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Autor: Albeverio, Sergio

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Scattering Theory in a Model of Quantum Fields II

by **Sergio Albeverio**[1] [2]

Department of Physics, Joseph Henry Laboratories,
Princeton University, Princeton, N.J. 08540

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Abstract. We pursue the study of the scattering theory in Nelson's model with relativistic kinematics ('Eckmann's model'). For the choice of mass renormalization yielding a renormalized Hamiltonian with the relativistic single-particle spectrum, we construct physical asymptotic nucleon fields as strong limits of dressed fields on dense sets of states with finitely many nucleons and arbitrarily many mesons. The commutation relations of the asymptotic fields among themselves, with the asymptotic meson fields and with the Hamiltonian are derived, as well as the asymptotic decomposition of the latter and the relation with the wave operators. An expression for the S-matrix is also given, which is then discussed for the case of the meson-nucleon scattering.

1. Introduction

In a previous investigation [1] (to which we shall always refer in this paper as I) we have constructed some basic quantities for the study of the scattering in two closely related models of quantum field theory, which had been previously renormalized and discussed by J. P. Eckmann [2], [3]. The present paper is based on I and uses everywhere the same notations and definitions. Let us recall briefly the definitions and results of I we shall use most.[3]) There are two kinds of 'bare particles' in the model, 'nucleons' (or 'b particles') and 'mesons' (or 'a particles'). Both nucleons and mesons are assumed to have strictly positive masses m_b, $m_a > 0$, to have spin 0 and Bose statistics, and to move in 3-space dimensions.[4]) The Fock space $\mathscr{H}$ is the tensor product of the individual Fock spaces for the two kinds of particles. Since the number of nucleons is conserved by the interaction, the model splits into dynamically independent sectors $\mathscr{H}^{(n)} \equiv \oplus_{m=0}^{\infty} \mathscr{H}^{(n,m)}$ with fixed number n of nucleons ($n = 0, 1, 2, \ldots$) and any number of mesons. The subspace $\mathscr{H}^{(n,m)}$ of the states with exactly n nucleons and m mesons is realized as the space $L_2^{(s)}(\mathbb{R}^{3(n+m)})$ of square integrable functions which are separately symmetric in their n nucleon arguments momenta and in their m meson arguments (momenta). Let $(,)$, $\|,\|$ be the scalar product resp. norm in $\mathscr{H}$.

The free kinetic energy operator is $H_0 \equiv H_0^{(a)} + H_0^{(b)}$, where

$$H_0^{(a)} \equiv \int \mu(k)\, a^*(k)\, a(k)\, dk,$$

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[1]) Supported by the Swiss National Foundation (Forschungsstipendium of the Eidgenössische Technische Hochschule, Zürich).

[2]) Present address: Institute of Mathematics, University of Oslo, Blindern, Oslo 3, Norway.

[3]) For more details we refer to I (even if not stated every time explicitly).

[4]) For remarks to these assumptions see I.

$$H_0^{(b)} \equiv \int \omega(k)\, b^*(k)\, b(k)\, dk,$$

$$\mu(k) \equiv (m_a^2 + |k|^2)^{1/2},$$

$$\omega(k) \equiv (m_b^2 + |k|^2)^{1/2},$$

$a^{\#}(k)$, $b^{\#}(k)$ being the usual formal creation-annihilation operators for a, respectively b, particles (always in this paper the symbol # attached to any operator c stands for the star * or its omission: $c^{\#} = c^*$ or c).

The formal euclidean invariant interaction is $\lambda V = \lambda(V^a + V^c)$, where λ is a real number ('coupling constant') and

$$V^c = \int \omega(k_1)^{-1/2}\, \omega(k_2)^{-1/2}\, \mu(k_3)^{-1/2}\, \delta(k_1 - k_2 + k_3)\, b^*(k_1)\, b(k_2)\, a^*(k_3)\, dk_1\, dk_2\, dk_3. \tag{1.1}$$

V^a is the formal adjoint of V^c.

Because of its ultraviolet divergence, V^c is not a well-defined operator in any $\mathscr{H}^{(n)}$, $n > 0$ and the renormalization is done by introduction of an ultraviolet cut-off σ in all momenta in the interaction and a suitable mass counter term $\mathscr{M}_\sigma$.

In I we discussed the possible choices of $\mathscr{M}_\sigma$. In the present paper we continue the study initiated in §3 of I of the 'full model' with mass renormalization $\mathscr{M}_\sigma = \hat{M}_\sigma$ chosen in such a way that it compensates, in a suitable resolvent limit, the ultraviolet divergence of V_σ for $\sigma \to \infty$ and, at the same time, determines the energy dependence on the momentum k of any physical nucleons associated with the Hamiltonian to be the relativistic one ($\omega(k)$) (so that bare mass equal physical mass, if we take the terminology of relativistic theories). This choice of mass renormalization is given explicitly ([3], I) as a power series $\hat{M}_\sigma(\lambda) = \sum_{k=1}^{\infty} \lambda^{2k} M_{\sigma 2k}$ in λ^2, convergent for $|\lambda|$ sufficiently small. The correspondent renormalized Hamiltonian $\hat{H}_\sigma \equiv H_0 + \lambda V_\sigma + \hat{M}_\sigma(\lambda)$ is, for any $\sigma < \infty$, self-adjoint on the domain $D(H_0)$ of H_0 and bounded from below in each $\mathscr{H}^{(n)}$. Moreover, by construction, $\hat{H}_\sigma$ acts as the free-energy operator H_0 on states with at most one nucleon, in the sense that $\hat{H}_\sigma(T_\sigma A_\sigma) = (T_\sigma A_\sigma) H_0$ on

$$(\mathscr{H}^{(1,0)} \cup \mathscr{H}^{(0)}) \cap D(H_0),$$

for a suitable invertible operator $(T_\sigma A_\sigma)$ (T_σ is the dressing transformation defined by (3.8) of I and A_σ is the amplitude renormalization defined by (3.9) of I: see also Theorem I3.1.[5]) $T_\sigma A_\sigma$ acts as the identity in $\mathscr{H}^{(0)}$.) This solves the 'one-body problem' for the nucleons and mesons.[6]) The dressing operator T_σ is itself a convergent power series in λ, given in [3] and I.

In I we introduced accordingly creation-annihilation operators for (dressed ≡ bare) mesons and dressed nucleons.

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[5]) We shall use the notations Theorem Ix, Lemma Iy, formula I(z), etc., for Theorem x in I, Lemma y in I, formula (z) in I, etc.

[6]) If there are no bound states for the Hamiltonian in the center of mass system (e.g. for $|\lambda|$ sufficiently small) (besides of course the vacuum, the physical one nucleon and the one meson states), then only physical particles with the relativistic energy-momentum dependence would be associated with the Hamiltonian. In any case the particular choice of mass renormalization can be looked upon as a tentative to mimic as far as possible a relativistic situation (the additional bound states would not necessarily have the relativistic energy-momentum dependence).

Their general form is:

$$B^*(h'_\sigma) = b^*(h'_\sigma) + \mathfrak{b}^*(h'_\sigma), \tag{1.2}$$

where $\mathfrak{b}^*(.)$ is a 'cloud term' which creates one bare nucleon and infinitely many mesons. The test-function $h'_\sigma(.)$ is obtained from an arbitrary $L_2(\mathbb{R}^3)$-function $h(.)$ with L_2-norm 1 by multiplication with an 'amplitude or field-strength renormalization' $\nu_\sigma^{-1/2}$:

$$h'_\sigma(k) \equiv h(k) . \nu_\sigma^{-1/2}(k) \quad \text{for all } k \in \mathbb{R}^3.$$

(A_σ acts as multiplication by $\nu_\sigma^{-1/2}(.)$ on one nucleon functions.) ν_σ is given in [3] and I ((3.9), (3.9′)) as a power series in λ^2, convergent for $|\lambda|$ sufficiently small. The introduction of h'_σ in (1.2) makes that the physical (dressed) one-nucleon states

$$B^*(h'_\sigma)\,\Omega_0 = T_\sigma A_\sigma b^*(h)\,\Omega_0,$$

Ω_0 being the vacuum, have norm 1, as the correspondent bare ones. In I (§3.1.1) we have shown that $B^*(h'_\sigma)$ is a linear operator defined on a dense set

$$D'^{(n)} \equiv \bigcup_{\alpha > (1/2)\ln 2} D^{(n),\alpha}$$

of states $\Phi^{(n)}$ in $\mathscr{H}^{(n)}$, the norm of whose components with high numbers m of mesons decreases exponentially: $\|\Phi^{(n,m)}\| = O(e^{-\alpha m})$, $\alpha > \frac{1}{2}\ln 2$, $m \to \infty$. We recall the definition

$$D^{(n),\alpha} \equiv \{\Phi^{(n)} \in \mathscr{H}^{(n)} |\, \sup e^{\alpha m} \|\Phi^{(n,m)}\| < \infty\}.$$

Moreover (I, §3.1.1),

$$B^*(h'_\sigma) \in \mathfrak{B}(D^{(n),\alpha}; D^{(n+1),\alpha'}),$$

for all $\alpha > \frac{1}{2}\ln 2$, $0 < \alpha' < \alpha - \frac{1}{2}\ln 2$, where we denote in general by $\mathfrak{B}(X, Y)$ the set of all bounded operators between Banach spaces X, Y with their respective norms.

In particular the n dressed nucleons, m mesons states

$$\prod_{j=1}^{n} B^*(h'^{(j)}_\sigma) \prod_{i=1}^{m} a^*(f^{(i)})\,\Omega_0$$

can be formed, for any $n, m = 0, 1, 2, \ldots$ $h^{(j)}, f^{(i)} \in L_2(\mathbb{R}^3)$ (the prime $'$ and the label σ associated simultaneously to an L_2-function denote always multiplication by the amplitude renormalization $\nu_\sigma^{-1/2}(.)$: $h'^{(j)}_\sigma \equiv \nu_\sigma^{-1/2} h^{(j)}_\sigma$). $T_\sigma A_\sigma$ maps the bare states $\prod_j b^*(h^{(j)}) \prod_i a^*(f^{(i)})\,\Omega_0$ into these correspondent dressed states:

$$T_\sigma A_\sigma \prod_j b^*(h^{(j)}) \prod_i a^*(f^{(i)})\,\Omega_0 = \prod_j B^*(h'^{(j)}_\sigma) \prod_i a^*(f^{(i)})\,\Omega_0.$$

The wave operators $\tilde{\Omega}^\pm_\sigma$ in $\mathscr{H}^{(n)}$ exist as the partial isometric extensions to $\mathscr{H}^{(n)}$ of the strong limits (in the $\mathscr{H}^{(n)}$-topology) of the operators $\Omega_\sigma(t) \equiv e^{it\hat{H}_\sigma} T_\sigma A_\sigma e^{-itH_0}$ on $D'^{(n)}$. They have been shown (Theorem I 3.4) to have the usual properties of wave operators $\tilde{\Omega}^{\pm *}_\sigma \tilde{\Omega}^\pm_\sigma = \mathbb{1}, \tilde{\Omega}^\pm_\sigma \tilde{\Omega}^{\pm *}_\sigma = P_\pm$, where $P_\pm$ are the projectors on the ranges $R^\pm_\sigma = \tilde{\Omega}^\pm_\sigma \mathscr{H}$, and $e^{it\hat{H}_\sigma} \tilde{\Omega}^\pm_\sigma = \tilde{\Omega}^\pm_\sigma e^{itH_0}$.

The Heisenberg picture adjusted dressed fields

$$\hat{b}^{\#}_{\sigma,t}(h'_\sigma) \equiv e^{it\hat{H}_\sigma} B^{\#}(e^{\mp it\omega} h'_\sigma)\, e^{-it\hat{H}_\sigma}$$

and

$$\hat{a}^{\#}_{\sigma,t}(f) \equiv e^{it\hat{H}_\sigma} a^{\#}(e^{\mp it\mu} f)\, e^{-it\hat{H}_\sigma}$$

(– going with *) have also been shown to be defined on $D'^{(n)}$ resp. $\bigcup_{\beta>0} D^{(n),\beta}$ and to belong to $\mathfrak{B}(D^{(n),\alpha};\ D^{(n\pm 1),\alpha'})$ resp. $\mathfrak{B}(D^{(n),\beta};\ D^{(n),\beta'})$ for any $0<\alpha'<\alpha-\frac{1}{2}\ln 2$, $0<\beta'<\beta$. Since

$$\prod_j \hat{b}^*_{\sigma,t}(h'^{(j)}_\sigma) \prod_i \hat{a}^*_{\sigma,t}(f^{(i)})\,\Omega_0 = \Omega_\sigma(t) \prod_j b^*(h^{(j)}) \prod_i a^*(f^{(i)})\Omega_0,$$

the strong convergence of $\Omega_\sigma(t)$ implies the strong convergence of the time dependent states

$$\prod_j \hat{b}^*_{\sigma,t}(h'^{(j)}_\sigma) \prod_i \hat{a}^*_{\sigma,t}(f^{(i)})\,\Omega_0$$

to

$$|h^{(1)}\ldots h^{(n)};f^{(1)}\ldots f^{(m)}\rangle_\pm \equiv \tilde{\Omega}^\pm_\sigma \prod_j b^*(h^{(j)}) \prod_i a^*(f^{(i)})\,\Omega_0.$$

We also proved the strong convergence of the fields $\hat{a}^\#_{\sigma,t}(f)$ in this model,[7]) derived their asymptotic properties and extended finally all the results to the case without cut-off. In this paper we first show (section 2) that the dressed nucleon fields $\hat{b}^\#_{\sigma,t}(h'_\sigma)$ also converge strongly as $t\to\pm\infty$ on dense domains, for all test functions h in $L_2(\mathbb{R}^3)$.

As described in the introduction of I, especially in relation with the opposite case of the space cut-off models, only the dressed fields $\hat{b}^\#_{\sigma,t}(h'_\sigma)$ and not the bare ones can give strong convergence (because of the translation invariance of the model and the related persistent effects by which the nucleon 'gets dressed'). The creation operators for dressed nucleons create, as described above, infinitely many mesons and are given by convergent power series in λ. Although they are not bounded even after multiplication by inverse powers of H_0 and/or the number operator N, by suitable analytic domination arguments we get control on the relevant power series for the time derivative of the adjusted Heisenberg picture dressed fields and in this way we show the strong convergence to asymptotic physical nucleon fields and establish the free commutation relations of these fields among themselves and with the meson fields as well as with the unitary time translation group generated by the Hamiltonian. Whereas the interacting dressed nucleon and meson fields do not satisfy canonical commutation relations (the commutators are in general not even c-numbers and there is a field-strength renormalization) the asymptotic fields do have the canonical commutation relations of free fields. Remarks on the case without cut-off are also given.

In Section 3 we derive an asymptotic decomposition of the Hamiltonian, establish the relations between asymptotic fields, states and wave operators and give an expression for the S-matrix.

In Section 4 we give a preliminary discussion of the scattering between a physical nucleon and meson, including an asymptotic series expansion in powers of the coupling constant for the asymptotic meson fields and S-matrix elements.

2. Strong convergence of dressed fields to asymptotic physical fields

Since (except for a few remarks) we shall always keep the ultra-violet cut-off $0\leqslant\sigma<\infty$ fixed, we shall drop the label σ from all quantities (most of them were introduced in I with a label σ). We also suppose accordingly that the coupling constant

[7]) And in a related model (I, section 2). For Nelson's model [4] see [5].

λ is chosen in such a way (generally dependent on σ) that all quantities we shall use and which were introduced in I are well defined (by the propositions of I). Thus we shall write in particular $\hat{H}$ for $\hat{H}_\sigma$ (defined by Theorem I 3.1), $\hat{b}_t^\#(h')$ for $\hat{b}_{\sigma,t}^\#(h'_\sigma)$ (defined by Theorem I 3.2b), etc. We shall also make the convention that all operators are restricted to a fixed $\mathscr{H}^{(n)}$ ($n = 0, 1, 2, \ldots$).

The following theorem gives the strong convergence of the dressed nucleon fields on dense domains in the Fock space $\mathscr{H}^{(n)}$ of n nucleons and arbitrarily many mesons (this theorem is the correspondent of the one proved in I for the mesons: Theorem I 3.5).

Theorem 2.1. *There exists a number* $\Lambda > 0$ *such that for any* $|\lambda| \leqslant \Lambda$ *and all* $h \in L_2(\mathbb{R}^3)$ *the following propositions hold:*

i) $$s - \lim_{t \to \pm\infty} \hat{b}_t^\#(h') = \hat{b}_{\pm\infty}^\#(h')$$

exist on a domain $\Delta^{(n)}$ *which is dense in* $\mathscr{H}^{(n)}$ *and contains the set* $\mathscr{E}(\hat{H}^2)$ *of all entire vectors*[8]) *for* $\hat{H}^2$.[9])

ii) Furthermore, for all $\Phi \in \mathscr{H}^{(0)}$,

$$\hat{b}_\pm(h')\,\Phi = \hat{b}_t(h')\,\Phi = 0,$$

and for any $\Psi \in \mathscr{H}^{(1,0)} \cup \mathscr{H}^{(0)}$:

$$\hat{b}_\pm^*(h')\,TA\Psi = s - \lim_{t\to\pm\infty} \hat{b}_t^*(h')\,TA\Psi = s - \lim_{t\to\pm\infty} \Omega(t)\,b^*(h)\,\Psi$$
$$= \tilde{\Omega}^\pm\, b^*(h)\,\Psi.$$

For $\Psi = \Omega_0$ *one has the correctly normalized physical one-nucleon states* $\hat{b}_\pm^*(h')\,\Omega_0 = TAb^*(h)\,\Omega_0$ *with* $\|\hat{b}_\pm^*(h')\,\Omega_0\| = 1$ *whenever* $\|h\| = 1$.

iii) $(\hat{b}_\pm(h'))^* = \hat{b}_\pm^*(\bar{h}')$ *on* $\Delta^{(n)}$ *($^-$ means complex conjugation). The operators* $\hat{b}_\pm^\#(.)$ *are closable, we shall denote their closures by the same symbols and call them the asymptotic physical nucleon creation and annihilation operators (or fields).*

iv) The Hamiltonian and the asymptotic fields satisfy the same commutation relations as do the free Hamiltonian and the bare fields in the sense that

$$e^{-it\hat{H}}\,\hat{b}_\pm^\#(h')\,e^{it\hat{H}} = \hat{b}_\pm^\#(e^{\mp it\omega}\,h')$$

on $\Delta^{(n)}$.

Remark 2.1. The set $\Delta^{(n)}$ contains a certain subset $\mathfrak{B}_{\alpha_0}^{(n)} \supset \mathscr{E}(\hat{H}^2)$ (dense in $\mathscr{H}^{(n)}$) of analytic vectors for $(\hat{H} + \beta)^2$, $\beta =$ infimum over all β such that $\hat{H} + \beta \geqslant 1$ (β exists since $\hat{H}$ is lower bounded, by Theorem I 3.1). More details on $\Delta^{(n)}$ will be given in the course of the proof of Theorem 1.1 (after Lemma 2.4).

We shall now state the theorem on the commutation relations of the asymptotic physical nucleon and meson fields and then devote the rest of the section to the proofs and some remarks. Note that the commutation relations of the meson fields among themselves have already been given in I, Theorem I 3.5, and shall not be repeated here.

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[8]) An entire vector for an operator A in a Hilbert space is any vector φ such that $\sum_{l=0}^{\infty} \|A^l \varphi\|/l!\; s^l < \infty$ for all $s > 0$ (see e.g. [6]).

[9]) See Remark 2.1.

Theorem 2.2. *For all* $f, g \in L_2(\mathbb{R}^3)$ *there exist:*

i) *a number* $0 < \Lambda_1' \leqslant \Lambda$ *such that for all* $|\lambda| \leqslant \Lambda_1'$ *the following strong commutation relations hold on a dense subset* $\Delta'^{(n)}$*:*

$[\hat{b}_\pm(f'), \hat{b}_\pm(g')] = 0 = [\hat{b}^*_\pm(f'), \hat{b}^*_\pm(g')], \qquad [\hat{b}_\pm(\bar{f}'), \hat{b}^*_\pm(g')] = (f, g).$

One has $\mathscr{E}(\hat{H}^2) \subset \Delta'^{(n)} \subset \Delta^{(n)}$; [10])

ii) *a number* $0 < \Lambda_2' \leqslant \Lambda$ *such that for all* $|\lambda| \leqslant \Lambda_2'$

$[\hat{b}_\pm(f'), \hat{a}^\#_\pm(g)] = 0 = [\hat{b}^*_\pm(f'), \hat{a}^\#_\pm(g)]$

on $\Delta^{(n)}$.

Proof of Theorem 2.1. We shall write the proof for the case of the creation operators (the case of the annihilation operators being treated in a completely similar way). We shall try to keep the proof short. More details can be found in [7].

To the notations for contractions between Wick monomials used in I (§3.1) we shall have to add the following one. Let $W^{(i)}$ be the basic Wick monomials we shall have to deal with, of the operator form $b^* a^{*i} b$ (where a^{*i} stands for the product of i consecutive a^* operators) ($W^{(i)}$ is defined more precisely in I (3.1)). We define $V^a_{\underline{a}}\, W^{(i)}$ as the sum of all i terms one obtains by contracting the a annihilator in V^a with any of the i a^* creators in $W^{(i)}$.[11])

Lemma 2.1. *Let* $h \in L_2(\mathbb{R}^3), \Phi, \Psi \in D^{(n),\alpha} \cap D(H_0)$, $\alpha > \frac{1}{2}\ln 2$. *Then the time derivative of* $L(t) \equiv (\Phi, b_t^*(h')\Psi)$ *exists, is continuous and equal to*

$$\frac{d}{dt} L(t) = i\lambda(\Phi, e^{it\hat{H}} \mathscr{B}(h,t)\, e^{-it\hat{H}}\, \Psi), \tag{2.1}$$

where

$$\mathscr{B}(h,t) \equiv \lambda\, V^a_{\underline{1}} b^*(e^{-it\omega} h') + \lambda : V^a_{\underline{a}} \Gamma(Q)_{\underline{1}} b^*(e^{-it\omega} h') : + \lambda : V^a_{\underline{1}}(\Gamma(Q)_{\underline{1}} b^*(e^{-it\omega} h')) :, \tag{2.2}$$

$\Gamma(Q)$ *being defined in I* §3.1.1 (*formula* (3.7)) *as the basic quantity which gives the dressing transformation* $T = :\exp \Gamma(Q):$.

Proof of Lemma 2.1. The function $(W(t')\Phi, B^*(h')\, W(t)\Psi)$, $W(t) \equiv e^{itH_0} e^{-it\hat{H}}$ is separately continuous differentiable with respect to t, t' (as seen using Lemmata I 3.1, I 3.2 and Theorem I 3.1). ■

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10) For more details on Λ_1', Λ_2', $\Delta'^{(n)}$ see the proof of Theorem 2.2.

11) $W^{(i)}$ is of the form (I(3.1)):

$$W^{(i)} = \int \chi(q; p_1 \cdots p_i)\, w^{(i)}\, (q; p_1 \cdots p_i)\, b^*\left(q - \sum_{j=1}^{i} p_j\right)\left(\prod_{j=1}^{i} a^*(p_j)\right)\, b(q)\, dq\, dp_1 \cdots dp_i$$

and

$$V^a_{\underline{a}}\, W^{(i)} = \sum_{l=1}^{i} \int \omega(k_1)^{-1/2}\, \omega(k_2)^{-1/2}\, \mu(k_1 - k_2)^{-1/2}\, \delta(k_1 - k_2 - p_l)\, \chi(q; p_1 \cdots p_i)\, w^{(i)}(q; p_1 \cdots p_i)$$

$$b^*(k_1) b^* \left(q - \sum_{j=1}^{i} p_j\right)\left(\prod_{\substack{j=1 \\ j \neq l}}^{i} a^*(p_j)\right) b(k_2)\, b(q)\, dk_1\, dk_2\, dq\, dp_1 \cdots dp_i.$$

In the following we shall make estimates using the following dense subset of $L_2(\mathbb{R}^3)$: $\mathscr{D}^0 \equiv \{f \mid f \in \mathscr{D}(\mathbb{R}^3); f \equiv 0$ in a neighbourhood U_f of the origin in the momentum space $\mathbb{R}^3\}$, $\mathscr{D}(\mathbb{R}^3)$ being Schwartz space of infinitely differentiable functions of compact support.

Lemma 2.2. *Assume $h \in \mathscr{D}^0$. There exists a number $\Lambda_0 > 0$ such that for all $|\lambda| \leqslant \Lambda_0$ and any $\Phi \in \mathscr{H}^{(n)}$:*

$$\|\mathscr{B}(h;t)\, e^{-N_a(1/2)\ln 2}\Phi\| \leqslant K_1(1+|t|^{3/2})^{-1}\|\Phi\|, \tag{2.3}$$

where $\mathscr{B}(h;t)$ is as in Lemma 2.1 and K_1 is some constant (dependent on (σ), λ, h but) independent of Φ, t. N_a is the meson number operator.

Proof of Lemma 2.2. From the definition we have:

$$\left.\begin{aligned}
&\mathscr{B}(h;t) = \mathscr{B}^{(1,1)}(h;t) + \sum_{\nu \geqslant i \geqslant 1}\ \sum_{\eta = a,b} \mathscr{B}^{(\eta)(\nu+1,i)}(h;t)\\
&\text{where}\\
&\mathscr{B}^{(1,1)}(h;t) \equiv \lambda V^{a}_{1} b^*(e^{-it\omega}h')\\
&\mathscr{B}^{(a)(\nu+1,i)}(h;t) \equiv \lambda : V^{a}_{a}(\Gamma(S_{\nu,i})_{1} b^*(e^{-it\omega}h')):\\
&\mathscr{B}^{(b)(\nu+1,i)}(h;t) \equiv \lambda : V^{a}_{1}(\Gamma(S_{\nu,i})_{1} b^*(e^{-it\omega}h')):,
\end{aligned}\right\} \tag{2.4}$$

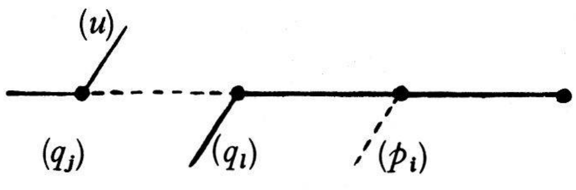
for all $\nu \geqslant i \geqslant 1$ (the terms $S_{\nu,i}$ are contributions of order ν, with i meson creators, and are defined in I §3.1). $\mathscr{B}^{(1,1)}$ has the operator form b^*a, $\mathscr{B}^{(a)(\nu+1,i)}$ the operator form $b^*b^*a^{*i-1}b$ and $\mathscr{B}^{(b)(\nu+1,i)}$ has the operator form $b^*a^{*i}a$. For the proof of the decay property (2.3) it is essential that each term is not a pure creation term but contains one uncontracted annihilation operator (a or b) and a contraction of a b operator with $b^*(e^{-it\omega}h')$. This yields an integration in the kernel over a function oscillating in time, which is then responsible for the decay (as shown for the similar case of the meson operators in I, section 2).[12] To give a few more details let us pick up an example (this simplifies the notation and gives the idea how to proceed in the general case).

Let us consider e.g. the term:

$$\mathscr{B}^{(a)(3,2)}\Psi = \lambda^3 : V^{a}_{a}\Gamma(V^{c}_{1}\Gamma V^{c})_{1} b^*(e^{-it\omega}h') : \Psi,$$

$$\Psi \in \mathscr{H}^{(n,m)}.$$

Its Friedrichs diagram [13]) is:



[12]) There is no decay in the case of adjusted Heisenberg picture bare nucleon fields, because here pure creation terms arise. In fact these bare fields cannot converge, as discussed in the introduction of I. Note that pure creation terms do not prevent the convergence in the case in which the interaction has a space cut-off in it; the space cut-off gives namely an additional integration (which is lost when translation invariance holds, due to the momentum-conservation δ-function). Hence in the space cut-off case bare nucleon fields (and of course also the meson fields) converge strongly (see Section 4). Plainly this is a common feature of all space cut-off models ([8], [9], [10], [11], [12], [13]).

[13]) Definitions are, e.g., in [14], [15].

where the solid lines are nucleon lines, the dotted lines are meson lines, ● are the vertices. There are 4 external lines, with momenta which we call q_j, q_l, resp. p_i (corresponding to creation of nucleons resp. mesons) ($j = 1 \ldots n+1; l = 1 \ldots n; i = 1, \ldots, m+1$) and u (corresponding to annihilation of a nucleon). Set $\mathbf{q} \equiv (q_1, \ldots, q_{n+1})$, $\mathbf{p} \equiv (p_1, \ldots, p_{m+1})$. It can easily be seen that $(\mathscr{B}^{(a)(3,2)}(h;t)\,\Psi)^{(n+1;m+1)}(\mathbf{q};\mathbf{p})$ is of the form $\lambda^3(m+1)^{-1/2}(n+1)^{-1/2}\sum_{j=1}^{n+1}\sum_{l=1}^{n}\sum_{i=1}^{m+1}\chi(q_j)\,\omega(q_j)^{-1/2}\quad \Phi_{j,l,i}^{(n+1,m+1)}(\mathbf{q};\mathbf{p})$, where $\chi(.)$ is the smooth cut-off function (χ_σ of I §2) and the $\Phi_{j,l,i}$, are, as functions of $q_j + q_l + p_i$, convolutions of the function

$$\chi^{1/2}(q_l)\,\chi^{1/2}(p_i)\,\chi(.)\,\omega(.)^{-1/2}\,\Psi^{(n,m)}(\mathbf{q}\hat{q}_j\,\hat{q}_l.\,;\mathbf{p}\hat{p}_i)$$

($\wedge$ standing for omission of the corresponding momenta) and of a function

$$g_t(.,q_l,p_i) \equiv e^{-it\omega(.)}\chi(.)\,h'(.)\,\omega(.)^{-1/2}\chi^{1/2}(q_l)\,\chi^{1/2}(p_i)\,\varphi(.,q_l,p_i),$$

where $\varphi(.,q_l,p_i)$ is a product of everywhere positive energy denominators times $(\omega(.-p_i) - \omega(.) + \mu(p_i))^{-1}$ (the integration variable (.) in the convolution is u).

Hence we have expressions of the same structure as I (2.20) and taking the Fourier transform with respect to the external variable q_j we obtain:

$$\|\mathscr{B}^{(a)(3,2)}(h;t)\,e^{-N_a(1/2)\ln 2}\,\Phi\| \leqslant K_2\|\tilde{g}_t\|_\infty\|\Phi\|, \tag{2.5}$$

where K_2 is a constant, independent of t, Φ and, setting $g_0(.) \equiv g_t(.)$ for $t = 0$,

$$\|\tilde{g}_t\|_\infty = \sup\left|\int dv\, e^{ix_jv}\, e^{-it\omega(v)}\, g_0(v,q_l,p_i)\right|$$

the sup being taken over all $x_j \in \mathbb{R}^3$, $|q_l| \leqslant \sigma$, $|p_i| \leqslant \sigma$. As a function of x_j the integrand is a smooth solution of the Klein-Gordon equation. But e.g. from [16] we have then, for all $t \neq 0$:

$$\sup_{x_j}|\tilde{g}_t(x_j,q_l,p_i)| \leqslant K_3\,|t|^{-3/2}\int dx_j\,F(x_j,q_l,p_i),$$

where K_3 is independent of t, q_l, p_i and

$$F(x_j,q_l,p_i) \equiv \left|\int dv\,e^{ix_jv}(\omega(v)+1)^3\,\omega(v)\,g_0(v,q_l,p_i)\right|.$$

Split now above integral over x_j in one over $|x_j| \leqslant \delta$ and one over $|x_j| > \delta$, for some $\delta > 0$. Estimate the second one after 4 times partial integrations with respect to $|v|$ (the boundary terms vanish since $h \in \mathscr{D}^0$) as

$$\leqslant |x_j|^{-4}\left|\int dv\left\{\frac{\partial^4}{\partial|v|^4}[(\omega(v)+1)^3\,\omega(v)\,g_0(v,q_l,p_i)]\right\}\right|.$$

$\|\tilde{g}_t\|_\infty \leqslant K_4(1+|t|^{3/2})^{-1}$ follows then observing that the integrand in $\{\cdots\}$ is bounded uniformly over the whole range of the variables (since there $\omega(v+q_l) - \omega(v+q_l+p_i) + \mu(p_i) \geqslant \Omega(\sigma) - \omega(\sigma) > 0$, with $\Omega(\sigma) \equiv [(m_a+m_b)^2+\sigma^2]^{1/2}$).

This and (2.5) prove then

$$\|\mathscr{B}^{(a)(3,2)}(h;t)\,e^{-N_a(1/2)\ln 2}\,\Phi\| \leqslant K^{(a)(3,2)}(1+|t|^{3/2})^{-1}\|\Phi\|, \tag{2.6}$$

where $K^{(a)(3,2)}$ is independent of t, Φ.

For the reason mentioned before each term $\mathscr{B}^{(\eta)(\nu+1,i)}$ can be estimated in the same way. The control over $\sum_{\nu,i} K^{(\eta)(\nu+1,i)}$ is obtained for $|\lambda|$ small enough, as in [3], since $K^{(\eta)(\nu+1,i)}$ can be obtained essentially by estimating the kernel of $\mathscr{B}^{(\eta)(\nu+1,i)}$ and some of its derivatives, using the condition $h \in \mathscr{D}^0$. This proves Lemma 2.2. For more details we refer to [7]. We would also like to stress that another proof of this Lemma is contained in the very interesting Thesis by J. Fröhlich [17] (proof of Theorem 9, §3.3). ■

Lemma 2.3. *Let* $\mathfrak{A}_\alpha^{(n)} \equiv \{\Psi \in \mathscr{H}^{(n)} | \sum_{l=0}^\infty (\alpha^l/l!) \|\hat{H}^l \Psi\| < \infty\}$ *be the set of all analytic vectors of* $\hat{H}$ *with radius of convergence not smaller than* α, $\alpha > 0$. *Then for any given* $\delta > 0$ *there exist numbers* $\Lambda_0(\delta) > 0$ *and* $\alpha_0 = \alpha_0(\lambda, \delta) > 0$ *such that, for all* $|\lambda| \leqslant \Lambda_0(\delta)$, $\mathfrak{A}_{\alpha_0}^{(n)} \subset D^{(n);(1/2)\ln+\delta}$ *and for any* $\Psi \in \mathfrak{A}_{\alpha_0}^{(n)}$, $h \in \mathscr{D}^0$:

$$\|\mathscr{B}(h;t)\, e^{-it\hat{H}}\, \Psi\| \leqslant K'(1+|t|^{3/2})^{-1}, \tag{2.7}$$

where K' *is a constant (independent of* t*).*

Remark 2.2. The set $\mathfrak{A}_{\alpha_0}^{(n)}$ is dense in $\mathscr{H}^{(n)}$, since it contains in particular all entire vectors for the self-adjoint operator $\hat{H}$ (restricted to $\mathscr{H}^{(n)}$).

Proof of Lemma 2.3. Following [6] we shall use the following notation, for any number z, operator A, vector χ:

$$\|e^{|z||A|}\chi\| \equiv \sum_{l=0}^{\infty} \frac{|z|^l}{l!} \|A^l \chi\|.$$

It is easily seen, using Lemma 2.2, that it suffices to prove

$$\|e^{N_a[(1/2)\ln 2+\delta]}\chi\| \leqslant \|e^{\alpha_0|\hat{H}+\beta|}\chi\|, \tag{2.8}$$

for any $\chi \in \mathfrak{A}_{\alpha_0}^{(n)}$ and some $\beta > 0$. We shall now give a short proof of (2.8). (For more details see [7]. An independent proof follows from [17] (§1.2, Corollary 1′ after Lemma 4)). We shall use following Theorem of Nelson [6] (Theorem 1):

Let A, B operators in some Hilbert space. Suppose $C^\infty(A) \subset C^\infty(B)$ (using the definition $C^\infty(T) \equiv \bigcap_{l=0}^\infty D(T^l)$ for any operator T).

Suppose, for all $\Psi \in C^\infty(A)$:

$$\|B\Psi\| \leqslant c\|A\Psi\|, \quad c < \infty$$

$$\|\operatorname{ad} B^l(A)\, \Psi\| \leqslant c_l \|A\Psi\|,$$

where $\operatorname{ad} B^l(A)$ is defined recursively by $\operatorname{ad} B^1(A) \equiv [B, A]$, $\operatorname{ad} B^{l+1}(A) \equiv [B, \operatorname{ad} B^l(A)]$ and c, c_l are constants.

Let $g(r) \equiv \sum_{l=1}^\infty (c_l/l!)\, r^l$, where the right-hand side is assumed to converge for all $r < r_0$, $r_0 > 0$.

Let

$$t(s') \equiv c \int_0^{s'} (1-g(r))^{-1}\, dr.$$

Then, for all $\Psi \in C^\infty(A)$,

$$\|e^{|s'||B|}\, \Psi\| \leqslant \|e^{|t(s')||A|}\, \Psi\|.$$

We shall apply this theorem for $B = N_a$, $A = \hat{H} + \beta$, where β is such[9]) that $\hat{H} + \beta \geqslant 1$.

One has the simple first order estimate (Lemma I 2.1 and Lemma I 3.1):

$$\|N_a \chi\| \leqslant K_0 \|(\hat{H} + \beta)\chi\|,$$

where $K_0 \equiv 2(m_a^{-1})[1 + |\beta| + (m_a/2) + (2/m_a)(2n|\lambda|L)^2 + \|\hat{M}\|]$. L is defined in Lemma I 2.1 and is independent of λ. $\|M\|$ is the norm of the restriction to $\mathscr{H}^{(n)}$ of the total mass renormalization. We remark that $C^\infty(\hat{H}) \subset C^\infty(N_a)$, as a consequence of $D(\hat{H}^l) = D(H_0^l)$, a property which can be proved using estimates of $\hat{H}^l$ in terms of $c_l'(H_0 + d_l)^l, c_l', d_l$ being constants (these estimates can be proved e.g. as in [18]).

For $\chi \in \mathfrak{A}_{\alpha_0}^{(n)} \subset C^\infty(\hat{H})$ we can therefore compute

$$\operatorname{ad} N_a^l(\hat{H} + \beta)\chi = \lambda(V^c + (-1)^l V^a)\chi.$$

Hence:

$$\|\operatorname{ad} N_a^l(\hat{H} + \beta)\chi\| \leqslant c_l \|(\hat{H} + \beta)\chi\|,$$

with

$$c_l \equiv 8n|\lambda|L(1 + K_0)^{1/2} \equiv K''.$$

Then, applying Nelson's theorem, (2.8) follows when $s' \leqslant (\frac{1}{2})\ln 2 + \delta$ and $t(s') < \infty$. For the latter it suffices to show $K'' < 1/(e^\delta \sqrt{2} - 1)$. This is certainly verified for given δ when λ is sufficiently small and K_0 is finite for all bounded $|\lambda|$. A look at the definition of K_0 shows that this is indeed the case (Theorem I 3.1). Setting $s' = \frac{1}{2}\ln 2 + \delta$ and $\alpha_0 = t(\frac{1}{2}\ln 2 + \delta)$ we have then (2.8) and the Lemma 2.3 is proved. ∎

Remark 2.3. The proof of Lemma 2.3 shows in particular that for any given $\delta > 0$ (and in fact also for $\delta = 0$) and $|\lambda|$ sufficiently small the inequality (2.8) holds for some $\alpha_0(\delta) > 0$, β and all $\chi \in \mathfrak{A}_{\alpha_0(\delta)}^{(n)}$. This implies

$$\mathfrak{A}_{\alpha_0(\delta)}^{(n)} \subset D(e^{N_a[(1/2)\ln 2 + \delta]}) \subset D^{(n);(1/2)\ln 2 + \delta} \subset D^{(n);(1/2)\ln 2}.$$

Furthermore, under above conditions, one has the uniform bound in t:

$$\|e^{N_a[(1/2)\ln 2 + \delta]} e^{-it\hat{H}} \chi\| \leqslant \|e^{\alpha_0(\delta)|\hat{H} + \beta|}\chi\|. \tag{2.9}$$

Remark 2.4. (2.9) together with Theorem I 3.2 a shows that, for all $\chi \in \mathfrak{A}_{\alpha_0(\delta)}^{(n)}$,

$$\|\hat{b}_t^\#(h')\chi\| \leqslant K''' \|h\|, \tag{2.10}$$

with K''' independent of h and t.[14])

In order to prove the strong continuity in t of the operator $e^{it\hat{H}}\mathscr{B}(h;t)e^{-it\hat{H}}$ we introduce the following domain:

$$\mathfrak{B}_{\alpha_0}^{(n)} \equiv \left\{ \Phi \in \mathscr{H}^{(n)} \,\middle|\, \sum_{l=0}^{\infty} \frac{\alpha_0^l}{l!} \|(\hat{H} + \beta)^{2l}\Phi\| < \infty \right\}, \tag{2.11}$$

where α_0, β are as before. $\mathfrak{B}_{\alpha_0}^{(n)}$ is dense in $\mathscr{H}^{(n)}$ and one has

$$\mathfrak{A}_{2\alpha_0}(\hat{H}^2) \subset \mathfrak{B}_{\alpha_0}^{(n)} \subset \mathfrak{A}_{\alpha_0}^{(n)}.$$

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[14]) The estimate (2.10) follows also from the already quoted Corollary 1′ after Lemma 4 (§1.2) of [17]. See Theorem 9 (§3.3) of [17] and our Remark 2.5 below.

Lemma 2.4. *Let* δ, $\alpha_0(\delta)$ *be as in Lemma 2.3. Then for all* $\Psi \in \mathfrak{B}^{(n)}_{\alpha_0(\delta)}$ *and* $h \in \mathscr{D}^0$ *one has:*

$$\hat{b}^*_t(h')\,\Psi = B^*(h')\,\Psi + i\lambda \int_0^t dt'\, e^{it'\hat{H}} \mathscr{B}(h;t')\, e^{-it'\hat{H}}\, \Psi, \tag{2.12}$$

where the integral on the right-hand side is to be understood in the strong sense.

Proof: Lemma 2.3 shows that the norm of the integrand is bounded by const. $(1+|t|^{3/2})^{-1}$. The strong continuity of the integrand is proved using Lemmata 2.2, 2.3 and estimating separately the strong continuity of the bounded operator $\mathscr{B}(h;t')\,e^{-N_a[(1/2)\ln 2+\delta]}$ and of $e^{N_a[(1/2)\ln 2+\delta]}\,e^{it'\hat{H}}$. The former follows by an estimate like the one in Lemma 2.2 together with Lebesgue's dominated convergence theorem, the latter is proved using Lemma 2.3. The equation (2.12) is then a consequence of Lemma 2.1. ∎

After these preliminary lemmata the proofs of the single points of the Theorem 2.1 follow now easily:

i) Require Λ to satisfy $0 < \Lambda \leqslant \lambda_1$, λ_1 being such that $\hat{b}^{\#}_t(h')$ are well defined (by Theorem I 3.2). Moreover fixe $\delta > 0$ and choose Λ such that for all $|\lambda| \leqslant \Lambda$:

$$K'' < (\sqrt{2}-1)^{-1}$$

(K'' is defined in the proof of Lemma 2.3). Fixe now λ such that $|\lambda| \leqslant \Lambda$. Let $\tau(\lambda)$ be any number in the interval $I(\lambda) \equiv (0, \ln[1+(K'')^{-1} - \tfrac{1}{2}\ln 2])$ and define

$$\alpha_0(\tau) \equiv t(\tfrac{1}{2}\ln 2 + \tau(\lambda)),$$

with $t(.)$ as in the proof of Lemma 2.3. Define furthermore

$$\Delta^{(n)} \equiv \bigcup_{\tau \in I(\lambda)} \mathfrak{B}^{(n)}_{\alpha_0(\tau)}, \text{ with } \mathfrak{B}^{(n)}_{\alpha_0(\tau)} \text{ as in (2.11)}$$

Note that, in particular, $\Delta^{(n)} \subset D^{(n);\alpha}$ for all $\alpha > \tfrac{1}{2}\ln 2$ and $\mathscr{E}(\hat{H}^2) \subset \mathfrak{B}^{(n)}_{\alpha_0(\tau)}$. Lemma 2.3 gives then the strong convergence of the right-hand side of (2.12) for $t \to \pm\infty$, which proves i), for $h \in \mathscr{D}^0$. The extension to all $h \in L_2(\mathbb{R}^3)$ follows from the uniform bound (2.10).

ii), iii), iv) follow also easily, from i) and estimates of I (Theorem I 3.2b, Theorem I 3.2a and Lemmata I 2.1, I 3.1). This concludes the proof of Theorem 2.1. ∎

Remark 2.5. Theorem 2.1 as it stands is limited to the case with ultraviolet cut-off σ in the interaction. From its proof we see that it remains true with small modifications once suitable higher order estimates of N_a^l in terms of $(\hat{H}_\sigma + \beta_\sigma)^l$ (as those involved in Lemma 2.3) are proved to hold also for the case $\sigma = \infty$ (with the usual restriction on all nucleon momenta to be in a ball of finite radius R in momentum space: see [3], [17], [1]). These estimates have now been proved (in this and related models) by J. Fröhlich [17] (§1.2, Corollary 1, after Lemma 4). For the correspondent application to the extension of our convergence result Theorem 2.1, i), we refer to the same reference [17] (§3.3, Theorem 9).

Proof of Theorem 2.2

i) Choose $0 < \Lambda_1' \leqslant \Lambda$ such that, for all $|\lambda| \leqslant \Lambda_1'$, one has $K'' < 1$. Fixe now λ, $|\lambda| \leqslant \Lambda_1'$. The domain $\Delta'^{(n)}$ can then be chosen as $\Delta'^{(n)} \equiv \bigcup_{\tau \in I'} \mathfrak{B}^{(n)}_{\alpha_0(\tau)}$, where

$$I' \equiv (\tfrac{1}{2}\ln 2, \ln[1+(K'')^{-1} - \tfrac{1}{2}\ln 2]).$$

Note that in particular $\mathscr{E}(\hat{H}^2) \subset \Delta'^{(n)} \subset D^{(n);\theta}$ for all $\theta > \ln 2$. We can now proceed similarly as in the proof of the meson-meson commutation relations (Theorems I 2.1 and I 3.5). We have that $\|\hat{b}_t^{\#}(h')\,\hat{b}_t^{\#}(g')\,\Psi\|$ (where h stands for f or $\bar{f}$) is uniformly bounded in t for any $\Psi \in \Delta'^{(n)}$, because $\|B^{\#}(e^{\mp it\omega}h')\,B^{\#}(e^{\mp it\omega}g')\,e^{-N_a \ln 2}\|$ and $\|e^{N_a \ln 2}\,e^{-it\hat{H}}\Psi\|$ are separately uniformly bounded in t, by Theorem I 3.2a and (2.9). Then it follows for $\Phi \in \Delta^{(n)}$: $(\hat{b}_t^{\#}(h')^*\Phi, \hat{b}_t^{\#}(g')\Psi) \leqslant E\|\Phi\|$, with E independent of t, Φ, and therefore $\hat{b}_\pm^{\#}(g')$ maps Ψ into the domain of $\hat{b}_\pm^{\#}(h')$. Then we can compute $(\Phi, [\hat{b}_\pm^{\#}(h'), \hat{b}_\pm^{\#}(g')]\,\Psi)$ as limits of $(\Phi, [\hat{b}_t^{\#}(h'), \hat{b}_t^{\#}(g')]\,\Psi)$. The cases where both $\#$ stand for creation or both for annihilation are trivial (I 3.19a). In the remaining case we have that $[\hat{b}_t(\bar{f}'), \hat{b}_t^*(g')]\,\Psi$ is the sum of the time-independent term $I \equiv (f', g')\,\Psi$, of the term $II \equiv e^{it\hat{H}}\mathscr{R}e^{-it\hat{H}}\,\Psi$, with $\mathscr{R} \equiv [b(e^{it\omega}\bar{f}') \smile \Gamma(Q)^*, \Gamma(Q) \smile b^*(e^{-it\omega}g')]$, and of terms which have the same good form as the terms $\mathscr{B}^{(\eta)(\nu+1,i)}(h;t)$ we discussed for the proof of Lemma 2.2. Using the same method we can show that these terms vanish for $|t| \to \infty$. For the same reason all graphs of $\mathscr{R}$ with more than two external lines give a vanishing contribution to $\|II\|$. It is then not difficult to verify that $I + II \to (f, g)\,\Psi$ strongly as $t \to \pm\infty$ (see [7] for details), which proves i). The proof of ii) is similar. The corresponding uniform bound is obtained[15]) by (2.10) and (2.9) and then use is made of the fact that

$$[\hat{b}_t^*(f'), \hat{a}_t(g)]\,\Psi = e^{it\hat{H}}\,\mathscr{R}'\,e^{-it\hat{H}}\,\Psi,$$

where

$$\mathscr{R}' \equiv a(e^{it\mu}g) \underset{a}{\smile} \Gamma(Q) \underset{1}{\smile} b^*(e^{-it\omega}f')$$

contains two contractions involving operators smeared with oscillating functions. ∎

3. Asymptotic decomposition of $\hat{\mathbf{H}}$. Connection between asymptotic states and fields. The S-matrix

3.1. *Asymptotic decomposition of $\hat{H}$.*

We can prove the asymptotic decomposition of $\hat{H}$ along the lines of [9], [10], exploiting the fact that both the commutation relations between all fields and with the unitary group generated by $\hat{H}$ hold on the subset $\mathscr{E}(\hat{H}^2)$ of $D(\hat{H})$, dense in $\mathscr{H}^{(n)}$. Let $V_\pm^{0(n)}$ be the closed linear spans of all vectors which are annihilated by all $\hat{b}_\pm(h')$, $h \in L_2(\mathbb{R}^3)$ and by all $\hat{a}_\pm(f), f \in L_2(\mathbb{R}^3)$. The $V_\pm^{0(n)}$ are subspaces of $\mathscr{H}^{(n)}$ which reduce $\hat{H}$. Proceeding then as in [9], [10], using the commutation relations, Theorem 2.1.iii) and Theorem 2.2 on $\mathscr{E}(\hat{H}^2)$ (instead of the domains of [9], [10]) we can construct for any $\phi_0 \in V_\pm^{0(n)}$ the asymptotic symmetric Fock spaces $\mathscr{H}_\pm^{(n)}(\phi_0)$, with ϕ_0 as cyclic vector, namely $\mathscr{H}_\pm^{(n)}(\phi_0) = \bigoplus_{m=0}^{\infty} \mathscr{H}_\pm^{\{n,m\}}(\phi_0)$, where $\mathscr{H}_\pm^{\{n,m\}}(\phi_0) \equiv$ closed linear hull of $\prod_{j=1}^{n} \hat{b}_\pm^*(g'^{(j)}) \prod_{i=1}^{m} \hat{a}_\pm^*(f^{(i)})\,\phi_0$, for all $f^{(i)}, g^{(j)} \in L_2(\mathbb{R}^3)$, $m = 0, 1, 2, \ldots$ ($\prod^0 \equiv \mathbb{1}$).

Clearly $\mathscr{H}_\pm^{(n)}(\phi_0) = \mathscr{H}_\pm^{(b)(n)}(\phi_0) \otimes \mathscr{H}_\pm^{(a)}(\phi_0)$, where $\mathscr{H}_\pm^{(b)(n)}(\phi_0) \equiv \mathscr{H}_\pm^{\{n,0\}}(\phi_0)$, $\mathscr{H}_\pm^{(a)}(\phi_0) \equiv \mathscr{H}_+^{(0)}(\phi_0)$. All these spaces reduce $\hat{H}$. For $\phi_0 = \Omega_0$ we get the asymptotic Fock spaces $\mathscr{H}_\pm^{(n)}(\Omega_0)$ for physical nucleons and mesons, which reduce $\hat{H}$ in such a way that $\hat{H}\,|\,_{\mathscr{H}_\pm^{(n)}(\Omega_0)} \cong H_0 \cong \mathbb{1} \otimes H_0^{(a)} + H_0^{(b)} \otimes \mathbb{1}$ ($\cong$ stands for unitarily equivalent). In particular the spectrum of $\hat{H}$ in $\mathscr{H}^{(n)}$, $n > 0$ has a continuum containing $[nm_b, \infty)$. In $\mathscr{H}^{(0)}$ it consists of course of the simple isolated eigenvalue 0 and the absolutely continuous part $[m_a, \infty)$. Defining $\mathscr{H}^{(n,m)} \equiv$ closure of $\bigcup_{\phi_0 \in V_\pm^{0(n)}} \mathscr{H}_\pm^{\{n,m\}}(\phi_0)$, we can prove that $\bigoplus_{m=0}^{\infty} \mathscr{H}_\pm^{(n,m)} = \mathscr{H}^{(n)}$.

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[15]) For details see [7].

This is done by an adaptation of the arguments in [8], [9], using essentially the fact that $\hat{H}$ is bounded from below in $\mathscr{H}^{(n)}$, all commutation relations hold for $\mathscr{E}(\hat{H}^2)$ and one has, for any $\Psi \in \mathscr{E}(\hat{H}^2)$, the t-uniform estimate (2.10) and the similar one for $\hat{a}_t^{\#}(h)$ (I (2.16)).[16]

It follows then in particular the following tensorial decomposition of $\hat{H}$ (on $\mathscr{H}^{(n)}$): $\hat{H} \cong H_0 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{H}|_{V_\pm^{0(n)}}$.

3.2 *Connections between asymptotic states, fields and the S-matrix.*

We shall briefly give some connections between the asymptotic states constructed in I through the wave operators and the asymptotic fields constructed in Section 2 above. It will be convenient for this to introduce the following unified notation for all fields.

Let η be a label taking the values a (for mesons) and b (for nucleons). Set, for any $h^{(\eta)} \in L_2(\mathbb{R}^3)$, $\hat{h}^{(\eta)} \equiv (\nu^{(\eta)})^{-1/2} h^{(\eta)}$, with $\nu^{(b)} \equiv \nu^{-1/2}$, $\nu^{(a)} \equiv \mathbb{1}$. Define then

$$\hat{c}_t^{(\eta)\#}(\hat{h}^{(\eta)}) \equiv \begin{cases} \hat{b}_t^{\#}(\nu^{-1/2} h^{(b)}) & \text{for } \eta = b \\ \hat{a}_t^{\#}(h^{(a)}) & \text{for } \eta = a \end{cases}$$

$$= e^{it\hat{H}} C^{(\eta)\#}(e^{\mp it\Omega^{(\eta)}} \hat{h}^{(\eta)}) e^{-it\hat{H}},$$

with

$$C^{(\eta)\#}(.) \equiv \begin{cases} B^{\#}(.) & \text{for } \eta = b \\ a^{\#}(.) & \text{for } \eta = a, \end{cases}$$

$$\Omega^{(\eta)}(.) \equiv \begin{cases} \omega(.) & \text{for } \eta = b \\ \mu(.) & \text{for } \eta = a. \end{cases}$$

Thus $\hat{c}_t^{(\eta)\#}(\cdot)$ are the time t Heisenberg picture's adjusted creators and annihilators for dressed b and a particles. Let finally

$$\Delta_\eta^{(n)} \equiv \begin{cases} \Delta^{(n)} & \text{for } \eta = a \\ D(H_0^{1/2}) \, (\cap \mathscr{H}^{(n)}) & \text{for } \eta = b \end{cases}$$

$$D_\eta^{(n)} \equiv \begin{cases} D''^{(n)} \equiv \bigcup_{\alpha > \ln 2} D^{(n),\alpha} & \text{for } \eta = b \\ D'^{(n)} \equiv \bigcup_{\alpha > (1/2)\ln 2} D^{(n),\alpha} & \text{for } \eta = a \end{cases}, \quad c^{(\eta)\#}(.) \equiv \begin{cases} b^{\#}(.) & \text{for } \eta = b \\ a^{\#}(.) & \text{for } \eta = a \end{cases}$$

Theorem 2.1, Theorem 2.2 and Theorem I 3.5 give the properties of the strong limits $\hat{c}_\pm^{(\eta)\#}(.)$.

Theorem 3.1. *Let λ be as in Theorem 2.1 (for $\eta = b$) resp. Theorem I 3.5 (for $\eta = a$) and let Φ be any vector in $D_\eta^{(n)}$. Let furthermore (as before and Theorem I 3.4)*

$$\Omega(t) \equiv e^{it\hat{H}} T A e^{-itH_0}.$$

16) For details see [7]. (2.10) gives the basis for the needed extension of Lemma 5 of [8] to the case of our unbounded operators.

Then for any $h^{(\eta)} \in L_2(\mathbb{R}^3)$:

1) $s - \lim_{t \to \pm\infty} \hat{c}_t^{(\eta)*}(\hat{h}^{(\eta)})\, \Omega(t)\, \Phi = \hat{c}_\pm^{(\eta)*}(\hat{h}^{(\eta)})\, \tilde{\Omega}^\pm\, \Phi$

Moreover $(\tilde{\Omega}^\pm)^* \hat{c}_\pm^{(\eta)*} \tilde{\Omega}^\pm = c^*(h^{(\eta)})$ *on* $D_\eta^{(n)}$.

2) *For any* $h_l^{(\eta_l)} \in L_2(\mathbb{R}^3)$, $\eta_l = a, b;\ l = 1, 2, \ldots$

$$s - \lim_{t \to \pm\infty} \hat{c}_t^{(\eta)*}(\hat{h}^{(\eta)}) \prod_l \hat{c}_t^{(\eta_l)*}(\hat{h}_l^{(\eta_l)})\, \Omega_0 = \hat{c}_\pm^{(\eta)*}(\hat{h}^{(\eta)}) \prod_l \hat{c}_\pm^{(\eta_l)*}(\hat{h}_l^{(\eta_l)})\, \Omega_0.$$

Moreover

$$\hat{c}_\pm^{(\eta)*}(\hat{h}^{(\eta)})\, s - \lim_{t \to \pm\infty} \prod_l \hat{c}_t^{(\eta_l)*}(\hat{h}_l^{(\eta_l)})\, \Omega_0 = \tilde{\Omega}^\pm\, c^{(\eta)*}(\hat{h}^{(\eta)}) \prod_l c^{(\eta_l)*}(h_l^{(\eta_l)})\, \Omega_0$$
$$= \hat{c}_\pm^{(\eta)*}(\hat{h}^{(\eta)})\, \tilde{\Omega}^\pm \prod_l \hat{c}^{(\eta_l)*}(\hat{h}_l^{(\eta_l)})\, \Omega_0.$$

This gives in particular the action of the asymptotic physical fields $\hat{c}_\pm^{(\eta)*}$ *on scattering states of the form*

$$|\cdots h_l^{(\eta_l)} \cdots\rangle_\pm = s - \lim_{t \to \pm\infty} \prod_l \hat{c}_t^{(\eta_l)*}(\hat{h}_l^{(\eta_l)})\, \Omega_0. \tag{3.1}$$

One has $\theta_\pm^{(n)} = \tilde{\Omega}^\pm \mathscr{H}^{(n)}$, *if* $\theta_\pm^{(n)}$ *denotes the closed linear span of all scattering states of the form* (3.1) *with exactly* n *indices* η_l *equal to* b.

3) *The scattering operator* S *is defined*[17]) *(as in I) by* $\hat{S} = (\tilde{\Omega}^+)^* \tilde{\Omega}^-$. *It is a contraction operator*[18]) *mapping* $\mathscr{H}^{(n)}$ *into* $\mathscr{H}^{(n)}$, *extending to all* $\mathscr{H}$ *and commuting with* e^{itH_0}. *Its matrix elements between bare states give the amplitude for scattering from the correspondent in and out physical asymptotic states (in* $\theta_\pm^{(n)}$*).*

In particular the amplitude for scattering from an incoming state

$$|g^{(1)} \cdots g^{(n)}; f^{(1)} \cdots \tilde{f}^{(m)}\rangle_- = \prod_{j=1}^n \hat{b}_-^*(g'^{(j)}) \prod_{i=1}^m \hat{a}_-^*(f^{(i)})\, \Omega_0$$

to an outgoing state $|\tilde{g}^{(1)} \ldots \tilde{g}^{(n)}; \tilde{f}^{(1)} \ldots \tilde{f}^{(m')}\rangle_+$ *(all distributions of momenta* $g^{(j)}, \ldots, \tilde{f}^{(k)}$ *being normalized functions of* $L_2(\mathbb{R}^3)$*) is*

$$(|\tilde{g}^{(1)} \cdots \tilde{g}^{(n)}; \tilde{f}^{(1)} \cdots \tilde{f}^{(m)}\rangle_+; |g^{(1)} \cdots g^{(n)}; f^{(1)} \cdots f^{(m)}\rangle_-)$$
$$= \Big(\tilde{\Omega}^+ \prod_j b^*(\tilde{g}^{(j)}) \prod_i a^*(\tilde{f}^{(i)})\, \Omega_0, \tilde{\Omega}^- \prod_j b^*(g^{(j)}) \prod_i a^*(f^{(i)})\, \Omega_0\Big)$$
$$= \Big(\prod_j b^*(\tilde{g}^{(j)}) \prod_i a^*(\tilde{f}^{(i)})\, \Omega_0, \hat{S} \prod_j b^*(g^{(j)}) \prod_i a^*(f^{(i)})\, \Omega_0\Big).$$

Remark 3.1. This is an improvement over the correspondent Theorem in I. Except for $\hat{S} = \mathbb{1}$ in $\mathscr{H}^{(0)}$ we expect $\hat{S}$ to be a nontrivial map ($\neq 0, \mathbb{1}$) from $\mathscr{H}^{(n)}$ into $\mathscr{H}^{(n)}$ for all $n > 0$. In section 4 we shall give preliminary remarks to this.

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[17]) In the channel in which all physical nucleons and mesons are free.

[18]) For remarks on the unitarity of $\hat{S}$ (equivalent with the equality of the ranges of $\tilde{\Omega}^+$ and $\tilde{\Omega}^-$) and asymptotic completeness see I.

Proof of Theorem 3.1. 1) has already been proved in I for the case $\eta = a$. The case $\eta = b$ is proved similarly, using the construction of section 3.1. 2) and 3) follow then easily. ■

4. Some remarks on the meson-nucleon scattering

We shall give a preliminary discussion of the scattering of one physical nucleon and one meson, from an asymptotic situation at $t = -\infty$, described by

$$|h^{(1)}; f^{(1)}>_- \equiv \hat{b}^*_-(h'^{(1)})\, \hat{a}^*_-(f^{(1)})\, \Omega_0,$$

to an asymptotic situation at $t = +\infty$, described by $|h^{(2)}; f^{(2)}>_+ \equiv \hat{b}^*_+(h'^{(2)})\, \hat{a}^*_+(f^{(2)})\, \Omega_0$, $\|h^{(i)}\| = \|f^{(i)}\| = 1$, $i = 1, 2$. The study of the S-matrix element for the transition, $\hat{S}(h^{(1)} f^{(1)}; h^{(2)} f^{(2)})$, has been already started in I, where we derived reduction formulae and in particular the formula:

$$\hat{S}(h^{(1)} f^{(1)}; h^{(2)} f^{(2)}) = (h^{(1)}, h^{(2)})(f^{(1)}, f^{(2)}) + \hat{S}^{(\mathrm{I})} + \hat{S}^{(\mathrm{II})},$$

with

$$\hat{S}^{(\mathrm{I})} \equiv i \int_{-\infty}^{+\infty} dt_2\, F^{(\mathrm{I})}(t_2), \quad \hat{S}^{(\mathrm{II})} = \int_{-\infty}^{+\infty} dt_2 \int_0^{-\infty} dt_1\, F^{(\mathrm{II})}(t_2, t_1)$$

$$F^{(\mathrm{I})}(t) \equiv (a^*(f^{(1)})\, TAb^*(h^{(1)})\, \Omega_0,\quad e^{it\hat{H}} \mathscr{A}^*(f^{(2)}; t)\, TAb^*(e^{-it\omega} h^{(2)})\, \Omega_0)$$

($\mathscr{A}^*(f; t) \equiv \mathscr{A}^*(f_t)$ is the bounded extension to $\mathscr{H}^{(n)}$ of the bounded, densely defined operator $\lambda\, [V, a^*(e^{-it\mu} f)]$) and

$$F^{(\mathrm{II})}(t_2, t_1) \equiv (\mathscr{A}^*(f^{(1)}; t_1)\, TAb^*(e^{it_1\omega} h^{(1)})\, \Omega_0,\quad e^{i(t_2 - t_1)\hat{H}} \mathscr{A}^*(f^{(2)}; t_2)$$
$$TAb^*(e^{it_2\omega} h^{(2)})\, \Omega_0).$$

We first prove that it is possible to expand the integrands $F^{(\mathrm{I})}(t_2)$, $F^{(\mathrm{II})}(t_2, t_1)$ for finite times $|t_2|, |t_1| < \infty$ into power series in λ, absolutely convergent for $|\lambda|$ sufficiently small, in such a way that each term contains only bare quantities and has a dependence on t given entirely in terms of factors of the form $e^{it\Omega^{(\eta)}}$. Let in fact $A^{(j)}(t_2)$ be the term of order j (in λ) in the Dyson expansion of $e^{-it_2 \hat{H}}$ given by Lemma I 3.2, convergent in the $\mathfrak{B}(D^{(n);\beta}; D^{(n);\gamma})$-topology, $0 < \beta < \gamma$.

Let moreover $B^{*(k)}(h')\Omega_0$ be the term of order k in the expansion of

$$B^*(h')\, \Omega_0 = TAb^*(h)\, \Omega_0$$

in powers of λ.

This expansion is strongly convergent for $|\lambda|$ sufficiently small (Lemma I 3.1).

Using $a^*(f) \in \mathfrak{B}(D^{(n);\alpha}; D^{(n);\alpha'})$, $0 < \alpha' < \alpha$, $\mathscr{A}^*(g; t) \in \mathfrak{B}(\mathscr{H}^{(n)}; \mathscr{H}^{(n)})$ for any $f, g \in L_2(\mathbb{R}^3)$ and Schwartz inequalities, we can show easily that the right-hand sides of

$$F^{(\mathrm{I})}(t_2) = \sum_{l=2}^{\infty} \lambda^l F^{(\mathrm{I})(l)}(t_2)$$

and

$$F^{(\mathrm{II})}(t_2, t_1) = \sum_{l=2}^{\infty} \lambda^l F^{(\mathrm{II})(l)}(t_2, t_1)$$

are absolutely convergent for $|\lambda|$ sufficiently small. Here

$$F^{(\mathrm{I})(l)}(t_2) \equiv \sum_{\substack{j,k,i\\ j+k=l-1-i}} (A^{(j)}(t_2)\, a^*(f^{(1)})\, B^{*(k)}(h'^{(1)})\, \Omega_0,$$

$$\mathscr{A}^*(f^{(2)};t_2)\, B^{*(i)}(e^{-it_2\omega}\, h'^{(2)})\, \Omega_0),$$

$$F^{(\mathrm{II})(l)}(t_2,t_1) \equiv \sum_{\substack{j,k,i\\ j+k=l-2-i}} (A^{(j)}(t_2)\mathscr{A}^*(f^{(1)},t_1)\, B^{*(k)}(e^{-it_1\omega}\, h'^{(1)})\, \Omega_0,$$

$$\mathscr{A}^*(f^{(2)};t_2)\, B^{*(i)}(e^{-it_2\omega}\, h'^{(2)})\, \Omega_0).$$

Thus before integration over times the terms giving the scattering amplitude are convergent power series in λ and can be computed one by one. The study of the resulting series for the scattering amplitude itself requires of course special care.

The non-triviality of the scattering would follow from a strong control on the series as a whole.[19]) As a preliminary remark in this direction we shall show, using methods of Høegh-Krohn [10], that, at least in the case in which the interaction has a space cut-off, the expansions of the asymptotic meson fields and of the scattering amplitudes in powers of the coupling constant λ are (strong) asymptotic series (for $|\lambda| \to 0$), for which the (strong) Borel transform exists, at least for $|\lambda|$ sufficiently small.[20]) In particular the non-triviality of the scattering is established.

Let $\lambda V(g)$ be the space cut-off interaction, defined from λV in the usual way (see e.g. [19], [15]), with a space cut-off function $g(x)$, $g(.) \in \mathscr{D}(\mathbb{R}^3)$. Let $\hat{H}(g)$ be the correspondent Hamiltonian, for which we have of course the same estimates as for $\hat{H}$. In this case all the asymptotic fields are obtained as strong limits of adjusted Heisenberg picture bare fields $c_{g,t}^{(\eta)\#}(h) \equiv e^{it\hat{H}(g)}\, c^{(\eta)\#}(e^{\mp it\Omega^{(\eta)}})\, e^{-it\hat{H}(g)}$. In an entirely similar way as in Theorem I 3.4 one can prove the convergence in the $\mathfrak{B}(D^{(n);\alpha}; D^{(n);\beta})$-topology, $0 < \alpha < \beta$, of the Dyson-Schwinger perturbation series for $c_{g,t}^{(\eta)\#}(h)$. In particular (as in I (3.27)):

$$a^{\#}_{g,t}(h)\, \Psi = s - \lim_{L\to\infty} \sum_{l=0}^{L} \lambda^l\, I^{(l)}(g;t)\, \Psi,$$

for all $\Psi \in C^\infty(N_a)$, where $I^{(l)}(g;t)$ are independent of λ, $I^{(0)}(g;t) \equiv a^{\#}(h)$ and $\lambda^l I^{(l)}(g;t)$ for $l = 1, 2, 3, \ldots$ is the sum of all the terms of order l (in the coupling constant) one can extract from

$$i^l \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{l-1}} dt_l\, [\mathrm{V}''(t_l),[\mathrm{V}''(t_{l-1}),\ldots,[\mathrm{V}''(t_1), a^{\#}(h)]\ldots],$$

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[19]) Of course the strongest one would be the convergence of the series. But also its asymptotic character would suffice, at least for $|\lambda|$ sufficiently small (since term by term $\hat{S}$ is non-trivial).

[20]) We call a (formal) power series expansion $\sum \lambda^l A_l$ of A, where A, A_l are operators defined on a dense domain D of a Hilbert space $\mathscr{H}$, strong asymptotic to A (on D, for $\lambda \to 0$) when $\|(A - \sum_{l=0}^{r} \lambda^l A_l)\,\Psi\| \leqslant |\lambda|^{r+1} C_{r+1}(\Psi)$, $r \geqslant 0$, for some $C_{r+1}(\Psi)$ independent of λ, all $|\lambda| > 0$ sufficiently small and all $\Psi \in D$. Similarly we call strong Borel transform of a series $\sum \lambda^l A_l$ on D the series $\sum_l (\lambda^l/l!)\, A_l$, whenever the latter is strongly convergent on D for all $|\lambda| > 0$ sufficiently small. If the A, A_l are constants we have the usual concept of asymptotic series resp. Borel transform (restricted to real λ), see e.g. [21]. Note that the above adjective 'strong' refers to the operator topology and its meaning should not be confused with the one in B. Simon's 'strong asymptotic series' [21].

with $V''(g) \equiv \lambda V(g) + \hat{M}(g)$, $V''(t) \equiv e^{itH_0} V''(g) e^{-itH_0}$. Moreover one can show, adapting methods of [10], that there exists a dense subset $\mathscr{D}_\gamma$ of $L_2(\mathbb{R}^3)$ and a constant $\Lambda_\gamma > 0$ such that, for all $h \in \mathscr{D}_\gamma$, $|\lambda| \leqslant \Lambda_\gamma$, one has:

$$\|I^{(l)}(g;\pm\infty)\,\Psi\| \leqslant C_0 C_1^l \|(N_a + l)^{1/2} \ldots (N_a + 1)^{1/2}\,\Psi\|, \tag{4.1}$$

for some constants C_0, C_1 (independent of l), $I^{(l)}(g;\pm\infty)\,\Psi$ being defined as the strong limit of $I^{(l)}(g;t)\,\Psi$ for $t \to \pm\infty$. In the following we shall always make use of the fact that in the present space cut-off case it is no restriction of generality to assume, in the discussion of scattering quantities, that the Hamiltonian $\hat{H}(g)$ is simply $H_0 + \lambda V(g)$ instead of $H_0 + \lambda V(g) + \hat{M}(g)$. Assume now $\Psi \in C^\infty(H_0)$.

Denote again by $I^{(l)}(g;\pm\infty)$, $a^{\#}_{g,t}(h)$ the quantities defined as before, but with $\hat{M}(g)$ replaced in their definitions by the operator multiplication by 0. We claim that the strong limits for $t \to \pm\infty$ of $a^{\#}_{g,t}(h)\,\Psi$ (which exist, as shown by the same methods as for Theorem I 3.5) admit the power series expansions

$$a^{\#}_{g,\pm\infty}(h)\,\Psi \equiv s - \lim_{t\to\pm\infty} a^{\#}_{g,t}(h)\,\Psi = \sum_{l=0}^{r} \lambda^l I^{(l)}(g;\pm\infty)\,\Psi + R_{g,(r)}(h;\pm\infty)\,\Psi, \tag{4.2}$$

with $R_{g,(0)}(h;\pm\infty) \equiv 0$ and, for $r = 1, 2, 3, \ldots$,

$$R_{g,(r)}(h;\pm\infty) \equiv \lambda^{r+1} i^{r+1} \int_0^{\pm\infty} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{r-1}} dt_r \int_0^{t_r} d\tau\, e^{i\tau\hat{H}(g)}\, e^{-i\tau H_0}$$

$$[V''(\tau), [V''(t_r), \ldots, [V''(t_1), a^{\#}(h)] \ldots]\, e^{i\tau H_0}\, e^{-i\tau\hat{H}(g)}.$$

This holds because the formal power series expansion can be justified here, using the information on the domains of the various operators involved (in particular estimates involving powers of N_a, $\hat{H}(g)$ and H_0) and (4.1):

$$\|R_{g,(r)}(h;\pm\infty)\,\Psi\| \leqslant |\lambda|^{r+1} C_2 C_3^{r+1} \sup_\tau \|(N_a + r + 1)^{1/2} \ldots (N_a + 1)^{1/2}\, e^{-i\tau\hat{H}(g)}\,\Psi\| \leqslant$$
$$\leqslant |\lambda|^{r+1} C_2 C_3^{r+1}\, \alpha_{(r+1)}(\Psi),$$

($\alpha_{(r+1)}$ resp. C_2, C_3 being constants, independent of λ resp. λ, Ψ, r).

The same estimate shows moreover that the power series $\sum_{l=0}^\infty \lambda^l I^{(l)}(h;\pm\infty)$ is asymptotic (in the strong sense[20]) to the asymptotic fields $a^{\#}_{g,\pm\infty}(h)$, on $C^\infty(H_0)$, for $\lambda \to 0$ along reals.

We have also an immediate application for the meson-nucleon S-matrix element, which in this space cut-off case is simply

$$\hat{S}_g(h^{(1)}, f^{(1)}; h^{(2)}, f^{(2)}) = (a^{*}_{g,-}(f^{(1)})\, b^{*}(h^{(1)})\, \Omega_0,\, a^{*}_{g,+}(f^{(2)})\, b^{*}(h^{(2)})\, \Omega_0). \tag{4.3}$$

Assuming $f^{(i)}$, $i = 1, 2$ in the dense subset $\mathscr{D}_\gamma$ of $L_2(\mathbb{R}^3)$, choosing again $|\lambda| \leqslant \Lambda_\gamma$ and inserting the asymptotic expansion (4.2) into (4.3) we obtain a power series expansion $\sum_{l=0}^\infty \lambda^l s_l$ in the coupling constant λ (s_l independent of λ) which is asymptotic to the S-matrix element (4.3) for $\lambda \to 0$ along reals.

From the same estimates we have also the existence, for $|\lambda|$ sufficiently small, of the strong Borel transform[20]), $\sum_{l=0}^\infty (\lambda^l/l!)\, I^{(l)}(h;\pm\infty)\,\Psi$ of the asymptotic meson fields. This implies the existence of the Borel transform $\sum_{l=0}^{\infty} (\lambda^l/l!)\, s_l$, for $|\lambda|$ sufficiently small, of the meson-nucleon S-matrix elements.[21])

In the same way one can prove that all the similar expansions in powers of the coupling constant for the S-matrix elements between states with finitely many nucleons and mesons are asymptotic series (for $|\lambda|$ sufficiently small and distributions of momenta in a dense set of $L_2(\mathbb{R}^3)$). The S-matrix is non-trivial. Moreover its asymptotic series have Borel transforms (for $|\lambda|$ sufficiently small).[22])

This concludes our discussion of the scattering in Eckmann's model. The methods can be applied, with due modifications, to other models as well, including Nelson's model and Lee-type models. For the discussions of many other problems (like e.g. existence of Green's functions, scattering theory along the lines of [22], infrared problems) we refer to the very interesting work by J. Fröhlich [17].

Dedication

Es ist eine grosse Freude diese Arbeit meinem verehrten Lehrer Herrn Professor M. Fierz zum 60. Geburtstag widmen zu dürfen. Professor M. Fierz ist für mich ein wunderbarer Lehrer und eine grosse ständige Quelle von wissenschaftlicher und kultureller Inspiration immer gewesen. Zudem war er mir durch sein liebenswürdiges Verständnis in schweren Zeiten meines Lebens eine wesentliche Hilfe. Möge diese Widmung als ein kleiner Ausdruck meiner tiefsten Dankbarkeit gelten.

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21) Strong growth estimates (in r) on the remainders hold for real λ. For $f(\lambda) \equiv \hat{S}_g(h^{(1)}, f^{(1)}; h^{(2)}, f^{(2)})$ to be the unique sum of $\sum_{l=0}^{\infty} \lambda^l s_l$ within a certain set $\mathfrak{S}$ of functions it would be sufficient to prove, e.g., the same growth estimates for the remainders for the case of complex λ in a sector $\mathfrak{C}(L, (\pi/2)+\theta) \equiv \{\lambda \mid 0<|\lambda|<L, |\arg\lambda|<(\pi/2)+\theta\}$, some $L>0$, $\theta>0$ and to prove, e.g., that $f(\lambda)$ belongs to the set $\mathfrak{R}(L, (\pi/2)+\theta)$ of all functions regular in $\mathfrak{C}(L, (\pi/2)+\theta)$. Then we would have ([21]) uniqueness in $\mathfrak{S} = \mathfrak{R}(L, (\pi/2)+\theta)$, i.e. that $\sum_{l=0}^{\infty} \lambda^l s_l$ is asymptotic to $f(\lambda)$ in $\mathfrak{C}(L, \theta)$ but to no other function in $\mathfrak{R}(L, (\pi/2)+\theta)$ and that $f(\lambda)$ is the (unique) Borel sum of $\sum_{l=0}^{\infty} \lambda^l s_l$.

22) This uses the fact that asymptotic nucleon fields $b^{\#}_{g,\pm}(h)$ have an asymptotic expansion of the same form as (4.2) (with $a^{\#}(h)$ replaced of course by $b^{\#}(h)$).

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