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Representations of the Gauge Groups of Electrodynamics and General Relativity

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Abstract. A semidirect product of the gauge groups of Electrodynamics and General Relativity is determined and unitarily represented on a Hilbertspace of the type $\mathfrak{L}_2(\mathscr{S}', \mu)$.

1. Introduction

Electrodynamics and General Relativity, being rest-mass zero theories of vector and tensor fields, enjoy special symmetries governed by gauge groups. Some implications of these symmetries have been intensively studied [1]. In fact a linearized Lorentz-covariant theory of gravitation gets uniquely promoted to Einstein's General Relativity with the help of the gauge group [2]. The generators of the gauge groups of Electrodynamics and General Relativity form a nuclear Lie algebra that we show is similar to a current algebra [3]. The Gel'fand-Vilenkin formalism [4] then gives representations of the integrated group, being the semidirect product $\mathscr{S}(\mathbb{R}^4) \wedge \mathrm{diff}_{\mathscr{S}}(\mathbb{R}^4)$, where $\mathscr{S}(\mathbb{R}^4)$ is the gauge group for Electrodynamics and $\mathrm{diff}_{\mathscr{S}}(\mathbb{R}^4)$ the gauge group for General Relativity. The semidirect product structure corresponds to the Klein-Kaluza formalism [5]. The representation spaces are of the type $\mathfrak{L}_2(\mathscr{S}', \mu)$, with μ a cylindrical measure on $\mathscr{S}'$. We might be enlightened in the physics of these representations by current investigations into a Wightman formalism for Quantum Electrodynamics and for the Theory of Gravitation [6].

2. The Gauge Lie Algebra for Electrodynamics and General Relativity

Let A_μ denote the potential in Electrodynamics and $g_{\mu\nu}$ the potential in General Relativity, $\mu, \nu = 1,2,3,4$. Let $\phi \in \mathscr{S}(\mathbb{R}^4)$ be real valued, then the transformations $E(\phi)$ defined by

$$E(\phi) A_\mu = A_\mu + \phi_{,\mu} \tag{2.1}$$

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constitute the gauge group of Electrodynamics. The Lie algebra of the gauge group in General Relativity [2] originates from infinitesimal coordinate transformations

$$T(\xi)x^{\mu} = x^{\mu} - \xi^{\mu}(x), \tag{2.2}$$

where ξ is a real valued vector field $\{\xi^{\mu}\}$ with $\xi^{\mu} \in \mathscr{S}(\mathbb{R}^4)$. The generators $T(\xi)$ transform A_{μ} and $g_{\mu\nu}$ as follows

$$T(\xi)A_{\mu} = A_{\mu} + A_{\alpha}\xi^{\alpha}_{,\mu} + \xi^{\alpha}A_{\mu,\alpha} \tag{2.3}$$

$$T(\xi)g_{\mu\nu} = g_{\mu\nu} + g_{\mu\beta}\xi^{\beta}_{,\nu} + g_{\alpha\nu}\xi^{\alpha}_{,\mu} + \xi^{\alpha}g_{\mu\nu,\alpha}. \tag{2.4}$$

The group theoretical commutators lead to the following Lie algebra $\mathfrak{L}$

$$[E(\phi), E(\psi)] = 0 \tag{2.5}$$

$$[E(\phi), T(\xi)] = -E(\xi\cdot\operatorname{grad}\phi) \tag{2.6}$$

$$[T(\xi), T(\eta)] = T([\xi,\eta]), \tag{2.7}$$

where

$$[\xi,\eta]^{\mu} = \eta^{\mu}_{,\nu}\xi^{\nu} - \xi^{\mu}_{,\nu}\eta^{\nu} \tag{2.8}$$

This Lie algebra is similar to the charge-current algebra of a nonrelativistic field theory, which has been discussed by G. A. Goldin [3]. Hence the Gel'fand-Vilenkin formalism for nuclear Lie groups [4] will give us unitary representations of a gauge group G, whose Lie algebra is $\mathfrak{L}$.

3. Representations of $\mathfrak{L}$ on the Field Algebra

Recall that the Field algebra $\mathfrak{U}$, as a topological vector space, is given by the topological direct sum $\mathfrak{U} = \bigoplus_{n=0}^{\infty} \mathscr{S}^n(\mathbb{R}^4)$, where $\mathscr{S}^0(\mathbb{R}^4) = \mathbb{C}$ [7]. We then get the following representation for the generators $E(\phi)$ and $T(\xi)$:

$$(\mathbb{E}(\phi)f)_n(x_1,\ldots,x_n) = \sum_{k=1}^{n}\phi(x_k)f_n(x_1,\ldots,x_n) \tag{3.1}$$

$$(\mathbb{T}(\xi)f)_n(x_1,\ldots,x_n) = \sum_{k=1}^{n}\left\{\xi^{\mu}(x_k)\frac{\partial}{\partial x^{\mu}_k}f_n(x_1,\ldots,x_n) + \frac{1}{2}\frac{\partial}{\partial x^{\mu}_k}\xi^{\mu}(x_k)\cdot f_n(x_1,\ldots,x_n)\right\} \tag{3.2}$$

It is readily found that $\mathbb{T}(\xi)$ is antihermitian with respect to the $\mathfrak{L}_2$-inner product on $\mathscr{S}^n(\mathbb{R}^4)$, and satisfies the commutation relation (2.7). $\mathbb{E}(\phi)$ is hermitian and satisfies (2.5) and (2.6).

4. The Gauge Group G

We first look at the one-parameter subgroup of G of the form

$$(e^{t\mathbb{T}(\xi)}f)_n(x_1,\ldots,x_n) \equiv (V(\Phi_t(\xi,\cdot))f)_n(x_1,\ldots,x_n). \tag{4.1}$$

Lemma 4.1. $(V(\Phi_t(\xi,\cdot))f)_n(x_1,\ldots,x_n)$

$$=f_n(\Phi_t(\xi,x_1),\ldots,\Phi_t(\xi,x_n))\prod_{k=1}^{n}\sqrt{\det\frac{\partial\Phi_t(\xi,x_k)}{\partial x_k}}, \tag{4.2}$$

where $\Phi_t(\xi,\cdot)$ is the flow generated by the vector field $\xi(x)$, i.e.

$$\frac{d}{dt}\Phi_t(\xi,x)=\xi(\Phi_t(\xi,x)),\quad \Phi_0(\xi,x)=x. \tag{4.3}$$

Proof: See Appendix A.

Since ξ is an $\mathscr{S}$-vector field, the corresponding flow $\Phi_t(\xi,x)$ is then C_∞ in x for all t. Let $\mathrm{diff}_{\mathscr{S}}(\mathbb{R}^4)$ stand for the C_∞-diffeomorphisms on $\mathbb{R}^4$, generated by the flows $\Phi_t(\xi,x)$. The composition of flows turns $\mathrm{diff}_{\mathscr{S}}(\mathbb{R}^4)$ into a group, the gauge group for General Relativity. Hence on the Field algebra $\mathfrak{U}$ we have for $\Phi\in\mathrm{diff}_{\mathscr{S}}(\mathbb{R}^4)$

$$(V(\Phi)f)_n(x_1,\ldots,x_n)=f_n(\Phi(x_1),\ldots,\Phi(x_n))\prod_{k=1}^{n}\sqrt{\det\frac{\partial\Phi(x_k)}{\partial x_k}}. \tag{4.4}$$

The gauge group for Electrodynamics is $\mathscr{S}(\mathbb{R}^4)$ under addition and its unitary representation on $\mathfrak{U}$ is given by

$$(e^{i\mathrm{E}(\phi)}f)_n(x_1,\ldots,x_n)\equiv(U(\phi)f)_n(x_1,\ldots,x_n) \tag{4.5}$$

$$(U(\phi)f)_n(x_1,\ldots,x_n)=e^{i\sum_{k=1}^{n}\phi(x_k)}f_n(x_1,\ldots,x_n). \tag{4.6}$$

The full group is now given by the semidirect product

$$G=\mathscr{S}(\mathbb{R}^4)\wedge\mathrm{diff}_{\mathscr{S}}(\mathbb{R}^4) \tag{4.7}$$

with the semidirect product map

$$\mathscr{S}(\mathbb{R}^4)\times\mathrm{diff}_{\mathscr{S}}(\mathbb{R}^4)\to\mathscr{S}(\mathbb{R}^4) \tag{4.8}$$

$$(\phi,\Phi)(x)=(\phi\circ\Phi)(x)=\phi(\Phi(x)). \tag{4.9}$$

The multiplication law of G thus is

$$(\phi_1\wedge\Phi_1)(\phi_2\wedge\Phi_2)=(\phi_1+\phi_2\circ\Phi_1)\wedge\Phi_2\circ\Phi_1. \tag{4.10}$$

For a unitary representation this reads

$$U(\phi_1)V(\Phi_1)U(\phi_2)V(\Phi_2)=U(\phi_1+\phi_2\circ\Phi_1)V(\Phi_2\circ\Phi_1). \tag{4.11}$$

The gauge group G inherits a nuclear topology from $\mathscr{S}(\mathbb{R}^4)$, and $\mathscr{S}$ is a normal subgroup of G. The operators $U(\phi)$ and $V(\Phi)$ are unitary on the Fockspace $\mathfrak{H}_f$ which is the completion of the field algebra $\mathfrak{U}$ in the norm

$$\|f\|^2=\sum_{k=1}^{\infty}\|f_n\|^2 \tag{4.12}$$

$$\|f_n\|^2=\int\overline{f_n(x_1,\ldots,x_n)}f_n(x_1,\ldots,x_n)\,dx_1\ldots dx_n \tag{4.13}$$

5. Representations of the Gauge Group G

Here we give a summary of the Gel'fand-Vilenkin method [4] as applied to G. For details see Ref. [3]. A continuous unitary representation $U(\phi)$ of the normal subgroup $\mathscr{S}(\mathbb{R}^4) \subset G$ on a Hilbertspace $\mathfrak{H}$ with cyclic vector Ω gives us a functional $L(\phi)$, defined by

$$L(\phi) = (\Omega, U(\phi)\Omega) \tag{5.1}$$

$L(\phi)$ has the following properties

1. L is continuous on $\mathscr{S}(\mathbb{R}^4)$ (5.2)

2. $L(0) = 1$ (5.3)

3. $\sum_{k,l=1}^{N} \bar{c}_k c_l L(\phi_k - \phi_l) \geqslant 0$ for any complex numbers $c_1, \ldots, c_N$ (5.4)

and hence by Bochner's Theorem there exists a unique cylinder measure μ on $\mathscr{S}'$ such that

$$L(\phi) = \int_{\mathscr{S}'} e^{i(T,\phi)}\, d\mu(T). \quad T \in \mathscr{S}'(\mathbb{R}^4) \tag{5.5}$$

A strongly continuous representation of $\mathscr{S}$ in $\mathfrak{H} = \mathfrak{L}_2(\mathscr{S}', \mu)$ is thus given by

$$U(\phi)\Psi(T) = e^{i(T,\phi)}\Psi(T) \tag{5.6}$$

and the cyclic vector Ω is realized by the unit function on $\mathscr{S}'$. For $\Phi \in \mathrm{diff}_{\mathscr{S}}(\mathbb{R}^4)$ let $\Phi^* : \mathscr{S}'(\mathbb{R}^4) \to \mathscr{S}'(\mathbb{R}^4)$ be defined by

$$(\Phi^* T, \phi) = (T, \phi \circ \Phi) \tag{5.7}$$

and define a transformed measure μ^{Φ^*} on $\mathscr{S}'$ by

$$\mu^{\Phi^*}(T) = \mu(\Phi^* T). \tag{5.8}$$

The group $\mathrm{diff}_{\mathscr{S}}(\mathbb{R}^4)$ is then represented on $\mathfrak{L}_2(\mathscr{S}', \mu)$ by

$$V(\Phi)\Psi(T) = \chi(\Phi, T)\Psi(\Phi^* T)\sqrt{\frac{d\mu^{\Phi^*}(T)}{d\mu(T)}} \tag{5.9}$$

where $d\mu^{\Phi^*}/d\mu$ is the Radon-Nikodym derivative and $\chi(\Phi, T)$ a complex valued function of modulus one, satisfying

$$\chi(\Phi_2, T)\chi(\Phi_1, \Phi_2 T) = \chi(\Phi_1 \circ \Phi_2, T). \tag{5.10}$$

The measure μ on $\mathscr{S}'$ is quasiinvariant under $\mathrm{diff}_{\mathscr{S}}(\mathbb{R}^4)$, i.e. μ and μ^{Φ^*} have the same set of measure zero. Thus the expectation functional $L(\phi)$ defines a representation of G up to a phase function.

Remarks:

1. The Fock representation of G on $\mathfrak{H}_F$ corresponds to a Gaussian measure. In the n-particle space $\mathfrak{H}_F^{(n)} \subset \mathfrak{H}_F$, μ is concentrated on the set $T = \{T_{x_1} + \cdots + T_{x_n}, x_i \neq x_k\}$, where $(T_x, \phi) = \phi(x)$.
$d\mu(T_{x_1} + \cdots + T_{x_n}) = \pi^{-2n} e^{-\|x_1\|^2 - \ldots - \|x_n\|^2} dx_1 \ldots dx_n$. Observe that this representation is on a Fockspace of scalar functions.
2. For the recovery of the infinitesimal generators $\mathbb{E}(\phi)$ and $\mathbb{T}(\xi)$, Goldin [3] gives sufficient conditions, expressed as properties of the measure μ.
3. Here we do not investigate if the Gel'fand-Vilenkin representations of G are physical. If they are, then operators commuting with the $U(\phi)$ are candidates for observables in Quantum Electrodynamics and operators commuting with $V(\Phi)$ are candidates for observables in a quantum theory of gravitation.

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Appendix A

We want to show that

$$(e^{t\mathbb{T}(\xi)} f)_n (x_1, \ldots, x_n) = f_n (\Phi_t (\xi, x_1), \ldots, \Phi_t (\xi, x_n)) \prod_{k=1}^{n} \sqrt{\det \frac{\partial \Phi_t (\xi, x_k)}{\partial x_k}} \tag{A.1}$$

where

$$(\mathbb{T}(\xi) f)_n (x_1, \ldots, x_n) = \sum_{k=1}^{n} \left\{ \xi^\mu (x_k) \frac{\partial}{\partial x_k^\mu} + \frac{1}{2} \frac{\partial \xi^\mu (x_k)}{\partial x_k^\mu} \right\} f_n (x_1, \ldots, x_n). \tag{A.2}$$

It suffices to give the proof for one variable x, i.e.

$$e^{t\mathbb{T}(\xi)} f(x) = f(\Phi_t (\xi, x)) \sqrt{\det \frac{\partial \Phi_t (\xi, x)}{\partial x}} \tag{A.3}$$

with

$$\mathbb{T}(\xi) f(x) = \left\{ \xi^\mu (x) \frac{\partial}{\partial x^\mu} + \frac{1}{2} \frac{\partial \xi^\mu (x)}{\partial x^\mu} \right\} f(x). \tag{A.4}$$

For fixed ξ and x let

$$F(f, t) = e^{t\mathbb{T}(\xi)} f(x) \tag{A.5}$$

$$G(f, t) = \sqrt{\det \frac{\partial \Phi_t (\xi, x)}{\partial x}} f(\Phi_t (\xi, x)). \tag{A.6}$$

Then $F(f,t)$ satisfies the following differential equation

$$\frac{\partial F(f,t)}{\partial t} = F(\mathbb{T}(\xi)f,t) \tag{A.7}$$

with the initial condition

$$F(f,0) = f. \tag{A.8}$$

Similarly, using the formula

$$\frac{\partial}{\partial t} \det A(t) = \det A(t) \cdot \mathrm{Tr}\left(A(t)^{-1} \frac{\partial A(t)}{\partial t}\right), \tag{A.9}$$

we get a differential equation for $G(f,t)$.

$$\begin{aligned}\frac{\partial G(f,t)}{\partial t} &= \frac{\partial f(\Phi_t(\xi,x))}{\partial \Phi_t(\xi,x)} \frac{\partial \Phi_t(\xi,x)}{\partial t} \sqrt{\det \frac{\partial \Phi_t(\xi,x)}{\partial x}} \\ &\quad + \tfrac{1}{2} f(\Phi_t(\xi,x)) \sqrt{\det \frac{\partial \Phi_t(\xi,x)}{\partial x}} \frac{\partial x^\mu}{\partial \Phi_t^\nu(\xi,x)} \frac{\partial}{\partial t} \frac{\partial \Phi_t^\nu(\xi,x)}{\partial x^\mu} \\ &= \sqrt{\det \frac{\partial \Phi_t(\xi,x)}{\partial x}} \left\{ \frac{\partial f(\Phi_t(\xi,x))}{\partial \Phi_t(\xi,x)} \cdot \xi(\Phi_t(\xi,x)) \right. \\ &\quad \left. + \tfrac{1}{2} f(\Phi_t(\xi,x)) \frac{\partial x^\mu}{\partial \Phi_t^\nu(\xi,x)} \cdot \frac{\partial}{\partial x^\mu} \xi^\nu(\Phi_t(\xi,x)) \right\} \\ &= G(\mathbb{T}(\xi)f,t).\end{aligned} \tag{A.10}$$

The initial condition is

$$G(f,0) = f. \tag{A.11}$$

Hence F and G satisfy the same differential equation of first order and the same initial condition, and thus have to be equal.

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