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Representations of the Gauge Groups of Electrodynamics and General Relativity

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(12. XI. 71)

Abstract. A semidirect product of the gauge groups of Electrodynamics and General Relativity is determined and unitarily represented on a Hilbertspace of the type $\mathfrak{L}_2(\mathscr{S}',\mu)$.

1. Introduction

Electrodynamics and General Relativity, being rest-mass zero theories of vector and tensor fields, enjoy special symmetries governed by gauge groups. Some implications of these symmetries have been intensively studied [1]. In fact a linearized Lorentz-covariant theory of gravitation gets uniquely promoted to Einstein's General Relativity with the help of the gauge group [2]. The generators of the gauge groups of Electrodynamics and General Relativity form a nuclear Lie algebra that we show is similar to a current algebra [3]. The Gel'fand-Vilenkin formalism [4] then gives representations of the integrated group, being the semidirect product $\mathscr{S}(\mathbb{R}^4)$ \wedge diff $_{\mathscr{S}}(\mathbb{R}^4)$, where $\mathscr{S}(\mathbb{R}^4)$ is the gauge group for Electrodynamics and diff $_{\mathscr{S}}(\mathbb{R}^4)$ the gauge group for General Relativity. The semidirect product structure corresponds to the Klein-Kaluza formalism [5]. The representation spaces are of the type $\mathfrak{L}_2(\mathscr{S}',\mu)$, with μ a cylindrical measure on \mathscr{S}' . We might be enlightened in the physics of these representations by current investigations into a Wightman formalism for Quantum Electrodynamics and for the Theory of Gravitation [6].

2. The Gauge Lie Algebra for Electrodynamics and General Relativity

Let A_{μ} denote the potential in Electrodynamics and $g_{\mu\nu}$ the potential in General Relativity, $\mu,\nu=1,2,3,4$. Let $\phi\in\mathcal{S}(\mathbb{R}^4)$ be real valued, then the transformations $E(\phi)$ defined by

$$E(\phi)A_{\mu} = A_{\mu} + \phi_{,\mu} \tag{2.1}$$

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constitute the gauge group of Electrodynamics. The Lie algebra of the gauge group in General Relativity [2] originates from infinitesimal coordinate transformations

$$T(\xi)x^{\mu} = x^{\mu} - \xi^{\mu}(x),$$
 (2.2)

where ξ is a real valued vector field $\{\xi^{\mu}\}$ with $\xi^{\mu} \in \mathcal{S}(\mathbb{R}^4)$. The generators $T(\xi)$ transform A_{μ} and $g_{\mu\nu}$ as follows

$$T(\xi)A_{\mu} = A_{\mu} + A_{\alpha} \, \xi^{\alpha}_{,\mu} + \xi^{\alpha} \, A_{\mu,\alpha} \tag{2.3}$$

$$T(\xi)g_{\mu\nu} = g_{\mu\nu} + g_{\mu\beta}\,\xi^{\beta}_{,\nu} + g_{\alpha\nu}\,\xi^{\alpha}_{,\mu} + \xi^{\alpha}\,g_{\mu\nu,\alpha}. \tag{2.4}$$

The group theoretical commutators lead to the following Lie algebra $\mathfrak L$

$$[E(\phi), E(\psi)] = 0 \tag{2.5}$$

$$[E(\phi), T(\xi)] = -E(\xi \cdot \operatorname{grad} \phi)$$
(2.6)

$$[T(\xi), T(\eta)] = T([\xi, \eta]),$$
 (2.7)

where

$$[\xi, \eta]^{\mu} = \eta^{\mu}_{,\nu} \xi^{\nu} - \xi^{\mu}_{,\nu} \eta^{\nu} \tag{2.8}$$

This Lie algebra is similar to the charge-current algebra of a nonrelativistic field theory, which has been discussed by G. A. Goldin [3]. Hence the Gel'fand-Vilenkin formalism for nuclear Lie groups [4] will give us unitary representations of a gauge group G, whose Lie algebra is \mathfrak{L} .

3. Representations of $\mathfrak L$ on the Field Algebra

Recall that the Field algebra \mathfrak{U} , as a topological vector space, is given by the topological direct sum $\mathfrak{U} = \bigoplus_{n=0}^{\infty} \mathscr{S}^n(\mathbb{R}^4)$, where $\mathscr{S}^0(\mathbb{R}^4) = \mathbb{C}$ [7]. We then get the following representation for the generators $E(\phi)$ and $T(\xi)$:

$$(\mathbb{E}(\phi)f)_{n}(x_{1},\ldots,x_{n}) = \sum_{k=1}^{n} \phi(x_{k})f_{n}(x_{1},\ldots,x_{n})$$
(3.1)

$$(\mathbb{T}(\xi)f)_{n}(x_{1},\ldots,x_{n}) = \sum_{k=1}^{n} \left\{ \xi^{\mu}(x_{k}) \frac{\partial}{\partial x_{k}^{\mu}} f_{n}(x_{1},\ldots,x_{n}) + \frac{1}{2} \frac{\partial}{\partial x_{k}^{\mu}} \xi^{\mu}(x_{k}) \cdot f_{n}(x_{1},\ldots,x_{n}) \right\}$$

$$(3.2)$$

It is readily found that $\mathbb{T}(\xi)$ is antihermitian with respect to the \mathfrak{Q}_2 -inner product on $\mathscr{S}^n(\mathbb{R}^4)$, and satisfies the commutation relation (2.7). $\mathbb{E}(\phi)$ is hermitian and satisfies (2.5) and (2.6).

4. The Gauge Group G

We first look at the one-parameter subgroup of G of the form

$$(e^{t\mathbb{T}(\xi)}f)_n(x_1,\ldots,x_n) \equiv (V(\Phi_t(\xi,\cdot))f)_n(x_1,\ldots,x_n). \tag{4.1}$$

Lemma 4.1. $(V(\Phi_t(\xi, \cdot))f)_n(x_1, ..., x_n)$

$$= f_n\left(\Phi_t\left(\xi, x_1\right), \ldots, \Phi_t\left(\xi, x_n\right)\right) \prod_{k=1}^n \sqrt{\det \frac{\partial \Phi_t\left(\xi, x_k\right)}{\partial x_k}}, \tag{4.2}$$

where $\Phi_t(\xi,\cdot)$ is the flow generated by the vector field $\xi(x)$, i.e.

$$\frac{d}{dt}\Phi_{t}\left(\xi,x\right) = \xi\left(\Phi_{t}\left(\xi,x\right)\right), \quad \Phi_{0}\left(\xi,x\right) = x. \tag{4.3}$$

Proof: See Appendix A.

Since ξ is an \mathscr{S} -vector field, the corresponding flow $\Phi_t(\xi,x)$ is then C_∞ in x for all t. Let $\mathrm{diff}_{\mathscr{S}}(\mathbb{R}^4)$ stand for the C_∞ -diffeomorphisms on \mathbb{R}^4 , generated by the flows $\Phi_t(\xi,x)$. The composition of flows turns $\mathrm{diff}_{\mathscr{S}}(\mathbb{R}^4)$ into a group, the gauge group for General Relativity. Hence on the Field algebra \mathfrak{U} we have for $\Phi \in \mathrm{diff}_{\mathscr{S}}(\mathbb{R}^4)$

$$(V(\Phi)f)_n(x_1,\ldots,x_n) = f_n(\Phi(x_1),\ldots,\Phi(x_n)) \prod_{k=1}^n \sqrt{\det \frac{\partial \Phi(x_k)}{\partial x_k}}.$$
 (4.4)

The gauge group for Electrodynamics is $\mathscr{S}(\mathbb{R}^4)$ under addition and its unitary representation on \mathfrak{U} is given by

$$(e^{i\mathbb{E}(\phi)}f)_n(x_1,\ldots,x_n) \equiv (U(\phi)f)_n(x_1,\ldots,x_n)$$
(4.5)

$$(U(\phi)f)_n(x_1,\ldots,x_n) = e^{i\sum_{k=1}^n \phi(x_k)} f_n(x_1,\ldots,x_n).$$
 (4.6)

The full group is now given by the semidirect product

$$G = \mathcal{S}(\mathbb{R}^4) \wedge \operatorname{diff}_{\mathscr{S}}(\mathbb{R}^4) \tag{4.7}$$

with the semidirect product map

$$\mathscr{S}(\mathbb{R}^4) \times \operatorname{diff}_{\mathscr{C}}(\mathbb{R}^4) \to \mathscr{S}(\mathbb{R}^4) \tag{4.8}$$

$$(\phi, \Phi)(x) = (\phi \circ \Phi)(x) = \phi(\Phi(x)). \tag{4.9}$$

The multiplication law of G thus is

$$(\phi_1 \wedge \Phi_1)(\phi_2 \wedge \Phi_2) = (\phi_1 + \phi_2 \circ \Phi_1) \wedge \Phi_2 \circ \Phi_1. \tag{4.10}$$

For a unitary representation this reads

$$U(\phi_1)V(\Phi_1)U(\phi_2)V(\Phi_2) = U(\phi_1 + \phi_2 \circ \Phi_1)V(\Phi_2 \circ \Phi_1). \tag{4.11}$$

The gauge group G inherits a nuclear topology from $\mathcal{S}(\mathbb{R}^4)$, and \mathcal{S} is a normal subgroup of G. The operators $U(\phi)$ and $V(\Phi)$ are unitary on the Fockspace \mathfrak{H}_f which is the completion of the field algebra \mathfrak{U} in the norm

$$||f||^2 = \sum_{k=1}^{\infty} ||f_n||^2 \tag{4.12}$$

$$||f_n||^2 = \int \overline{f_n(x_1, \dots, x_n)} f_n(x_1, \dots, x_n) dx_1 \dots dx_n$$
 (4.13)

5. Representations of the Gauge Group G

Here we give a summary of the Gel'fand-Vilenkin method [4] as applied to G. For details see Ref. [3]. A continuous unitary representation $U(\phi)$ of the normal subgroup $\mathscr{S}(\mathbb{R}^4) \subset G$ on a Hilbertspace \mathfrak{H} with cyclic vector Ω gives us a functional $L(\phi)$, defined by

$$L(\phi) = (\Omega, U(\phi)\Omega) \tag{5.1}$$

 $L(\phi)$ has the following properties

1.
$$L$$
 is continuous on $\mathcal{S}(\mathbb{R}^4)$ (5.2)

2.
$$L(0) = 1$$
 (5.3)

3.
$$\sum_{k,l=1}^{N} \bar{c}_k c_l L(\phi_k - \phi_l) \ge 0 \quad \text{for any complex numbers } c_1, \dots, c_N$$
 (5.4)

and hence by Bochner's Theorem there exists a unique cylinder measure μ on \mathscr{S}' such that

$$L(\phi) = \int_{\mathscr{S}'} e^{i(T,\phi)} d\mu (T). \quad T \in \mathscr{S}'(\mathbb{R}^4)$$
 (5.5)

A strongly continuous representation of \mathcal{S} in $\mathfrak{H} = \mathfrak{L}_2(\mathcal{S}', \mu)$ is thus given by

$$U(\phi)\Psi(T) = e^{i(T,\phi)}\Psi(T)$$
(5.6)

and the cyclic vector Ω is realized by the unit function on \mathscr{G}' . For $\Phi \in \mathrm{diff}_{\mathscr{S}}(\mathbb{R}^4)$ let $\Phi^* : \mathscr{S}'(\mathbb{R}^4) \to \mathscr{S}'(\mathbb{R}^4)$ be defined by

$$(\Phi^* T, \phi) = (T, \phi \circ \Phi) \tag{5.7}$$

and define a transformed measure μ^{Φ^*} on \mathscr{S}' by

$$\mu^{\Phi^*}(T) = \mu(\Phi^*T). \tag{5.8}$$

The group $\mathrm{diff}_{\mathscr{S}}(\mathbb{R}^4)$ is then represented on $\mathfrak{L}_2(\mathscr{S}',\mu)$ by

$$V(\Phi) \Psi(T) = \chi(\Phi, T) \Psi(\Phi^* T) \sqrt{\frac{d\mu^{\Phi^*}(T)}{d\mu(T)}}$$
(5.9)

where $d\mu^{\Phi^*}/d\mu$ is the Radon-Nikodym derivative and $\chi(\Phi, T)$ a complex valued function of modulus one, satisfying

$$\chi(\Phi_2, T)\chi(\Phi_1, \Phi_2 T) = \chi(\Phi_1 \circ \Phi_2, T). \tag{5.10}$$

The measure μ on \mathscr{S}' is quasiinvariant under diff_{\mathscr{S}}(\mathbb{R}^4), i.e. μ and μ^{Φ^*} have the same set of measure zero. Thus the expectation functional $L(\phi)$ defines a representation of G up to a phase function.

Remarks:

- 1. The Fock representation of G on \mathfrak{H}_F corresponds to a Gaussian measure. In the n-particle space $\mathfrak{H}_F^{(n)} \subset \mathfrak{H}_F$, μ is concentrated on the set $T = \{T_{x_1} + \cdots + T_{x_n}, x_i \neq x_k\}$, where $(T_x, \phi) = \phi(x)$. $d\mu(T_{x_1} + \cdots + T_{x_n}) = \pi^{-2n} e^{-\|x_1\|^2 \cdots \|x_n\|^2} dx_1 \dots dx_n$. Observe that this representation is on a Fockspace of scalar functions.
- 2. For the recovery of the infinitesimal generators $\mathbb{E}(\phi)$ and $\mathbb{T}(\xi)$, Goldin [3] gives sufficient conditions, expressed as properties of the measure μ .
- 3. Here we do not investigate if the Gel'fand-Vilenkin representations of G are physical. If they are, then operators commuting with the $U(\phi)$ are candidates for observables in Quantum Electrodynamics and operators commuting with $V(\Phi)$ are candidates for observables in a quantum theory of gravitation.

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Appendix A

We want to show that

$$(e^{t \operatorname{T}(\xi)} f)_{n}(x_{1}, \ldots, x_{n}) = f_{n}(\Phi_{t}(\xi, x_{1}), \ldots, \Phi_{t}(\xi, x_{n})) \prod_{k=1}^{n} \sqrt{\det \frac{\partial \Phi_{t}(\xi, x_{k})}{\partial x_{k}}}$$
(A.1)

where

$$(\mathbb{T}(\xi)f)_{n}(x_{1},\ldots,x_{n}) = \sum_{k=1}^{n} \left\{ \xi^{\mu}(x_{k}) \frac{\partial}{\partial x_{k}^{\mu}} + \frac{1}{2} \frac{\partial \xi^{\mu}(x_{k})}{\partial x_{k}^{\mu}} \right\} f_{n}(x_{1},\ldots,x_{n}). \tag{A.2}$$

It suffices to give the proof for one variable x, i.e.

$$e^{t T(\xi)} f(x) = f(\Phi_t(\xi, x)) \sqrt{\det \frac{\partial \Phi_t(\xi, x)}{\partial x}}$$
(A.3)

with

$$\mathbb{T}(\xi)f(x) = \left\{ \xi^{\mu}(x) \frac{\partial}{\partial x^{\mu}} + \frac{1}{2} \frac{\partial \xi^{\mu}(x)}{\partial x^{\mu}} \right\} f(x). \tag{A.4}$$

For fixed ξ and x let

$$F(f,t) = e^{t \operatorname{T}(\xi)} f(x) \tag{A.5}$$

$$G(f,t) = \sqrt{\det \frac{\partial \Phi_t(\xi, x)}{\partial x}} f(\Phi_t(\xi, x)). \tag{A.6}$$

Then F(f,t) satisfies the following differential equation

$$\frac{\partial F(f,t)}{\partial t} = F(\mathbb{T}(\xi)f,t) \tag{A.7}$$

with the initial condition

$$F(f,0) = f. (A.8)$$

Similarly, using the formula

$$\frac{\partial}{\partial t} \det A(t) = \det A(t) \cdot \operatorname{Tr}\left(A(t)^{-1} \frac{\partial A(t)}{\partial t}\right), \tag{A.9}$$

we get a differential equation for G(f,t).

$$\frac{\partial G(f,t)}{\partial t} = \frac{\partial f(\Phi_{t}(\xi,x))}{\partial \Phi_{t}(\xi,x)} \frac{\partial \Phi_{t}(\xi,x)}{\partial t} \sqrt{\det \frac{\partial \Phi_{t}(\xi,x)}{\partial x}} \\
+ \frac{1}{2} f(\Phi_{t}(\xi,x)) \sqrt{\det \frac{\partial \Phi_{t}(\xi,x)}{\partial x}} \frac{\partial x^{\mu}}{\partial \Phi_{t}^{\nu}(\xi,x)} \frac{\partial}{\partial t} \frac{\partial \Phi_{t}^{\nu}(\xi,x)}{\partial x^{\mu}} \\
= \sqrt{\det \frac{\partial \Phi_{t}(\xi,x)}{\partial x}} \left\{ \frac{\partial f(\Phi_{t}(\xi,x))}{\partial \Phi_{t}(\xi,x)} \cdot \xi(\Phi_{t}(\xi,x)) + \frac{1}{2} f(\Phi_{t}(\xi,x)) \frac{\partial x^{\mu}}{\partial \Phi_{t}^{\nu}(\xi,x)} \cdot \frac{\partial}{\partial x^{\mu}} \xi^{\nu} (\Phi_{t}(\xi,x)) \right\} \\
= G(\mathbb{T}(\xi)f,t). \tag{A.10}$$

The initial condition is

$$G(f,0) = f. (A.11)$$

Hence F and G satisfy the same differential equation of first order and the same initial condition, and thus have to be equal.

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