L are replaced by $F_{\beta,\mu}^L(\{r_i, i \in J\})$ fails, and we will eventually provide a counter-example to it. Hence, we introduce the following definitions:

Definition 1: $\rho_{\beta,\mu}$ *has normal fluctuations around* $\alpha^{(i)}_{\beta,\mu}$, $i \in J$, if for all subsets $\{r_i, i \in J\}$ of Z_+,

$$\exists \lim_{L\to\infty} \rho_{\beta,\mu}(F^L_{\beta,\mu}(\{r_i, i \in J\})). \tag{2.14}$$

Definition 2: $\rho_{\beta,\mu}$ *has a normal approximation with respect to the set* $S_L(\beta,\mu)$ if, for all subsets $\{r_i, i \in J\}$ of Z_+,

$$\exists \lim_{L\to\infty} \rho^L_{\beta,\mu}(F^L_{\beta,\mu}(\{r_i, i \in J\})). \tag{2.15}$$

We call the limits (2.14) and (2.15), respectively, when they exist, 'normal fluctuation' and 'normally approximating fluctuation'.

Definitions 1 (which is already contained in [2]) and 2 both make precise, in slightly different ways, the statement that an extensive observable fluctuates around its average value in the equilibrium state of a system of an infinite number of degrees of freedom, at a certain (β,μ), by a quantity growing no faster than the square root of the volume.

We refer to [2] for further general properties connected with Definition 1, in particular for the theorem of stability of normality, as described by Definition 1, under time-evolution by a Hamiltonian of type $H_L = LP(\alpha^{(1)}_L, \ldots, \alpha^{(l)}_L)$, and for the precise statement that the operators obtained, in correspondence to the fluctuation operators from the limits (2.14), satisfy boson commutation relations.

In the following Sections **3**, **4** and **5** we study the above definitions for three soluble quantum statistical models. Each section will contain a summary of the relevant facts about the model considered, one or two theorems proving (2.14) and/or (2.15) for a suitable ((β,μ)-dependent) set of operators, and a section about other properties of the normal and normally approximating fluctuations. Some general conclusions will be summarized in the remarks in each section. We refer in particular to Remarks **3.2** and **4.1**.

3. Strong-Coupling B.C.S. Model

The Hamiltonian of the strong-coupling B.C.S. model is [3, 4]:

$$H_L = \mathscr{E}(L - 2S^3_L) - (4\lambda/L)\, S^-_L S^+_L \tag{3.1}$$

where

$$S^\pm_L = S^1_L \pm i S^2_L,$$

$0 < \mathscr{E} < 2\lambda$, and $L =$ volume $=$ number of Cooper pairs (fixing the pair density as unity). The critical temperature is defined by

$$\mathscr{E}/2\lambda = \tanh(\mathscr{E}/T_c). \tag{3.2}$$

We work in the canonical ensemble. It follows from standard methods that

$$f_\beta = \lim_{L\to\infty} f^L_\beta = \begin{cases} f_1(\beta) & \text{if } \beta \geq \beta_c \\ f_2(\beta) & \text{if } \beta \leq \beta_c \end{cases}$$

where

$$f_1(\beta) = -\frac{\mathscr{E}^2}{4\lambda} - 4\lambda\sigma_1(\beta)^2 + \frac{1}{2\beta}\log\frac{1-(2\sigma_1(\beta))^2}{4} + \frac{2\sigma_1(\beta)}{\beta}\operatorname{arctanh}(2\sigma_1(\beta))$$

$$f_2(\beta) = -2\mathscr{E}\sigma_2(\beta) + \frac{1}{2\beta}\log\frac{1-(2\sigma_2(\beta))^2}{4} + \frac{2\sigma_2(\beta)}{\beta}\operatorname{arctanh}(2\sigma_2(\beta))$$

where

$$2\sigma_2(\beta) = \tanh(\beta\mathscr{E})$$

and $\sigma_1(\beta)$ is the unique positive root of the 'gap equation'

$$2\sigma_1(\beta) = \tanh(2\lambda\beta(2\sigma_1(\beta))).$$

From these formulae, it follows that at $\beta = \beta_c$ a second-order phase transition occurs (see [5] for a thorough discussion of the phase transition using spin-waves). It also follows [4, 7] that

$$S_\beta^3 = \begin{cases} \mu_1 = \mathscr{E}/4\lambda & \text{if } \beta \geqslant \beta_c \quad (3.3a) \\ \sigma_2(\beta) & \text{if } \beta \leqslant \beta_c. \quad (3.3b) \end{cases}$$

We define on $\mathscr{H}_L^c$ operators H_L^1 and $H_{L\beta}^{2,\Phi}$, $\varphi \in [0, 2\pi]$, which yield linear Heisenberg equations of motion for the local and intensive observables, and whose corresponding Gibbs states, $\rho_{\beta 1}^L(\cdot)$ and $\rho_{\beta 2\varphi}^L(\cdot)$, $\varphi \in [0, 2\pi]$, are such that, for any local or intensive $A \in \mathscr{A}_\Gamma$,

$$\rho_\beta(A) \equiv \lim_{L\to\infty} \rho_\beta^L(A) = \begin{cases} \lim\limits_{L\to\infty} \rho_{\beta 1}^L(A) & \text{if } \beta \in (0, \beta_c) \quad (3.4a) \\ \lim\limits_{L\to\infty} \displaystyle\int_0^{2\pi} \frac{d\varphi}{2\pi}\, \rho_{\beta 2\varphi}^L(A) & \text{if } \beta \in (\beta_c, \infty). \quad (3.4b) \end{cases}$$

It also follows that, if $f_{\beta 1}^L$ and $f_{\beta 2\varphi}^L$, $\varphi \in [0, 2\pi]$, arbitrary, are the unit-volume free energies constructed from H_L^1 and $H_{L\beta}^{2,\varphi}$, respectively,

$$f_\beta = \begin{cases} \lim\limits_{L\to\infty} f_{\beta 1}^L & \text{if } \beta \in (0, \beta_c) \\ \lim\limits_{L\to\infty} f_{\beta 2\varphi}^L & \text{if } \beta \in (\beta_c, \infty). \end{cases}$$

Hence, these Hamiltonians are called 'asymptotically exact'. Their precise form, both for the B.C.S. and the Dicke maser model, are given in Appendix A. We denote the limits of the states $\rho_{\beta 1}^L$ and $\rho_{\beta 2\varphi}^L$, which exist in the sense of (2.5), by omitting the superscript L.

In the theorems proved, both for the B.C.S. and the Dicke model, it will be necessary to take $S_L(\beta)$ consisting of gauge-invariant operators (i.e., commuting with all operators which commute with the Hamiltonian) for $\beta \in (\beta_c, \infty)$. This is a consequence of the fact that, in both models, there is a spontaneously broken symmetry below T_c, accompanied by a continuous family of ground-state representations of the quasi-local spin algebra in Hilbert spaces $\mathscr{H}^{2,\varphi}$, labelled by a continuous parameter

φ, which are unitarily inequivalent for different φ [1, 4]. To see the reason for this most concretely and clearly, we give a short proof of the following theorem, which is in fact a consequence of more general results stated in Ref. [2]:

Theorem 3.1: Let

$$S_L(\beta) = \begin{cases} \{S_L^{(1)}, S_L^{(2)}, S_L^{(3)}\} & \text{if } \beta \in (0, \beta_c) \\ \{S_L^{(3)}, \mathbf{S}_L^2\} & \text{if } \beta \in (\beta_c, \infty). \end{cases}$$

Then ρ_β has normal fluctuations around $\{S_\beta^1 = S_\beta^2 = 0, S_\beta^3\}$ if $\beta \in (0, \beta_c)$ and around $\{S_\beta^3, \sigma_1(\beta)^2\}$ if $\beta \in (\beta_c, \infty)$.

Proof: Let $\varphi \in [0, 2\pi] \to \tau_{\beta\varphi}(\cdot)$ be defined by

$$\tau_{\beta\varphi}(\cdot) = \begin{cases} \rho_{\beta 1}(\cdot) & \text{if } \beta \in (0, \beta_c) \\ \rho_{\beta 2\varphi}(\cdot) & \text{if } \beta \in (\beta_c, \infty). \end{cases}$$

Then [3, 4, 7]

$$\tau_{\beta\varphi}(S_{p_1}^{(i_1)} \dots S_{p_m}^{(i_m)}) = \begin{cases} (\tfrac{1}{2}\tanh \beta\mathscr{E})^m \mathbf{n}^{(i_1)} \dots \mathbf{n}^{(i_m)}, & \text{where } \mathbf{n} \equiv (0,0,1) \qquad (3.5\text{a}) \\ (\sigma_1(\beta))^m \mathbf{n}_{\varphi\beta}^{(i_1)} \dots \mathbf{n}_{\varphi\beta}^{(i_m)}, & \text{where } \mathbf{n}_{\varphi\beta} \text{ is given by (A.2)} \quad (3.5\text{b}) \end{cases}$$

where $S_p^{(i)} = \frac{1}{2}\sigma_p^{(i)}$. Let $\sigma_L^{(i)} = S_L^{(i)}/L$, and $S_{L\beta}^{(i)}$ be the fluctuation operators associated to $S_L^{(i)}$ by (2.12). We have, for all $r \in Z_+$,

$$\tau_{\beta\varphi}((S_{L\beta}^{(i)})^r) = L^{-r/2} \sum_{k=0}^{r} (-1)^k \binom{r}{k} (L\sigma_1(\beta))^k \, \tau_{\beta\varphi}\left(\sum_{i_1=1}^{L} \dots \sum_{i_{r-k}=1}^{L} S_{i_1}^{(i)} \dots S_{i_{r-k}}^{(i)} \right). \qquad (3.6)$$

Using the product structure of $\tau_{\beta\varphi}$, given by (3.5a, b), $\sigma_p^{(i)2} = \mathbb{1}$ and the symmetry of the binomial coefficients, one may see easily that the left-hand side of (3.6) is zero for r odd. For r even, only the term in $L^{r/2}$ in the sum over k in (3.6) contributes, due again to the product structure of (3.5a, b), $\sigma_p^{(i)2} = \mathbb{1}$, the symmetry of the binomial coefficients and the combinatorial identity

$$\sum_{k=m}^{n} (-1)^k \binom{n}{k} \binom{k}{m} = (-1)^m \delta_{nm} \quad n \geqslant m$$

Together with the fact that $\mathbf{S}_L^2$ and $S_L^{(3)}$ are gauge-invariant, i.e., their average values are φ-independent, this proves (2.14) in the special case $k = 1$. The proof for general k is identical, and only slightly different if monomials in the fluctuation operators of type $((\mathbf{S}_L^2 - L^2\sigma_1(\beta)^2)/L)^r$ are included. ■

Remark 3.1: We may obtain a result similar to Theorem 3.1 using the same set $S_L(\beta) = \{S_L^{(1)}, S_L^{(2)}, S_L^{(3)}\}$ for all $\beta \in (0, \infty)$ if we use a different definition of normal fluctuations, following the ideas of N. N. Bogoliubov Jr. [14]. This will be developed in Appendix B. ■

We now prove that ρ_β has a normal approximation with respect to the set

$$S_L(\beta) = \begin{cases} \{S_L^+, S_L^-, S_L^3\} & \text{if } \beta < \beta_c \qquad (3.7a) \\ \{\sqrt{\mathbf{S}_L^2}, S_L^3\} & \text{if } \beta > \beta_c. \qquad (3.7b) \end{cases}$$

It might have seemed more natural to take $S_L(\beta) = \{\mathbf{S}_L^2, S_L^3\}$ for $\beta > \beta_c$ instead of (3.7b). However, from Theorems 3.1 and 3.3, and the identity

$$(\mathbf{S}_L^2 - L^2\sigma_1^2)/L = ((\sqrt{\mathbf{S}_L^2} - L\sigma_1)/\sqrt{L})^2 + 2\sigma_1\sqrt{L}((\sqrt{\mathbf{S}_L^2} - L\sigma_1)/\sqrt{L})$$

it follows immediately that

$$\nexists \lim_{L\to\infty} \rho_\beta^L(((\mathbf{S}_L^2 - L^2\sigma_1(\beta)^2)/L)^p), \quad \forall p \in Z_+$$

(and also that

$$\nexists \lim_{L\to\infty} \rho_\beta(((\sqrt{\mathbf{S}_L^2} - L\sigma_1(\beta))/\sqrt{L})^p), \quad \forall p \in Z_+).$$

Following the lines of Theorem 3.4 of [1], it is easy to prove

Theorem 3.2: Let

$$H_\beta = 2\mathscr{E} b^* b - 4\lambda\sqrt{2\sigma_2(\beta)}\,(b + b^*)$$

be a Hamiltonian on $\mathscr{F} = L^2(\mathbb{R}^3)$, the one-boson Fock space. Then if $\beta \in (0, \beta_c)$,

$$G_\beta(r_1, r_2, r_3) \equiv \lim_{L\to\infty} \rho_\beta^L((S_L^+/\sqrt{L})^{r_1}\,(S_L^-/\sqrt{L})^{r_2}\,((S_L^3 - L\sigma_2(\beta))/\sqrt{L})^{r_3})$$

$$= \delta_{r_1 r_2} \frac{\mathrm{tr}_{\mathscr{F}}(b^{*r_1} b^{r_1} e^{-\beta H_\beta})}{\mathrm{tr}_{\mathscr{F}}\, e^{-\beta H_\beta}} \cdot (2\sigma_2(\beta))^{r_1}(\tfrac{1}{4} - \sigma_2(\beta)^2)^{r_3/2} \cdot \tau(r_3) \qquad (3.8)$$

where

$$t \in Z_+ \to \tau(t) = \begin{cases} 0 & \text{if } t \text{ is odd} \\ \dfrac{t!}{2^t(t/2)!} & \text{if } t \text{ is even.} \end{cases} \qquad (3.9)$$

Theorem 3.3: If $\beta \in (\beta_c, \infty)$,

$$G_\beta(r_1, r_2) \equiv \lim_{L\to\infty} \rho_\beta^L(((\sqrt{\mathbf{S}_L^2} - L\sigma_1(\beta))/\sqrt{L})^{r_1} \cdot ((S_L^3 - L\mu_1)/\sqrt{L})^{r_2})$$

$$= \tau(r_1) \cdot \tau(r_2) \cdot (8\beta\lambda)^{-r_2/2}\,(2/\beta g''(\sigma_1))^{r_1/2} \qquad (3.10)$$

where $\sigma_1 = \sigma_1(\beta)$, and g is the function defined on the interval $[0, \frac{1}{2}]$ by

$$g(x) = -y(x) - 4\lambda\beta^2 x^2 \qquad (3.11a)$$

whereby

$$x \in [0, \tfrac{1}{2}] \to y(x) = -(\tfrac{1}{2} + x)\log(\tfrac{1}{2} + x) - (\tfrac{1}{2} - x)\log(\tfrac{1}{2} - x). \qquad (3.11b)$$

Proof: The idea of the proof is simple. Let

$$P_L(x) = \frac{\frac{1}{L}\, e^{-L(\varphi(x)-\varphi(\bar{x}))}}{\frac{1}{L}\sum_{i=1}^{L} e^{-L(\varphi(x_i)-\varphi(\bar{x}))}} \tag{3.12a}$$

($x_i = i/L$, $i \in [1, L]$) be one-dimensional probability measures on the interval $[0,1]$, where φ is a function known to have a unique minimum at $x = \bar{x} \in (0,1)$, essentially by Thirring's results [4]. The proof may be reduced to demonstrating that the steps in the following relation

$$\sum_{i=1}^{L} P_L(x_i)\left(\frac{x_i - \bar{x}}{\sqrt{L}}\right)^p \xrightarrow[L\to\infty]{} \frac{\int_{-\infty}^{+\infty} dx\, e^{-1/2x^2\varphi''(\bar{x})}\, x^p}{\int_{-\infty}^{+\infty} dx\, e^{-1/2x^2\varphi''(\bar{x})}}, \tag{3.12b}$$

which one expects intuitively, may be made rigorous.[2])

Consider firstly the case $r_1 = 0$. Let

$$H_\beta^L(r_2) = \rho_\beta^L(((S_L^3 - L\mu_{1L})/\sqrt{L})^{r_2} - (8\beta\lambda)^{-r_2/2}\,\tau(r_2)) \tag{3.13}$$

where $\mu_{1L} = \mu_1 - 1/(2L)$ (see [4]). We prove that

$$\lim_{L\to\infty} H_\beta^L(r_2) = 0 \quad \forall r_2 \in Z_+. \tag{3.14}$$

Using the results of [4], it follows in a straightforward way that

$$H_\beta^L(r_2) = \frac{\frac{(8\lambda\beta)^{-r_2/2}}{2L^2}\sum_{\sigma\in S_1}\beta(L,\sigma)\,M_{L\sigma}(r_2)}{Z_L} \tag{3.15}$$

where the sets S_1, $S_{2\sigma}$ form just a net in the triangle

$$\{\{\sigma, \mu(\sigma)\};\ \sigma \in [0, \tfrac{1}{2}],\ |\mu(\sigma)| \leqslant \sigma\}$$

given in Appendix A ((A.15), (A.16)), and

$$\beta(L,\sigma) = e^{-L(g(\sigma)-g(\sigma_1))}\,\phi_L(\sigma) \tag{3.16}$$

$$M_{L\sigma}(r_2) = \sum_{\mu\in S_{2\sigma}} \exp\left[-\left(-\frac{\mu-\mu_{1L}}{\delta_L}\right)^2\right]\left[\left(\frac{\mu-\mu_{1L}}{\delta_L}\right)^{r_2} - \tau(r_2)\right] \tag{3.17}$$

[2]) This method relies on the fact that the function φ assumes its minimum value in the interior of the interval, for $\beta \in (\beta_c, \infty)$. For $\beta \in (0, \beta_c)$ the minimum is assumed at the boundary, and this is the reason why we were not able to give a unified proof of this theorem for both ranges of temperature and had to rely on the method of Ref. [1] for $\beta \in (0, \beta_c)$ (Theorem 3.2).

$$Z_L = \frac{1}{2L^2} \sum_{\sigma \in S_1} \beta(L, \sigma) \sum_{\mu \in S_{2\sigma}} \exp\left[-\left(\frac{\mu - \mu_{1L}}{\delta_L}\right)^2\right] \tag{3.18}$$

where

$$\delta_L = (2\sqrt{2L\lambda\beta})^{-1} \tag{3.19}$$

and $\phi_L(\cdot)$ is defined in Appendix A ((A.17)). The only properties we shall need are:

$$\phi_L(\sigma) = \frac{2\sigma e^{4\beta\lambda\sigma}}{e^3(1-2\sigma)^{1/2}(1+2\sigma)^{3/2}} + \mathscr{E}(L) \tag{3.20a}$$

where

$$\lim_{L\to\infty} \mathscr{E}(L) = 0. \tag{3.20b}$$

It is readily shown that g has a unique minimum at $\sigma = \sigma_1(\beta)$. Hence, and since the strict inequality $\beta > \beta_c$ is assumed, we may choose $\delta > 0$ so small that for $|\sigma - \sigma_1| \leqslant \delta$,

$$\tfrac{1}{4}(\sigma - \sigma_1)^2 g''(\sigma_1) \leqslant g(\sigma) - g(\sigma_1) \leqslant (\sigma - \sigma_1)^2 g''(\sigma_1) \tag{3.21}$$

and

$$\sigma_1 - \delta > \mu_1. \tag{3.22}$$

We also define

$$\Gamma = \{\sigma \in [0, \tfrac{1}{2}];\ |\sigma - \sigma_1| \leqslant \delta\}. \tag{3.23}$$

Separating the contributions to $H_\beta^L(r_2)$ from Γ and S_1/Γ, we have

$$H_\beta^L(r_2) = R_L/Z_L + K_L \tag{3.24}$$

where

$$K_L = \frac{1}{Z^L} \frac{(8\beta\lambda)^{-r_2/2}}{2L^2} \sum_{\sigma \in S_1 - (S_1 \cap \Gamma)} \beta(L, \sigma) M_{L\sigma}(r_2) \leqslant \text{const. } L^2 \exp(-L\delta^2 g''(\sigma_1)/4) \tag{3.25}$$

as is easily seen from (3.21), and

$$R_L = \frac{(8\beta\lambda)^{-r_2/2}}{2L^2} \sum_{\sigma \in S_1 \cap \Gamma} \beta(L, \sigma) M_{L\sigma}(r_2). \tag{3.26}$$

Now, let

$$\overline{M}_{L\sigma}(r_2) = (8\beta\lambda)^{-r_2/2} \int_{-\sigma}^{\sigma} d\mu \exp\left[-\left(\frac{\mu - \mu_{1L}}{\delta_L}\right)^2\right]\left[\left(\frac{\mu - \mu_{1L}}{\delta_L}\right)^{r_2} - \tau(r_2)\right]. \tag{3.27}$$

Let $x_0 = a_L$ and $x_{i+1} = x_i + (b_L - a_L)/n$, $i \in [0, n-1]$, $\bar{x}_i \in [x_i, x_{i+1})$, and $\{g_L\}_{L \in Z_+}$ be a sequence of functions defined on (a_L, b_L) such that:

a) $\exists p < \infty \ni (a_L, b_L] = \text{disjoint}\bigcup_{i=1}^{p} \text{union}\,(a_{iL}, b_{iL}]$, and g_L is monotone in $(a_{iL}, b_{iL}]$ for all $i \in [1, p]$, for all $L \in Z_+$;

b) $\exists c > 0 \in |g_L(x)| \leqslant c \;\forall x \in (a_L, b_L)$, $\forall L \in Z_+$.

Then

$$\sum_{i=0}^{n} g_L(x_i)\,(b_L - a_L)/n = \int_{a_L}^{b_L} dx\, g_L(x) + \alpha_n^L$$

where, for some $d > 0$ independent of L,

$$\alpha_n^L \leqslant d/n.$$

The functions

$$x \to e^{-x^2}(x^{r_2} - \tau(r_2))$$

$$x \to e^{-Lg''(\sigma_1)(x-\sigma_1)^2/4}$$

satisfy a) and b). Hence, by (3.21), we obtain the bound

$$\begin{aligned} R_L &\leqslant a_L\{ \sup_{\sigma \in [\sigma_1-\delta, \sigma_1+\delta]} |\overline{M}_{L\sigma}(r_2)| \left(\int_{\sigma_1-\delta}^{\sigma_1+\delta} d\sigma\, e^{-Lg''(\sigma_1)(\sigma-\sigma_1)^2/4} + |\alpha_L^{(1)}| \right) \\ &\quad + |\alpha_L^{(2)}| \left(\int_{\sigma_1-\delta}^{\sigma_1+\delta} d\sigma\, e^{-Lg''(\sigma_1)(\sigma-\sigma_1)^2/4} + |\alpha_L^{(1)}| \right) \\ &\leqslant \text{const.}\, L^{-1/2} \sup_{\sigma \in [\sigma_1-\delta, \sigma_1+\delta]} |\overline{M}_{L\sigma}(r_2)| + \text{const.}\, L^{-3/2} \end{aligned} \tag{3.28}$$

where

$$a_L = \sup_{\sigma \in [\sigma_1-\delta, \sigma_1+\delta]} \phi_L(\sigma) \leqslant \text{const.}$$

by (3.20a, b) and

$$|\alpha_L^{(i)}| \leqslant \text{const.}/L, \quad i = 1, 2.$$

Let $\delta_L^1 = (Lg''(\sigma_1))^{-1/2}$. Choosing L so large that $\delta_L < \delta$ and $\delta_L^1 < \delta$, we get, by (3.21) and (3.22),

$$\begin{aligned} Z_L &\geqslant \left(\frac{1}{L} \sum_{\mu \in [\mu_{1L}-\delta, \mu_{1L}+\delta]} \exp\left[-\left(\frac{\mu - \mu_{1L}}{\delta_L} \right)^2 \right] \right) \cdot \left(\int_{\sigma_1-\delta}^{\sigma_1+\delta} d\sigma \exp\left[-\left(\frac{\sigma - \sigma_1}{\delta_L'} \right)^2 \right] + \beta_L \right) \\ &\geqslant \left(\int_{\mu_{1L}-\delta}^{\mu_\mu+\delta} d\mu \exp\left(-((\mu - \mu_{1L})/\delta_L)^2\right) + \gamma_L \right) \cdot \left(\int_{\sigma_1-\delta}^{\sigma_1+\delta} d\sigma \exp\left[-\left(\frac{\sigma - \sigma_1}{\delta_L'} \right)^2 \right] + \beta_L \right). \end{aligned}$$

$$\geqslant 4\delta_L \delta_L' e^{-2} r_L + 2\gamma_L \delta_L' e^{-1} r_L + 2\delta_L e^{-1} \beta_L + \beta_L \gamma_L \geqslant f/L, \quad \text{with } f > 0 \text{ independent of } L\text{, for sufficiently large } L \tag{3.29}$$

where

$$r_L = \inf_{|\sigma-\sigma_1|\leqslant \delta_L'} \phi_L(\sigma) \geqslant \text{const.} > 0$$

by (3.20a, b), and

$$|\beta_L| \leqslant \text{const.}/L, \ |\gamma_L| \leqslant \text{const.}/L.$$

(3.28) and (3.29) yield

$$R_L/Z_L \leqslant \text{const.}\ L^{1/2} \sup_{\sigma\in[\sigma_1-\delta,\sigma_1+\delta]} |\overline{M}_{L\sigma}(r_2)| + \text{const.}\ L^{-1/2}. \tag{3.30}$$

Now,

$$\overline{M}_L\sigma(r_2) = \delta_L \int_{-(\sigma+\mu_{1L})/\delta_L}^{(\sigma-\mu_{1L})/\delta_L} d\mu\, e^{-\mu^2}(\mu^{r_2} - \tau(r_2)) = \delta_L \left\{ \left[\int_{-\infty}^{+\infty} d\mu\, e^{-\mu^2}(\mu^{r_2} - \tau(r_2)) \right] - \int_{(\sigma-\mu_{1L})/\delta_L}^{\infty} d\mu\, e^{-\mu^2}(\mu^{r_2} - \tau(r_2)) - \int_{-\infty}^{-(\sigma+\mu_{1L})/\delta_L} d\mu\, e^{-\mu^2}(\mu^{r_2} - \tau(r_2)) \right\}. \tag{3.31}$$

Since

$$\int_{-\infty}^{+\infty} d\mu\, e^{-\mu^2}(\mu^{r_2} - \tau(r_2)) = 0$$

by definition (3.9) of τ, and

$$\int_{x^2}^{\infty} dy\, e^{-y} y^{r_2} \leqslant C_{r_2} e^{-x^2} x^{2r_2},$$

(3.31) entails

$$\sup_{\sigma\in[\sigma_1-\delta,\sigma_1+\delta]} |\overline{M}_{L\sigma}(r_2)| \leqslant \delta_L C_{r_2} \sup_{\sigma\in[\sigma_1-\delta,\sigma_1+\delta]} \{e^{-((\sigma+\mu_{1L})/\delta_L)^2}[((\sigma+\mu_{1L})/\delta_L)^{r_2}+1] + e^{-((\sigma-\mu_{1L})/\delta_L)^2}[((\sigma-\mu_{1L})/\delta_L)^{r_2}+1]\}$$
$$\leqslant \text{const.}\ L^{(r_2-1)/2} \exp(-8\lambda\beta\delta^2 L) \tag{3.32}$$

since, by (3.22) and the definition of μ_{1L}, $\sigma_1 - \delta - \mu_{1L} > 0\ \forall L \in Z_+$. (3.32) in (3.30) implies

$$\lim_{L\to\infty} R_L/Z_L = 0, \tag{3.33}$$

(3.33) and (3.25) in (3.24) imply (3.14). We now prove that

$$F_\beta^L(r_1, r_2) \equiv \rho_\beta^L((\sqrt{\mathbf{S}_L^2} - L\sigma_1)/\sqrt{L})^{r_1}((S_L^3 - L\mu_{1L})/\sqrt{L})^{r_2}) \xrightarrow[L\to\infty]{} \alpha_\beta(r_1)\,\tau(r_2)\,(8\beta\lambda)^{-r_2/2} \tag{3.34}$$

where

$$\alpha_\beta(r_1) = \frac{\int_{-\infty}^{+\infty} dx\, e^{-1/2 g''(\sigma_1) x^2} x^{r_1}}{\int_{-\infty}^{+\infty} dx\, e^{-1/2 g''(\sigma_1) x^2}} = \tau(r_1)\,(2/(\beta g''(\sigma_1)))^{r_1/2}. \tag{3.35}$$

By (3.14),

$$\frac{L^{r_1/2}}{Z_L}\,\frac{1}{2L^2}\sum_{\sigma\in S_1\cap\Gamma} \beta(L,\sigma)\left(-\frac{\alpha_\beta(r_1)}{L^{r_1/2}}\right) M_{L_\sigma}(r_2) \xrightarrow[L\to\infty]{} \tau(r_2)\,(8\beta\lambda)^{-r_2/2}\,\alpha_\beta(r_1).$$

Hence, to prove (3.34), it is easily found that it suffices to prove that

$$N_\beta^L(r_1, r_2) \equiv \frac{L^{r_1/2}}{Z_L}\,\frac{1}{2L^2}\sum_{\sigma\in S_1\cap\Gamma} \beta(L,\sigma)\cdot[(\sqrt{\sigma(\sigma+1/L)} - \sigma_1)^{r_1} - \alpha_\beta(r_1)/L^{r_1/2}]$$

$$\cdot M_{L\sigma}(r_2) \xrightarrow[L\to\infty]{} 0.$$

Now, by (3.21) and (3.29),

$$|N_\beta^L(r_1, r_2)| = \left| \frac{L^{r_1/2}}{Z_L}\,\frac{1}{Z_L}\sum_{S_1\cap\Gamma} \beta(L,\sigma)\left\{\left[\sigma - \sigma_1 + \sum_{k=1}^{\infty} \frac{\prod_{l=0}^{k-1}(\frac{1}{2} - l)}{k!}\left(\frac{1}{L\sigma}\right)^k\right]^{r_1}\right.\right.$$

$$\left.\left. - \frac{\alpha_\beta(r_1)}{L^{r_1/2}}\right\} M_{L\sigma}(r_2)\right| \leqslant \text{const.} \sup_{\sigma\in[\sigma_1-\delta,\sigma_1+\delta]} (L^{-1/2}|M_{L\sigma}(r_2)|)$$

$$\cdot\left(\int_{-L^{1/2}\delta}^{L^{1/2}\delta} d\sigma' \exp[-(\tfrac{1}{2}g''(\sigma_1)\,\sigma'^2 + O(L^{-1/2}))]\cdot[(\sigma' + O(L^{-1}))^{r_1} - \alpha_\beta(r_1)]\right)$$

$$+ |C_L|/\sqrt{L} \xrightarrow[L\to\infty]{} 0$$

where $|C_L| \leqslant$ const., by (3.35) and the fact that, by the Lebesgue bounded convergence theorem,

$$\int_{-L^{1/2}\delta}^{L^{1/2}\delta} d\sigma' \exp[-\tfrac{1}{2}g''(\sigma_1)\,\sigma'^2 + O(L^{-1/2})]\cdot[\sigma' + O(L^{-1})]^{r_1} \xrightarrow[L\to\infty]{} \int_{-\infty}^{+\infty} d\sigma'\,\sigma'^{r_1} \exp[-\tfrac{1}{2}g''(\sigma_1)\,\sigma'^2]$$

for all $r_1 \in Z_+$. This proves (3.34). From (3.34) we get (3.10) immediately from the expansion

$$\rho_\beta^L(((\sqrt{\mathbf{S}_L^2} - L\sigma_1)/\sqrt{L})^{r_1}((S_L^3 - L\mu_1)/\sqrt{L}))^{r_2}) = \sum_{k=0}^{r_2} \binom{r_2}{k} \left(-\frac{1}{2\sqrt{L}}\right)^k F_L^\beta(r_1, r_2 - k). \blacksquare$$

Other properties

Let $s_{L\beta}(r) = ((S_L^3 - Ls_\beta^3)/L^{1/2})^r$, and

$$F_{\beta 1}^L(r) = \rho_{\beta 1}^L(S_{L\beta}(r))$$

$$\varphi \in [0, 2\pi] \to F_{\beta 2\varphi}^L(r) = \rho_{\beta 2\varphi}^L(S_{L\beta}(r))$$

$$F_\beta^L(r) = \rho_\beta(S_{L\beta}(r))$$

$$G_\beta^L(r) = G_\beta^L(0, r) = \rho_\beta^L(S_{L\beta}(r)).$$

The limits of these quantities, if they exist, will be denoted by omitting the superscript L.

Lemma 3.4: Let r be even. If $\beta < \beta_c$,

$$F_\beta(r) = F_{\beta 1}(r) = G_\beta(r). \tag{3.36a}$$

If $\beta_c < \beta < \infty$,

$$F_\beta(r) = \int_0^{2\pi} \frac{d\varphi}{2\pi} F_{\beta 2\varphi}(r) \neq G_\beta(r). \tag{3.36b}$$

At $\beta = \beta_c$ and $\beta = \infty$ we have, respectively:

a) $F_\beta(r)$ is continuous at $\beta = \beta_c$, while $G_\beta(r)$ is, in general, discontinuous at this point; (3.36c)

b) $\lim_{\beta \to \infty} G_\beta(r) = 0$, while, in general, $\lim_{\beta \to \infty} F_\beta(r) > 0$. (3.36d)

Proof: (3.36a) follows from Theorem 3.2. The continuity of $F_\beta(r)$ follows from the continuity of s_β^3, defined by (3.3a, b). To prove the remaining assertions, it suffices to consider the special case $r = 2$. Now, if $s_i^3 = \frac{1}{2}\sigma_i^3$,

$$F_\beta(2) = \lim_{L\to\infty} \rho_\beta(((S_L^3 - LS_\beta^3)/\sqrt{L})^2) = \lim_{L\to\infty} \frac{1}{L} \rho_\beta\left(\left(\sum_{i=1}^{L} S_i^3 - LS_\beta^3\right)^2\right)$$

$$= \lim_{L\to\infty}\left\{\frac{1}{L}\left[(L^2 - L)\, S_\beta^{32} + L^2 S_\beta^{32} - 2L^2\, S_\beta^{32} + \frac{L}{4}\right]\right\} = \tfrac{1}{4} - S_\beta^{32} \tag{3.37a}$$

whence, by (3.3a, b),

$$\beta > \beta_c => F_\beta(2) = \tfrac{1}{4} - \mu_1^2 \quad \text{(independent of } \beta). \tag{3.37b}$$

(3.36b) follows from (3.37b) by comparison with Theorem 3.3. (Theorem 3.3 yields $G_\beta(2) = \text{const.}/\beta$ if $\beta > \beta_c$.) (3.37b) also entails (3.36d). To prove the second statement of (3.36c), let $\mathscr{E} < \sqrt{2}\lambda$. Then

$$G_{\beta_c+0}(2) = \frac{T_c}{16\lambda} < \tfrac{1}{8} < G_{\beta_c-0}(2) = \tfrac{1}{4} - \mu_1^2. \quad \blacksquare$$

Remark 3.2: (3.36b) of Lemma 3.4 shows that, for a fixed set $S_L(\beta) = \{A_L^{(i)}, i \in J\}$ ($\{S_L^3\}$ in Lemma 3.4), the two notions, namely, that a state ρ_β has normal fluctuations around $a_\beta^{(i)}, i \in J$ (Definition 1), and that the same state ρ_β has a normal approximation with respect to the same set $S_L(\beta)$ (Definition 2) are distinct, in the sense that the normal fluctuation and the normally approximating fluctuation need not coincide. Actually, Lemma 3.4 shows that only the normally approximating fluctuation has the properties:

A) $\lim_{\beta \to \infty} G_\beta(r) = 0$

B) $G_\beta(r)$ is discontinuous at $\beta = \beta_c$.

A) is expected on physical grounds (absence of thermodynamic fluctuations of – in this case – the Cooper pair energy at absolute zero, for reasonably defined 'fluctuations'), and a property like B) (some 'anomalous' behaviour of the fluctuations at the critical temperature) is also expected. It seems therefore more significant to consider the normally approximating fluctuations. $\blacksquare$

Remark 3.3: The idea of the proofs of the theorems on the normal approximation property that follow (Theorem 4.1 and Theorem 5.3, whose proof will be omitted) is the same as that stated at the beginning of the proof of Theorem 3.3, and which actually conforms to the conventional ideas of statistical mechanics. The proofs are designed to write the finite-volume fluctuations (given by the expression on which the lim operates upon, in (2.15)) in the form (3.12a), and then to prove (3.12b). $\blacksquare$

4. Imperfect Bose Gas

Let $\Lambda \subset \mathbb{R}^3$ be an open region of unit volume and smooth boundary, and for $L \geqslant 1$ let

$$\Lambda_L = \{Lx : x \in \Lambda\}. \tag{4.1}$$

$$\mathscr{H}_L = \text{symmetric Fock Space constructed from } L^2(\Lambda_L). \tag{4.2}$$

Let S_L be a self-adjoint Hamiltonian on $L^2(\Lambda_L)$ with discrete spectrum and eigenvalues

$$0 = L^{-2}E_0 < L^{-2}E_1 \leqslant L^{-2}E_2 \ldots \tag{4.3}$$

such that

$$\lim_{x \to \infty} [x^{-3/2} \max\{m : E_m \leqslant x\}] = \sqrt{2}/(3\pi^2) \tag{4.4}$$

and let the corresponding eigenvectors be

$$\phi_k^L : k = 0, 1, 2, \ldots. \tag{4.5}$$

(4.4) is satisfied for $S_L = -\frac{1}{2}\Delta$ for a large class of regions of unit volume with various boundary conditions [8]. Let the number operator on $\mathscr{H}_L$ be defined by

$$N_L = \sum_{k=1}^{\infty} n_{Lk}$$
$$n_{Lk} = \psi^*(\phi_k^L)\,\psi(\bar{\phi}_k^L)$$

whereby $\psi(f)$, $\psi^*(f)$, $f \in L^2(\Lambda_L)$, are the standard annihilation and creation operators on $\mathscr{H}_L$, and let H_{0L} be the free Hamiltonian constructed from S_L in the usual way. The Hamiltonian we consider is [9]

$$H_L = H_{0L} + L^3 f(N_L/L^3) \tag{4.6}$$

where

$$f \text{ is } C^\infty \text{ on } (0,\infty), f(0) = 0, \lim_{x\to\infty} f'(x) = +\infty, \quad \text{and } f''(x) > 0 \text{ for all } x \in (0,\infty). \tag{4.7}$$

We shall work in the grand-canonical ensemble, and observe that the region L described in Section 2 is now Λ_L of volume L^3. The mean particle density in the infinite-volume limit is

$$\rho(\beta,\mu) = \lim_{L\to\infty} L^{-3}\,\mathrm{tr}_L\,(\sigma_{\beta,\mu}^L N_L). \tag{4.8}$$

Let μ_0 be some negative real number, and P_n the projection of $\mathscr{H}_L$ onto its n-particle subspace. Define the function γ_L by interpolation from the formula

$$\mathrm{tr}_L\,(P_n\,\eta_{\beta,\mu_0}^L) = \exp[-\beta L^3 \gamma_L(n/L^3)] \tag{4.9}$$

where

$$\eta_{\beta,\mu_0}^L = \frac{\exp[-\beta(H_{0L} - \mu_0 N_L)]}{\mathrm{tr}_L \exp[-\beta(H_{0L} - \mu_0 N_L)]}. \tag{4.10}$$

Let γ be the function defined on $[0,\infty)$ by

$$\gamma(x) = \begin{cases} (\mu(x) - \mu_0)\,x - \beta^{-1}(2\pi\beta)^{-3/2}[g_{5/2}(e^{\beta\mu(x)}) - g_{5/2}(e^{\beta\mu_0})] \\ \quad \text{if } 0 \leqslant x \leqslant \rho_c \\ \gamma(\rho_c) - \mu_0(x - \rho_c) \quad \text{if } \rho_c \leqslant x < \infty \end{cases} \tag{4.11}$$

where $g_\alpha(\cdot)$, ρ and $\mu(\cdot)$ are defined by

$$g_\alpha(x) = \sum_{n=1}^{\infty} x^n/n^\alpha \quad \text{if } \alpha > 1 \text{ and } x \in [0,1]$$
$$\rho_c = (2\pi\beta)^{-3/2} g_{3/2}(1)$$
$$x = (2\pi\beta)^{-3/2} g_{3/2}(e^{\beta\mu(x)}).$$

We take β fixed. ρ_c is the critical density for Bose–Einstein condensation [9, 11].

We shall take in this model

$$S_L = S_L(\rho) = \{N_L\}$$

and study the normal approximation property with respect to this set:

Theorem 4.1: Let $\mu = \mu(L)$ be defined by

$$\bar{\rho} = L^{-3}\,\mathrm{tr}_L\,(N_L\,\sigma_{\beta,\mu(L)}^L) \tag{4.12}$$

where $0 < \bar{\rho} \neq \rho_c$ is the mean density, and $0 < \beta < \infty$. Let

$$G^L_{\beta,\bar{\rho}}(r) = \rho^L_{\beta,\mu(L)}(((N_L - L^3\bar{\rho})/L^{3/2})^r). \tag{4.13}$$

Then

$$G_{\beta,\bar{\rho}}(r) \equiv \lim_{L\to\infty} G^L_{\beta,\rho}(r) = \left(\frac{2}{\beta(\gamma''(\bar{\rho}) + f''(\bar{\rho}))}\right)^{r/2} \tau(r) \tag{4.14}$$

where τ is given by (3.9).

Proof: By (4.7) it follows that (Theorem 4.1 of [9]) if $\rho(\beta,\mu)$ is defined as the point of minimum of the function

$$g_\mu(x) = \gamma(x) + f(x) + (\mu_0 - \mu)x \tag{4.15}$$

then $0 < \rho(\beta,\mu) < \infty$ is continuous in μ, monotonically increasing in μ, $\lim_{\mu\to-\infty} \rho(\beta,\mu) = 0$ and $\lim_{\mu\to\infty} \rho(\beta,\mu) = +\infty$. Hence, there exists a unique μ_1 such that

$$L^{-3}\,\mathrm{tr}_L(N_L\,\sigma^L_{\beta,\mu(L)}) = \bar{\rho} = \rho(\beta,\mu_1).$$

By Theorem 4.2 of [9], $\lim_{L\to\infty} \mathrm{tr}_L(N_L\,\sigma^L_{\beta,\mu(L)}) = \rho(\beta,\mu_1)$ and it follows that

$$\lim_{L\to\infty} \mu(L) = \mu_1. \tag{4.16}$$

By definitions (4.9) and (4.10),

$$\exp[-\beta L^3 \gamma_L(n/L^3)] = \sum_{m=0}^{\infty} (1 - \exp[-\beta(L^{-2}E_m - \mu_0)]) \times \exp\left[-L^3\left(\frac{n}{L^3}\right)\beta(L^{-2}E_m - \mu_0)\right]. \tag{4.17}$$

By (4.17) we see that, for each fixed L, γ_L may be interpolated to a convex C^∞ function on $(0,\infty)$, which we shall denote, without confusion, again by γ_L. For $\bar{\rho} \neq \rho_c$, γ is real analytic, and by [9, Lemmata 3.5–3.8] each $\bar{\rho}$ has a neighbourhood where $\gamma_L \to \gamma$ uniformly. It thus follows easily that

$$\gamma_L^{(n)}(x) \xrightarrow[L\to\infty]{} \gamma^{(n)}(x) \quad \forall x \neq \rho_c \cdot \forall n \in Z_+. \tag{4.18}$$

Let, for $x \in (0,\infty)$,

$$g_L(x) = \gamma_L(x) + f(x) + (\mu_0 - \mu(L))\,x. \tag{4.19}$$

We have

$$G^L_{\beta,\bar{\rho}}(r) = L^{3r/2} \sum_{m=0}^{\infty} \left(\frac{m}{L^3} - \bar{\rho}\right)^r \mathrm{tr}_L(P_m\,\sigma^L_{\beta,\mu(L)})$$

$$= \frac{L^{3r/2}\,L^{-3/2} \sum_{n\in T_L} (n - \bar{\rho})^r \exp[-\beta L^3(g_L(n) - g_L(n_L))]}{L^{-3/2} \sum_{n\in T_L} \exp[-\beta L^3(g_L(n) - g_L(n_L))]} \tag{4.20}$$

where n_L is the point of minimum of g_L, and $T_L = \{n/L^3, n \in Z_+ \cup |0|\}$. Now, $\bar{\rho}$ is the point of minimum of g_{μ_1}, whence, given $\mathscr{E} > 0$, $\exists \delta > 0$ such that, if $|x - \bar{\rho}| \geqslant \mathscr{E}$,

$$g_{\mu_1}(x) - g_{\mu_1}(\bar{\rho}) \geqslant \delta \tag{4.21}$$

and, if $a > \bar{\rho} + \mathscr{E}$, $\exists B > 0$ such that, $\forall x \geqslant a$,

$$g_{\mu_1}(x) - g_{\mu_1}(\bar{\rho}) \geqslant \delta + B(x-a). \tag{4.22}$$

By (4.15), (4.16) and (4.19), every $\bar{\rho}$ has a neighbourhood where

$$g_L \to g_{\mu_1} \quad \text{uniformly} \tag{4.23a}$$

and it follows that

$$n_L \to \bar{\rho}. \tag{4.23b}$$

We may thus assume that (4.21) and (4.22) hold for g_L, for sufficiently large L, with n_L replacing $\bar{\rho}$. Hence, we may restrict summations in (4.20) to the set

$$T_{L\mathscr{E}} = \{n \in T_L : |n - n_L| \leqslant \mathscr{E}/2\}$$

for some $\mathscr{E} > 0$, without affecting the limit $L \to \infty$. Let $T_{\mathscr{E}} = \{x : |x - \bar{\rho}| \leqslant \mathscr{E}\}$. By (4.18) and (4.23b),

$$\left.\frac{d^2 g_L}{dx^2}\right|_{x=n_L} \xrightarrow[L\to\infty]{} \left.\frac{d^2 g}{dx^2}\right|_{x=\bar{\rho}} = 0. \tag{4.24}$$

By continuity of g''_{μ_1}, $\exists \delta > 0$ and $\mathscr{E} > 0$ such that

$$\frac{d^2 g}{dx^2} > \delta \ \forall x \in T_{\mathscr{E}}.$$

By (4.23b), $\exists L(\mathscr{E}) < \infty$ such that $\forall L \geqslant L(\mathscr{E})$, $T_{L\mathscr{E}} \subseteq T_{\mathscr{E}}$. Hence, by (4.18), $\exists L(\mathscr{E}) \leqslant L(\delta, \mathscr{E}) < \infty$ such that, $\forall L \geqslant L(\delta, \mathscr{E})$,

$$\left.\frac{d^2 g_L}{dx^2}\right|_{x=\bar{x}} > \frac{\delta}{2} \ \forall \bar{x} \in T_{L\mathscr{E}}. \tag{4.25}$$

We make in both numerator and denominator of (4.20) the change of variable $n' = L^{3/2}(n - n_L)$ to get

$$G^L_{\beta,\bar{\rho}}(r) = \frac{L^{-3/2} \sum_{n' \in T'_{L\varepsilon}} f_L(r, n', \beta) + O(e^{-\alpha L^3})}{L^{-3/2} \sum_{n' \in T'_{L\varepsilon}} f_L(0, n', \beta) + O(e^{-\alpha L^3})} \tag{4.26}$$

for some $\alpha > 0$, and

$$T'_{L\mathscr{E}} = \{[-\mathscr{E} L^{3/2}] + k/L^{3/2}; \quad k \in [O, [L^3]]\}$$

$$f_L(r, n', \beta) = n'^r \exp\left\{-\beta\left[\frac{1}{2}\left.\frac{d^2 g_L}{dx^2}\right|_{x=n_L} n'^2 + O(L^{-3/2})\right]\right\}. \tag{4.27}$$

By Taylor's theorem, we may also write, $\forall L \geqslant L(\delta, \mathscr{E})$,

$$f_L(r, n', \beta) = n'^r \exp\left[-\beta \frac{1}{2} \frac{d^2 g_L}{dx^2}\bigg|_{x=\bar{x}\in TL_\varepsilon} \cdot n'^2\right]. \tag{4.28}$$

(4.28) and (4.25) imply

$$f_L(r, n', \beta) \leqslant n'^r \exp(-\beta/4\, \delta n'^2)\ \forall L \geqslant L(\delta, \mathscr{E}). \tag{4.29}$$

From (4.26), (4.29) and the elementary argument to estimate the remainder given in Theorem 3.3,

$$G^L_{\beta,\bar{\rho}}(r) = \left(\int_{-\infty}^{+\infty} dx\, f_L(r, x, \beta) + C_L^{(1)}/L^{3/2}\right)\Bigg/\left(\int_{-\infty}^{+\infty} dx\, f_L(0, x, \beta) + C_L^{(2)}/L^{3/2}\right)$$

where $|C_L^{(i)}| \leqslant$ const. $\forall L \geqslant 1, i = 1, 2.$ (4.30)

(4.24) and (4.27) in (4.30) entail, by (4.29) and the Lebesgue bounded convergence theorem,

$$G_{\beta,\bar{\rho}}(r) = \frac{\int_{-\infty}^{+\infty} dx\, x^r \exp[-\beta/2(\gamma''(\bar{\rho}) + f''(\bar{\rho}))\, x^2]}{\int_{-\infty}^{+\infty} dx \exp[-\beta/2(\gamma''(\bar{\rho}) + f''(\bar{\rho}))\, x^2]} \tag{4.31}$$

from which (4.14) follows. ■

Other properties

Lemma 4.2:

$$\lim_{\beta\to\infty} G_{\beta,\bar{\rho}}(r) = 0 \tag{4.32}$$

$G_{\beta,\bar{\rho}}(r)$ is continuous at $\bar{\rho} = \rho_c$. (4.33)

Proof: Using $f''(x) > 0\ \forall x > 0$, $\gamma''(\rho_c) = 0$, the continuity of γ'' at ρ_c and the Lebesgue bounded convergence theorem in (4.31), we get

$$\lim_{\bar{\rho}\to\rho_c+0} G_{\beta,\bar{\rho}}(r) = \lim_{\bar{\rho}\to\rho_c-0} G_{\beta,\bar{\rho}}(r) = \left(\frac{2}{\beta f''(\rho_c)}\right)^{r/2} \tau(r)$$

which proves (4.33). Now,

$$0 < \bar{\rho} < \rho_c \Rightarrow \gamma''(\bar{\rho}) = \frac{\beta^{1/2}}{(2\pi)^{-3/2} g_{1/2}(\exp(\beta\mu(\bar{\rho})))}.$$

As $\mu(\bar{\rho}) < 0$ for all $0 < \bar{\rho} < \rho_c$, we get $\lim_{\beta\to\infty} g_{1/2}(e^{\beta\mu(\bar{\rho})}) = 0$ pointwise in $(0, \rho_c)$, which yields, together with (4.14), (4.32) for $0 < \bar{\rho} < \rho_c$, while, for $\bar{\rho} \geqslant \rho_c$, $\gamma''(\bar{\rho}) = 0$ and, since $f''(\bar{\rho}) > 0$, (4.32) follows immediately from (4.14). ■

Remark 4.1: Unlike the ideal Bose gas, this 'imperfect Bose gas' has the same behaviour for the canonical and grand-canonical ensembles, and is moreover stable under small perturbations [9]. The behaviour with regard to fluctuations is also in sharp contrast to the ideal Bose gases, which is given in Appendix C for comparison.

Also, we remark that the phase transition (Bose–Einstein condensation) is characterized by a discontinuity of γ''' at $\rho = \rho_c$, while γ'' remains continuous at this point. (4.33) is an immediate consequence of this fact. ■

5. Dicke Maser Model

The Dicke Hamiltonian for finite volume is [1]

$$H_L = a^* a + \mathscr{E} S_L^3 + \frac{\lambda}{\sqrt{L}} (S_L^+ a + S_L^- a^*)$$

where $\lambda > 0$, $0 < \mathscr{E} < \lambda^2$, on $\mathscr{H}_L = \mathscr{H}_L^C \otimes \mathscr{H}_{\mathrm{ph}}$, where $\mathscr{H}_L^C$ is given by (2.7) and $\mathscr{H}_{\mathrm{ph}}$ is the Fock space for one boson (photon). The operators $\mathbf{S}_L^2 = \sum_{i=1}^3 (S_L^{(i)})^2$ and $C_L = a^*a + S_L^3$ commute with each other and with H_L. In the subspace of $\mathscr{H}_L$ consisting of vectors of extensive (i.e., proportional to L) energy of the system (photons + molecules), the $a\#/L^{1/2}$ will play the role of *intensive* observables and the $a\#$, the role of fluctuation observables. With this understanding, we denote by $\sigma = S/L$ and $\gamma = C/L$ the 'intensive' quantities corresponding to the eigenvalues $S(S+1)$ and C of $\mathbf{S}_L^2$ and C_L, respectively.

Let $\mu \in [-\sigma, [\gamma, \sigma]]$ (where $[a, b]$ is the smallest of a and b) and

$$e(\sigma, \gamma, \mu) = \gamma + (\mathscr{E} - 1)\mu - 2\lambda\sqrt{\gamma - \mu}\sqrt{\sigma^2 - \mu^2}$$
$$f(\sigma, \gamma, \mu) = e(\sigma, \gamma, \mu) - \beta^{-1} y(\sigma)$$

where y is given by (3.11b).

We work in the canonical ensemble. It is shown in Ref. [1] that

$$f(\beta) = \lim_{L\to\infty} f_\beta^L = \min_{0 \leqslant \sigma \leqslant 1/2} \; \min_{\mu \leqslant \gamma \leqslant \zeta} \; \min_{-\sigma \leqslant \mu \leqslant \sigma} f(\sigma, \gamma, \mu)$$
$$= \begin{cases} f(\sigma_1(\beta), \gamma_1(\beta), \mu_1(\beta)) & \text{if } \beta \leqslant \beta_c \\ f(\sigma_2(\beta), \gamma_2(\beta), \mu_2(\beta)) & \text{if } \beta \geqslant \beta_c \end{cases} \tag{5.1}$$

where ζ is a fixed real number, for which a lower bound may be obtained from Lemma 2.2 of [1], and β_c, $\sigma_i(\beta)$, $\gamma_i(\beta)$, $\mu_i(\beta)$, $i = 1, 2$ are defined in Appendix A. From these formulae it follows [1] that at $\beta = \beta_c$ a second-order phase-transition (from normal to 'super-radiance') occurs.

Let $\rho_{\beta 1}^L$ and $\rho_{\beta 2\varphi}^L$ be the Gibbs states associated (as in the B.C.S. model) to the 'asymptotically exact' Hamiltonians for the Dicke model, given in Appendix A, and let us denote by the superscript R the restriction of these states to $\mathscr{A}^R \equiv B(\mathscr{H}_L^C)$, and the limits of these states, if they exist in the sense of (2.5), by omitting the superscript L.

Theorem 5.1: Let

$$S_L(\beta) = \begin{cases} \{S_L^-, S_L^+, S_L^3\} & \text{if } \beta \in (0, \beta_c) \\ \{\mathbf{S}_L^2\} & \text{if } \beta \in (\beta_c, \infty). \end{cases}$$

Then ρ_β^R has normal fluctuations around $\{S_\beta^-, S_\beta^+, S_\beta^3\}$ ($S_\beta^\pm = 0$), if $\beta \in (0, \beta_c)$, and around $\sigma_2(\beta)^2$ if $\beta \in (\beta_c, \infty)$.

Proof: By Ref. [1], Theorems 3.4 and 3.13, we have

$$\rho_\beta^R(S_{p_1}^{(i_1)} \dots S_{p_m}^{(i_m)}) = \begin{cases} \rho_{\beta 1}^R(S_{p_1}^{(i_1)} \dots S_{p_m}^{(i_m)}) & \text{if } \beta < \beta_c \\ \displaystyle\int_0^{2\pi} \frac{d\varphi}{2\pi} \rho_{\beta 2\varphi}^R(S_{p_1}^{(i_1)} \dots S_{p_m}^{(i_m)}) & \text{if } \beta > \beta_c \end{cases}$$

where $\rho_{\beta 1}^R$ and $\rho_{\beta 2\varphi}^R$ have *product structures* analogous to (3.5a, b). The proof is then identical to that of Theorem 3.1, since $\mathbf{S}_L^2$ is gauge-invariant. ■

Theorem 5.2: Let $\beta \in (0, \beta_c)$ and

$$S_L(\beta) = \{a, a^*, S_L^-, S_L^+, S_L^3\}, \quad \text{if } \beta \in (0, \beta_c).$$

Then ρ_β has a normal approximation with respect to $S_L(\beta)$.

Proof: This is Theorem 3.4 of [1]. ■

Remark 5.1: Theorem 3.4 of [1] proves that the limit (2.15) for the above set may be evaluated by comparison with the termal averages for the Hamiltonian

$$\overline{H}_\beta = a^* a + \mathscr{E} b^* b + \bar{\lambda}(b^* a + a^* b)$$

on the Fock space of two bosons, where $\bar{\lambda} = \lambda\sqrt{2\sigma_2(\beta)} < \sqrt{\mathscr{E}}$ iff $\beta < \beta_c$. This condition on $\bar{\lambda}$ is necessary to prove Lemma 3.8 of [1], which is an essential tool in the proof. For $\beta > \beta_c$ this argument does not, therefore, hold and it seems that more information on the spectrum of H_L is needed to prove the existence of (2.15) for $S_L(\beta) = \{C_L, \mathbf{S}_L^2\}$ if $\beta > \beta_c$. This is an open problem. ■

For $\beta \in (\beta_c, \infty)$ we were only able to prove

Theorem 5.3: Let $(\mathscr{E}, \lambda)$ satisfy, besides $0 < \mathscr{E} < \lambda^2$, the condition

$$1/\lambda^4(\mathscr{E} + \mathscr{E}^2/4) > \tfrac{1}{4} \tag{5.2}$$

and let

$$G_\beta^L(r) = \rho_\beta^L(((C_L - L\gamma_2(\beta))/L^{1/2})^r). \tag{5.3}$$

Then, for all r even, for all $\beta \in (\beta_c, \infty)$, there exist two strictly positive constants $c_1(\beta, r)$ and $c_2(\beta, r)$ independent of L such that

$$c_1(\beta, r) \leqslant G_\beta^L(r) \leqslant c_2(\beta, r). \tag{5.4}$$

In particular, there exists a subsequence $\{i_L\}_{L \in Z_+}$ of Z_+ such that

$$\lim_{L \to \infty} G_\beta^{i_L}(r) > 0.$$

Proof: We omit the proof, which follows, except for a few details, the lines of Theorem 4.1. ■

APPENDIX A

B.C.S. model:

$$H_{L\beta}^{2,\varphi} = -4\lambda\sigma_1(\beta) \sum_{p=1}^{L} \boldsymbol{\sigma}_p \cdot \mathbf{n}_{\varphi\beta} \quad (\text{on } \mathscr{H}_L^c) \tag{A.1}$$

$$\mathbf{n}_{\varphi\beta} = ((1 - (\mathscr{E}/4\lambda\sigma_1(\beta))^2)^{1/2}\cos\varphi, (1 - (\mathscr{E}/4\lambda\sigma_1(\beta))^2)^{1/2}\sin\varphi, \mathscr{E}/4\lambda\sigma_1(\beta)) \tag{A.2}$$

$$H_L^1 = -2\mathscr{E}S_L^3 \quad (\text{on } \mathscr{H}_L^c). \tag{A.3}$$

Dicke maser model:

$$H_{L\beta}^{2,\varphi} = b_L^* b_L + \tfrac{1}{2}\rho(\beta) \sum_{p=1}^{L} (1 - \boldsymbol{\sigma}_p \cdot \mathbf{e}_2) + L(|\alpha(\varphi,\beta)|^2 - \rho(\beta)/2) \quad (\text{on } \mathscr{H}_L^c \otimes \mathscr{H}_{\text{ph}}) \tag{A.4}$$

$$b_L^* = a^* - L\alpha^*(\varphi,\beta); \quad b_L = a - L\alpha(\boldsymbol{\varphi},\beta) \tag{A.5}$$

$$\rho(\beta) = 2\lambda^2\sigma_2(\beta) \tag{A.6}$$

$$\alpha(\varphi,\beta) = -\frac{\rho(\beta)}{2\lambda}(e_2^{(1)}(\varphi) - ie_2^{(2)}(\varphi)) \tag{A.7}$$

$$\mathbf{e}_2(\varphi) \equiv ((1 - \mathscr{E}^2/\rho^2)^{1/2}\cos\varphi, (1 - \mathscr{E}^2/\rho^2)^{1/2}\sin\varphi, -\mathscr{E}/\rho). \tag{A.8}$$

For the Dicke maser model we have

$$\beta_c = \frac{2}{\mathscr{E}} \operatorname{arctanh} \frac{\mathscr{E}}{\lambda^2} \tag{A.9}$$

$$\sigma_1(\beta) = \tfrac{1}{2}\tanh\frac{\beta\mathscr{E}}{2} \tag{A.10}$$

$$\gamma_1(\beta) = \mu_1(\beta) = -\sigma_1(\beta) \tag{A.11}$$

and $\sigma_2(\beta)$ is the unique positive root of the 'gap equation'

$$2\sigma_2(\beta) = \tanh(\beta\lambda^2\sigma_2(\beta)) \tag{A.12}$$

$$\gamma_2(\beta) = \mu_2 + \lambda^2(\sigma_2(\beta)^2 - \mu_2^2) \tag{A.13}$$

$$\mu_2 = -\frac{\mathscr{E}}{2\lambda^2}. \tag{A.14}$$

B.C.S. model:

$$S_1 = \{k/2L;\ k \in [0, L]\} \tag{A.15}$$

$$S_{2\sigma} = \{-\sigma + k/L;\ k \in [0, 2\sigma L]\}, \quad \sigma \in S_1 \tag{A.16}$$

$$\phi_L(\sigma) = \frac{\begin{array}{l}(2\sigma + 1/L)\exp \\ \times \left\{4\lambda\sigma\beta - \int_0^\infty dt \left[\operatorname{arctg}\frac{t}{L/2(1+2\sigma)+2} + \operatorname{arctg}\frac{t}{L/2(1-2\sigma)+1}\right]\frac{1}{e^{2\pi t}-1}\right\}\end{array}}{\begin{array}{l}(1 - 2\sigma + 2/L)^{1/2}(1 + 2\sigma + 4/L)^{3/2} \\ \times \left(1 + \frac{2}{L(1-2\sigma)}\right)^{[L(1-2\sigma)]/2}\left(1 + \frac{4}{L(1+2\sigma)}\right)^{[L(1+2\sigma)]/2}\end{array}}. \tag{A.17}$$

APPENDIX B

As observed in Remark 3.1, we may obtain a result similar to Theorem 3.1 using the same set $S_L(\beta) = \{S_L^{(1)}, S_L^{(2)}, S_L^{(3)}\}$ for all $\beta \in (0, \infty)$, if we use a different definition of normal fluctuations, with the aid of the concept of quasi-averages, due to N. N. Bogoliubov, and developed extensively by N. N. Bogoliubov Jr. [14]. Let

$$\Gamma_L^t(C) = \mathscr{E}(L - 2S_L^3) - 4\lambda(CS_L^+ + C^* S_L^-) + 4\lambda|C|^2 \tag{B.1}$$

$$\Gamma_L(\tau, C) = \mathscr{E}(L - 2S_L^3) - \frac{4\lambda}{L}(1-\tau) S_L^- S_L^+ - 4\lambda\tau(CS_L^+ + C^* S_L^-) \tag{B.2}$$

where t stands for 'trial' in (B.1), $0 < \tau < 1$, $C \in \mathbb{C}$, and let $\rho_{1\beta}^{LC}$ and $\rho_{2\beta}^{L,C,\tau}$ be the Gibbs states associated to $\Gamma_L^t(C)$ and $\rho_{2\beta}^{L,C,\tau}$, respectively. The limits of these states, if they exist in the sense of (2.5), will be denoted by omitting the superscript L. Clearly, $\Gamma_L(0, C) = H_L$, the Hamiltonian for the strong-coupling B.C.S. model of Section 3, and $\Gamma_L(\tau, C)$ contains symmetry-breaking terms for $\tau > 0$.

Let $\bar{C}$ be a point of absolute minimum of the infinite-volume free-energy function for $\Gamma_L^t(C)$ ([14], p. 105):

$$f_\infty(\Gamma^t(C)) = -\beta^{-1} \lim_{L\to\infty} \frac{1}{L} \log \operatorname{tr}_L e^{-\beta\Gamma_L^t(C)}.$$

Theorem B.1: Let $S_L(\beta) = \{S_L^{(1)}, S_L^{(2)}, S_L^{(3)}\}$ $\forall \beta \in (0, \infty)$. Then

$$\exists \lim_{\tau\to 0+} \lim_{L\to\infty} \rho_{2\beta}^{\bar{C},\tau}(F_L^{\beta,\tau,\bar{C}}(\mathbf{r})) \quad \mathbf{r} \equiv \{r_1, r_2, r_3\} \tag{B.3}$$

for all subsets $\{r_i, i \in J\} \subset Z_+$, where

$$F_L^{\beta,\tau,\bar{C}}(\mathbf{r}) = \prod_{i=1}^{3} ((S_L^{(i)} - LS_\beta^{(i)}(\tau, \bar{C}))/\sqrt{L})^{r_i} \tag{B.4}$$

and

$$\lim_{L\to\infty} \rho_{2\beta}^{\bar{C},\tau}(\sigma_L^{(i)}) = S_\beta^{(i)}(\tau, \bar{C}), \quad i \in [1, 3]. \tag{B.5}$$

Proof: It is proven in Ref. [14], p. 129, that for all local elements $\gamma \in \mathscr{A}$,

$$\lim_{L\to\infty} \rho_{2\beta}^{L,\bar{C},\tau}(\gamma) = \rho_{2\beta}^{\bar{C},\tau}(\gamma) = \lim_{L\to\infty} \rho_{1\beta}^{L,\bar{C}}(\gamma) = \rho_{1\beta}^{\bar{C}}(\gamma). \tag{B.6}$$

Hence, $S_\beta^{(i)}(\tau, \bar{C}) = S_\beta^{(i)}(\bar{C})$ are independent of $\tau \in (0, 1)$, hence also $F_L^{\beta,\tau,\bar{C}}(\mathbf{r})$, and writing $F_L^{\beta,\bar{C}}(\mathbf{r}) = F_L^{\beta,\tau,\bar{C}}(\mathbf{r})$, (B.6) yields

$$\rho_{2\beta}^{\bar{C},\tau}(F_L^{\beta,\bar{C}}(\mathbf{r})) = \rho_{1\beta}^{\bar{C}}(F_L^{\beta,\bar{C}}(\mathbf{r})) \tag{B.7}$$

and

$$\exists \lim_{L\to\infty} \rho_{1\beta}^{\bar{C}}(F_L^{\beta,\bar{C}}(\mathbf{r}))$$

by arguments identical to those of Theorem 3.1. Hence, by (B.7),

$$\exists \lim_{L\to\infty} \rho_{2\beta}^{\bar{C},\tau}(F_L^{\beta,\bar{C}}(\mathbf{r})) = \lim_{L\to\infty} \rho_{1\beta}^{\bar{C}}(F_L^{\beta,\bar{C}}(\mathbf{r})). \tag{B.8}$$

But $\Gamma_L^t(\bar{C})$ does not depend on τ, hence the right-hand side of (B.8) is independent of τ, whence, trivially

$$\exists \lim_{\tau\to 0+} \lim_{L\to\infty} \rho_{2\beta}^{\bar{C},\tau}(F_L^{\beta,\bar{C}}(\mathbf{r})) = \lim_{L\to\infty} \rho_{1\beta}^{\bar{C}}(F_L^{\beta,\bar{C}}(\mathbf{r})). \quad \blacksquare$$

APPENDIX C

In this appendix, we consider the fluctuations of the set $S_L = \{N_L\}$ for the *ideal Bose gas*, to be compared with Theorem 4.1. The notation is the same as that of Section 4, with f identically zero.

Let $z(L) = e^{\beta\mu(L)}$ be fixed by

$$\bar{\rho} = L^{-3} \sum_{k=1}^{\infty} z(L)/(e^{\beta E_k^L} - z(L)).$$

Since

$$0 < \mathrm{tr}_L\,(n_k^L \sigma_{\beta,\mu(L)}) < \infty \quad \forall k \in Z_+ \cup \{0\}$$

we must require

$$0 \leqslant z(L) < 1.$$

The properties of $z(L)$ are given by

Theorem C.1 [11]: For $\bar{\rho} < \rho_c$,

$$L^{-3} z(L)/(1 - z(L)) \to 0 \quad \text{and} \quad z(L) \to \zeta \tag{C.1}$$

where $0 < \zeta < 1$ is the unique root of the equation

$$\bar{\rho} = (2\pi\beta)^{-3/2} g_{3/2}(\zeta). \tag{C.2}$$

For

$$\bar{\rho} \geqslant \rho_c,\ L^{-3} z(L)/(1 - z(L)) \to \bar{\rho} - \rho_c \quad \text{and} \quad z(L) \to 1. \tag{C.3}$$

Here, g_α and ρ_c are given in Section 4. Let $G_{\beta,\bar{\rho}}^L(r)$ be given by (4.13) of Section 4, with all quantities replaced by the ones for the ideal Bose gas. It is easy to prove that, given any $1 \leqslant L < \infty$, $p \geqslant 2$, there exists a real neighbourhood of zero, $N_{Lp}(0) \neq \{0\}$, such that, $\forall h \in N_{Lp}(0)$, one may define

$$\begin{aligned}
\sigma_{\beta h}^L &= \exp[-\beta(H_L - (\mu + h/\beta) N_L)] \\
Z_{\beta h}^L &= \mathrm{tr}_L\, \sigma_{\beta h}^L \\
\langle N_L \rangle_h &= \mathrm{tr}_L(\sigma_{\beta h}^L N_L)/Z_{\beta h}^L \\
C_p^L(h) &= \langle (N_L - \langle N_L \rangle_h)^p \rangle_h \quad p \geqslant 2 \\
f_L(h) &= L^{-3} \log Z_{\beta h}^L
\end{aligned}$$

and where

$$|d^p f_L(h)/dh^p| < \infty \quad \forall p \geqslant 0 \cdot \forall h \in N_{Lp}(0).$$

Under these conditions, one can prove

Lemma C.2: The mean p-variances $C_p^L(h)$ may be expressed

$$\forall h \in \bigcap_{k \geqslant 2}^{p} N_{Lk}(0) \neq \{0\}$$

in terms of the successive derivatives of f_L by means of the set of recurrence relations:

$$C_2^L(h)/L^3 = d^2 f_L(h)/dh^2 \tag{C.4}$$

$$\frac{d}{dh} C_p^L(h) = C_{p+1}^L(h) - p C_{p-1}^L(h)\, C_2^L(h). \tag{C.5}$$

Using this lemma, one may now prove

Theorem C.3: Let $0 < \beta < \infty$ be fixed. If r is even,

$$G_{\beta,\bar{\rho}}(r) = \lim_{L \to \infty} G_{\beta,\bar{\rho}}^L(r) = \begin{cases} \alpha(\bar{\rho}, \beta) > 0 & \text{if } \bar{\rho} < \rho_c & \text{(C.6a)} \\ +\infty & \text{if } \bar{\rho} > \rho_c. & \text{(C.6b)} \end{cases}$$

If r is odd and $\bar{\rho} < \rho_c$,

$$G_{\beta,\bar{\rho}}(r) = 0. \tag{C.6c}$$

Furthermore, if $\bar{\rho} < \rho_c$,

$$G_{\beta,\bar{\rho}}(2) = (2\pi\beta)^{-3/2} g_{1/2}(\zeta) \tag{C.6d}$$

where ζ is defined by (C.2).

Proof: It is easy to prove that, by (C.4), Lemma 2.1 of [9] and bounded convergence,

$$\begin{aligned} G_{\beta,\bar{\rho}}^L(2) &= C_2^L(0)/L^3 = (d^2 f_L(h)/dh^2)_{h=0} \\ &= L^{-3} z(L)/(1 - z(L))^2 - \int_0^\infty \frac{d}{dx} \left\{ \frac{e^{-\beta(x-\mu(L))}}{(1 - e^{-\beta(x-\mu(L))})^2} \right\} \cdot F_L(x)\, dx \end{aligned} \tag{C.7}$$

where

$$F_L(x) = L^{-3}[\max\{m : L^{-2} E_m \leqslant x\} - 1]. \tag{C.8a}$$

Hence,

$$\lim_{L \to \infty} F_L(x) = \frac{\sqrt{2}}{3\pi^2} x^{3/2} \tag{C.8b}$$

and $\exists k$ independent of L and x such that

$$F_L(x) \leqslant k x^{3/2}. \tag{C.8c}$$

If $\bar{\rho} < \rho_c$, then $\mu(L) \to \log \zeta < 0$ by (C.1), and, from (C.7), (C.1), (C.8b, c) and dominated convergence, it follows that

$$G_{\beta,\bar{\rho}}(2) = (2\pi\beta)^{-3/2} g_{1/2}(\zeta) \tag{C.9}$$

which proves (C.6d) and, *a fortiori*, (C.6a) for $r = 2$. If $\bar{\rho} > \rho_c$, since

$$\frac{d}{dx}\left\{\frac{e^{-\beta(x-\mu(L))}}{(1-e^{-\beta(x-\mu(L))})^2}\right\} \leqslant 0 \quad \forall L \geqslant 1$$

it follows that

$$G^L_{\beta,\bar{\rho}}(r) \geqslant \left(L^{-3}\frac{z(L)}{1-z(L)}\right)^2 \cdot \frac{L^3}{z(L)} \xrightarrow[L\to\infty]{} +\infty$$

by (C.3), proving (C.6b) for $r = 2$. The remaining assertions follow from application of the recurrence relation (C.5). In fact, by (C.5), if r is even, $G^L_{\beta,\bar{\rho}}(r)$ always contains a dominating term

$$\text{const. } (>0)\, L^{-3r/2} C^L_2(0)^{r/2} \xrightarrow[L\to\infty]{} \begin{cases} \text{const. } (>0)\, ((2\pi\beta)^{-3/2} g_{1/2}(\zeta))^{r/2} > 0 & \text{if } \bar{\rho} < \rho_c \\ +\infty & \text{if } \bar{\rho} > \rho_c \end{cases}$$

proving (C.6a, b) for general r even. (C.6c) follows from the structure of (C.5), which relates $C^L_p(h)$, p odd, to $C^L_{p-2}(h), \ldots, C^L_3(h)$, with coefficients tending to finite (possibly zero) limits. Since

$$\frac{C^L_3(0)}{L^{9/2}} = \frac{1}{L^{3/2}} \frac{d^3 f_L}{dh^3}\bigg|_{h=0} \xrightarrow[L\to\infty]{} 0$$

the result follows by induction. ∎

Acknowledgments

I am greatly indebted to Prof. K. Hepp for his suggestion of this work, very helpful observations and continuous encouragement. I should also like to thank Dr. G. L. Sewell for remarks and conversations.

REFERENCES

[1] K. Hepp and E. H. Lieb, Ann. Phys. *76*, 360 (1973).
[2] K. Hepp and E. H. Lieb, 'Phase Transitions in Reservoir Open Systems with Applications to Lasers and Superconductors', preprint E.T.H. 1973, submitted to Helvetica Physica Acta (to appear).
[3] W. Thirring and A. Wehrl, Comm. Math. Phys. *4*, 1 (1967).
[4] W. Thirring, Comm. Math. Phys. *7*, 181 (1968).
[5] A. Wehrl, Comm. Math. Phys. *23*, 319 (1971).
[6] D. A. Dubin and G. L. Sewell, Jour. Math. Phys. *11*, 2990 (1970).
[7] F. Jelinek, Comm. Math. Phys. *9*, 169 (1968).
[8] H. P. McKean and I. M. Singer, Jour. Diff. Geom. *1*, 43 (1967).
[9] E. B. Davies, Comm. Math. Phys. *28*, 69 (1972).
[10] M. Kac, Informal Lecture Notes, Rockefeller University, 1970–1971.
[11] J. T. Lewis and J. Pulè, Oxford University preprint 1972, to appear.
[12] D. Ruelle, *Statistical Mechanics* (Benjamin 1969).
[13] A. Muenster, *Statistical Thermodynamics* (Springer 1969).
[14] N. N. Bogoliubov Jr., *A Method for Studying Model Hamiltonians*, translated by P. J. Shepherd (Pergamon Press, Oxford 1972).
[15] D. Ruelle, Comm. Math. Phys. *3*, 133 (1966).
[16] D. Kastler and D. W. Robinson, Comm. Math. Phys. *3*, 151 (1966).