Zeitschrift: Helvetica Physica Acta
Band: 45 (1972)
Heft: 5

Artikel: Transformation properties of observables

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DOI: https://doi.org/10.5169/seals-114418

Helvetica Physica Acta
Vol. 45, 1972.

Birkhäuser Verlag Basel

Transformation Properties of Observables

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(27. III. 72)

Abstract. It is well known that the position observable considered as a projection-valued measure (decision observable) is uniquely determined by its transformation properties within an irreducible representation ($m \neq 0$) of the Poincaré [1] and the Galilei group [2]. The notion of an observable, however, has been generalized by several authors considering positive-operator-valued measures (POV measures) [3] or corresponding subsets of the lattice of projections [4]. Using a theorem of Neumark [5, 6] we prove a theorem, which allows the construction of all such generalized observables with the desired transformation properties. As an example we discuss position observables in an irreducible representation of the Galilei group. It is shown that every position observable is the convolution of the usual position observable and a suitable measure in the spin Hilbert space and thus the generalized position observable is by no means unique.

In Reference [17] we discussed the notion of an observable in a quantum mechanical system described by a dual pair of Banach spaces B, B'. In this paper we shall continue the discussion assuming that B is the space of Hermitean trace class operators and B' is the space of Hermitean bounded operators in a separable Hilbert space H. $K = \{V \in B \mid V \geqslant 0,\ \operatorname{tr} V = 1\}$ denotes the ensembles and $\hat{L} = \{F \in B' \mid 0 \leqslant F \leqslant 1\}$ denotes the effects, i.e. the yes–no experiments of the quantum mechanical system [3]. $\operatorname{tr}(VF)$ is the probability for measuring the effect of F in the ensemble V.

Following References [3, 7] an observable $(Q, F(q))$ is a Boolean algebra Q and an effective $\hat{L}$-valued measure $F(q)$ (a mapping $F: Q \to \hat{L}$ such that $F(q_1 \vee q_2) = F(q_1) + F(q_2)$ if $q_1 \wedge q_2 = 0$, $q_1, q_2 \in Q$ and $F(q) = 0$ implies $q = 0$) such that $F(1) = 1$ and Q is u_K-complete and u_K-separable.

u_K is the uniform structure on Q, a base of which is given by the sets:

$$\{(q_1, q_2) \in Q \times Q \mid q_1 + q_2 \in U_{V_1, \dots, V_n, \epsilon}\}$$

$q_1 + q_2 = (q_1 \wedge q_2^*) \vee (q_1^* \wedge q_2)$ being the symmetric difference in Q and $U_{V_1, \dots V_n, \epsilon}$ being the neighbourhoods of $0 \in Q$:

$$U_{V_1, \dots, V_n, \epsilon} = \{q \in Q \mid \operatorname{tr}(V_i F(q)) \leqslant \epsilon, \quad V_i \in K, \quad i = 1, \dots, n\}$$

This uniform structure has a distinct physical meaning discussed in Reference [7].

$(Q, F(q))$ is called a decision observable if $F(q)$ is a projection operator for all $q \in Q$.

In Reference [7] the mapping $F: Q \to \hat{L}$ was proved to be uniformly continuous if Q is equipped with u_K and $\hat{L}$ is equipped with the uniform structure induced by the $\sigma(B', B)$ topology.

Q is a Boolean σ-algebra and $m_V(q) = \operatorname{tr}(VF(q))$ is a σ-additive measure on Q for all $V \in K$.

An $\hat{L}$-valued measure on a Boolean σ-algebra Q is called σ-additive (POV-measure in the terminology of Ref. [4]) if

$$\sum_{i=1}^{\infty} F(q_i) = F\left(\bigvee_{i=1}^{\infty} q_i\right)$$

holds in the strong operator topology for $q_i \wedge q_k = 0,\ i \neq k$.

1. Lemma. An effective $\hat{L}$ valued measure on a Boolean algebra Q with $F(1) = 1$ is an observable if and only if Q is a Boolean σ-algebra generated by a countable Boolean algebra $Q_c \subset Q$ and $F(q)$ is σ-additive.

Proof. Suppose $(Q, F(q))$ is an observable. There is a countable subset and thus a countable Boolean subalgebra (not necessarily a σ-algebra) $Q_c \subset Q$, which is u_K-dense in Q. We consider a base $\{\varphi_i\}_{i=1,\dots}$ in H and $\lambda_i > 0$ such that $\sum_{i=1}^{\infty} \lambda_i = 1$. $V_0 = \sum_{i=1}^{\infty} P_{\varphi_i}$ P_{φ_i} being the projection onto φ_i, is an effective ensemble ($\mathrm{tr}(V_0 F) = 0$ implies $F = 0$) and $m_{V_0}(q)$ is an effective σ-additive scalar measure on Q. Thus $\delta(q_1, q_2) = m_{V_0}(q_1 + q_2)$ is a metric on Q generating the uniform structure u_K [7].

For every $q \in Q$ there is a sequence $q_i \in Q_c$ such that $m_{V_0}(q + q_i) \to 0$. This implies

$$\text{o-lim}\, q_i = \bigwedge_{m=1}^{\infty} \bigvee_{i=m}^{\infty} q_i = \bigwedge_{m=1}^{\infty} \bigvee_{i=m}^{\infty} q_i = q \qquad ([8], \text{II}, 2.4)$$

and therefore Q is generated by Q_c. In order to prove the σ-additivity of $F(q)$ assume $q_i \in Q$ with $q_i \wedge q_k = 0$ for $i \neq k$. Since $m_V(q)$ is σ-additive for all $V \in K$, $F_n = \sum_{i=1}^{n} F(q_i)$ converges to $F_0 = F(\vee_{i=1}^{\infty} q_i)$ with respect to the $\sigma(B', B)$ topology, which is equivalent to the weak operator topology on $\hat{L}$. As $0 \leqslant F_0 - F_n \leqslant 1$ we have $(F_0 - F_n)^2 \leqslant F_0 - F_n$ and therefore $\|(F_0 - F_n)\varphi\|^2 \leqslant \langle\varphi|(F_0 - F_n)\varphi\rangle$ for all $\varphi \in H$ and F_n converges to F_0 with respect to the strong operator topology.

Now suppose that Q is a Boolean σ-algebra generated by a countable Boolean algebra Q_c and $F(q)$ is a countably additive $\hat{L}$-valued measure on Q with $F(1) = 1$. $F(q)$ is countably additive with respect to the $\sigma(B', B)$ topology and $m_{V_0}(q) = \mathrm{tr}(V_0 F(q))$ is an effective σ-additive scalar measure on Q, V_0 being an effective ensemble. Thus Q is u_K-complete ([8], II, 2.4). The u_K closure $\bar{Q}_c$ of Q_c is also u_K-complete and $F(q)$ is an effective $\hat{L}$-valued measure on $\bar{Q}_c$. Hence $\bar{Q}_c$ is a Boolean σ-algebra [7] and our assumptions imply $Q = \bar{Q}_c$ and the proof is complete.

Jauch and Piron, too, consider general yes–no experiments at the beginning of their foundations of quantum mechanics [9]. But they introduce such a strong equivalence relation between yes–no experiments that every equivalence class may be represented by a projection operator in H. This equivalence relation applied to $\hat{L}$ reads $F_1 \sim \mathrm{F}_2$ if ($\mathrm{tr}(VF_1) = 1$ is equivalent to $\mathrm{tr}(VF_2) = 1$ for $V \in K$), i.e. $F_1 \sim F_2$ if $P_1(F_1) = P_1(F_2)$, $P_1(F)$ denoting the projection onto the eigenspace of eigenvalue 1 of F. Thus in every equivalence class C of $\hat{L}$ lies one and only one projection operator $P_1(F)$, $F \in C$.

If $F(q)$ is an $\hat{L}$-valued measure with $F(1) = 1$ on a Boolean algebra Q the corresponding projections $P_1(F(q))$ perform a generalized observable in the sense of Reference [4] (compare also Reference [10]):

$$P_1(F(0)) = 0, \qquad P_1(F(1)) = 1, \qquad P_1(F(q_1)) \cdot P_1(F(q_2)) = 0$$

if $q_1 \wedge q_2 = 0$ and

$$P_1(F(q_1)) \wedge P_1(F(q_2)) = P_1(F(q_1 \wedge q_2)).$$

Therefore every observable as defined above determines a generalized observable in the sense of Reference [4].

In the sequel we consider an observable $(Q, F(q))$ and a topological group G which acts as a continuous transformation group on Q and has a unitary continuous representation $g \mapsto U_g$ in H such that

$$U_g F(q) U_g^* = F(g(q)) \qquad \text{for all } g \in G,\, q \in Q. \tag{1}$$

As an application of Neumark's theorem [5, 6] we shall prove that there is a decision observable $(Q, E(q))$ in an enlarged Hilbert space $\tilde{H}$ such that $F(q)$ is the restriction of $E(q)$ to the subspace H of $\tilde{H}$ and that there is a (up to unitary equivalence) unique unitary representation of G in $\tilde{H}$ which transforms $(Q, E(q))$ according to (1). The detailed statements are as follows.

2. Theorem. Let $(Q, F(q))$ be an observable in a separable Hilbert space H. Let $\varphi: G \to \mathrm{Aut}(Q)$ be a homomorphism of a topological group (G, τ) into the group of automorphisms of Q such that the mapping $\varphi_q: G \to Q$ given by $g \mapsto g(q) = \varphi(g)\, q$ is $\tau - u_K$-continuous for all $q \in Q$. If $g \mapsto U_g$ is a weakly continuous unitary representation of G in H such that $U_g F(q) U_g^* = F(g(q))$, the following statements are true:

i) There is a Hilbert space $\tilde{H}$ containing H as a subspace and a projection-valued (additive) measure $E(q)$ on Q such that $F(q) = P_0 E(q) P_0$, where P_0 denotes the projection onto H. $\tilde{H}$ is the closed linear hull of $\{E(q)\varphi \,/\, \varphi \in H, q \in Q\}$ ($\tilde{H} = \overline{\mathrm{lin}}\{E(q)\varphi \,/\, \varphi \in H, q \in Q\}$).

ii) $(Q, E(q))$ is a decision observable and $\tilde{H}$ is separable.

iii) There is a weakly continuous unitary representation $g \mapsto \tilde{U}_g$ of G in $\tilde{H}$ such that $\tilde{U}_g E(q) \tilde{U}_g^* = E(g)q))$ for all $g \in G$, $q \in Q$ and $U_g = P_0 \tilde{U}_g P_0$. ($g \mapsto U_g$ is a subrepresentation of $g \mapsto \tilde{U}_g$).

iv) This construction is unique in the following sense: If $\tilde{H}$, P_0, $E(q)$, $\tilde{U}_g$ and $\tilde{H}'$, P_0', $E'(q)$, $\tilde{U}_g'$ satisfy i) and iii) there exists an isometric mapping T of $\tilde{H}$ onto $\tilde{H}'$ with $P_0' = T^{-1} P_0' T$, $E(q) = T^{-1} E'(q)\, T$ and $\tilde{U}_g = T^{-1} \tilde{U}_g' T$.

Proof. Statement i) (and partially ii) and iv)) was proved by Neumark [5]. We shall verify that i) implies all the other statements.

If $E(q)$ is countably additive with respect to the strong or equivalently the weak operator topology $(Q, E(q))$ is a decision observable according to Lemma 1. Since $E(q)$ is linear, $\|E(q)\| \leqslant 1$ for all $q \in Q$ and $\tilde{H} = \overline{\mathrm{lin}}\{E(q)\varphi \,/\, q \in Q, \varphi \in H\}$ the σ-additivity is proved by

$$\begin{aligned}\sum_i \langle E(q)\varphi | E(q_i)\, E(q)\varphi\rangle &= \sum_i \langle \varphi | P_0 E(q \wedge q_i)\, P_0 \varphi\rangle \\ = \sum_i \langle \varphi | F(q \wedge q_i)\, \varphi\rangle &= \langle \varphi | F(q \wedge (\textstyle\bigvee_i q_i))\, \varphi\rangle \\ &= \langle E(q)\varphi | E(\textstyle\bigvee_i q_i)\, E(q)\varphi\rangle.\end{aligned}$$

$\tilde{H}$ is separable: Q is $u_{\tilde{K}}$-separable with $\tilde{K} = \{\tilde{V} \in B(\tilde{H}) \,/\, \tilde{V} \geqslant 0,\ \mathrm{tr}\,\tilde{V} = 1\}$ (compare Lemma 1). Hence there is a countable subalgebra $Q_c \subset Q$ which is $u_{\tilde{K}}$-dense in Q. Moreover there is a countable subset $S_c \subset H$ dense in H. Since $(Q, E(q))$ is an observable the mapping $q \mapsto E(q)$ is continuous, Q equipped with the $u_{\tilde{K}}$-topology and $B(\tilde{H})$ equipped with the weak operator topology. It is verified by an easy estimation that $\{E(q)\varphi \,/\, q \in Q_c, \varphi \in S_c\}$ is dense in $\{E(q)\varphi \,/\, q \in Q, \varphi \in H\}$.

We shall now construct the representation $g \mapsto \tilde{U}_g$ in $\tilde{H}$: Since

$$\begin{aligned}\|\sum_{i=1}^{n} E(q_i)\varphi_i\|^2 &= \sum_{i,k=1}^{n} \langle\varphi_i|E(q_i \wedge q_k)\varphi_k\rangle \\ &= \sum_{i,k=1}^{n} \langle\varphi_i|F(q_i \wedge q_k)\varphi_k\rangle \\ &= \sum_{i,k=1}^{n} \langle U_g\varphi_i|F(g(q_i) \wedge g(q_k))\,U_g\varphi_k\rangle \\ &= \|\sum_{i=1}^{n} E(g(q_i))\,U_g\varphi_i\|^2 \qquad \text{for all } \varphi_i \in H,\ q_i \in Q \end{aligned} \tag{2}$$

an operator $\tilde{U}_g$ is well defined on $\operatorname{lin}\{E(q)\varphi \;/\; q \in Q,\ \varphi \in H\}$ by

$$\tilde{U}_g\left(\sum_{i=1}^{n} E(q_i)\varphi_i\right) = \sum_{i=1}^{n} E(g(q_i))\,U_g\varphi_i.$$

$\tilde{U}_g$ is obviously linear and is isometric by Equation (2).

Thus $\tilde{U}_g$ has a unique linear isometric extension onto

$$\tilde{H} = \overline{\operatorname{lin}}\{E(q)\varphi \;/\; \varphi \in H,\ q \in Q\}.$$

As P_0, U_g, $\tilde{U}_g$, $F(q)$, $E(q)$ are all bounded linear operators it is sufficient to verify operator identities in the sequel on vectors of the form $E(q)\varphi$, $q \in Q$, $\varphi \in H$.

The representation property of $g \mapsto \tilde{U}_g$ is proved by

$$\tilde{U}_{g_1}\tilde{U}_{g_2}E(q)\varphi = \tilde{U}_{g_1}E(g_2(q))\,U_{g_2}\varphi = E(g_1g_2(q))\,U_{g_1}U_{g_2}\varphi = \tilde{U}_{g_1g_2}E(q)\varphi.$$

Moreover this equation shows that the operators $\tilde{U}_g$ are unitary ($U_e = 1$).
$P_0\tilde{U}_gP_0 = U_g$ is verified by

$$\tilde{U}_g\varphi = \tilde{U}_gE(1)\varphi = E(g(1))\,U_g\varphi = U_g\varphi \qquad \text{for all } \varphi \in H.$$

$\tilde{U}_gE(q')\tilde{U}_g^* = E(g(q'))$ is verified by

$$\begin{aligned}\tilde{U}_gE(q')\tilde{U}_g^*E(q)\varphi &= \tilde{U}_gE(q')E(g^{-1}(q))\,U_g^*\varphi \\ &= \tilde{U}_gE(q' \wedge g^{-1}(q))\,U_g^*\varphi \\ &= E(g(q') \wedge q)\varphi = E(g(q'))E(q)\varphi.\end{aligned}$$

It remains to prove the weak continuity of the representation $g \mapsto \tilde{U}_g$. Since $\|\tilde{U}_g\| = 1$ for all $g \in G$ and $\tilde{U}_g$ is linear it is sufficient to check the continuity for all vectors of the form $E(q)\varphi$, $q \in Q$, $\varphi \in H$, $\|\varphi\| = 1$:

$$\begin{aligned}|\langle E(q)\varphi|(\tilde{U}_g - 1)E(q)\varphi\rangle| &= |\langle\varphi|(E(q \wedge g(q))\,U_g - E(q))\varphi\rangle| \\ &\leqslant |\langle\varphi|E(q \wedge g(q))\,(U_g - 1)\varphi\rangle| + |\langle\varphi|(E(q \wedge g(q)) - E(q))\varphi\rangle| \\ &\leqslant \|(U_g - 1)\varphi\| + \langle\varphi|E(q \wedge g(q^*))\varphi\rangle \\ &\leqslant \|(U_g - 1)\varphi\| + \langle\varphi|F(q + g(q))\varphi\rangle.\end{aligned}$$

The first term of the estimation tends to 0 if g converges to the unit element e of the group since $g \mapsto U_g$ is assumed to be a continuous representation of G. The second term tends to 0 for $g \to e$ since $q \mapsto g(q)$ is assumed to be u_K-continuous and the mapping $q \mapsto F(q)$ is continuous according to the remark before Lemma 1.

To complete the proof assume $\tilde{H}$, P_0, $E(q)$, $\tilde{U}_g$ and $\tilde{H}'$, P_0, $E'(q)$, $\tilde{U}'_g$ to satisfy i) and iii). We may identify the subspaces of $\tilde{H}$ and $\tilde{H}'$ which are isometric to H.

Since

$$\| \sum_{i=1}^{n} E(q_i)\,\varphi_i\|^2 = \sum_{i,k=1}^{n} \langle \varphi_i | F(q_i \wedge q_k)\,\varphi_k\rangle = \| \sum_{i=1}^{n} E'(q_i)\,\varphi_i\|^2 \tag{3}$$

for all $q_i \in Q$, $\varphi_i \in H$, a mapping T of $\operatorname{lin}\{E(q)\,\varphi \,/\, q \in Q, \varphi \in H\}$ onto $\operatorname{lin}\{E'(q)\,\varphi \,/\, q \in Q, \varphi \in H\}$ is well defined by

$$T\left(\sum_{i=1}^{n} E(q_i)\,\varphi_i\right) = \sum_{i=1}^{n} E'(q_i)\,\varphi_i.$$

T is obviously linear and is isometric by (3). Thus T has a unique linear isomeric extension to a mapping of $\tilde{H}$ onto $\tilde{H}'$.

The identities $P_0 = T^{-1} P' T$, $E(q) = T^{-1} E'(q)\,T$ and $\tilde{U}_g = T^{-1} U'_g T$ are easily verified.

In the next lemma we discuss the continuity assumption of Theorem 2 concerning the transformation of Q by G. This lemma provides a possibility to check the continuity assumption in concrete examples. Moreover, statement ii) of the lemma shows the necessity of this assumption in Theorem 2.

3. Lemma. Let $(Q, F(q))$ be an observable in H, (G, τ) a topological group and $\varphi: G \to \operatorname{Aut}(Q)$ a homomorphism, $\varphi_q: G \to Q$ the mapping given by $g \mapsto g(q) = \varphi(g)\,q$. If $g \mapsto U_g$ is a continuous representation of G such that $U_g F(q)\, U_g^* = F(g(q))$ we have

i) If φ_q is $\tau - u_K$-continuous at the unit element $e \in G$ then φ_q is continuous on G.
ii) If $(Q, F(q))$ is a decision observable then φ_q is continuous on G for all $q \in Q$.
iii) φ_q is continuous on G for all $q \in Q$, if there is a subset $S \subset Q$ which generates Q such that φ_q is continuous for all $q \in S$.

Proof. The first statement is a consequence of

$$\begin{aligned} &\operatorname{tr}(V F(g_0(q) + g(q))) \\ &\quad = \operatorname{tr}(V U_{g_0} F(q + g_0^{-1} g(q))\, U_{g_0}^*) \\ &\quad = \operatorname{tr}((U_{g_0}^* V U_{g_0})\, F(q + g_0^{-1} g(q))) \end{aligned}$$

To prove the second statement consider the equation

$$\begin{aligned} F(q + g(q)) &= F(q^* \wedge g(q)) + F(q \wedge g(q^*)) \\ &= F(q^*)\, U_g F(q)\, U_g^* + F(q)\, U_g F(q^*)\, U_g^*. \end{aligned} \tag{4}$$

Since $g \mapsto U_g$ is a weakly continuous representation both terms of (4) are continuous functions of g with respect to the weak operator topology and thus are $\sigma(B', B)$-continuous functions of g.

To prove iii) we shall show that the set $Q_{\text{con}} = \{q \in Q \,/\, \varphi_q \text{ is continuous}\}$ is a σ-subalgebra of Q. Q_{con} is a subalgebra of Q since the algebraic operations in Q are u_K-continuous [7] and the following identities hold

$$\begin{aligned} &\varphi_{q_1 \wedge q_2}(g) = g(q_1 \wedge q_2) = g(q_1) \wedge g(q_2) = \varphi_{q_1}(g) \wedge \varphi_{q_2}(g) \\ &\quad \varphi_{q^*}(g) = (\varphi_q(g))^*. \end{aligned}$$

To see that Q_{con} is a σ-subalgebra consider a sequence $q_i \in Q_{\text{con}}$ with $q_{i+1} \leqslant q_i$. If $q = \bigwedge_{i=1}^{\infty} q_i, q_i$ converges to q with respect to the u_K-topology. The estimation

$$\begin{aligned} F(q+g(q)) &\leqslant F(q+q_i) + F(q_i+g(q_i)) + F(g(q_i+q)) \\ &= 2F(q+q_i) + F(q_i+g(q_i)) + (F(g(q_i+q)) - F(q_i+q)) \end{aligned}$$

proves the continuity of φ_q. The first term tends to 0 because $q_i \to q$, the second because of the continuity of φ_{q_i} and the third because of the continuity of the transformation $U_g F(q_i+q) U_g^*$.

As an application of Theorem 2 we consider an elementary quantum mechanical system with mass m and spin s which is described by a continuous irreducible representation up to a factor $(m \neq 0)\, g \mapsto U_g$ of the Galilei group G in a separable Hilbert space H. To simplify the notation we choose $m = \hbar = 1$.

Let Σ be the Boolean σ-algebra of Lebesgue measurable sets of the 3-dimensional Euclidean space R^3. If J_0 is the ideal of sets of Lebesgue measure zero, denote by $Q = \Sigma/J_0$ the quotient algebra on which the Lebesgue measure is effective and denote by χ the canonical surjection $\chi:\Sigma \to Q$. If $g \in G, g = (R, a, v, \tau)$, $R \in SO(3)$ is the rotation, a is the space-, v velocity-, and τ time-translation.

$$g(x) = Rx + a + vt, \qquad x \in R^3, \qquad g \in G \tag{5}$$

defines a homeomorphism of R^3 for fixed real parameters t.

$$g(\sigma) = \{y \in R^3 / y = g(x), x \in \sigma\}$$

defines an automorphism of Σ which leaves J_0 invariant and thus $g(\chi(\sigma)) = \chi(g(\sigma))$ is an automorphism of Q for all $g \in G$ (6). We have $g_1(g_2(q)) = (g_1 g_2)(q)$, and the mapping $\varphi: G \to \text{Aut}(Q)$ defined by (6) is a homomorphism. An observable $(Q, F(q))$ which is transformed in the Schrödinger picture according to

$$F(g(q)) = U_g F(q) U_g^* \qquad \text{with } t = 0 \text{ in (5)} \tag{7}$$

will be called position observable. The transformation property of $F_t(q)$ in the Heisenberg picture is determined by the transformation property of $F_0(q)$, i.e. by (7), since

$$U_g F_t(q) U_g^* = U_{\tilde{g}} F_0(q) U_{\tilde{g}}^* \qquad \text{with } \tilde{g} = (R,\, a + vt,\, v,\, \tau + t) \tag{8}$$

If we consider the description of a particle in an external field there is a representation of the Galilei group without time-translation G_0 in H. In this case (7) remains valid, only (8) has to be changed.

In the sequel we shall determine all possible realization of (7) by means of Theorem 2. First we shall check the continuity of the mapping $\varphi_q: G \to Q$ defined by $g \mapsto g(q)$ by the help of iii) of Lemma 3:

Consider a sequence $g_n \in G$ converging to e. If σ is an open subset of R^3 there is a neighbourhood $U(e)$ in G for every $x \in \sigma$ such that $g^{-1}(x) \in \sigma$ for all $g \in U(e)$. (This is a consequence of the continuity of the representation of G in R^3.) This implies

$$\bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} (g_n(\sigma) \wedge \sigma) = \sigma$$

hence

$$\bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} (\chi(g_n(\sigma) \wedge \sigma) = \chi(\sigma)$$

Since $m_V(q) = \mathrm{tr}(VF(q))$ is a σ-additive measure on Q for all $V \in K$ we have

$$\lim_{n\to\infty} m_V\Big(\bigwedge_{n=k}^{\infty} \chi(g_n(\sigma) \wedge \sigma)\Big) = m_V(\chi(\sigma))$$

hence

$$\lim_{n\to\infty} m_V(\chi(g_n(\sigma) \wedge \sigma)) = m_V(\chi(\sigma))$$

and

$$\lim_{n\to\infty} m_V(\chi(g_n(\sigma^*) \wedge \sigma)) = 0. \tag{9}$$

$S = \{q \in Q \,/\, q = \chi(\sigma),\ \sigma$ a bounded open sphere in $R^3\}$ generates Q. If σ is a bounded open sphere in R^3 the boundary of σ is of Lebesgue measure zero and thus

$$\chi(\sigma^{*0}) = \chi(\sigma)^* = q^*,$$

where σ^{*0} denotes the inner of the complement of σ and $q = \chi(\sigma)$.

Equation (9) implies

$$\lim_{n\to\infty} (m_V(g_n(q^*) \wedge q) + m_V(g_n(q) \wedge q^*))$$
$$= \lim_{n\to\infty} m_V(g_n(q) + q) = 0$$

Thus $g_n(q)$ converges to q with respect to the u_K-topology and φ_q is continuous for all $q \in S$, hence for all $q \in Q$.

The Galilei group G, the position observable $(Q, F(q))$ and the representation up to a factor $g \mapsto U_g$ satisfy the assumptions of Theorem 2. (It is well known that the representation up to a factor of G is a representation of a central extension of G.) We may conclude that there is a separable Hilbert space $\tilde{H}$ containing H as a subspace, a decision observable $(Q, E(q))$ in $\tilde{H}$ and an extension $g \mapsto \tilde{U}_g$ of the representation of G with all the properties described in Theorem 2.

Let us discuss the form of the extended representation of G. We shall write $\tilde{U}_R$ instead of $\tilde{U}_g$ if $g = (R,0,0,0)$ and analogously $\tilde{U}_a$, $\tilde{U}_v$. The operators $\tilde{U}_a$ and $\tilde{U}_v$ provide a continuous representation of the Weyl commutation relations in the separable Hilbert space $\tilde{H}$. $\tilde{H}$ may be decomposed into a direct product $\tilde{H} = H' \times H''$ such that $\tilde{U}_a = U'_a \times 1$ and $\tilde{U}_v = U'_v \times 1$ and U'_a and U'_v form an irreducible representation of the Weyl commutation relations in H' [11]. If $R \mapsto U'_R$ is the usual representation of $SO(3)$ given in an irreducible representation of the Weyl commutation relations we have

$$(U'_R \times 1 \cdot \tilde{U}^*_R)\, U'_a \times 1 (U'_R \times 1 \cdot \tilde{U}^*_R)^* = U'_a \times 1$$
$$(U'_R \times 1 \cdot \tilde{U}^*_R)\, U'_v \times 1 (U'_R \times 1 \cdot \tilde{U}^*_R)^* = U'_v \times 1$$

It follows $U'_R \times 1 \cdot \tilde{U}^*_R = 1 \times U''_R$, hence $\tilde{U}_R = U'_R \times U''_R$, where $R \mapsto U''_R$ is a representation of $SO(3)$ in H''.

In this decomposition of $\tilde{H}$ the subspace H is of the form $H = H' \times H_s$, the spin space H_s being a $(2s+1)$-dimensional subspace of H'' in which U''_R acts as an irreducible representation of $SO(3)$ of weight s.

According to Theorem 2 the transformation property (7) of $(Q, F(q))$ is equivalent to

$$\tilde{U}_g E(q)\, \tilde{U}^*_g = E(g(q)) \qquad q \in Q, g \in G. \tag{10}$$

$E(\chi(\sigma))$ in short $E(\sigma)$ is a σ-additive projection-valued measure on R^3 and we can define

$$S_v = \int \exp(i(vx))\, dE(x),$$

where (vx) denotes the inner product of $x, v \in R^3$. $v \mapsto S_v$ is a weakly continuous representation of the translation group $T(3)$ and (10) implies the following commutation relations

$$\begin{aligned} \tilde{U}_a S_v \tilde{U}_a^* &= \exp(-i(av))\, S_v \\ \tilde{U}_{v'} S_v \tilde{U}_{v'}^* &= S_v \\ \tilde{U}_R S_v \tilde{U}_R^* &= S_{Rv}. \end{aligned} \tag{11}$$

As the measure $E(\sigma)$ is uniquely determined by the unitary operators S_v (theorem of Bochner) the Equations (11) are even equivalent to (10). Considerations similar to those concerning the representation of $R \to \tilde{U}_R$ of $SO(3)$ show that

$$S_v = U_v' \times U_v'' \tag{12}$$

and $v \to U_v''$ is a weakly continuous unitary representation of $T(3)$ in H''. From (11) follows

$$U_R'' U_v'' U_R''^* = U_{Rv}'' \tag{13}$$

Thus $(v, R) \mapsto U_v' U_R''$ is a representation of the Euclidean group (the semidirect product $T(3)$ ⓢ $SO(3)$) in H''. The equations (12) and (13) are equivalent to (11) hence to (10) and (7).

The minimality property $\tilde{H} = \overline{\text{lin}}\{E(q)\varphi \mid q \in Q,\ \varphi \in H\}$ of Theorem 2 can be expressed by

$$\tilde{H} = \overline{\text{lin}}\{S_v \varphi \mid v \in T(3),\ \varphi \in H\}$$

or

$$H'' = \overline{\text{lin}}\{U_v'' \varphi \mid v \in T(3),\ \varphi \in H_s\}$$

This is the case if and only if there is no proper subspace of H'' containing H_s which is invariant under the representation of the Euclidean group in H''. (14)

Summarizing every position observable in an irreducible representation of the Galilei group in $H = H' \times H_s$ can be constructed in the following manner: In a continuous unitary representation of the Euclidean group in a separable Hilbert space H'' we consider an irreducible subrepresentation of $SO(3)$ of weight s in a subspace H_s of H''. (We may confine ourselves to the case in which (14) holds.) In $\tilde{H} = H' \times H''$ we consider the projection-valued measure $E(\sigma)$, $\sigma \in \Sigma$, generated by the representation $v \mapsto U_v' \times U_v''$ of $T(3)$. U_v' is given by the Galilei-transformations in H' and U'' by the space-translations of the Euclidean group in H''. The restriction $F(\sigma) = P_0 E(\sigma) P_0$ of $E(\sigma)$ to the subspace H of $\tilde{H} = H' \times H''$ is transformed by the representation of the Galilei group according to (7). However, it remains to show that in every case $F(\sigma)$ is an effective $\hat{L}$-valued measure on $Q = \Sigma/J_0$, i.e. $F(\sigma)$ vanishes exactly on sets of Lebesgue measure zero. If $E'(\sigma)$ denotes the usual position measure in H' defined by the Galilei transformations U_v', $F(\sigma)$ is the convolution of $E'(\sigma) \times 1_s$ and the measure $1 \times F''(\sigma) = 1 \times P_s E''(\sigma) P_s$, P_s being the projection onto the subspace H_s of H'' and $E''(\sigma)$ being the projection-valued measure generated by the representation $v \to U_v''$ of $T(3)$ in H''. If $E' \times F''(\rho)$ (ρ Borel set in R^6) denotes the product measure of $E'(\sigma) \times 1_s$

and $1 \times F''(\sigma)$ and $f: R^6 \to R^3$ denotes the mapping defined by $(x,y) \mapsto x+y$ then $F(\sigma)$ is obtained by $F(\sigma) = E' * F''(\sigma) = E' \times F''(f^{-1}(\sigma))$. Since $(f^{-1}(\sigma))_y = \{x \in R^3 / (x,y) \in f^{-1}(\sigma)\} = \sigma - y$ for $y \in R^3$, $\sigma \in \Sigma$ we have $F(\sigma) = 0$ if and only if $E'(\sigma - y) = 0$ for F''-almost all $y \in E_3$. As $F''(E_3) \neq 0$ we have $F(\sigma) = 0$ if and only if $E'(\sigma) = 0$. Thus $F(q) = F(\chi(\sigma))$ is an effective measure on $Q = \Sigma/J_0$, hence a position observable.

Let us discuss briefly the special case of a spin-independent position observable $(Q, F(q))$. The case $s = 0$ is included in this discussion. The assumption

$$\langle \varphi \times u | F(q)\, \varphi \times u \rangle = \langle \varphi \times u' | F(q)\, \varphi \times u' \rangle$$

for all $u, u' \in H_s$, $\varphi \in H'$, $q \in Q$ implies $\langle u | U''_v u \rangle = \langle u' | U''_v u' \rangle$ for all $u, u' \in H_s$, $v \in T(3)$. Hence $F''(\sigma) = m(\sigma)\, P_s$, where $m(\sigma)$ is a scalar σ-additive measure on R^3 which is rotational invariant because of $m(R\sigma)\, P_s = U''_R\, (m(\sigma)\, P_s)\, U''^*_R = m(\sigma)\, P_s$. A spin-independent position observable is the convolution of the usual position observable $E'(q) \times 1_s$ and a rotational invariant scalar measure $m(\sigma)$ on R^3. We have $E'(q) \times 1_s = E' * m(q)$ for all $q \in Q$ if and only if $m(\sigma)$ is concentrated on the point $0 \in R^3$. If $m(\sigma)$ is concentrated in a ball centred by $0 \in R^3$ the position observable $E'(q) \times 1_s$ is smeared by convolution with $m(\sigma)$. If this ball is bounded $P_1(E' * m(q))$ does not vanish for all bounded regions $q \in Q$ (notation as in the beginning of this paper). But if $E' * m(q) \neq E'(q) \times 1_s$ for some $q \in Q$ there are regions $q' \in Q$ such that $P_1(E' * m(q')) = 0$.

This last remark holds also for spin-dependent position observables.

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