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The Parametrization of $\pi^- p$ Scattering Experiments

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Abstract. The difficulty of making a precise statement of charge independence for pion-nucleon scattering is pointed out. We emphasize that in taking account of electromagnetic effects in the analysis of $\pi^- p$ scattering experiments, it is necessary to consider the radiative capture process $\pi^- p \to \gamma n$. A procedure for doing this is given. The unsatisfactory aspects of recent statements about time-reversal violation and possible isotensor current contributions in pion photoproduction and radiative capture are pointed out.

1. Introduction

In the analysis of pion-nucleon scattering experiments it is generally assumed that the hypothesis of charge independence holds. What this means can only be specified precisely within the framework of a specific model for electromagnetic effects. Looking first at $\pi^+ p$ scattering in the elastic region, it is well known that the nuclear scattering amplitude can be approximated by a truncated partial-wave expansion and can be parametrized by means of one real phase shift for each state of fixed total angular momentum and parity. The long-range electromagnetic effects of the Coulomb interaction are taken into account in the analysis of experiments by adding the point-charge Coulomb amplitude to a modified nuclear amplitude. Further electromagnetic corrections to the phase-shifts determined from experiment are then necessary if the statistical errors on these phase-shifts are much smaller than, or of the order of magnitude of, these residual electromagnetic effects. Methods for making these corrections have been worked out by several authors using different models [1]–[6].

Turning now to the case of $\pi^- p$ elastic and charge exchange scattering, we can again approximate the nuclear scattering amplitudes by truncated partial-wave expansions. In the charge independent limit, one needs two real phase shifts for each state of fixed total angular momentum J and parity P, corresponding to the two possibilities for the total isospin, $T = 1/2$ and $T = 3/2$. Furthermore, charge independence implies that the $T = 3/2$ phase shifts can be taken from the analysis of $\pi^+ p$ experiments. Again, the long range electromagnetic effects of the Coulomb interaction are taken into account by including the point-charge Coulomb amplitude (for $\pi^- p$ elastic scattering) and modifying the nuclear amplitudes. If one does not make the charge independence hypothesis, the situation for the analysis of $\pi^- p$ scattering data changes drastically, and the number of parameters to be determined for each (J, P) state is increased. Following earlier work by Chiu [7], it was emphasized by Oades and Rasche

[8] that *three* real parameters (two phase shifts and a mixing angle) are necessary for each (J, P) state, to fit the $\pi^- p$ elastic and charge exchange data. This is correctly taken into account in approximate models, such as the method for making 'outer' Coulomb corrections proposed by van Hove [1].

The fact that in the $\pi^- p$ case three parameters are necessary to specify the scattering amplitudes for each (J, P) state is independent of the specific model used to calculate the mixing angle and the electromagnetic corrections to the phases. Attempts to make models for these corrections arise from the situation that the experimental data are not accurate enough to determine these three parameters directly. Each model tries to express the three parameters in terms of two charge independent phase shifts. Then, if one accepts the model, these two parameters can be determined from experiment with reasonable accuracy. The present models all use non-relativistic potential theory in one form or another. This might be satisfactory in the very low energy range, but it certainly is not reliable in the region of the $(3/2, 3/2)$ resonance. This, and other difficulties in the analysis, make the numbers for the parameters of the $(3/2, 3/2)$ resonance, as quoted for example by Carter et al. [9], very doubtful.[1])

This paper is mainly concerned to point out another electromagnetic effect which hitherto has been neglected completely in the analysis of $\pi^- p$ data. As soon as one takes into account corrections to charge independence, the influence of the radiative capture process

$$\pi^- p \to \gamma n$$

has to be considered as well. The value of the Panofsky ratio indicates that at low energies at least there is quite a high probability for this process. As we shall show, this means that for each (J, P) state, we require ten real parameters, except for the $(1/2, \pm 1)$ states, where the number of parameters reduces to six.

It should be noted that any conclusion about 'charge independent' phase shifts, or parameters obtained from them (such as scattering lengths) is invalid if one has not already taken into account this increased number of parameters in the analysis of experiments. This remark applies in particular to numerical calculations for testing charge independence, methods for which have been developed by Törnqvist [10]. If, because of the limited statistics of the experiments, it is not possible to include these parameters in some way or other in the analysis of the data, then all electromagnetic corrections are doubtful and no firm conclusions can be reached concerning the charge independent phases.

In Section 2, we write down the consequences of the unitarity of the S-operator for the case of n two-body channels, at energies where no other channels are open. We also point out the consequences of time-reversal invariance and parity conservation. A very simple example with two channels only is provided by np elastic scattering and radiative capture and we discuss this in Section 3. In particular, we point out the modification of the low-energy behaviour of the np elastic scattering cross-section. In Section 4, we turn to the case of $(\pi^- p)$, $(\pi^0 n)$ and (γn) and show that, for each (J, P) state, except those with $(1/2, \pm 1)$, it is necessary to consider a 4×4 unitary,

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[1]) *Explicit* introduction of a potential can be avoided by using a certain ansatz for the wave function, but this ansatz can be justified only from non-relativistic potential theory. Auvil [8] has used the Klein-Gordon equation, which involves making the static approximation for the target motion. His work is purely speculative and suggests *one* possible extension of the results from the Schrödinger equation to relativistic situations.

symmetric matrix. We then indicate how an approximate knowledge of pion photoproduction multipole amplitudes can be used to reduce to three the number of parameters required for each (J, P) state to fit $\pi^- p$ experiments. Section 5 will summarize our conclusions.

2. Unitarity

Consider the case of n two-body channels, $i = 1, \ldots, n$, at an energy where no other channels are open. With plane wave states normalized so that $\langle \mathbf{p}' | \mathbf{p} \rangle = \delta^{(3)}(\mathbf{p}' - \mathbf{p})$ we define the quantity S_{ji} for the process $(i) \to (j)$ as follows:

$$\langle \mathbf{p}_j^{(1)} \lambda_j'^{(1)}; \mathbf{p}_j^{(2)} \lambda_j'^{(2)} | S | \mathbf{p}_i^{(1)} \lambda_i^{(1)}; \mathbf{p}_i^{(2)} \lambda_i^{(2)} \rangle$$

$$= -(2\pi)^{-2} \delta^{(4)}(p_j^{(1)} + p_j^{(2)} - p_i^{(1)} - p_i^{(2)}) \frac{N_i N_j}{[E_i^{(1)} E_i^{(2)} E_j^{(1)} E_j^{(2)}]^{1/2}}$$

$$\times S_{ji}(E^2 - \mathbf{P}^2; \mathbf{n}_j \mathbf{n}_i; \lambda_j'^{(1)} \lambda_j'^{(2)} \lambda_i^{(1)} \lambda_i^{(2)}).$$

Here $\mathbf{P}$ is the total three-momentum, E the total energy and $\mathbf{n}_i$ and $\mathbf{n}_j$ are unit vectors. The vector $\mathbf{n}_i$ is defined as the unit vector in the direction of the vector $(m_i^{(1)} + m_i^{(2)})^{-1} \cdot (m_i^{(2)} \mathbf{p}_i^{(1)} - m_i^{(1)} \mathbf{p}_i^{(2)})$, and similarly for $\mathbf{n}_j$. N_i and N_j are products of two factors, one for each particle, the factors being $(\text{mass})^{1/2}$ for a fermion and $2^{-1/2}$ for a boson. The superscripts (1), (2) label the two particles in each channel and the quantities $\lambda_i^{(1)}$, $\lambda_i^{(2)}, \lambda_j'^{(1)}, \lambda_j'^{(2)}$ are the helicities of the respective particles. We now go to the centre-of-momentum system (CMS), so that $\mathbf{P} = 0$ and $\mathbf{n}_i$ may be identified as the unit vector in the direction of motion of particle (1) in channel (i). The magnitude of the three-momentum of either particle will be denoted by q_i. The invariant quantity $(E^2 - \mathbf{P}^2)$ on which S_{ji} depends is just the Mandelstam variable s.

Now by a modification of the argument leading to eq. (30) of Jacob and Wick [11] we have the following partial wave decomposition:

$$S_{ji}(s; \mathbf{n}_j \mathbf{n}_i; \lambda_j'^{(1)} \lambda_j'^{(2)} \lambda_i^{(1)} \lambda_i^{(2)})$$

$$= -\frac{2\pi s^{1/2}}{N_i N_j (q_i q_j)^{1/2}} \sum_{J, M} (J + \tfrac{1}{2}) S^J_{j\lambda_j'^{(1)}\lambda_j'^{(2)}, i\lambda_i^{(1)}\lambda_i^{(2)}}(s) \, \bar{D}^J_{M\lambda_j'}(\phi_j, \theta_j, 2\pi - \phi_j) \, D^J_{M\lambda_i}(\phi_i, \theta_i, 2\pi - \phi_i)$$

In this equation $\lambda_i = \lambda_i^{(1)} - \lambda_i^{(2)}$, $\lambda_j' = \lambda_j'^{(1)} - \lambda_j'^{(2)}$ and (θ_i, ϕ_i), (θ_j, ϕ_j) are the polar angles of $\mathbf{n}_i$, $\mathbf{n}_j$ respectively in some established system of spherical polar coordinates. We may now deduce from the unitarity of the S operator,

$$S^\dagger S = \mathbb{1},$$

that

$$\sum_{k=1}^{n} \sum_{\lambda_k''^{(1)}, \lambda_k''^{(2)}} \overline{S^J_{k\lambda_k''^{(1)}\lambda_k''^{(2)}, j\lambda_j'^{(1)}\lambda_j'^{(2)}}(s)} \, S^J_{k\lambda_k''^{(1)}\lambda_k''^{(2)}, i\lambda_i^{(1)}\lambda_i^{(2)}}(s) = \delta_{ij} \, \delta_{\lambda_i^{(1)}\lambda_j'^{(1)}} \, \delta_{\lambda_i^{(2)}\lambda_j'^{(2)}}$$

This means that if, for fixed s and J, we construct a finite square matrix

$$S^J_{j\lambda_j'^{(1)}\lambda_j'^{(2)}, i\lambda_i^{(1)}\lambda_i^{(2)}}(s),$$

labelling the rows and columns by means of the possible combinations of helicities for each of the n channels in turn, then this matrix is unitary.

It is shown by Jacob and Wick [11] that from the time-reversal invariance property of the S-operator it follows that the matrix $S^J_{j\lambda_j^{(1)}\lambda_j^{(2)},i\lambda_i^{(1)}\lambda_i^{(2)}}(s)$ is symmetric. This means that, if it is an $N \times N$ matrix, it may be specified in terms of $\frac{1}{2}N(N+1)$ real parameters, which may be chosen as $\frac{1}{2}N(N-1)$ moduli of matrix elements and N phases.

The consequences of conservation of parity are a little more complicated to state. Again using the calculation of Jacob and Wick, we find that

$$S^J_{j-\lambda_j^{(1)}-\lambda_j^{(2)},i-\lambda_i^{(1)}-\lambda_i^{(2)}}(s) = \eta_j \eta_i^{-1} S^J_{j\lambda_j^{(1)}\lambda_j^{(2)},i\lambda_i^{(1)}\lambda_i^{(2)}}(s)$$

where η_i is a phase factor which depends only on the particles in channel (i). As Jacob and Wick point out, this means that the number of independent matrix elements is reduced by 'roughly' a factor 2. The word 'roughly' takes account of the possibility that, if there are two or more channels involving only massive bosons, there will be entries in the matrix which correspond to all four helicities being zero, and conservation of parity says nothing about these entries. In the example of Section 4, we show how the 8×8 matrix corresponding to a given J can be transformed into the direct sum of two 4×4 submatrices (still unitary and symmetric) which can be considered separately.

Note finally that if there are further conserved quantum numbers (such as total isospin), the number of independent matrix elements is further reduced. Appendix A gives expressions for the differential cross-section for the binary process $(i) \to (j)$ and for the optical theorem.

3. The Influence of Radiative Capture on Elastic *np* Scattering

If one considers np elastic scattering, even at low energies, there is another two-body channel, namely (γd), and the exothermic reaction $np \to \gamma d$ is possible (d denotes the deuteron). We are interested in the influence of the (γd) channel on the $(J=0, L=0)$ scattering length for np scattering. In considering this, it is convenient to neglect non-central forces; this means that we assume that, as well as the total angular momentum J, the orbital angular momentum L is also a good quantum number. It is obvious that for our present purpose this approximation is adequate; the $(J=1, L=2)$ state admixture has a very low probability in the deuteron.

We look at the np initial state with $J=0$, $L=0$. The final nuclear state consists of the deuteron, with $J=1$, $L=0$. According to the well-known selection rules for γ-emission, the emitted γ must be magnetic dipole radiation. Thus, we are left with a 2×2 matrix S_{ji}, $i, j = 1, 2$, where 1 refers to the (np) channel, 2 to the (γd). We do not consider the possibility of triplet scattering and look only at the contribution to the cross-sections from the $(J=0, L=0)$ state. From Appendix A,

$$\sigma_{11} = \pi q^{-2} |S_{11} - 1|^2,$$
$$\sigma_{21} = \pi q^{-2} |S_{21}|^2,$$

where σ_{11} is the total cross-section for np elastic scattting, σ_{21} the total cross-section for radiative capture $np \to \gamma d$ and q is the magnitude of the three-momentum of either n or p in the CMS.

It can easily be seen that a unitary, symmetric 2×2 matrix can be written, using the three real parameters ρ, α and β, in the following way:

$$S_{11} = \rho e^{2i\alpha}, \quad S_{22} = \rho e^{2i\beta},$$
$$S_{12} = S_{21} = i(1-\rho^2)^{1/2} e^{i(\alpha+\beta)}.$$

The parametrization has been chosen in such a way that in the limiting case of neglecting the (γd) channel $(\rho = 1)$, α is the conventional phase shift for np elastic scattering in the $(J = 0,\ L = 0)$ state. Thus ρ can be interpreted as an inelasticity parameter. One easily checks that

$$\sigma_{11} = \pi q^{-2}[(1-\rho)^2 + 4\rho\sin^2\alpha], \quad \sigma_{21} = \pi q^{-2}(1-\rho^2).$$

The cross-section σ_{21} can be calculated using certain approximations [12]. Since we are only interested in the very low energy behaviour, we write

$$\sigma_{21} = \pi C q^{-1},$$

where C is a constant which can be taken from the calculation of Ref. [12]. Then

$$\rho^2 = 1 - Cq.$$

Assuming that the low-energy behaviour of the phase α is given by

$$\alpha = -qs + 0(q^3),$$

as suggested by effective range theory, we obtain

$$\sigma_{11} = 4\pi[s^2 + \tfrac{1}{4}C^2 - 2C(s^2 - \tfrac{1}{4}C^2)q + 0(q^2)] \tag{*}$$

The interesting point to note is that the low-energy behaviour of σ_{11} is modified; a term linear in q appears, which would not be present if $C = 0$.

The phase α is not to be confused with the real phase α_N which would describe np scattering if the γd channel were absent (we neglect all other corrections to nuclear scattering, in particular those due to the magnetic moment interaction). The 'purely nuclear' scattering length s_N is defined via the low-energy behaviour of α_N:

$$\alpha_N = -qs_N + 0(q^3)$$

To get the experimental value for the parameter s, C is taken either from Ref. [12] or from experiments on radiative capture [13]. Then eq. (*) can be fitted to experiments by adjusting the parameter s. It turns out that the numerical value of s changes only very little compared with the statistical error if one puts $C = 0$ in eq. (*). This does not necessarily mean that $(s - s_N)$ is small.

Since we are really interested in s_N (for example, to test the charge independence of the nucleon-nucleon interaction), it is necessary to resort to a model in order to determine it. This can be done by fitting C and s in a phenomenological way by means of a complex potential. Having determined the parameters of the potential, the imaginary part is put equal to zero and s_N calculated using the real part only. Making a simple numerical estimate in this way, it turns out that $(s - s_N)$ is much smaller than the experimental error on s [14], so that at the present experimental accuracy the influence of the (γd) channel on the determination of s_N can be completely neglected.

4. $\pi^- p$ and Related Channels

We consider now the system consisting of the three channels $(\pi^- p)$, $(\pi^\circ n)$ and (γn), at energies below the threshold for π° production. We make the approximation of neglecting higher order electromagnetic processes in which more than one photon is involved. Furthermore, we neglect Bremsstrahlung, since we confine ourselves to two-particle final states.

For the three channels the parity factors may be chosen as -1, -1 and $+1$, respectively. When time-reversal invariance and parity conservation have been taken into account, the 8×8 matrix for fixed J has just twenty independent entries; we now write it out fully. The numbers in brackets which are given for each row and column are the helicities of the particles in the channel concerned.

$$
\begin{array}{cc}
 & \begin{array}{cccccccc}
\multicolumn{2}{c}{\overbrace{\hphantom{(0\tfrac12)\quad(0-\tfrac12)}}^{(\pi^- p)}} & \multicolumn{2}{c}{\overbrace{\hphantom{(0\tfrac12)\quad(0-\tfrac12)}}^{(\pi^0 n)}} & \multicolumn{4}{c}{\overbrace{\hphantom{(1\tfrac12)\quad(-1-\tfrac12)\quad(1-\tfrac12)\quad(-1\tfrac12)}}^{(\gamma n)}} \\
(0\tfrac12) & (0-\tfrac12) & (0\tfrac12) & (0-\tfrac12) & (1\tfrac12) & (-1-\tfrac12) & (1-\tfrac12) & (-1\tfrac12)
\end{array} \\
\begin{array}{rr}
(\pi^- p)\left\{\begin{array}{r} (0\tfrac12) \\ (0-\tfrac12) \end{array}\right. & \\
(\pi^0 n)\left\{\begin{array}{r} (0\tfrac12) \\ (0-\tfrac12) \end{array}\right. & \\
(\gamma n)\left\{\begin{array}{r} (1\tfrac12) \\ (-1-\tfrac12) \\ (1-\tfrac12) \\ (-1\tfrac12) \end{array}\right. &
\end{array}
&
\begin{bmatrix}
S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 \\
S_2 & S_1 & S_4 & S_3 & -S_6 & -S_5 & -S_8 & -S_7 \\
S_3 & S_4 & S_9 & S_{10} & S_{11} & S_{12} & S_{13} & S_{14} \\
S_4 & S_3 & S_{10} & S_9 & -S_{12} & -S_{11} & -S_{14} & -S_{13} \\
S_5 & -S_6 & S_{11} & -S_{12} & S_{15} & S_{16} & S_{17} & S_{18} \\
S_6 & -S_5 & S_{12} & -S_{11} & S_{16} & S_{15} & S_{18} & S_{17} \\
S_7 & -S_8 & S_{13} & -S_{14} & S_{17} & S_{18} & S_{19} & S_{20} \\
S_8 & -S_7 & S_{14} & -S_{13} & S_{18} & S_{17} & S_{20} & S_{19}
\end{bmatrix}
\end{array}
$$

Now multiply this matrix on the right by the matrix R and on the left by R^T, where R is the real orthogonal matrix given by

$$
R = \frac{1}{\sqrt{2}}
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

The transformed matrix, which is also unitary and symmetric, now takes the form

$$\begin{bmatrix}
S_1+S_2 & S_3+S_4 & S_5-S_6 & S_7-S_8 & 0 & 0 & 0 & 0 \\
S_3+S_4 & S_9+S_{10} & S_{11}-S_{12} & S_{13}-S_{14} & 0 & 0 & 0 & 0 \\
S_5-S_6 & S_{11}-S_{12} & S_{15}-S_{16} & S_{17}-S_{18} & 0 & 0 & 0 & 0 \\
S_7-S_8 & S_{13}-S_{14} & S_{17}-S_{18} & S_{19}-S_{20} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & S_1-S_2 & S_3-S_4 & S_5+S_6 & S_7+S_8 \\
0 & 0 & 0 & 0 & S_3-S_4 & S_9-S_{10} & S_{11}+S_{12} & S_{13}+S_{14} \\
0 & 0 & 0 & 0 & S_5+S_6 & S_{11}+S_{12} & S_{15}+S_{16} & S_{17}+S_{18} \\
0 & 0 & 0 & 0 & S_7+S_8 & S_{13}+S_{14} & S_{17}+S_{18} & S_{19}+S_{20}
\end{bmatrix}$$

Each of the 4×4 submatrices is thus unitary and symmetric. The upper submatrix corresponds to parity $(-1)^{J+1/2}$, the lower to parity $(-1)^{J-1/2}$. The transformation with the matrix R corresponds to the well-known transition from helicity eigenstates to electromagnetic multipoles.

Corresponding to each fixed total angular momentum J and parity P, we thus have a 4×4 matrix, except for the states with $J = \frac{1}{2}$, for which the matrix is 3×3. A convenient notation for the elements of the matrix is

$$\begin{array}{c c}
 & \begin{array}{cccc} (\pi^- p) & (\pi^0 n) & (\gamma n, 1) & (\gamma n, 2) \end{array} \\
\begin{array}{c} (\pi^- p) \\ (\pi^0 n) \\ (\gamma n, 1) \\ (\gamma n, 2) \end{array} &
\begin{bmatrix}
S_{--} & S_{0-} & S_{1-} & S_{2-} \\
S_{0-} & S_{00} & S_{10} & S_{20} \\
S_{1-} & S_{10} & S_{11} & S_{12} \\
S_{2-} & S_{20} & S_{12} & S_{22}
\end{bmatrix}
\end{array}$$

Now for the energies under consideration, there is an approximate dispersion theory of pion photoproduction which yields the four complex numbers S_{1-}, S_{2-}, S_{10}, S_{20}, which are of first order in the proton charge e. The multipole amplitudes $E_{l\pm}^{(0,1,3)}$, $M_{l\pm}^{(0,1,3)}$ are tabulated by Berends, Donnachie and Weaver [15] and in Appendix B explicit formulae are given for computing S_{1-}, S_{2-}, S_{10}, S_{20} from them.

Knowing these four numbers, the three complex numbers S_{--}, S_{0-}, S_{00} required for the analysis of the reactions $\pi^- p \to \pi^- p$, $\pi^- p \to \pi^\circ n$, $\pi^\circ n \to \pi^\circ n$ respectively may be obtained in terms of *three* real parameters. If we write $S_{--} = |S_{--}|e^{2i\alpha}$, $S_{0-} = i|S_{0-}|e^{i\gamma}$, $S_{00} = |S_{00}|e^{2i\beta}$, and α, β, and $|S_{0-}|$ are left as parameters to be determined, then $|S_{--}|$, $|S_{00}|$ and γ may be obtained from the following three equations:

$$|S_{--}|^2 = 1 - |S_{0-}|^2 - |S_{1-}|^2 - |S_{2-}|^2,$$

$$|S_{00}|^2 = 1 - |S_{0-}|^2 - |S_{10}|^2 - |S_{20}|^2,$$

$$-i|S_{--}||S_{0-}|e^{i(2\alpha-\gamma)} + i|S_{00}||S_{0-}|e^{i(\gamma-2\beta)} + S_{1-}\bar{S}_{10} + S_{2-}\bar{S}_{20} = 0.$$

If in the last equation we introduce δ defined by

$$\gamma = \alpha + \beta + \delta$$

we obtain the two equations

$$\cos\delta|S_{0-}|(|S_{00}| - |S_{--}|) = \mathrm{Re}[ie^{-i(\alpha-\beta)}(S_{1-}\bar{S}_{10} + S_{2-}\bar{S}_{20})],$$

$$\sin\delta|S_{0-}|(|S_{00}| + |S_{--}|) = \mathrm{Im}[ie^{-i(\alpha-\beta)}(S_{1-}\bar{S}_{10} + S_{2-}\bar{S}_{20})].$$

The second of these equations shows that δ is of order e^2, so that the two equations become, keeping terms of order e^2 only,

$$|S_{0-}|(|S_{1-}|^2 + |S_{2-}|^2 - |S_{10}|^2 - |S_{20}|^2) \approx 2(1 - |S_{0-}|^2)^{1/2}\,\mathrm{Re}[ie^{-i(\alpha-\beta)}(S_{1-}\bar{S}_{10} + S_{2-}\bar{S}_{20})],$$

$$\delta \approx \tfrac{1}{2}[|S_{0-}|(1 - |S_{0-}|^2)^{1/2}]^{-1}\,\mathrm{Im}[ie^{-i(\alpha-\beta)}(S_{1-}\bar{S}_{10} + S_{2-}\bar{S}_{20})]$$

If the photoproduction amplitudes were known with sufficient accuracy, the first of these equations could be taken as a relation between $|S_{0-}|$, α and β, so that only two parameters would be required for each (J, P) state in order to fit the experimental data on $\pi^- p$ elastic and charge exchange scattering. However, the photoproduction theory is not reliable enough for this purpose; it uses Watson's final state theorem, 'charge independent' pion nucleon phases, equal masses for the pions and for the nucleons and the approximate solution of integral equations. It would be better to say that very accurate $\pi^- p$ experiments might eventually be used to give a relation between the photoproduction amplitudes and thus a check on the reliability of photoproduction theory.

What we have pointed out is that, instead of the ten parameters required to specify the full 4×4 unitary symmetric matrix for each (J, P) state, it is possible to use our approximate knowledge of the pion photoproduction multipole amplitudes to express the two amplitudes S_{--}, S_{0-} required for the analysis of $\pi^- p$ elastic and charge exchange scattering in terms of three parameters $|S_{0-}|$, α and β, via the equations

$$S_{--} = |S_{--}|e^{2i\alpha}, \quad |S_{--}| \approx (1 - |S_{0-}|^2)^{1/2} - \tfrac{1}{2}(1 - |S_{0-}|^2)^{-1/2}(|S_{1-}|^2 + |S_{2-}|^2),$$

$$S_{0-} = i|S_{0-}|e^{i\gamma}, \quad \gamma = \alpha + \beta + \delta,$$

$$\delta \approx \tfrac{1}{2}[|S_{0-}|(1 - |S_{0-}|^2)^{1/2}]^{-1}\,\mathrm{Im}[ie^{-i(\alpha-\beta)}(S_{1-}\bar{S}_{10} + S_{2-}\bar{S}_{20})].$$

It is perhaps not too much to hope that some day $\pi^- p$ experiments can be performed which are sufficiently accurate for three parameters for each (J, P) state to be reliably determined.

So far, we have not taken account of Coulomb corrections. This can be done only in a non-relativistic way. The formalism has been worked out by Oades and Rasche [16]. The formulae of Appendix 1 of that paper can be taken over, with the $S_{l\pm}^{--}$ and $S_{l\pm}^{0-}$ appearing there replaced by the S_{--}^{JP} and S_{0-}^{JP} of the present paper, and $J = (l \pm \frac{1}{2})$, $P = (-1)^{l+1}$. Note that, whereas in Ref. [16] it was shown, using real pion-nucleon potentials, that the submatrix $S_{l\pm}^{ij}$ $(i, j = 0, -)$ is unitary, it is clear from the discussion of this section, that this is no longer the case when the γn channel is taken into account. This would also be true in a phenomenological model in which the pion-nucleon potentials are taken to be complex.

5. Conclusions

Until recently, the statistics of $\pi^- p$ experiments were so limited that it was only possible to determine just *one* parameter for each (J, P) state. In the evaluation of most of the experiments, the electromagnetic effects have been taken into account only via the additive Coulomb amplitude. The nuclear amplitude was parametrized in a charge independent way by introducing two purely nuclear phases, for $T = \frac{1}{2}$ and $T = 3/2$, the latter being taken from the analysis of $\pi^+ p$ experiments.

The better statistics available in the experiments of Carter et al. [9] force one to think more carefully about the effect of the electromagnetic interactions. We would like to emphasize that this must be done already at a very early stage, namely as soon as one extracts phases from the differential cross-sections. Further, we emphasize that already at that stage, one has to take into account the presence of the γn channel, by introducing more parameters in the analysis of $\pi^- p$ experiments. In this paper, we have shown how some of the parameters can be fixed by an approximate knowledge of pion photoproduction multipole amplitudes, so that in the analysis of the data only three parameters have to be included for each (J, P) state.

Unfortunately, even for the presently available $\pi^- p$ experiments with the best statistical accuracy, it is impossible to determine three parameters for each (J, P) state. On the other hand, it is clear that taking one parameter from the analysis of $\pi^+ p$ experiments and including a second parameter for fitting $\pi^- p$ experiments is not adequate to obtain a good statistical fit, unless an attempt is made, using a model, to obtain 'purely nuclear' phases from the 'nuclear' phases extracted from the analysis. We are thus in an awkward intermediate stage where it is necessary to resort to some kind of 'charge independent' model, however inadequate, in order to fit the $\pi^\pm p$ experiments, but we cannot determine enough parameters to test any such model. To conclude, then, we have proposed a method for parametrizing $\pi^- p$ experiments which requires three parameters to be determined at each energy for a given (J, P) state. Since these three parameters cannot be reliably determined from the presently available experiments, it is necessary to use a specific charge independent model which enables $\pi^\pm p$ experiments at a given energy below the threshold for pion production to be analysed in terms of just two parameters for each (J, P) state. However, since the model cannot be tested as to its reliability, the 'charge independent' phases which are extracted from the analysis of the experiments are open to considerable uncertainty.

One corollary of this negative conclusion is that models for time reversal violation in pion photoproduction and radiative capture, and evidence for an isotensor term in the electromagnetic current [18, 19] must be viewed with suspicion. Apart from the different conclusions of these references, we wish to emphasize the unsatisfactory features of current photoproduction models which are listed in Section 4. These models use 'charge independence' in a very crude form. No account is taken of Coulomb or mass difference effects and, most important, the 'charge independent' phases which are used are uncertain to a degree which makes the unambiguous isolation of residual effects like an isotensor current impossible.

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Appendix A

The expression for the differential cross-section for the process $(i) \to (j)$ is

$$\frac{d\sigma}{d\Omega_{\mathbf{n}_j}}(s;\mathbf{n}_j\mathbf{n}_i;\lambda_j'^{(1)}\lambda_j'^{(2)}\lambda_i^{(1)}\lambda_i^{(2)}) = \frac{q_j}{q_i}\frac{N_i^2N_j^2}{4\pi^2 s}|T_{ji}(s;\mathbf{n}_j\mathbf{n}_i;\lambda_j'^{(1)}\lambda_j'^{(2)}\lambda_i^{(1)}\lambda_i^{(2)})|^2,$$

where T_{ji} is defined by

$$\langle\mathbf{p}_j^{(1)}\lambda_j'^{(1)};\mathbf{p}_j^{(2)}\lambda_j'^{(2)}|S-\mathbb{1}|\mathbf{p}_i^{(1)}\lambda_i^{(1)};\mathbf{p}_i^{(2)}\lambda_i^{(2)}\rangle$$

$$= -i(2\pi)^{-2}\,\delta^{(4)}(p_j^{(1)}+p_j^{(2)}-p_i^{(1)}-p_i^{(2)})\frac{N_iN_j}{[E_i^{(1)}E_i^{(2)}E_j^{(1)}E_j^{(2)}]^{1/2}}$$

$$\times\, T_{ji}(s;\mathbf{n}_j\mathbf{n}_i;\lambda_j'^{(1)}\lambda_j'^{(2)}\lambda_i^{(1)}\lambda_i^{(2)}).$$

In terms of partial wave decompositions, this means that

$$S^J_{j\lambda_j'^{(1)}\lambda_j'^{(2)},i\lambda_i^{(1)}\lambda_i^{(2)}}(s)-\delta_{ij}\,\delta_{\lambda_i^{(1)}\lambda_j'^{(1)}}\delta_{\lambda_i^{(2)}\lambda_i'^{(2)}} = iT^J_{j\lambda_j'^{(1)}\lambda_j'^{(2)},i\lambda_i^{(1)}\lambda_i^{(2)}}(s).$$

It is customary to take a system of axes for which $\mathbf{n}_i$ is in the direction of the polar axis ($\mathbf{n}_i = \mathbf{e}$). Since

$$D^J_{M\lambda_i}(\phi_i, 0, 2\pi-\phi_i) = e^{-2\pi\lambda_i i}\,\delta_{M\lambda_i},$$

we have

$$T_{ji}(s;\mathbf{n}_j\,\mathbf{e};\lambda_j'^{(1)}\lambda_j'^{(2)}\lambda_i^{(1)}\lambda_i^{(2)})$$

$$= -\frac{2\pi s^{1/2}}{N_iN_j\sqrt{q_iq_j}}e^{i(\lambda_i-\lambda_j')\phi_j}\sum_J(J+\tfrac{1}{2})\,T^J_{j\lambda_j'^{(1)}\lambda_j'^{(2)},i\lambda_i^{(1)}\lambda_i^{(2)}}(s)\,d^J_{\lambda_i\lambda_j'}(\theta_j),$$

since $(\lambda_i-\lambda_j')$ is always an integer. Integrating over all directions $\mathbf{n}_j$, we obtain the total cross-section σ_{ji} for the process $(i)\to(j)$:

$$\sigma_{ji}(s;\lambda_j'^{(1)}\lambda_j'^{(2)}\lambda_i^{(1)}\lambda_i^{(2)}) = \frac{2\pi}{q_i^2}\sum_J(J+\tfrac{1}{2})|T^J_{j\lambda_j'^{(1)}\lambda_j'^{(2)},i\lambda_i^{(1)}\lambda_i^{(2)}}(s)|^2.$$

Using an obvious notation for the matrices, it follows from

$$(S^J)^\dagger S^J = \mathbb{1},\quad (S^J)^T = S^J,\quad S^J-\mathbb{1} = iT^J$$

that

$$2 \operatorname{Im} T^J = (T^J)^\dagger T^J.$$

In particular,

$$2 \operatorname{Im} T^J_{i\lambda_i^{(1)}\lambda_i^{(2)}, i\lambda_i^{(1)}\lambda_i^{(2)}}(s) = \sum_{j=1}^{n} \sum_{\lambda_j'^{(1)}\lambda_j'^{(2)}} |T^J_{j\lambda_j'^{(1)}\lambda_j'^{(2)}, i\lambda_i^{(1)}\lambda_i^{(2)}}(s)|^2$$

and thus

$$\sum_{j=1}^{n} \sum_{\lambda_j'^{(1)}\lambda_j'^{(2)}} \sigma_{ji}(s; \lambda_j'^{(1)} \lambda_j'^{(2)} \lambda_i^{(1)} \lambda_i^{(2)})$$

$$= \frac{4\pi}{q_i^2} \sum_J (J + \tfrac{1}{2}) \operatorname{Im} T^J_{i\lambda_i^{(1)}\lambda_i^{(2)}, i\lambda_i^{(1)}\lambda_i^{(2)}}(s)$$

$$= -\frac{2N_i^2}{q_i s^{1/2}} \operatorname{Im} T_{ii}(s; \mathbf{ee}; \lambda_i^{(1)} \lambda_i^{(2)} \lambda_i^{(1)} \lambda_i^{(2)}).$$

This is the optical theorem; to express it in manifestly covariant form, one simply notes that

$$q_i s s^{/2} = \tfrac{1}{2}[\quad - (m_i^{(1)} + m_i^{(2)})^2]^{1/2} [s - (m_i^{(1)} - m_i^{(2)})^2]^{1/2}.$$

Appendix B

We now give the formulae connecting the pion photoproduction amplitudes S_{1-}^{JP}, S_{2-}^{JP}, S_{10}^{JP}, S_{20}^{JP} with the usual electric and magnetic multipole amplitudes. The details of the calculation can be reconstructed from Refs. [15, 17].

We shall use q_-, q_0, q_γ to denote the magnitude of the three-momentum of either particle in the CMS, for the channels $(\pi^- p)$, $(\pi^\circ n)$, (γn) respectively. Then

$$S_{i-}^{JP} = \sqrt{2 q_- q_\gamma} \left[\sqrt{2} A_i^{JP(0)} - \frac{\sqrt{2}}{3} A_i^{JP(1)} + \frac{\sqrt{2}}{3} A_i^{JP(3)} \right],$$

$$S_{i0}^{JP} = \sqrt{2 q_0 q_\gamma} [-A_i^{JP(0)} + \tfrac{1}{3} A_i^{JP(1)} + \tfrac{2}{3} A_i^{JP(3)}],$$

where $i = 1, 2$ and

$$A_1^{JP(k)} = -(J + \tfrac{3}{2}) E^{(k)}_{(J-\frac{1}{2})+} - (J - \tfrac{1}{2}) M^{(k)}_{(J-\frac{1}{2})+},$$

$$A_2^{JP(k)} = \sqrt{(J - \tfrac{1}{2})(J + \tfrac{3}{2})} [E^{(k)}_{(J-\frac{1}{2})+} - M^{(k)}_{(J-\frac{1}{2})+}],$$

when $P = (-1)^{J+1/2}$ and $k = 0, 1, 3$, while

$$A_1^{JP(k)} = (J - \tfrac{1}{2}) E^{(k)}_{(J+\frac{1}{2})-} - (J + \tfrac{3}{2}) M^{(k)}_{(J+\frac{1}{2})-},$$

$$A_2^{JP(k)} = \sqrt{(J - \tfrac{1}{2})(J + \tfrac{3}{2})} [E^{(k)}_{(J+\frac{1}{2})-} + M^{(k)}_{(J+\frac{1}{2})-}],$$

when $P = (-1)^{J-1/2}$ and $k = 0, 1, 3$.

REFERENCES

[1] L. VAN HOVE, Phys. Rev. *88*, 1358 (1952).
[2] J. HAMILTON and W. S. WOOLCOCK, Phys. Rev. *118*, 291 (1960).
[3] J. REIGNIER, Bull. Acad. Roy. Belg. (Classe des Sci.) *48*, 179 (1962).
[4] H. J. SCHNITZER, Nuovo Cim. *28*, 752 (1963).
[5] E. SAUTER, Nuovo Cim. *61A*, 515 (1969).
[6] G. RASCHE, Nuovo Cim. *62A*, 229 (1969).
[7] H. Y. CHIU, Phys. Rev. *111*, 1170 (1958).
[8] G. C. OADES and G. RASCHE, Helv. phys. Acta *44*, 5 (1971). See also the later papers by P. R. AUVIL, Phys. Rev. *D4*, 240 (1971); E. SAUTER, Nuovo Cim. *6A*, 335 (1971).
[9] A. A. CARTER, J. R. WILLIAMS, D. V. BUGG, P. J. BUSSEY and D. R. DANCE, Nucl. Phys. *B26*, 445 (1971).
[10] N. A. TÖRNQVIST, Nucl. Phys. *B6*, 187 (1968); M. KORKEA-AHO and N. A. TÖRNQVIST, Nucl. Phys. *B33*, 239 (1971).
[11] M. JACOB and G. C. WICK, Ann. Phys. *7*, 404 (1959).
[12] J. M. BLATT and V. F. WEISSKOPF, *Theoretical Nuclear Physics* (John Wiley, New York, 1952), chapter 12, sect. 4.
[13] R. J. ADLER, B. T. CHERTOK and H. C. MILLER, in: *High Energy Physics and Nuclear Structure*, ed. S. Devons (Plenum Press, New York, 1970) 541.
[14] C. YALCIN, Nucl. Phys. *A169*, 201 (1971).
[15] F. A. BERENDS, A. DONNACHIE and D. L. WEAVER, Nucl. Phys. *B4*, 1, 54, 103 (1967).
[16] G. C. OADES and G. RASCHE, Helv. phys. Acta *44*, 160 (1971).
[17] G. RASCHE and W. S. WOOLCOCK, Nucl. Phys. *87*, 51 (1966).
[18] W. PFEIL and D. SCHWELA, Lett. al Nuovo Cim. *1*, 1061 (1971).
[19] A. DONNACHIE and G. SHAW, Phys. Letters *35B*, 419 (1971).