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Representations of the Gauge Groups of Electrodynamics and General Relativity

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Abstract. A semidirect product of the gauge groups of Electrodynamics and General Relativity is determined and unitarily represented on a Hilbertspace of the type $\mathfrak{L}_2(\mathcal{S}', \mu)$.

1. Introduction

Electrodynamics and General Relativity, being rest-mass zero theories of vector and tensor fields, enjoy special symmetries governed by gauge groups. Some implications of these symmetries have been intensively studied [1]. In fact a linearized Lorentz-covariant theory of gravitation gets uniquely promoted to Einstein's General Relativity with the help of the gauge group [2]. The generators of the gauge groups of Electrodynamics and General Relativity form a nuclear Lie algebra that we show is similar to a current algebra [3]. The Gel'fand-Vilenkin formalism [4] then gives representations of the integrated group, being the semidirect product $\mathcal{S}(\mathbb{R}^4) \wedge \text{diff}_{\mathcal{S}}(\mathbb{R}^4)$, where $\mathcal{S}(\mathbb{R}^4)$ is the gauge group for Electrodynamics and $\text{diff}_{\mathcal{S}}(\mathbb{R}^4)$ the gauge group for General Relativity. The semidirect product structure corresponds to the Klein-Kaluza formalism [5]. The representation spaces are of the type $\mathfrak{L}_2(\mathcal{S}', \mu)$, with μ a cylindrical measure on \mathcal{S}' . We might be enlightened in the physics of these representations by current investigations into a Wightman formalism for Quantum Electrodynamics and for the Theory of Gravitation [6].

2. The Gauge Lie Algebra for Electrodynamics and General Relativity

Let A_μ denote the potential in Electrodynamics and $g_{\mu\nu}$ the potential in General Relativity, $\mu, \nu = 1, 2, 3, 4$. Let $\phi \in \mathcal{S}(\mathbb{R}^4)$ be real valued, then the transformations $E(\phi)$ defined by

$$E(\phi)A_\mu = A_\mu + \phi_{,\mu} \quad (2.1)$$

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constitute the gauge group of Electrodynamics. The Lie algebra of the gauge group in General Relativity [2] originates from infinitesimal coordinate transformations

$$T(\xi)x^\mu = x^\mu - \xi^\mu(x), \quad (2.2)$$

where ξ is a real valued vector field $\{\xi^\mu\}$ with $\xi^\mu \in \mathcal{S}(\mathbb{R}^4)$. The generators $T(\xi)$ transform A_μ and $g_{\mu\nu}$ as follows

$$T(\xi)A_\mu = A_\mu + A_\alpha \xi_{,\mu}^\alpha + \xi^\alpha A_{\mu,\alpha} \quad (2.3)$$

$$T(\xi)g_{\mu\nu} = g_{\mu\nu} + g_{\mu\beta} \xi_{,\nu}^\beta + g_{\alpha\nu} \xi_{,\mu}^\alpha + \xi^\alpha g_{\mu\nu,\alpha}. \quad (2.4)$$

The group theoretical commutators lead to the following Lie algebra \mathfrak{L}

$$[E(\phi), E(\psi)] = 0 \quad (2.5)$$

$$[E(\phi), T(\xi)] = -E(\xi \cdot \text{grad } \phi) \quad (2.6)$$

$$[T(\xi), T(\eta)] = T([\xi, \eta]), \quad (2.7)$$

where

$$[\xi, \eta]^\mu = \eta_{,\nu}^\mu \xi^\nu - \xi_{,\nu}^\mu \eta^\nu \quad (2.8)$$

This Lie algebra is similar to the charge-current algebra of a nonrelativistic field theory, which has been discussed by G. A. Goldin [3]. Hence the Gel'fand-Vilenkin formalism for nuclear Lie groups [4] will give us unitary representations of a gauge group G , whose Lie algebra is \mathfrak{L} .

3. Representations of \mathfrak{L} on the Field Algebra

Recall that the Field algebra \mathfrak{U} , as a topological vector space, is given by the topological direct sum $\mathfrak{U} = \bigoplus_{n=0}^{\infty} \mathcal{S}^n(\mathbb{R}^4)$, where $\mathcal{S}^0(\mathbb{R}^4) = \mathbb{C}$ [7]. We then get the following representation for the generators $E(\phi)$ and $T(\xi)$:

$$(\mathbb{E}(\phi)f)_n(x_1, \dots, x_n) = \sum_{k=1}^n \phi(x_k) f_n(x_1, \dots, x_n) \quad (3.1)$$

$$\begin{aligned} (\mathbb{T}(\xi)f)_n(x_1, \dots, x_n) = \sum_{k=1}^n \left\{ \xi^\mu(x_k) \frac{\partial}{\partial x_k^\mu} f_n(x_1, \dots, x_n) \right. \\ \left. + \frac{1}{2} \frac{\partial}{\partial x_k^\mu} \xi^\mu(x_k) \cdot f_n(x_1, \dots, x_n) \right\} \end{aligned} \quad (3.2)$$

It is readily found that $\mathbb{T}(\xi)$ is antihermitian with respect to the \mathfrak{L}_2 -inner product on $\mathcal{S}^n(\mathbb{R}^4)$, and satisfies the commutation relation (2.7). $\mathbb{E}(\phi)$ is hermitian and satisfies (2.5) and (2.6).

4. The Gauge Group G

We first look at the one-parameter subgroup of G of the form

$$(e^{i\mathbb{T}(\xi)}f)_n(x_1, \dots, x_n) \equiv (V(\Phi_t(\xi, \cdot))f)_n(x_1, \dots, x_n). \quad (4.1)$$

Lemma 4.1. $(V(\Phi_t(\xi, \cdot))f)_n(x_1, \dots, x_n)$

$$= f_n(\Phi_t(\xi, x_1), \dots, \Phi_t(\xi, x_n)) \prod_{k=1}^n \sqrt{\det \frac{\partial \Phi_t(\xi, x_k)}{\partial x_k}}, \quad (4.2)$$

where $\Phi_t(\xi, \cdot)$ is the flow generated by the vector field $\xi(x)$, i.e.

$$\frac{d}{dt} \Phi_t(\xi, x) = \xi(\Phi_t(\xi, x)), \quad \Phi_0(\xi, x) = x. \quad (4.3)$$

Proof: See Appendix A.

Since ξ is an \mathcal{S} -vector field, the corresponding flow $\Phi_t(\xi, x)$ is then C_∞ in x for all t . Let $\text{diff}_{\mathcal{S}}(\mathbb{R}^4)$ stand for the C_∞ -diffeomorphisms on \mathbb{R}^4 , generated by the flows $\Phi_t(\xi, x)$. The composition of flows turns $\text{diff}_{\mathcal{S}}(\mathbb{R}^4)$ into a group, the gauge group for General Relativity. Hence on the Field algebra \mathfrak{U} we have for $\Phi \in \text{diff}_{\mathcal{S}}(\mathbb{R}^4)$

$$(V(\Phi)f)_n(x_1, \dots, x_n) = f_n(\Phi(x_1), \dots, \Phi(x_n)) \prod_{k=1}^n \sqrt{\det \frac{\partial \Phi(x_k)}{\partial x_k}}. \quad (4.4)$$

The gauge group for Electrodynamics is $\mathcal{S}(\mathbb{R}^4)$ under addition and its unitary representation on \mathfrak{U} is given by

$$(e^{iE(\phi)}f)_n(x_1, \dots, x_n) \equiv (U(\phi)f)_n(x_1, \dots, x_n) \quad (4.5)$$

$$(U(\phi)f)_n(x_1, \dots, x_n) = e^{i \sum_{k=1}^n \phi(x_k)} f_n(x_1, \dots, x_n). \quad (4.6)$$

The full group is now given by the semidirect product

$$G = \mathcal{S}(\mathbb{R}^4) \ltimes \text{diff}_{\mathcal{S}}(\mathbb{R}^4) \quad (4.7)$$

with the semidirect product map

$$\mathcal{S}(\mathbb{R}^4) \times \text{diff}_{\mathcal{S}}(\mathbb{R}^4) \rightarrow \mathcal{S}(\mathbb{R}^4) \quad (4.8)$$

$$(\phi, \Phi)(x) = (\phi \circ \Phi)(x) = \phi(\Phi(x)). \quad (4.9)$$

The multiplication law of G thus is

$$(\phi_1 \wedge \Phi_1)(\phi_2 \wedge \Phi_2) = (\phi_1 + \phi_2 \circ \Phi_1) \wedge \Phi_2 \circ \Phi_1. \quad (4.10)$$

For a unitary representation this reads

$$U(\phi_1)V(\Phi_1)U(\phi_2)V(\Phi_2) = U(\phi_1 + \phi_2 \circ \Phi_1)V(\Phi_2 \circ \Phi_1). \quad (4.11)$$

The gauge group G inherits a nuclear topology from $\mathcal{S}(\mathbb{R}^4)$, and \mathcal{S} is a normal subgroup of G . The operators $U(\phi)$ and $V(\Phi)$ are unitary on the Fockspace \mathfrak{H}_f which is the completion of the field algebra \mathfrak{U} in the norm

$$\|f\|^2 = \sum_{k=1}^{\infty} \|f_k\|^2 \quad (4.12)$$

$$\|f_n\|^2 = \int \overline{f_n(x_1, \dots, x_n)} f_n(x_1, \dots, x_n) dx_1 \dots dx_n \quad (4.13)$$

5. Representations of the Gauge Group G

Here we give a summary of the Gel'fand-Vilenkin method [4] as applied to G . For details see Ref. [3]. A continuous unitary representation $U(\phi)$ of the normal subgroup $\mathcal{S}(\mathbb{R}^4) \subset G$ on a Hilbertspace \mathfrak{H} with cyclic vector Ω gives us a functional $L(\phi)$, defined by

$$L(\phi) = (\Omega, U(\phi)\Omega) \quad (5.1)$$

$L(\phi)$ has the following properties

$$1. \quad L \text{ is continuous on } \mathcal{S}(\mathbb{R}^4) \quad (5.2)$$

$$2. \quad L(0) = 1 \quad (5.3)$$

$$3. \quad \sum_{k,l=1}^N \bar{c}_k c_l L(\phi_k - \phi_l) \geq 0 \quad \text{for any complex numbers } c_1, \dots, c_N \quad (5.4)$$

and hence by Bochner's Theorem there exists a unique cylinder measure μ on \mathcal{S}' such that

$$L(\phi) = \int_{\mathcal{S}'} e^{i(\tau, \phi)} d\mu(T). \quad T \in \mathcal{S}'(\mathbb{R}^4) \quad (5.5)$$

A strongly continuous representation of \mathcal{S} in $\mathfrak{H} = \mathfrak{L}_2(\mathcal{S}', \mu)$ is thus given by

$$U(\phi)\Psi(T) = e^{i(\tau, \phi)} \Psi(T) \quad (5.6)$$

and the cyclic vector Ω is realized by the unit function on \mathcal{S}' . For $\Phi \in \text{diff}_{\mathcal{S}}(\mathbb{R}^4)$ let $\Phi^*: \mathcal{S}'(\mathbb{R}^4) \rightarrow \mathcal{S}'(\mathbb{R}^4)$ be defined by

$$(\Phi^* T, \phi) = (T, \phi \circ \Phi) \quad (5.7)$$

and define a transformed measure μ^{Φ^*} on \mathcal{S}' by

$$\mu^{\Phi^*}(T) = \mu(\Phi^* T). \quad (5.8)$$

The group $\text{diff}_{\mathcal{S}}(\mathbb{R}^4)$ is then represented on $\mathfrak{L}_2(\mathcal{S}', \mu)$ by

$$V(\Phi)\Psi(T) = \chi(\Phi, T) \Psi(\Phi^* T) \sqrt{\frac{d\mu^{\Phi^*}(T)}{d\mu(T)}} \quad (5.9)$$

where $d\mu^{\Phi^*}/d\mu$ is the Radon-Nikodym derivative and $\chi(\Phi, T)$ a complex valued function of modulus one, satisfying

$$\chi(\Phi_2, T)\chi(\Phi_1, \Phi_2 T) = \chi(\Phi_1 \circ \Phi_2, T). \quad (5.10)$$

The measure μ on \mathcal{S}' is quasiinvariant under $\text{diff}_{\mathcal{S}}(\mathbb{R}^4)$, i.e. μ and μ^{Φ^*} have the same set of measure zero. Thus the expectation functional $L(\phi)$ defines a representation of G up to a phase function.

Remarks:

1. The Fock representation of G on \mathfrak{H}_F corresponds to a Gaussian measure. In the n -particle space $\mathfrak{H}_F^{(n)} \subset \mathfrak{H}_F$, μ is concentrated on the set $T = \{T_{x_1} + \cdots + T_{x_n}, x_i \neq x_k\}$, where $(T_x, \phi) = \phi(x)$. $d\mu(T_{x_1} + \cdots + T_{x_n}) = \pi^{-2n} e^{-\|x_1\|^2 - \cdots - \|x_n\|^2} dx_1 \dots dx_n$. Observe that this representation is on a Fockspace of scalar functions.
2. For the recovery of the infinitesimal generators $\mathbb{E}(\phi)$ and $\mathbb{T}(\xi)$, Goldin [3] gives sufficient conditions, expressed as properties of the measure μ .
3. Here we do not investigate if the Gel'fand-Vilenkin representations of G are physical. If they are, then operators commuting with the $U(\phi)$ are candidates for observables in Quantum Electrodynamics and operators commuting with $V(\Phi)$ are candidates for observables in a quantum theory of gravitation.

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Appendix A

We want to show that

$$(e^{t\mathbb{T}(\xi)} f)_n(x_1, \dots, x_n) = f_n(\Phi_t(\xi, x_1), \dots, \Phi_t(\xi, x_n)) \prod_{k=1}^n \sqrt{\det \frac{\partial \Phi_t(\xi, x_k)}{\partial x_k}} \quad (\text{A.1})$$

where

$$(\mathbb{T}(\xi) f)_n(x_1, \dots, x_n) = \sum_{k=1}^n \left\{ \xi^\mu(x_k) \frac{\partial}{\partial x_k^\mu} + \frac{1}{2} \frac{\partial \xi^\mu(x_k)}{\partial x_k^\mu} \right\} f_n(x_1, \dots, x_n). \quad (\text{A.2})$$

It suffices to give the proof for one variable x , i.e.

$$e^{t\mathbb{T}(\xi)} f(x) = f(\Phi_t(\xi, x)) \sqrt{\det \frac{\partial \Phi_t(\xi, x)}{\partial x}} \quad (\text{A.3})$$

with

$$\mathbb{T}(\xi) f(x) = \left\{ \xi^\mu(x) \frac{\partial}{\partial x^\mu} + \frac{1}{2} \frac{\partial \xi^\mu(x)}{\partial x^\mu} \right\} f(x). \quad (\text{A.4})$$

For fixed ξ and x let

$$F(f, t) = e^{t\mathbb{T}(\xi)} f(x) \quad (\text{A.5})$$

$$G(f, t) = \sqrt{\det \frac{\partial \Phi_t(\xi, x)}{\partial x}} f(\Phi_t(\xi, x)). \quad (\text{A.6})$$

Then $F(f, t)$ satisfies the following differential equation

$$\frac{\partial F(f, t)}{\partial t} = F(\mathbb{T}(\xi)f, t) \quad (\text{A.7})$$

with the initial condition

$$F(f, 0) = f. \quad (\text{A.8})$$

Similarly, using the formula

$$\frac{\partial}{\partial t} \det A(t) = \det A(t) \cdot \text{Tr} \left(A(t)^{-1} \frac{\partial A(t)}{\partial t} \right), \quad (\text{A.9})$$

we get a differential equation for $G(f, t)$.

$$\begin{aligned} \frac{\partial G(f, t)}{\partial t} &= \frac{\partial f(\Phi_t(\xi, x))}{\partial \Phi_t(\xi, x)} \frac{\partial \Phi_t(\xi, x)}{\partial t} \sqrt{\det \frac{\partial \Phi_t(\xi, x)}{\partial x}} \\ &\quad + \frac{1}{2} f(\Phi_t(\xi, x)) \sqrt{\det \frac{\partial \Phi_t(\xi, x)}{\partial x}} \frac{\partial x^\mu}{\partial \Phi_t^\nu(\xi, x)} \frac{\partial}{\partial t} \frac{\partial \Phi_t^\nu(\xi, x)}{\partial x^\mu} \\ &= \sqrt{\det \frac{\partial \Phi_t(\xi, x)}{\partial x}} \left\{ \frac{\partial f(\Phi_t(\xi, x))}{\partial \Phi_t(\xi, x)} \cdot \xi(\Phi_t(\xi, x)) \right. \\ &\quad \left. + \frac{1}{2} f(\Phi_t(\xi, x)) \frac{\partial x^\mu}{\partial \Phi_t^\nu(\xi, x)} \cdot \frac{\partial}{\partial x^\mu} \xi^\nu(\Phi_t(\xi, x)) \right\} \\ &= G(\mathbb{T}(\xi)f, t). \end{aligned} \quad (\text{A.10})$$

The initial condition is

$$G(f, 0) = f. \quad (\text{A.11})$$

Hence F and G satisfy the same differential equation of first order and the same initial condition, and thus have to be equal.

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