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A Note on Symmetry Operations in Quantum Mechanics

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1. Introduction

In the analysis of the structure of quantum mechanics, Wigner's theorem [1] on symmetry operations plays a fundamental role: from the postulated invariance of transition probabilities it derives that symmetry operations act as *linear* (or antilinear) *transformations in Hilbert space* (superposition principle). Another characterization of symmetry operations is due to Kadison [2]: it states that a symmetry operation (acting on the states of a quantum mechanical system) commutes with the operation of *mixing*. This is a necessary condition for any operation describing the kinematical or dynamical behaviour of a system.

Unfortunately, Kadison's work (and a related paper by Roberts and Roepstorff [3]) is written for experts in C^*-algebras and obscures to others the quite elementary nature of the theorem. As a teacher I have tried to find a proof using tools available to physics students. The result is presented in this note, which I dedicate to Markus Fierz as a contribution to our discussions on the teaching of quantum mechanics.

2. Statement of the Theorems

Let $\mathscr{H}$ and $\mathscr{H}'$ be complex Hilbert spaces of dimensions $\geqslant 2$. $\Pi(\mathscr{H})$ denotes the set of all one-dimensional projections π on $\mathscr{H}$. The set of all finite convex combinations of elements $\pi \in \Pi(\mathscr{H})$ is called $E(\mathscr{H})$. The theorems state the equivalence of the following definitions:

I. A symmetry operation is a linear or antilinear isometry U of $\mathscr{H}$ into $\mathscr{H}'$.

II. A symmetry operation is a mapping $S: \pi \to \pi'$ of $\Pi(\mathscr{H})$ into $\Pi(\mathscr{H}')$ such that (Tr = trace)

$$\operatorname{Tr} \pi_1' \pi_2' = \operatorname{Tr} \pi_1 \pi_2. \tag{1}$$

III. A symmetry operation is a one-to-one mapping $S: A \to A'$ of $E(\mathscr{H})$ into $E(\mathscr{H}')$ such that for $0 \leqslant a \leqslant 1$ and all $A_1, A_2 \in E(\mathscr{H})$

$$(aA_1 + (1-a)A_2)' = aA_1' + (1-a)A_2'. \tag{2}$$

The equivalence I $\sim$ II (Wigner's theorem) and I $\sim$ III (Kadison's theorem, adapted to ordinary quantum mechanics) is to be understood in the sense

$$\pi' = U\pi U^{-1} \quad \text{and} \quad A' = UAU^{-1}.$$

In both cases, U is determined by S up to a complex factor of modulus 1 (phase). In quantum mechanics, $\mathcal{H}$ and $\mathcal{H}'$ are coherent subspaces and S is required to be a mapping *onto*. Then U is unitary or antiunitary. We now turn to the proof of Kadison's theorem.

3. Preliminary Remarks

$\Pi(\mathcal{H})$ is the set of the extremal elements of the convex set $E(\mathcal{H})$. Therefore, since S is one-to-one, S maps $\Pi(\mathcal{H})$ into $\Pi(\mathcal{H}')$. S has a unique linear extension to the *real*-linear span $\bar{E}(\mathcal{H})$ of $E(\mathcal{H})$ (or of $\Pi(\mathcal{H})$), which is the real vector space of all symmetric operators of finite rank. This extended mapping $A \to A'$ has the properties:

$$\begin{aligned} &A' \neq 0 \quad \text{if } A \neq 0, \\ &A' \geqslant 0 \quad \text{if } A \geqslant 0, \\ &\operatorname{Tr} A' = \operatorname{Tr} A. \end{aligned} \tag{3}$$

Later we shall choose between the (unique) linear or antilinear further extension of S to the *complex*-linear span $\bar{\bar{E}}(\mathcal{H})$ of $E(\mathcal{H})$, which is the *algebra* of all operators of finite rank. Then we have

$$(A')^* = (A^*)', \tag{4}$$

$$\operatorname{Tr} A' = \begin{cases} \operatorname{Tr} A & \text{for the linear extension,} \\ \overline{\operatorname{Tr} A} & \text{for the antilinear extension.} \end{cases} \tag{5}$$

4. Proof for $\dim \mathcal{H} = \dim \mathcal{H}' = 2$

We identify $\mathcal{H}$ and $\mathcal{H}'$ with C^2 by introducing an orthonormal basis in $\mathcal{H}$ and $\mathcal{H}'$. S then becomes a one-to-one mapping of $E(C^2)$ into itself satisfying (2). The elements $\pi \in \Pi(C^2)$ are the 2×2-matrices

$$\pi = \tfrac{1}{2}(1 + \vec{e} \cdot \vec{\sigma})$$

with $\vec{e} \in R^3$, $|\vec{e}| = 1$, where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the set of Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\bar{E}(C^2)$ is the space of all hermitian 2×2-matrices

$$A = \tfrac{1}{2}(a_0 1 + \vec{a} \cdot \vec{\sigma}), \quad a = (a_0, \vec{a}) \in R^4.$$

S therefore extends to a linear mapping of R^4 onto itself which leaves the planes $\operatorname{Tr} A = a_0 = \text{const.}$ invariant and which maps, in the plane $a_0 = 1$, the sphere $|\vec{a}| = 1$ onto itself. It follows that S has the form:

$$S: (a_0, \vec{a}) \to (a_0, R\vec{a}) \quad \text{with} \quad R \in O(3).$$

From the theory of spin 1/2, we know that this implies

$$A' = UAU^{-1},$$

where U is unitary (if $\det R = \det S = +1$) or antiunitary (if $\det S = -1$), and is determined by S up to a phase. Note that our definition of $\det S$ is *intrinsic*, i.e. independent of the choice of the basis in $\mathcal{H}$ and $\mathcal{H}'$.

5. Reduction to Wigner's Theorem

Lemma 1. *Let M be a 2-dimensional subspace of $\mathscr{H}$. Then S maps $E(M)$ onto $E(M')$, where M' is a 2-dimensional subspace of $\mathscr{H}'$.*

Proof: If $P(M)$ denotes the projection onto M, the statement $A \in \bar{E}(M)$ is equivalent to the two conditions $\pm A \leqslant cP(M)$ for some $c \geqslant 0$. Let $P(M) = \pi_1 + \pi_2, \pi_i \in \Pi(\mathscr{H})$. Then it follows from (3) that $\pm A' \leqslant c(\pi_1' + \pi_2') \leqslant 2cP(M')$, where M' is the subspace of $\mathscr{H}'$ spanned by the ranges of π_1' and π_2'. Hence $A' \in \bar{E}(M')$, and since $\dim M' \leqslant 2$ and S is nonsingular, we have $\dim M' = 2$.

Corollary: The restriction $S(M)$ of S to $\bar{E}(M)$ is of the form

$$A \to A' = U(M) A U(M)^{-1}, \tag{6}$$

where $U(M)$ is determined up to a phase as a unitary/antiunitary mapping of M onto M' if $\det S(M) = +1/-1$. In particular,

$$\pi_i' = U(M) \pi_i U(M)^{-1} \tag{7}$$

for any pair $\pi_1, \pi_2 \in \Pi(\mathscr{H})$, where M is a 2-dimensional subspace containing the ranges of π_1 and π_2. Therefore (1) is satisfied.

*

Having thus reduced Kadison's theorem to Wigner's theorem, we could now refer the reader to Bargmann's proof [4], for example. But since we already have the tools at hand, we complete the proof.

Lemma 2. $\det S(M)$ *is independent of M (and from now on denoted by $D(S)$).*

Proof: It suffices to show that $\det S(M_1) = \det S(M_2)$ for $M_1 \cap M_2 \neq \{0\}$. Then we can rotate M_1 continuously into M_2 in the at most 3-dimensional subspace N spanned by M_1 and M_2. Since S is linear, it is continuous on $\bar{E}(N)$, therefore M_1 is rotated continuously into M_2'. It follows that $\det S(M_1)$ is continuous under this rotation and therefore constant.

Definition: S is now defined on $\bar{\bar{E}}(\mathscr{H})$ by linear/antilinear extension if $D(S) = +1/-1$.

Lemma 3. *For all $A_1, A_2 \in \bar{\bar{E}}(\mathscr{H})$ we have $(A_1 A_2)' = A_1' A_2'$.*

Proof: Since S is linear or antilinear, it suffices to consider the case $A_i = \pi_i \in \Pi(\mathscr{H})$. By (7) we have

$$\pi_1' \pi_2' = U(M) \pi_1 \pi_2 U(M)^{-1}.$$

On the other hand, (6) extends to all $A \in \bar{\bar{E}}(M)$ since both sides are either linear or antilinear in A. Therefore, $(\pi_1 \pi_2)' = \pi_1' \pi_2'$.

6. Construction of U

1. Choose a fixed $\pi \in \Pi(\mathscr{H})$ and unit vectors e, e' in the ranges of π, π'.
2. Let $a \in \mathscr{H}$ be arbitrary. If $a = ce$ (c complex), define

$$U(a) = a' = \begin{cases} ce' \text{ if } D(S) = +1 \\ \bar{c}e' \text{ if } D(S) = -1. \end{cases}$$

Otherwise, let M be the 2-dimensional subspace spanned by e and a. Define (the phase of) $U(M)$ by $U(M)e = e'$ and then $U(a)$ by

$$U(a) = a' = U(M)a.$$

By construction, U is isometric and reduces to $U(M)$ on any 2-dimensional subspace M containing e. By Lemma 3, S preserves orthogonality of projections, therefore U preserves orthogonality of vectors. It follows that

$$A' = UAU^{-1} \tag{8}$$

for all $A \in \overline{\overline{E}}(M)$, M being any 2-dimensional subspace containing e. In particular,

$$\pi' = U\pi U^{-1}$$

for all $\pi \in \Pi(\mathscr{H})$. It remains to show that U is linear or antilinear, or equivalently, that

$$(a_1', a_2') = \begin{cases} (a_1, a_2) & \text{if } D(S) = +1 \\ \overline{(a_1, a_2)} & \text{if } D(S) = -1 \end{cases}$$

for all $a_1, a_2 \in \mathscr{H}$. Let M_i be the subspace spanned by e and a_i. Then the linear operator $A_i \in \overline{\overline{E}}(M_i)$ defined by

$$A_i u = \begin{cases} a_i & \text{for } u = e \\ 0 & \text{for } (u, e) = 0 \end{cases}$$

satisfies (8). Therefore,

$$A_i' u' = \begin{cases} a_i' & \text{for } u' = e' \\ 0 & \text{for } (u', e') = 0, \end{cases}$$

and it follows from (4), (5) and Lemma 3 that

$$(a_1', a_2') = (A_1' e', A_2' e') = \operatorname{Tr}(A_1')^* A_2' = \begin{cases} \operatorname{Tr} A_1^* A_2 = (a_1, a_2) & \text{if } D(S) = +1 \\ \overline{\operatorname{Tr} A_1^* A_2} = \overline{(a_1, a_2)} & \text{if } D(S) = -1. \end{cases}$$

Finally, it is clear that U is determined by S up to a phase, since this is true for the restriction of U to any 2-dimensional subspace.

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