**Zeitschrift:** Helvetica Physica Acta

**Band:** 45 (1972)

**Heft:** 7

**Artikel:** On the spectral properties of some one-particle Schrödinger Hamiltonians

**Autor:** Thomas, Lawrence E.

**DOI:** https://doi.org/10.5169/seals-114426

Helvetica Physica Acta
Vol. 45, 1972. Birkhäuser Verlag Basel

# On the Spectral Properties of Some One-Particle Schrödinger Hamiltonians

by **Lawrence E. Thomas**

Forschungsinstitut für Mathematik, ETH, Zürich

(6. VII. 72)

*Abstract.* We consider a class of relatively compact perturbations $\{V\}$ of $H_0 = p_1^2 + p_2^2 + p_3^2$ acting in momentum space, $L^2(\mathbb{R}^3, d^2p)$. Resolvent matrix elements $[\phi(1/H_0 + V - z)\phi]$ are shown to be meromorphic in a neighborhood of the positive real axis, $\phi$ belonging to a dense set. Absolute continuity of the continuous spectrum follows.

## 1. Introduction

In this article we discuss spectral properties of some one-particle Schrödinger Hamiltonians. We consider a class of perturbations $\{V\}$ of $H_0 = p_1^2 + p_2^2 + p_3^2$ acting in momentum space, $L^2(\mathbb{R}^3, d^3p)$, for which the following spectral properties of $H = H_0 + V$ are shown;

i) the absolutely continuous part of the spectrum of $H$ and the spectrum of $H_0$ coincide,
ii) the eigenvalues of $H$ are isolated from one another except perhaps at the origin, where they may accumulate,
iii) $H$ has no singular continuous part.

Each of these spectral properties is probably desirable in a mathematically rigorous scattering theory. This is particularly true in the time-dependent perturbation scheme, in which one wishes to establish the existence and completeness of wave operators (defined in some canonical way), effecting a unitary transformation between $H_0$ and the absolutely continuous part of $H$ [1]. Property i) is in fact a necessary condition for the existence of such operators. Properties ii) and iii) bear on the boundary value behavior of resolvent matrix elements and hence on the analytic properties of the $S$-matrix itself.

The perturbations considered are relatively compact, from which it follows that the essential spectra of $H_0$ and $H$ coincide. Of course, there exist compact perturbations of $H_0$ transforming the continuous spectrum of $H_0$ into a discrete spectrum for $H$. There also exist second-order ordinary differential operators with singular continuous spectrum [2]. But by imposing additional analytic conditions on $V$ we can rule out these pathologies and attain the above spectral properties.

We prove the above spectral properties for the class of perturbations $\{V\}$ by exhibiting a dense set of vectors $\mathscr{D}$ for which the resolvent matrix elements $[\psi(1/H - z)\phi]$ $\phi, \psi \in \mathscr{D}$ are meromorphic in $z$ as $z$ crosses the positive real axis (the essential spectrum of $H$ minus the origin) and travels into the second sheet. Aguilar and Combes [3] have given a proof that meromorphy of the resolvent matrix elements from a dense set implies the spectral properties. We do not repeat that argument but only show the meromorphy.

The method described here accommodates perturbations which are not necessarily short range [4], repulsive [5], spherically symmetric [6], or dilatation analytic [3]. In addition, the method is applicable to a wider class of problems, for example the description of spectral properties of multiparticle Hamiltonians and the discussion of positive bound states and resonances. These applications will be reported elsewhere.

Section 2 introduces the important notion of bounded contour distortion and discusses the resolvent meromorphy for a restricted class of perturbations (which includes some short-range potentials). Section 3 extends the results on meromorphy to perturbations (including some long-range potentials) which are limiting cases of perturbations in Section 2. Section 4 summarizes basic applications of the theory.

## 2. Second Sheet Continuation of Resolvent Matrix Elements

We will be working throughout in three-dimensional momentum space $\mathbb{R}^3$, and three-dimensional complex space $\mathbb{C}^3$. Let $\mathscr{H} = L^2(\mathbb{R}^3, d^3p)$ and let $\mathscr{D} = \{\phi \in \mathscr{H} | \phi$ is entire in $\mathbb{C}^3\}$. $\mathscr{D}$ is dense in $\mathscr{H}$. We set $H_0 = \vec{p}^2 = p_1^2 + p_2^2 + p_3^2$ and $H = H_0 + V$ where $V$ is the convolution by a function $v(\vec{p})$ with properties described below. A point in $\mathbb{C}^3$ (as well as in $\mathbb{R}^3 \subset \mathbb{C}^3$) will be designated by $\vec{p}$. The complex valued function $p_1^2 + p_2^2 + p_3^2$ on $\mathbb{C}^3$ is simply written $p^2$. We denote $|p_1^2| + p_2^2| + |p_3^2|$ defined on $\mathbb{C}^3$ by $|\vec{p}|^2$.

The convolution function $v(\vec{p})$ is assumed in this section to have the following properties:

i) $v(\vec{p})$ is an analytic function on an open set $\chi$ of $\mathbb{C}^3$ containing $\mathbb{R}^3$,
ii) for any $\vec{p} \in \chi$ there exists a real $M(\vec{p}) \geqslant 0$ such that

$$\int_{\mathbb{R}^3 \cap \{\vec{k} | |\vec{k}| > M(p)\}} v(\overrightarrow{p - k}) v^*(\overrightarrow{p - k}) d^3 k < \infty.$$

*Example 1.* $v(\vec{p}) = \cos \alpha p / p^2 + m^2$, $\alpha$ a real number. For $\alpha = 0$, $V$ is just the Yukawa potential. For $\alpha \neq 0$, $v(\vec{p})$ satisfies the above conditions but is not dilatation analytic.

*Example 2.* $v(\vec{p}) = \sin p^2 / p^2 + m^2$. This function is cited as an example which satisfies conditions i), but not ii). Hence it will not satisfy the hypotheses of the theorem below.

Let $U$ be a simply connected open set of the complex plane $\mathbb{C}$.

*Definition 1*: Bounded contour distortion. Let $\sigma(z, \vec{r}) : U \times \mathbb{R}^3 \to \mathbb{C}^3$ be a continuous function, and let $\sum(z)$ be the range of $\sigma$ for fixed $z$. $\sum(z)$ is a *bounded contour distortion* if for fixed $z$

i) $\sigma$ maps $\mathbb{R}^3$ to $\sum(z)$ homeomorphically, $\sum(z)$ is piecewise smooth, and the (complex valued) Jacobian $\partial\sigma/\partial r = \partial(\vec{p})/\partial(\vec{r})$ is bounded and bounded away from zero almost everywhere,
ii) there is an $M(z) > 0$ such that if $|\vec{r}| > M(z)$, $\sigma(z, \vec{r}) = \vec{r}$.

**Theorem 1.** *Let $\sum(z)$ be a bounded contour distortion defined in an open set $U_s$ intersecting the quadrant $\mathbb{C}_{++} = \{z | \mathrm{re}\, z > 0,\ \mathrm{im}\, z > 0\}$ such that*

*i) for some open set $N \subset U_s \cap \mathbb{C}_{++}$, $\sum(z) = \mathbb{R}^3$, $z \in N$;*

*ii) for $z$ fixed in $U_s$, $p^2 - z \neq 0$ for each $\vec{p} \in \sum(z)$ and for each $\vec{q} \in \sum(z)$, $v(\vec{p} - \vec{q})$ is analytic in $\vec{p}$ for $\vec{p}$ in a $\mathbb{C}^3$ neighborhood containing $\sum(z)$.*

*Then the resolvent matrix elements $[\psi(1/H - z)\phi]\ \phi$, $\psi \in \mathscr{D}$ may be meromorphically continued throughout $U_s$.*

We begin the proof of this theorem by defining a family of (separable) Hilbert spaces $\mathscr{H}_z$, $z \in U_s$. Let $\mathscr{H}_z$ be the space of square integrable functions defined on $\sum(z)$, with inner product

$$(\phi, \psi)_{\mathscr{H}_z} = \int_{\Sigma(z)} \psi^*(\vec{p})\phi(p)\,|d^3 p| = \int_{\mathbb{R}^3} \phi^*(\sigma(z,\vec{r}))\phi(\sigma(z,r)) \left| \frac{\partial\sigma}{\partial r} d^3 r \right|.$$

$\mathscr{H}_z$ is just $\mathscr{H}$ for $z$ in $N \subset V_s \cap \mathbb{C}_{++}$.

In each $\mathscr{H}_z$ we define the integral operator $K_z(\delta): \mathscr{H}_z \to \mathscr{H}_z$, depending on the complex variable $\delta$, as

$$(K_z(\delta)\phi)(\vec{p}) = \int_{\Sigma(z)} \frac{v(\vec{p} - \vec{q})}{q^2 - z - \delta}\, \phi(\vec{q})\, d^3 q.$$

(The reader should note that no absolute value signs appear around the differential form $d^3 q = (\partial\sigma/\partial r) d^3 r$. It is in general complex valued). It is clear that for $z$ in the neighborhood $N$ and $|\delta|$ sufficiently small, $K_z(\delta)$ is just $V(1/H_0 - z - \delta)$.

Lemma 1. *For sufficiently small $\eta(z) > 0$, $K_z(\delta)$ is compact analytic, $|\delta| < \eta(z)$.*

*Proof*: Choose $\eta(z) = \frac{1}{2} \min |q^2 - z|$. Then $q^2 - z - \delta \neq 0$, $\vec{q} \in \sum(z)$, and $K_z(\delta)$ is Hilbert–Schmidt since

$$\int_{\Sigma(z) \times \Sigma(z)} \left| \frac{v(\vec{p} - \vec{q})}{q^2 - z - \delta} \right|^2 |d^3 p\, d^3 q| < \infty$$

by the definition of $\sum(z)$, assumptions ii) of the theorem and i) on $v$. $K$ clearly depends analytically on $\delta$.

We next introduce a linear mapping $A^c_{wz}: \mathscr{H}_z \to \mathscr{H}_w$, $z, w \in U_s$. Let $c$ be a smooth curve running from $z$ to $w$ in $U_s$. Let $\mathscr{D}(A^c_{wz}) = \{\phi \in \mathscr{H}_z | \exists$ a $\mathbb{C}^3$ neighborhood $W$ containing $\bigcup_{x \in c} \sum(x)$ and $\phi$ is analytic in $W\}$. Then we define $A^c_{wz}\phi = \phi|_{\Sigma(w)}$. Hence $A^c_{wz}$ is analytic continuation of $\phi$ from $\sum(z)$ to $\sum(w)$. Note that $A^{c\,-1}_{wz} = A^c_{zw}$ and that this inverse is defined on the range of $A^c_{wz}$. If $x$ is a point on the curve $c$, we have $A^c_{zw} = A^c_{zx} A^c_{xw}$, for elements $\phi \in \mathscr{D}(A^c_{zw})$.

Let $z$ be a point in $U_s$ and let $\theta_z$ be the connected part of $\{z' \in U_s | \, |z' - z| < \eta(z)\}$, $\eta(z)$ the same as in Lemma 1.

Lemma 2. *For $z + \delta \in \theta_z$ and any path $c$ from $z$ to $z + \delta$ lying in $\theta_z$, $K_z(\delta)\phi = A^c_{z,z+\delta} K_{z+\delta}(0) A^c_{z+\delta,z}\phi$, $\phi \in \mathscr{D}(A^c_{z+\delta,z})$.*

*Proof*: We have

$$(K_z(\delta)\phi)|_{\Sigma(z)} = \int\limits_{\Sigma(z)} \frac{v(\vec{p}-\vec{q})}{q^2 - z - \delta}\,\phi(\vec{q})d^3q.$$

By condition ii) of the theorem, we may choose a complex $z_1 \in c$, $z_1 - z \neq 0$ such that $v(\vec{p}-\vec{q})\phi(\vec{q})$ is nonsingular for $\vec{p},\vec{q}$ ranging independently over a $\mathbb{C}^3$ neighborhood containing $\bigcup\limits_{x \in I_1} \sum(x)$, where $I_1$ is the interval on $c$ from $z$ to $z_1$. $[v(\vec{p}-\vec{q})/q^2 - z - \delta]\phi(\vec{q})d^3q$ is an analytic closed differential form in $\vec{q}$ on this neighborhood. Using the complex form of Stokes' theorem [7], we may replace the integration path $\sum(z)$ of the above integral by $\sum(z_1)$ to get

$$(K_z(\delta)\phi)|_{\Sigma(z)} = \int\limits_{\Sigma(z_1)} \frac{v(\vec{p}-\vec{q})}{q^2 - z - \delta}\,\phi(\vec{q})d^3q = A^c_{z,z_1}K_{z_1}(\delta - z_1 + z)A^c_{z_1,z}(\phi|_{\Sigma(z)}).$$

(Note that because $\sum(x)$ is a *bounded* contour distortion, $\sum(z_1) - \sum(z)$ is compact. $\sum(z_1) - \sum(z)$ may be regarded as the boundary of a four-dimensional region in the domain of analyticity of the differential form. This allows application of Stokes' theorem.) We next choose $z_2 \in c$ such that $v(\vec{p}-\vec{q})\phi(\vec{q})$ is nonsingular, $\vec{p}$, $\vec{q}$ ranging independently over a neighborhood of $\bigcup\limits_{x \in I_2} \sum(x)$, $I_2$ the portion of $c$ from $z_1$ to $z_2$. It follows in a similar manner that

$$(K_{z_1}(\delta - z_1 + z)\phi)|_{\Sigma(z_1)} = A^c_{z_1,z_2}K_{z_2}(\delta - z_2 + z)A^c_{z_2,z}(\phi|_{\Sigma(z_1)}).$$

Combining this equation with the previous one, we get

$$(K_z(\delta)\phi)|_{\Sigma(z)} = A^c_{z,z_2}K_{z_2}(\delta - z_2 + z)A^c_{z_2,z}(\phi|_{\Sigma(z)}).$$

By repeated application of this process, a finite set $z_1, z_2, \ldots, z_k$ can be obtained such that $z_k = z + \delta$, and

$$K_z(\delta)\phi = A^c_{z,z+\delta}K_{z+\delta}(0)A^c_{z+\delta,z}\,\phi.$$

Only a finite number of $z_j$'s are required since otherwise one could conclude the existence of a point $z_s \in c$ such that $v(\vec{p}-\vec{q})$ would be singular for $\vec{p}$, $\vec{q}$ ranging over $\sum(z_s)$.

Lemma 3. *Let $\phi \in \mathscr{D}$ and let $\psi$ be a solution to the integral equation*

$$\Psi + K_z(\delta)\psi = \phi|_{\Sigma(z)}, \quad z + \delta \in \theta_z.$$

*Then $\psi \in \mathscr{D}(A^c_{z+\delta,z})$ and $\psi|_{\Sigma(z+\delta)} = A^c_{z+\delta,z}\psi$ satisfies*

$$\psi|_{\Sigma(z+\delta)} + K_{z+\delta}(0)(\psi|_{\Sigma(z+\delta)}) = \phi|_{\Sigma(z+\delta)},$$

*where $c$ is any path in $\theta_z$ from $z$ to $z + \delta$.*

*Proof*: The proof of this lemma closely resembles that of Lemma 2. Condition ii) of the theorem and the entirety of $\phi \in \mathscr{D}$ imply the existence of a $z_1 \in c$, $z_1 - z \neq 0$, such that

$$\psi(\vec{p}) = -\int_{\Sigma(z)} \frac{v(\vec{p}-\vec{q})}{q^2 - z - \delta} \psi(\vec{q}) d^3 q + \phi(\vec{p})$$

is analytic in a $\mathbb{C}^3$ neighborhood of $\bigcup_{x \in I_1} \sum(x)$, $I_1$ the interval of $c$ from $z$ to $z_1$. Using Lemma 2 we may then write

$$\psi|_{\Sigma(z_1)} = A^c_{z_1,z} K_z(\delta)\psi + \phi|_{\Sigma(z_1)}$$

$$= K_{z_1}(\delta - z_1 + z)(\psi_{\Sigma(z_1)}) + \phi|_{\Sigma(z_1)}.$$

We next choose a $z_2 \in c$ such that $\psi(\vec{p})$ is analytic in a neighborhood of $\bigcup_{x \in I_2} \sum(x)$, $I_2$ the interval of $c$ from $z_1$ to $z_2$. Again by repeated application of this process we can obtain a finite set $z_1, z_2, \ldots, z_k$, $z_k = z + \delta$ and

$$\psi|_{\Sigma(z+\delta)} + K_{z+\delta}(0)(\psi|_{\Sigma(z+\delta)}) = \phi|_{\Sigma(z+\delta)}.$$

Only a finite number of $z_j$'s are required since otherwise there would be a $z_s \in c$ such that $\psi(\vec{p})$ would be singular on $\sum(z_s)$, and yet nonsingular on $\sum(z_s - \rho)$, $z_s - \rho \in c$, $\rho \neq 0$. But this is impossible since $\psi$ has the representation

$$\psi(\vec{p}) = -\int_{\Sigma(z_s - \rho)} \frac{v(\vec{p}-\vec{q})}{q^2 - z - \delta} \psi|_{\Sigma(z_s-\rho)}(\vec{q}) d^3 q + \phi(\vec{p})$$

which for $|\rho|$ sufficiently small surely is analytic in a $\mathbb{C}^3$ neighborhood of $\sum(z_s)$. Since $\psi$ is analytic in a neighborhood of $\sum(x)$ for any $x \in \theta_z$, the analytic continuation of $\psi$ to $\psi|_{\Sigma(z+\delta)}$ is path independent.

We are now able to prove Theorem 1. We show that the meromorphic continuation of $[\psi(1/H - z)\phi]$ $\phi$, $\psi \in \mathscr{D}$ throughout $U_s$ is given by

$$\mathscr{M}(z) = \int_{\Sigma(z)} (\psi^*|_{\Sigma(z)})(\vec{p}) \frac{1}{p^2 - z} (1 + K_z(0))^{-1} (\phi|_{\Sigma(z)})(\vec{p}) d^3 p.$$

($\psi^*|_{\Sigma(z)}$ is the analytic continuation of $\psi^*$ from $\mathbb{R}^3$ to $\sum(z)$.) First note that if $z \in N \subset U_s \cap \mathbb{C}_{++}$, the integral expression on the right-hand side is

$$\mathscr{M}(z) = \int_{\mathbb{R}^3} \psi^*(p) \frac{1}{H_0 - z} \left(1 + V \frac{1}{H_0 - z}\right)^{-1} \phi(\vec{p}) d^3 p = [\psi\,(1/H - z)\phi]_{\mathscr{H}},$$

i.e., $\mathscr{M}(z)$ is the resolvent matrix element for $z$ in $N$. It remains only to check the meromorphy of $\mathscr{M}(z')$, $z' \in \theta_z$, $z \in U_s$. By Lemma 3 the integrand of $\mathscr{M}(z')$ may be analytically continued from $\sum(z')$ to $\sum(z)$. Applying Stokes' theorem as in Lemma 2, we obtain

$$\mathscr{M}(z') = \int_{\Sigma(z')} (\psi^*|_{\Sigma(z')})(\vec{p}) \frac{1}{p^2 - z'} (1 + K_{z'}(0))^{-1} \phi|_{\Sigma(z')}(\vec{p}) d^3 p$$

$$= \int_{\Sigma(z)} (\psi^*|_{\Sigma(z)})(\vec{p}) \frac{1}{p^2 - z'} (A_{zz'}(1 + K_{z'}(0))^{-1} \phi|_{\Sigma(z')})(\vec{p}) d^3 p$$

$$= \int_{\Sigma(z)} (\psi^*|_{\Sigma(z)})(\vec{p}) \frac{1}{p^2 - z'} (1 + K_z(z' - z))^{-1} \phi|_{\Sigma(z)}(\vec{p}) d^3 p.$$

But from Lemma 1, $K_z(z'-z)$ is compact and analytic in $z'$. Hence the latter expression is meromorphic in $\theta_z$ [8]. This completes the proof.

*Example 3.* $v(\vec{p}) = \cos\alpha p/p^2+m^2$, $\alpha$ a real number. We discuss the meromorphy domains of the resolvent in two cases by constructing bounded contour distortions. In both cases $z$ starts from a neighborhood $N \subset \mathbb{C}_{++}$, crosses over the positive real axis and travels into the second sheet.

*Case 1.* The matrix elements of the resolvent $[\psi(1/H-z)\phi]$ $\phi, \psi \in \mathscr{D}$ will be meromorphic in the second sheet for $\operatorname{im}\sqrt{z} > -\frac{1}{2}m$, $z \neq 0$. In the complex plane $\mathbb{C}$, let $S_z(t)$ $0 \leqslant t < \infty$ be a simple smooth curve depending continuously on $z$ which originates at the origin, avoids the points $\pm\sqrt{z}$, and lies in the strip $|\operatorname{im} x| < \frac{1}{2}m$. In addition, let the locus of $S_z(t)$ be the positive real axis for all but a finite part of the curve, and for a neighborhood $N \subset \mathbb{C}_{++}$, let the locus be the entire positive real axis, $z \in N$. $S_z(t)$ is parameterized in such a way that $S_z(t) = t$ for $t$ sufficiently large. Then the mapping $\sigma(z,\vec{r})$ for $\sum(z)$ is given by $\sigma(z,\vec{r}) = S_z(r)(\vec{r}/r)$. One can verify that $\sum(z)$ satisfies the conditions of Theorem 1. In particular, for $\vec{p},\vec{q} \in \sum(z)$, $p^2 - z \neq 0$, and $v(\vec{p}-\vec{q})$ is analytic for $\vec{p}$ in a neighborhood of $\sum(z)$, $\vec{q} \in \sum(z)$.

*Case 2.* The matrix elements of the resolvent $[\psi(1/H-z)\phi]$ $\phi, \psi \in \mathscr{D}$ will be meromorphic in the second sheet region $\arg z > -\pi/2$. Let $x(z)$ be a point in $\mathbb{C}$ depending continuously on $z$ which lies on the positive real axis for $z$ in a neighborhood $N \subset \mathbb{C}_{++}$, and otherwise lies on the vertical line $\operatorname{Re} x = \operatorname{Re}\sqrt{z}$, $-\operatorname{Re}\sqrt{z} < \operatorname{im} x(z) < \operatorname{im}\sqrt{z}$. Let $S_z(t)$ $0 \leqslant t < \infty$ be the piecewise smooth curve with the locus of points consisting of the three straight line segments, $[0, x(z)]$, $[x(z), 2\operatorname{Re}\sqrt{z}]$, $[2\operatorname{Re}\sqrt{z}, +\infty]$ in $\mathbb{C}$. Again assume $S_z(t) = t$ for $t$ sufficiently large. (Note that $S_z(t)$ is so constructed that for any two points $x_1, x_2 \in S_z(t)$, $|\operatorname{re}(x_1-x_2)| > |\operatorname{im}(x_1-x_2)|$.) Then the mapping $\sigma(z,\vec{r})$ for $\sum(z)$ is $\sigma(z,\vec{r}) = S_z(r)(\vec{r}/r)$. Again one can verify that $\sum(z)$ satisfies the conditions of Theorem 1.

## 3. Continuation of Resolvent Matrix Elements for Long-Range Potentials

The results in the previous section concerning the meromorphy of resolvent matrix elements may be extended to a larger class of perturbations. This class consists of potentials which are limits, in a sense defined below, of potentials considered in Theorem 1. The class includes certain long-range potentials.

Let $V_n$, $n = 1, 2, \ldots$ be a sequence of potentials with corresponding convolution functions $v_n(\vec{p})$ and assume the $v_n$ satisfy the conditions given in Section 2. Let $V$ be a potential with convolution function $v(\vec{p})$.

**Theorem 2.** Suppose

i) *there is a bounded contour distortion $\sum(z)$, independent of $n$, defined throughout an open neighborhood $U_s$ satisfying the conditions of Theorem 1 for each $v_n(\vec{p})$;*

ii) *the $V_n$ converge to $V$ in the sense that the integral operators $K_{nz}: \mathscr{H}_z \to \mathscr{H}_z$, $K_z: \mathscr{H}_z \to \mathscr{H}_z$,*

$$(K_{nz}(\delta)\phi)(\vec{p}) = \int_{\Sigma(z)} \frac{v_n(\vec{p}-\vec{q})}{q^2-z-\delta}\phi(q)d^3q,$$

$$(K_z(\delta)\phi)(\vec{p}) = \int_{\Sigma(z)} \frac{v(\vec{p}-\vec{q})}{q^2 - z - \delta} \phi(q) d^3 q$$

*satisfy*

$$\lim_{n\to\infty} K_{nz}(0) \to \mathrm{K}_z(0)$$

*in norm, uniformly in z.*

*Then* $[\psi(1/H_0 + V - z)\phi]$ $\phi, \psi \in \mathscr{D}$ *can be meromorphically continued throughout* $U_s$.

*Proof*: Again let $\theta_z$ be the connected neighborhood of $z$ defined above Lemma 2 in Section 2. $K_z(0)$ is compact since it is the limit in norm of compact operators. $K_z(\delta)$ is compact analytic in $\delta$, $z + \delta \in \theta_z$ since it can be written as the composition of a bounded analytic (multiplication) operator and a compact operator,

$$K_z(\delta)\phi(p) = \int_{\Sigma(z)} \frac{v(\vec{p}-\vec{q})}{(q^2 - z)} \frac{1}{[1 - (\delta/q^2 - z)]} \phi(\vec{q}) d^3 q.$$

Note that $K_{nz}(\delta)$ converges uniformly to $K_z(\delta)$, $z + \delta \in \theta$. Now set

$$\mathscr{M}(z) = \int_{\Sigma(z)} \psi^*|_{\Sigma(z)}(\vec{p}) \frac{1}{p^2 - z} (1 + K_z(0))^{-1} \phi(\vec{p}) d^3 p, \quad \phi, \psi \in \mathscr{D}.$$

For $z$ in $N$, $\mathscr{M}(z)$ is just equal to $[\psi(1/H - z)\phi]$. $\mathscr{M}(z')$ is meromorphic about the point $z, z' \in \theta_z$ because

$$\mathscr{M}(z') = \lim_{n\to\infty} \int_{\Sigma(z')} \psi^*|_{\Sigma(z')}(\vec{p}) \frac{1}{p^2 - z'} (1 + K_{nz}(0))^{-1} \phi|_{\Sigma(z')}(\vec{p}) d^3 p$$

$$= \lim_{n\to\infty} \int_{\Sigma(z)} \psi^*|_{\Sigma(z)}(\vec{p}) \frac{1}{p^2 - z'} (1 + K_{nz}(z' - z))^{-1} \phi|_{\Sigma(z)}(\vec{p}) d^3 p$$

$$= \int_{\Sigma(z)} \psi^*|_{\Sigma(z)}(\vec{p}) \frac{1}{p^2 - z'} (1 + K_z(z' - z))^{-1} \phi|_{\Sigma(z)}(\vec{p}) d^3 p.$$

The latter expression is meromorphic in $z'$ since $K_z(z' - z)$ is compact analytic. This proves the theorem.

*Example* 4. $v = \dfrac{\cos \alpha p}{p^2}$, $v_n = \dfrac{\cos \alpha p}{p^2 + (1/n)}$, $\alpha$ a nonnegative real number. In configuration space, $V\alpha(r)$ is

$$V\alpha(r) = \begin{matrix} 0 & r < \alpha \\ \dfrac{2\pi^2}{r} & r > \alpha\,. \end{matrix}$$

Case 2 of example 3 in the previous section provides a contour distortion $\sum(z)$ in the region $R_2 = \{z \in \mathbb{C} | \arg z > -(\pi/2)\}$ for which all $v_n$ satisfy the conditions of Theorem 1. (Note that in Case 2, the contour distortion did not depend on $m$.) To establish the meromorphy of the matrix elements $[\psi(1/H_0 + V - z)\phi]$ $\phi, \psi \in \mathscr{D}$ in $R_2$, one must show the uniform convergence $K_{nz} \to K_z$ for $z$ in any compact neighborhood $B \subset R_2$. We show only the boundedness of $K_z$ in $B$, by writing $K_z$ as the sum of two operators, $K_z = K_z^1 + K_z^2$,

$$K_z \phi(p) = \int_{\Sigma(z)} \frac{\cos\alpha(p-q)}{(p-q)^2} \frac{1}{q^2 - z} \phi(\vec{q}) d^3 q$$

$$= \int_{\Sigma(z) \cap \{\vec{q} | |\vec{p}-\vec{q}| \leqslant M\}} \frac{\cos\alpha(p-q)}{(p-q)^2} \frac{1}{q^2 - z} \phi(q) d^3 q$$

$$+ \int_{\Sigma(z) \cap \{\vec{q} | |\vec{p}-\vec{q}| \geqslant M\}} \frac{\cos\alpha(p-q)}{(p-q)^2} \frac{1}{q^2 - z} \phi(q) d^3 q,$$

where $M$ is an arbitrary positive constant. The first term is bounded since the integration has kernel satisfying the Holmgren criteria for boundedness of the operation. (Namely, if

$$T\psi = \int K(x,y)\psi(y) d\mu(y),\ s_1 = \sup_x \int |K(x,y)| d\mu(y),\ s_2 = \sup_y \int |K(x,y)| d\mu(x),$$

then $|T| \leqslant (s_1 s_2)^{1/2}$ [9].) The second operation is bounded since it is Hilbert–Schmidt. The uniform convergence $K_{nz} \to K_z$ may be similarly demonstrated by breaking up the path of integration for the operator $(K_z - K_{nz})$ into the two parts again and showing the uniform convergence of the $K_z^1 - K_{nz}^1$ and $K_z^2 - K_{nz}^2$ separately.

## 4. Concluding Remarks

In this section we make some remarks concerning conditions for $V$ in configuration space in order that the convolution function $v$ for $V$ in momentum space permit applications of Theorem 1 or 2, for $z$ in a neighborhood of the positive real axis. If $V$ is multiplication by an $L^2$-function of compact support, then $V$ is convolution by an entire function in momentum space. Theorem 1 may be applied in this case to show that the resolvent matrix elements of $\mathscr{D}$ are meromorphic on an (infinitely sheeted, in general) Riemann surface $\{z | -\infty < \arg z < \infty, z \neq 0\}$. If $V$ is multiplication by an $L^2$-function $w$ such that $\int w e^{m|r|} d^3 r < \infty$ for some $m > 0$, $V$ will be convolution by a function $u$ analytic in the region $\operatorname{im} |\vec{p}| < m$. Theorem 1 will give meromorphy of the resolvent matrix elements in the region $\{z \in \mathbb{C} | \lim \sqrt{z}| < m/2,\ z \neq 0\}$. This latter result is that of Dolph, McLeod and Thoe [4]. Theorem 2 and the example following it show resolvent meromorphy in a neighborhood of the positive real axis for $V_\alpha$ multiplication by

$$w_\alpha = \begin{Bmatrix} 0 & r < \alpha \\ \frac{1}{r} & r > \alpha \end{Bmatrix}$$

in configuration space. One can show as well resolvent meromorphy in a neighborhood of the positive real axis for $V = V_1 + V_2$ where $V_1$ is multiplication by a function

$$w^1 \in L^2(\mathbb{R}^3), \quad \int w^1 e^{m|r|} d^3 r < \infty,$$

and $V_2$ is multiplication by

$$w^2(\vec{r}) = \sum_{i=1}^{N} a_i w_{\alpha^i}(\overrightarrow{r - r_0^i}),$$

$a_i$ real, $w_{\alpha^i}$ defined above. Considerably more general conditions on $V$ in configuration space can be given, so that the convolution function $v$ has appropriate analytic properties in momentum space for application of Theorem 2. The proof of the sufficiency of these conditions, however, requires a rather detailed examination of the Fourier transform of the potential and so we do not describe the conditions here.

## Acknowledgments

I am indebted to Dr. W. G. Faris, Battelle Advanced Studies Center, Geneva, for many helpful discussions. The work was completed while the author was a guest at the Forschungsinstitut für Mathematik, ETH, Zürich.

REFERENCES

[1] T. Kato, *Perturbation Theory for Linear Operators*, chap. X (Springer-Verlag, New York Inc., 1966).
[2] T. Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, New York Inc., 1966), pp. 244, 518, 523.
[3] J. Aguilar and J. M. Combes, Comm. Math. Phys. *22*, 269 (1971).
[4] C. L. Dolph, J. B. McLeod and D. Thoe, J. math. Analysis Applic. *16*, 311 (1966).
[5] R. Lavine, Comm. Math. Phys. *20*, 301 (1971).
[6] J. Weidmann, Math. Z. *98*, 268 (1967).
[7] V. S. Vladimirov, *Methods of the Theory of Functions of Many Complex Variables* (MIT Press, Cambridge 1966), p. 198.
[8] N. Dunford and J. T. Schwartz, *Linear Operators*, part I (Interscience Publishers, Inc., New York 1958), p. 592.
[9] K. O. Friedrichs, *Perturbation of Spectra in Hilbert Space* (American Mathematical Society, Providence, 1965), p. 107.