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Minimum Radius of Particles with Spin

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According to the generalized first law (momentum-energy and centre of energy-angular momentum), the density tensor of any insulated system Σ_{00} satisfies, in restricted relativity (r.r.)

$$\partial_\alpha \Theta^{\alpha\beta}(x) = 0 \quad (\Theta^{\alpha\beta} = \Theta^{(\alpha\beta)})(x). \tag{1}$$

The energy is

$$H = \int_\infty (d^3\, Vh)(\vec{x}\breve{t}) = \int_\infty (d^3\, V\Theta^{44})(x) = \int_{\tau(y)=0} (d\breve{\sigma}_\alpha \breve{\Theta}^{\alpha 4})(y) \tag{2}$$

and the momentum

$$\vec{\breve{\Pi}} = \overleftarrow{\breve{\Pi}} = \{\breve{\Pi}_i\} \quad \breve{\Pi}_i = \int_{\tau(y)=0} (d\breve{\sigma}_\alpha \Theta^\alpha_i)(y) = \int_\infty (d^3\, V\pi_i)(\vec{x}\breve{t}).$$

The angular momentum $\breve{\vec{M}} = \{\mathring{M}^i = M_{kl},\ ikl \circlearrowleft 123\}$ has the components

$$\breve{M}_{ik} = \int_{\tau(y)=0} (d\breve{\sigma}_\alpha(y_i \Theta^\alpha_k - y_k \Theta^\alpha_i))(y) = \int_\infty (d^3\, V(x_i \pi_k - x_k \pi_i))(\vec{x}\breve{t}) \tag{3}$$

and the centre of energy is determined by

$$\vec{M} = \overleftarrow{M} = \left\{ M_i = M_{i4} = \int (d\breve{\sigma}_\alpha(y_i \breve{\Theta}^{\alpha 4} - \breve{y}^4 \Theta^\alpha_i))(y) = \int_\infty (d^3\, Vx_i h)(\vec{x}\breve{t}) - \breve{t}\breve{\Pi}_i \right\} \tag{4}$$

the $\breve{M}_{\alpha\beta} = \breve{M}_{(\alpha\beta)}$ being constants.

$$\left.\begin{aligned} \vec{z}(\breve{t}) &= \frac{\vec{M}}{H} + \breve{t}\frac{\vec{\breve{\Pi}}}{H} = \frac{1}{H}\int_\infty (d^3\, V\vec{x}h)(\vec{x}\breve{t}) \\ \breve{z}^4(\breve{t}) &= \breve{t} \end{aligned}\right\} \tag{5}$$

$d\vec{z}(\breve{t})/d\breve{t} = \vec{\breve{\Pi}}/H \equiv \breve{\vec{v}}$ is the *3-velocity of the centre of energy* (c.e.) $\vec{z}(\breve{t})$.

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[1]) Paper presented at the Swiss Physical Society Meeting of Lausanne, 1 May, 1971, see: Helv. phys. Acta, *44*, 593 (1971).

In this frame τ(= proper time) $\overset{*}{=} \check{t}$, and $\vec{w} \overset{*}{=} \check{\vec{v}}$; $\check{w}^4 \overset{*}{=} +1$. So we may write (5) in a covariant way

$$\left.\begin{aligned} z^i(\tau) &= M^i + \tau w^i, \\ \check{z}^4(\tau) &= \tau \check{w}^4. \end{aligned}\right\} \tag{6}$$

Now we may choose in this frame $\vec{M} = 0$. Then the *world-line* (w.l.) $x^\alpha = z^\alpha(\tau)$ coincides with the $\check{t} = \check{x}^4$-axis in this particular frame. Let $\overset{\curvearrowright}{M} \neq 0$ be the intrinsic angular momentum (spin: $|\overset{\curvearrowright}{M}|^2 = \hbar^2 S(S+1)$) in this frame. Then, in any other frame $'x = Lx$, obtained by homogeneous Lorentz transformations L[2]) from x, the w.l. $'x = {}'z(\tau)$ is *parallel* (or *anti-parallel*) to the w.l. $x = z(\tau)$ (depending whether *'not' or 'yes' time inversion* is *included*). $'\vec{M}$ however, as soon as $\overset{\curvearrowright}{M} \neq 0$, no longer disappears in general. That means *these (anti-)parallels are shifted by an amount*, which can be easily obtained by L:

$$'M'^{4}_{\ i} = {}'M'^{i'4} (\equiv {}'M'^i) = (L'^i_{\ i} L'^4_{\ k} - L'^i_{\ k} L'^4_{\ i})\, \mathring{M}^l \quad (ikl \circlearrowleft 123), \tag{7}$$

where, taking (without time inversion sig $('\check{t})$ = sig $(\check{t})$)

$$\left.\begin{aligned} 'x'^1 &= x^1 \\ 'x'^2 &= x^2 \\ 'x'^3 &= (1 - v_0^2)^{-1/2} (x^3 - \check{v}_0 \check{t}) \\ '\check{t} = {}'\check{x}'^4 &= (1 - v_0^2)^{-1/2} (-\check{v}_0 x^3 + \check{t}) \end{aligned}\right\} \tag{8}$$

which determines

$$L'^3_{\ 3} = L'^4_{\ 4} = (1 - v_0^2)^{-1/2} \qquad \check{L}'^3_{\ 4} = -\check{v}_0 (1 - v_0^2)^{-1/2} = \check{L}'^4_{\ 3} \tag{9}$$

all other L'^α_β being zero or one. One thus obtains ($v_0^3 = v_0$; $v_0^2 = v_0^1 = 0$), taking account of $'H = M(1 - v_0^2)^{-1/2}$, for the displacement of the c.e.:

$$\frac{'\vec{M}}{'H} = -\frac{[\vec{v}_0 \wedge \overset{\curvearrowright}{M}]}{M} \tag{10}$$

The set of all c.e.'s fills a disk, normal to $\vec{v}_0$ and to $\overset{\curvearrowright}{M}$, whose radius R is given by $|\vec{v}_0| = 1 - \epsilon$, $\epsilon \to +0$:

$$R = \frac{|\overset{\curvearrowright}{M}|}{M} (1 - \epsilon) \Rightarrow \frac{\hbar}{M} \overset{+}{\sqrt{s(s+1)}}, \quad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots \tag{11}$$

Now $\hbar/M = h/2\pi M = \lambda_c$ ($c = 1$) the *Compton wave length* of a particle with mass M. The result is quite satisfactory, because the optimum location of a particle of mass

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[2]) We choose

$$g^{ll} = g_{ll} = -g^{44} = -g_{44} = 1; \quad g_{\alpha\beta} = g^{\alpha\beta} = 0 \text{ for } \alpha \neq \beta \tag{7a}$$

$'x = Lx$ is given by

$$'x'^\alpha = L'^\alpha x^\alpha \quad 'g'^{\alpha'\beta} = L'^\alpha_{\ \alpha} L'^\beta_{\ \beta} g^{\alpha\beta} \equiv g'^{\alpha'\beta} \tag{7b}$$

M is given by the invariant wave packet $D_\kappa^+(x)(\kappa = \hbar^{-1} M = \lambda_c^{-1})$, which decreases exponentially for space-like events $x^2 = x_\alpha x^\alpha > 0$

$$D_\kappa^+(x) \propto |x|^{-3} \exp\left(-\frac{|x|}{\lambda_c}\right), \quad |x| > \lambda_c \tag{12}$$

while for time-like events, it behaves $t^2 = -x^2 = -x_\alpha x^\alpha > 0$

$$D_\kappa^+(x) \propto |t|^{-3} \exp\left(+i\frac{|\breve{t}|}{\lambda_c}\right), \quad |\breve{t}| > \lambda_c \tag{13}$$