Zeitschrift: Helvetica Physica Acta

Band: 45 (1972)

Heft: 2

Artikel: A note on symmetry operations in quantum mechanics

Autor: Hunziker, Walter

DOI: https://doi.org/10.5169/seals-114380

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A Note on Symmetry Operations in Quantum Mechanics

by Walter Hunziker

Seminar für Theoretische Physik, ETH Hönggerberg, 8049 Zürich, Switzerland

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1. Introduction

In the analysis of the structure of quantum mechanics, Wigner's theorem [1] on symmetry operations plays a fundamental role: from the postulated invariance of transition probabilities it derives that symmetry operations act as linear (or antilinear) transformations in Hilbert space (superposition principle). Another characterization of symmetry operations is due to Kadison [2]: it states that a symmetry operation (acting on the states of a quantum mechanical system) commutes with the operation of mixing. This is a necessary condition for any operation describing the kinematical or dynamical behaviour of a system.

Unfortunately, Kadison's work (and a related paper by Roberts and Roepstorff [3]) is written for experts in C*-algebras and obscures to others the quite elementary nature of the theorem. As a teacher I have tried to find a proof using tools available to physics students. The result is presented in this note, which I dedicate to Markus Fierz as a contribution to our discussions on the teaching of quantum mechanics.

2. Statement of the Theorems

Let \mathscr{H} and \mathscr{H}' be complex Hilbert spaces of dimensions $\geqslant 2.\Pi(\mathscr{H})$ denotes the set of all one-dimensional projections π on \mathscr{H} . The set of all finite convex combinations of elements $\pi \in \Pi(\mathscr{H})$ is called $E(\mathscr{H})$. The theorems state the equivalence of the following definitions:

- I. A symmetry operation is a linear or antilinear isometry U of \mathscr{H} into \mathscr{H}' .
- II. A symmetry operation is a mapping $S:\pi\to\pi'$ of $\Pi(\mathscr{H})$ into $\Pi(\mathscr{H}')$ such that $(\mathrm{Tr}=\mathrm{trace})$

$$\operatorname{Tr} \pi_1' \pi_2' = \operatorname{Tr} \pi_1 \pi_2. \tag{1}$$

III. A symmetry operation is a one-to-one mapping $S: A \to A'$ of $E(\mathcal{H})$ into $E(\mathcal{H}')$ such that for $0 \le a \le 1$ and all $A_1, A_2 \in E(\mathcal{H})$

$$(aA_1 + (1-a)A_2)' = aA_1' + (1-a)A_2'. (2)$$

The equivalence I \sim II (Wigner's theorem) and I \sim III (Kadison's theorem, adapted to ordinary quantum mechanics) is to be understood in the sense

$$\pi' = U\pi U^{-1}$$
 and $A' = UAU^{-1}$.

In both cases, U is determined by S up to a complex factor of modulus 1 (phase). In quantum mechanics, \mathcal{H} and \mathcal{H}' are coherent subspaces and S is required to be a mapping onto. Then U is unitary or antiunitary. We now turn to the proof of Kadison's theorem.

3. Preliminary Remarks

 $\Pi(\mathscr{H})$ is the set of the extremal elements of the convex set $E(\mathscr{H})$. Therefore, since S is one-to-one, S maps $\Pi(\mathscr{H})$ into $\Pi(\mathscr{H}')$. S has a unique linear extension to the *real*-linear span $\overline{E}(\mathscr{H})$ of $E(\mathscr{H})$ (or of $\Pi(\mathscr{H})$), which is the real vector space of all symmetric operators of finite rank. This extended mapping $A \to A'$ has the properties:

$$A' \neq 0$$
 if $A \neq 0$,
 $A' \geqslant 0$ if $A \geqslant 0$,
 $\operatorname{Tr} A' = \operatorname{Tr} A$.

Later we shall choose between the (unique) linear or antilinear further extension of S to the *complex*-linear span $\overline{\overline{E}}(\mathscr{H})$ of $E(\mathscr{H})$, which is the *algebra* of all operators of finite rank. Then we have

$$(A')^* = (A^*)',$$
 (4)

$$\operatorname{Tr} A' = \begin{cases} \operatorname{Tr} A & \text{for the linear extension,} \\ \overline{\operatorname{Tr} A} & \text{for the antilinear extension.} \end{cases}$$
 (5)

4. Proof for dim $\mathcal{H} = \dim \mathcal{H}' = 2$

We identify \mathscr{H} and \mathscr{H}' with C^2 by introducing an orthonormal basis in \mathscr{H} and \mathscr{H}' . S then becomes a one-to-one mapping of $E(C^2)$ into itself satisfying (2). The elements $\pi \in \Pi(C^2)$ are the 2×2 -matrices

$$\pi = \frac{1}{2}(1 + \vec{e} \cdot \vec{\sigma})$$

with $\vec{e} \in \mathbb{R}^3$, $|\vec{e}| = 1$, where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the set of Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

 $\overline{E}(C^2)$ is the space of all hermitian 2×2 -matrices

$$A = \frac{1}{2}(a_0 \mathbf{1} + \vec{a} \cdot \vec{\sigma}), \quad a = (a_0, \vec{a}) \in \mathbb{R}^4.$$

S therefore extends to a linear mapping of R^4 onto itself which leaves the planes $\operatorname{Tr} A = a_0 = \operatorname{const.}$ invariant and which maps, in the plane $a_0 = 1$, the sphere $|\vec{a}| = 1$ onto itself. It follows that S has the form:

$$S:(a_0,\vec{a})\to(a_0,R\vec{a}) \quad \text{with} \quad R\in O(3).$$

From the theory of spin 1/2, we know that this implies

$$A' = UAU^{-1},$$

where U is unitary (if $\det R = \det S = +1$) or antiunitary (if $\det S = -1$), and is determined by S up to a phase. Note that our definition of $\det S$ is *intrinsic*, i.e. independent of the choice of the basis in \mathscr{H} and \mathscr{H}' .

5. Reduction to Wigner's Theorem

Lemma 1. Let M be a 2-dimensional subspace of \mathcal{H} . Then S maps E(M) onto E(M'), where M' is a 2-dimensional subspace of \mathcal{H}' .

Proof: If P(M) denotes the projection onto M, the statement $A \in \overline{E}(M)$ is equivalent to the two conditions $\pm A \leqslant cP(M)$ for some $c \geqslant 0$. Let $P(M) = \pi_1 + \pi_2, \pi_i \in \Pi(\mathscr{H})$. Then it follows from (3) that $\pm A' \leqslant c(\pi_1' + \pi_2') \leqslant 2cP(M')$, where M' is the subspace of \mathscr{H}' spanned by the ranges of π_1' and π_2' . Hence $A' \in \overline{E}(M')$, and since $\dim M' \leqslant 2$ and S is nonsingular, we have $\dim M' = 2$.

Corollary: The restriction S(M) of S to $\overline{E}(M)$ is of the form

$$A \to A' = U(M)AU(M)^{-1},\tag{6}$$

where U(M) is determined up to a phase as a unitary/antiunitary mapping of M onto M' if $\det S(M) = +1/-1$. In particular,

$$\pi_i' = U(M)\pi_i U(M)^{-1} \tag{7}$$

for any pair $\pi_1, \pi_2 \in \Pi(\mathcal{H})$, where M is a 2-dimensional subspace containing the ranges of π_1 and π_2 . Therefore (1) is satisfied.

*

Having thus reduced Kadison's theorem to Wigner's theorem, we could now refer the reader to Bargmann's proof [4], for example. But since we already have the tools at hand, we complete the proof.

Lemma 2. $\det S(M)$ is independent of M (and from now on denoted by D(S)).

Proof: It suffices to show that $\det S(M_1) = \det S(M_2)$ for $M_1 \cap M_2 \neq \{0\}$. Then we can rotate M_1 continuously into M_2 in the at most 3-dimensional subspace N spanned by M_1 and M_2 . Since S is linear, it is continuous on $\overline{E}(N)$, therefore M_1 is rotated continuously into M_2 . It follows that $\det S(M_1)$ is continuous under this rotation and therefore constant.

Definition: S is now defined on $\overline{\overline{E}}(\mathcal{H})$ by linear/antilinear extension if D(S) = +1/-1.

Lemma 3. For all $A_1, A_2 \in \overline{\overline{E}}(\mathcal{H})$ we have $(A_1 A_2)' = A_1' A_2'$.

Proof: Since S is linear or antilinear, it suffices to consider the case $A_i = \pi_i \in \Pi(\mathcal{H})$. By (7) we have

$$\pi_1'\pi_2' = U(M)\pi_1\pi_2 U(M)^{-1}.$$

On the other hand, (6) extends to all $A \in \overline{\overline{E}}(M)$ since both sides are either linear or antilinear in A. Therefore, $(\pi_1\pi_2)' = \pi_1'\pi_2'$.

6. Construction of U

- 1. Choose a fixed $\pi \in \Pi(\mathcal{H})$ and unit vectors e, e' in the ranges of π, π' .
- 2. Let $a \in \mathcal{H}$ be arbitrary. If a = ce (c complex), define

$$U(a) = a' = \begin{cases} ce' & \text{if } D(S) = +1\\ \overline{c}e' & \text{if } D(S) = -1. \end{cases}$$

Otherwise, let M be the 2-dimensional subspace spanned by e and a. Define (the phase of) U(M) by U(M)e=e' and then U(a) by

$$U(a) = a' = U(M)a$$
.

By construction, U is isometric and reduces to U(M) on any 2-dimensional subspace M containing e. By Lemma 3, S preserves orthogonality of projections, therefore U preserves orthogonality of vectors. It follows that

$$A' = UAU^{-1} \tag{8}$$

for all $A \in \overline{\overline{E}}(M)$, M being any 2-dimensional subspace containing e. In particular,

$$\pi' = U\pi U^{-1}$$

for all $\pi \in \Pi(\mathcal{H})$. It remains to show that U is linear or antilinear, or equivalently, that

$$(a_1', a_2') = \begin{cases} (a_1, a_2) & \text{if } D(S) = +1 \\ \hline (a_1, a_2) & \text{if } D(S) = -1 \end{cases}$$

for all $a_1, a_2 \in \mathcal{H}$. Let M_i be the subspace spanned by e and a_i . Then the linear operator $A_i \in \overline{E}(M_i)$ defined by

$$A_{i} u = \begin{cases} a_{i} & \text{for } u = e \\ 0 & \text{for } (u, e) = 0 \end{cases}$$

satisfies (8). Therefore,

$$A'_{i}u' = \begin{cases} a'_{i} & \text{for } u' = e' \\ 0 & \text{for } (u', e') = 0, \end{cases}$$

and it follows from (4), (5) and Lemma 3 that

$$(a_1', a_2') = (A_1'e', A_2', e') = \operatorname{Tr}(A_1') * A_2' = \begin{cases} \operatorname{Tr}A_1^*A_2 = (a_1, a_2) & \text{if } D(S) = +1 \\ \operatorname{Tr}A_1^*A_2 = \overline{(a_1, a_2)} & \text{if } D(S) = -1. \end{cases}$$

Finally, it is clear that U is determined by S up to a phase, since this is true for the restriction of U to any 2-dimensional subspace.

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