

Zeitschrift: Helvetica Physica Acta

Band: 44 (1971)

Heft: 7

Artikel: Perturbations and non-normalizable Eigenvectors

Autor: Faris, William C.

DOI: <https://doi.org/10.5169/seals-114319>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 03.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Perturbations and Non-Normalizable Eigenvectors

by William G. Faris

Battelle Institute, Advanced Studies Center, Geneva, Switzerland

(10. V. 71)

Abstract. A spectral representation of a self-adjoint operator acting in a Hilbert space is given by eigenvectors of an extension of the operator to a suitable space containing the original Hilbert space. A perturbation argument shows the extended operator has no eigenvalues that do not belong to the spectrum of the original operator. The abstract result is applied to Schrödinger operators $-\Delta + V$.

1. Introduction

The spectral theory of self-adjoint operators may be treated without ever mentioning non-normalizable eigenvectors. In fact, the spectral theorem may be stated as follows [1].

Theorem. Let A be a self-adjoint operator acting in the Hilbert space H . Then A is unitarily equivalent to a multiplication operator. That is, there is a Hilbert space $L^2(M, \mu)$, a real measurable function α on M , and a unitary operator $U:H \rightarrow L^2(M, \mu)$ such that f is in the domain of A if and only if $\alpha U f$ is in $L^2(M, \mu)$, and such that $U A f = \alpha U f$.

Here μ is a positive measure and $L^2(M, \mu)$ is the Hilbert space of all measurable complex functions h on M such that $\int |h(p)|^2 d\mu(p) < \infty$. (Functions which are equal almost everywhere are identified.) Such a unitary equivalence of A with a multiplication operator is called a spectral representation of A . The spectral theorem asserts the existence but not the uniqueness of spectral representations.

Let ϕ be a Borel measurable function defined on the real numbers. Then $\phi(A)$ may be defined by the spectral theorem as the operator acting in H which is unitarily equivalent to multiplication by $\phi(\alpha)$ [1]. This definition is independent of the spectral representation.

Non-normalizable eigenvectors enter the picture only when one attempts to describe the form of the unitary operator U . A suitable space K^* containing H is chosen. Vectors in K^* which are not in H are called non-normalizable vectors. (The norm under consideration is that of H , of course.) The self-adjoint operator A acting in H has an extension to an operator \tilde{A} acting in K^* . When A has continuous spectrum U is given in terms of non-normalizable eigenvectors of \tilde{A} .

G. I. Kac has given an elegant criterion for showing that K^* is large enough to contain all the non-normalizable eigenvectors necessary for any spectral representation [2]. It is desirable to choose K^* as small as possible consistent with this requirement, since this allows the widest class of perturbations and gives the sharpest estimates on non-normalizable eigenvectors. Also, if K^* is reasonably small, the eigenvalues of \tilde{A} may give a good idea of the spectrum of A . (In general, since \tilde{A} acting in K^* is not self-adjoint, it may even have non-real eigenvalues.) Here an abstract perturbation theory is developed to show that K^* does not contain unwanted non-normalizable eigenvectors of \tilde{A} . This is applied to Schrödinger operators $-\Delta + V$. In this case the results may be interpreted as estimates on growth at infinity of eigenfunctions of the Schrödinger operator. (Such results have also been obtained by partial differential equations methods [3].) Much stronger assumptions on the interaction would be needed in order to apply the theory of wave operators and scattering.

2. Non-Normalizable Eigenvectors

Let H be a Hilbert space. The inner product of g and f in H will be denoted $\langle g, f \rangle$. The convention adopted here is that the inner product is conjugate linear in the first variable and linear in the second variable. The norm of f in H is $\|f\| = \langle f, f \rangle^{1/2}$.

We wish to consider a situation where there is given another Hilbert space K which is a dense linear subspace of H . If f is an element of K , the norm of f as an element of K is written $\|f\|_K$. We shall assume that the injection of K into H is continuous. Thus there is a constant $c > 0$ such that $\|f\| \leq c \|f\|_K$.

Let K^* be the set of all bounded linear functions from K to the complex numbers.

Proposition 1. Let H be a Hilbert space. Let $K \subset H$ be another Hilbert space. Assume that K is dense in H and that the injection is continuous. To each g in H associate the linear function $f \rightarrow \langle g, f \rangle$ in K^* . Then this correspondence is injective, and H may be identified with a dense subspace of K^* , so that we have continuous inclusions of Hilbert spaces $K \subset H \subset K^*$.

Proof: If g is an element of H , then $|\langle g, f \rangle| \leq \|g\| \|f\| \leq c \|g\| \|f\|_K$, so the function which assigns to f in K the inner product $\langle g, f \rangle$ is in K^* . If the inner product $\langle g, f \rangle = 0$ for all f in K , then $g = 0$, since K is dense in H . Thus each element g in H determines a unique element of K^* . We identify each g in H with the corresponding element of K^* .

We wish to give K^* the structure of a Hilbert space in such a way that the injection of H into K^* is linear. If ψ is in K^* and f is in K , we write $\langle \psi, f \rangle$ for the value of ψ on f . If ψ_1 and ψ_2 are in K^* , we define $\psi_1 + \psi_2$ by $\langle \psi_1 + \psi_2, f \rangle = \langle \psi_1, f \rangle + \langle \psi_2, f \rangle$. If ψ is in K^* and a is a complex number, it is convenient to define the product of a with ψ to be given by $\langle a\psi, f \rangle = a^* \langle \psi, f \rangle$. With this convention, if ψ happens to be in H this coincides with scalar multiplication in H . With the definition $\|\psi\|_{K^*} = \sup\{|\langle \psi, f \rangle| : \|f\|_K \leq 1\}$, K^* becomes a Hilbert space.

Note that we have $\|g\|_{K^*} = \sup\{|\langle g, f \rangle| : \|f\|_K \leq 1\} \leq c \sup\{|\langle g, f \rangle| : \|f\| \leq 1\} = c \|g\|$. Hence the inclusion of H into K^* is continuous. It is also not hard to see that H is dense in K^* . This completes the proof.

Warning: Having identified $H \subset K^*$, it is no longer permissible to identify K^* with K .

Definition. Let H be a Hilbert space and let $K \subset H$ be another Hilbert space. Assume that K is dense in H and that the inclusion is continuous. Then the triple $K \subset H \subset K^*$ will be called a scale of Hilbert spaces.

From now on we shall assume that all Hilbert spaces under consideration have a countable orthonormal basis. (This is equivalent to their being separable metric spaces.) This allows us to consider only measure spaces which are σ -finite.

We now recall the theorem of Kac [2].

Theorem 1. Let A be a self-adjoint operator acting in H . Assume that $U: H \rightarrow L^2(M, \mu)$ is a unitary operator which gives a spectral representation of A . Let $K \subset H \subset K^*$ be a scale of Hilbert spaces. Assume that there is a Borel measurable function β which is bounded on the spectrum of A and which does not vanish on the spectrum of A such that $\beta(A)$ is a Hilbert-Schmidt operator from K to H . Then there is a function ψ from M to K^* such that for every f in K , $Uf(\rho) = \langle \psi(\rho), f \rangle$ for almost every ρ in M .

For the convenience of the reader we sketch a proof.

Proof: $U\beta(A) = \beta(\alpha) U: K \rightarrow L^2(M, \mu)$ is a Hilbert-Schmidt operator. Represent K as a space $L^2(N, \nu)$. Then there is an s in $L^2(M \times N, \mu \times \nu)$ such that for f in K , $\beta(\alpha(\rho)) Uf(\rho) = \int s(\rho, q) f(q) d\nu(q)$ for almost every ρ [4]. By Fubini's theorem, $s(\rho, q)$ is in $L^2(N, \nu)$ as a function of q for almost every ρ . Thus for these ρ we may define $\langle \psi(\rho), f \rangle = (1/\beta(\alpha(\rho))) \int s(\rho, q) f(q) d\nu(q)$. $\psi(\rho)$ is in K^* by the Schwarz inequality.

Remark. In practice the most useful choices of β are $\beta(a) = (a - z)^{-k}$ for some integer $k = 1, 2, 3, \dots$ and z not in the spectrum of A . Another possibility is $\beta(a) = 1$. In this case the condition is simply that the injection of K into H is Hilbert-Schmidt.

Note. If f and g are in H , and f is in the domain of A , we have

$$\langle g, Af \rangle = \int Ug(\rho)^* \alpha(\rho) Uf(\rho) d\mu(\rho).$$

In particular, under the conditions of Theorem 1, if f and g are in K , $\langle g, Af \rangle = \int \langle g, \psi(\rho) \rangle \alpha(\rho) \langle \psi(\rho), f \rangle d\mu(\rho)$. (In keeping with the usage in physics, we have written $\langle g, \psi(\rho) \rangle$ for $\langle \psi(\rho), g \rangle^*$.)

Definition. Let $K \subset H \subset K^*$ be a scale. Let A be a self-adjoint operator acting in H with domain D . Let $D_0 = \{f \text{ in } K \cap D : Af \text{ is in } K\}$. Assume D_0 is dense in K . Then the scale extension \tilde{A} of A is defined as the operator acting in K^* which is the adjoint of A restricted to D_0 .

Explicitly, if g is in K^* , g is in the domain of \tilde{A} if there is an h in K^* with $\langle h, f \rangle = \langle g, Af \rangle$ for all f in D_0 . Since D_0 is dense in K , h is uniquely determined and we set $\tilde{A}g = h$. \tilde{A} is clearly an extension of the original self-adjoint operator A .

Theorem 2 [2]. Let A be a self-adjoint operator acting in H . Let $K \subset H \subset K^*$. Let U be a spectral representation of A and assume that there is a function ψ from M to K^* such that $(Uf)(\rho) = \langle \psi(\rho), f \rangle$ for almost every ρ in M . Assume that the scale exten-

sion \tilde{A} of A is defined. Then $\psi(p)$ is in the domain of definition of \tilde{A} and $\tilde{A}\psi(p) = \alpha(p)\psi(p)$ for almost every p .

Proof: Let f be in D_0 . Then $\langle \psi(p), Af \rangle = UAf(p) = \alpha(p)Uf(p) = \langle \alpha(p)\psi(p), f \rangle$ for almost every p . Thus for any countable subset of D_0 , $\langle \psi(p), Af \rangle = \langle \alpha(p)\psi(p), f \rangle$ for f in the subset for almost every p . Now the graph of the operator A restricted to D_0 is a subspace of the direct sum of K with itself. Since a subspace of a separable metric space is separable, the graph is separable. Thus there is a countable subset of D_0 such that for each f in D_0 , there is a subsequence f_n in the subset with $f_n \rightarrow f$ and $Af_n \rightarrow Af$, in the norm of K . We conclude that $\langle \psi(p), Af \rangle = \langle \alpha(p)\psi(p), f \rangle$ for all f in D_0 for almost every p . In other words, for these p , $\tilde{A}\psi(p) = \alpha(p)\psi(p)$.

3. Perturbation Theory

Theorem 3. Let $K \subset H \subset K^*$ be a scale of Hilbert spaces. Let A be a self-adjoint operator acting in H . Assume that $(A - z)^{-1}$ is a Hilbert-Schmidt operator from K to H for some z not in the spectrum of A . Assume that B is a self-adjoint operator whose domain contains the domain of A and such that $A + B$ is self-adjoint with the same domain as A . Then if z is also not in the spectrum of $A + B$, $(A + B - z)^{-1}$ is a Hilbert-Schmidt operator from K to H .

Proof: The two resolvents are related by

$(A + B - z)^{-1} = [1 - (A + B - z)^{-1}B] (A - z)^{-1}$. Let T be the closure of $1 - (A - B - z)^{-1}B$. Then its adjoint $T^* = 1 - B(A + B - z^*)^{-1}$. Since $A + B$ has the same domain as A , T^* is defined on all of H . But any adjoint has a closed graph. So T^* is a bounded operator from H to H , by the closed graph theorem. Hence T is also bounded from H to H . The identity $(A + B - z)^{-1} = T(A - z)^{-1}$ thus exhibits $(A + B - z)^{-1}$ as a Hilbert-Schmidt operator from K to H followed by a bounded operator from H to H .

Proposition 2. Let A be a self-adjoint operator acting in the Hilbert space H . Let $K \subset H \subset K^*$ be a scale of Hilbert spaces. Assume that the scale extension of A to an operator \tilde{A} acting in K^* exists. Let λ be a complex number which is not in the spectrum of A . Then if $(A - \lambda)^{-1}$ sends K into K , λ is not an eigenvalue of \tilde{A} .

Proof: Assume that $(\tilde{A} - \lambda)\psi = 0$ for some ψ in K^* . If $(A - \lambda)^{-1}$ sends K into K , then the range of $A - \lambda$ restricted to D_0 is K . Hence $\langle \psi, (A - \lambda)f \rangle = 0$ for f in D_0 implies $\psi = 0$.

Definition. Let T be a positive self-adjoint operator acting in H with bounded inverse $T^{-1}: H \rightarrow H$. For $0 \leq s < \infty$, let K_s be the domain of T^s with the norm $\|f\|_s = \|T^s f\|$. Then the family $K_s \subset H \subset K_s^*$, $0 \leq s < \infty$, of scales is called an analytic scale.

There are interpolation theorems which apply to analytic scales. We shall need only the following special case [5].

Proposition 3. Let the spaces $K_s \subset H$, $0 \leq s < \infty$, define an analytic scale. Let $R: H \rightarrow H$ be a bounded operator and assume that it has a bounded restriction $R: K_a \rightarrow K_a$. Then $R: K_s \rightarrow K_s$ is bounded for $0 \leq s \leq a$.

Theorem 4. Let H be a Hilbert space. Let A be a self-adjoint operator acting in H with domain D . Let B be a self-adjoint operator acting in H whose domain contains D . Assume that $A + B$ is self-adjoint on D . Let the spaces $K_s \subset H$, $0 \leq s < \infty$, determine an analytic scale and set $K_1 = K$. Assume that for some $\varepsilon > 0$ and all s , $0 \leq s \leq 1$, $B: D \cap K_s \rightarrow K_{s+\varepsilon}$ is bounded. Then if λ is not in the spectrum of A or of $A + B$, and if the restriction $(A - \lambda)^{-1}: K \rightarrow K$ is bounded, then the restriction $(A + B - \lambda)^{-1}: K \rightarrow K$ is bounded.

Proof: We have $K = K_1 \subset K_s \subset K_0 = H$ for $0 \leq s \leq 1$. The space $D \cap K_s$ may be given the norm $(\|A f\|^2 + \|f\|_s^2)^{1/2}$, where $\|f\|_s$ is the norm on K_s .

$$\text{Write } (A + B - \lambda)^{-1} = \sum_{n=0}^{r-1} (-1)^n ((A - \lambda)^{-1} B)^n (A - \lambda)^{-1} \\ + (-1)^r ((A - \lambda)^{-1} B)^r (A + B - \lambda)^{-1}.$$

Consider the first r terms in the sum. Since $(A - \lambda)^{-1}: K \rightarrow D \cap K$ and $B: K \cap D \rightarrow K$ are bounded, each term is bounded from K to $D \cap K \subset K$.

To treat the final term in the sum, we use interpolation. Since $(A - \lambda)^{-1}: H \rightarrow H$ and $(A - \lambda)^{-1}: K \rightarrow K$ are bounded, it follows from Proposition 3 that $(A - \lambda)^{-1}: K_s \rightarrow K_s$ is bounded for $0 \leq s \leq 1$. Take r so large that $1/r < \varepsilon$. Then $B: D \cap K_{(n-1)/r} \rightarrow K_{n/r}$ is bounded, $n = 1, 2, 3, \dots, r$. Since $(A - \lambda)^{-1}: K_{n/r} \rightarrow D \cap K_{n/r}$ and $(A + B - \lambda)^{-1}: H \rightarrow D$ are bounded, the final term is bounded from $H \supset K$ to $D \cap K \subset K$.

4. Schrödinger Operators

Let $H = L^2(\mathbf{R}^3, dx)$. Let $\varrho \geq 0$ be a real function on \mathbf{R}^3 which is bounded and never zero. Let $K = L^2(\mathbf{R}^3, \varrho(x)^{-1} dx)$. Then $K \subset H$, the injection is continuous, and K is dense in H . So $K \subset H \subset K^*$ is a scale of Hilbert spaces. The nice feature of this case is that K^* has a natural realization as a space of functions. It should be considered as $K^* = L^2(\mathbf{R}^3, \varrho(x) dx)$. Then if g is in K^* and f is in K , $\langle g, f \rangle = \int g(x)^* f(x) dx$ and $|\langle g, f \rangle| \leq \|g\|_{K^*} \|f\|_K$.

In the following we shall require that ϱ be bounded away from zero on compact sets. This ensures that the K , H , and K^* norms are equivalent on any set of f with fixed compact support.

Example. Consider the Laplace operator Δ . Δ is a self-adjoint operator acting in H and one spectral representation is given by the Fourier transform $F: H \rightarrow L^2(\mathbf{R}^3, (2\pi)^{-3} dk)$. $(F \Delta f)(k) = -k^2 Ff(k)$, so Δ is isomorphic to multiplication by $\alpha(k) = -k^2$. Assume now that ϱ is in $L^1(\mathbf{R}^3, dx)$. Notice that for f in K , $(F f)(k) = \int \exp(-ikx) f(x) dx = \langle \psi(k), f \rangle$, where $\psi(k)$ is the function $\exp(ikx)$ in K^* . The $\psi(k)$ are non-normalizable eigenvectors: $\Delta \exp(ikx) = -k^2 \exp(ikx)$.

Definition. Let V be a real function on \mathbf{R}^3 such that $V = V_1 + V_2$, where V_1 is in $L^\infty(\mathbf{R}^3, dx)$ and V_2 is in $L^2(\mathbf{R}^3, dx)$. Then V will be said to satisfy the Kato condition.

Proposition 4 [6]. Let $H = L^2(\mathbf{R}^3, dx)$. Assume that V is a real function on \mathbf{R}^3 which satisfies the Kato condition. Then $-\Delta + V$ is a self-adjoint operator acting in H with the same domain as that of $-\Delta$.

Theorem 5. Let $\varrho \geq 0$ be a function on \mathbf{R}^3 which is bounded on \mathbf{R}^3 and bounded away from zero on compact subsets of \mathbf{R}^3 . Assume that ϱ is in $L^1(\mathbf{R}^3, dx)$. Let $K \subset H \subset K^*$ be the scale $L^2(\mathbf{R}^3, \varrho(x)^{-1} dx) \subset L^2(\mathbf{R}^3, dx) \subset L^2(\mathbf{R}^3, \varrho(x) dx)$. Let V be a real function on \mathbf{R}^3 which satisfies the Kato condition. Then for any spectral representation $U: H \rightarrow L^2(M, \mu)$ of $-\Delta + V$ there is a function ψ from M to K^* such that for each f in K , $(Uf)(\rho) = \langle \psi(\rho), f \rangle$ for almost every ρ in M . The $\psi(\rho)$ are (possibly non-normalizable) eigenvectors of the scale extension of $-\Delta + V$ for almost every ρ in M .

Proof: If $z > 0$, $(z - \Delta)^{-1}$ is an integral operator acting in H with kernel $(4\pi|x - y|)^{-1} \exp(-z^{1/2}|x - y|)$. Now $(4\pi|x - y|)^{-1} \exp(-z^{1/2}|x - y|) \varrho(y)^{1/2}$ is in $L^2(\mathbf{R}^6, dx dy)$. Hence it is the kernel of a Hilbert-Schmidt operator from H to H . Next note that multiplication by $\varrho^{-1/2}$ is an isomorphism from K to H . It follows that $(z - \Delta)^{-1}$ is a Hilbert-Schmidt operator from K to H .

If $z > 0$ is sufficiently positive, then $-z$ will not be in the spectrum of $-\Delta + V$ [6]. Hence Theorem 3 implies that $(z - \Delta + V)^{-1}$ is Hilbert-Schmidt from K to H . Thus Theorem 1 applies to $-\Delta + V$.

We now show that $-\Delta + V$ has a scale extension. First note that the $L^2(\mathbf{R}^3, dy)$ norm of $(4\pi|x - y|)^{-1} \exp(-z^{1/2}|x - y|)$ is finite and independent of x . It follows that $(z - \Delta)^{-1} g$ is in $L^\infty(\mathbf{R}^3, dx)$ for g in H . Hence the domain of definition of Δ is contained in $L^\infty(\mathbf{R}^3, dx)$. Now consider the space D_1 of functions in the domain of Δ which have compact support. Clearly D_1 is dense in K . $-\Delta$ sends D_1 into K . Multiplication by V_1 leaves K invariant. On the other hand, since the domain of Δ is contained in $L^\infty(\mathbf{R}^3, dx)$, multiplication by V_2 sends D_1 into K . Thus $-\Delta + V$ sends the dense set D_1 into K . This implies that the scale extension of $-\Delta + V$ exists and hence that Theorem 2 applies.

Surprisingly, Theorem 5 does not imply that the non-normalizable eigenfunctions are bounded. The exceptional set of ρ in M for which the $\psi(\rho)$ are not eigenvectors in K^* will depend in general on the choice of the scale. Maslov [7] has given an example of a bounded continuous V for which the non-normalizable eigenfunctions for some interval of energy are unbounded at infinity. Berezanskii [8] has given estimates on their rate of increase.

In the following we write $r = |x|$.

Definition. Let V be a real measurable function on \mathbf{R}^3 . Assume that $V = V_1 + V_2$, where $|V_1(x)| \leq c(1+r^2)^{-\epsilon/2}$ for some $\epsilon > 0$ and some c , and V_2 is in $L^2(\mathbf{R}^3, dx)$ and has compact support. Then V will be said to satisfy the condition of slight decrease.

Note that a function V of slight decrease satisfies the Kato condition. In addition it is a relatively compact perturbation of $-\Delta$, so that the essential spectrum of $-\Delta + V$ is $[0, \infty)$ [9]. In particular, the spectrum of $-\Delta + V$ contains the spectrum of $-\Delta$.

A particularly convenient choice of the function ϱ defining the scale is $\varrho(x) = (1+r^2)^{-s/2}$, $s \geq 0$. The condition that ϱ is in $L^1(\mathbf{R}^3, dx)$ is satisfied provided $s > 3$.

Theorem 6. Fix s , $0 \leq s < \infty$. Let $K_s \subset H \subset K_s^*$ be the scale $L^2(\mathbf{R}^3, (1+r^2)^{s/2} dx) \subset L^2(\mathbf{R}^3, dx) \subset L^2(\mathbf{R}^3, (1+r^2)^{-s/2} dx)$. Let V be a real function on \mathbf{R}^3 which satisfies the condition of slight decrease. Then the scale extension of $-\Delta + V$ to an operator in K_s^* has the complex number λ as an eigenvalue only if λ is real and in the spectrum of the self-adjoint operator $-\Delta + V$ acting in H .

Proof: Let T be the operator given by multiplication by $(1 + r^2)^{1/4}$. Then the scale in the theorem is the analytic scale associated with T ; K_s is the domain of T^s , $s \geq 0$.

In order to apply Theorem 4 we first show that for $z > 0$ or z not real, $(z + \Delta)^{-1}$: $K_s \rightarrow K_s$ is continuous, $s \geq 0$. Start with the case when s is an integer multiple of 4, $s = 4k$. By taking Fourier transforms we see that this is equivalent to showing that $(z - k^2)^{-1}$ is a continuous multiplication operator from $D(\Delta^k)$ to $D(\Delta^k)$. For f in $D(\Delta^k)$, expand $\Delta^k((z - k^2)^{-1}f)$ as a sum of products of partial derivatives of $(z - k^2)^{-1}$ and of f . The partial derivatives of f can be estimated in L^2 norm in terms of $\|\Delta^k f\|_2$ and $\|f\|_2$. On the other hand, the partial derivatives of $(z - k^2)^{-1}$ are all bounded functions (since they are Fourier transforms of integrable functions). Thus we have an estimate on $\|\Delta^k((z - k^2)^{-1}f)\|_2$, which disposes of the case when k is an integer. The general case now follows from Proposition 3.

The other hypothesis of Theorem 4 follows from the assumption of slight decrease. Multiplication by V_1 is bounded from K_s to $K_{s+\epsilon}$ for some $\epsilon > 0$. On the other hand, V_2 is bounded from D to K_s for all $s \geq 0$, since $D \subset L^\infty(\mathbb{R}^3, dx)$ and V_2 has compact support. Thus $V: K_s \cap D \rightarrow K_{s+\epsilon}$ is bounded.

So if λ is not in the spectrum of $-\Delta + V$, $(-\Delta + V - \lambda)^{-1}: K_s \rightarrow K_s$ is bounded. The theorem then follows from Proposition 2.

REFERENCES

- [1] A straightforward proof may be found in E. NELSON, *Topics in Dynamics I: Flows* (Princeton University Press, Princeton, N. J., 1969), p. 67–73.
- [2] Ju. M. BEREZANSKII, *Expansions in Eigenfunctions of Selfadjoint Operators*, Translations of Mathematical Monographs, Vol. 17, Amer. Math. Soc., Providence, R.I., 1968, Chap. V. This book contains an extensive bibliography on non-normalizable eigenvectors.
- [3] I. M. GLAZMAN, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, Israel Program for Scientific Translations, Jerusalem (1965), Chap. V., § 54.
- [4] T. KATO, *Perturbation Theory for Linear Operators* (Springer-Verlag, Berlin 1966), Chap. V, § 2.4.
- [5] S. G. KREIN and Ju. I. PETUNIN, *Scales of Banach Spaces*, Uspehi Mat. Nauk 21/2, 89 (1966), trans. in Russian Math. Surveys 21/2, 85 (1966).
- [6] See Reference [4], Chap. V, § 5.3.
- [7] V. MASLOV, *On Asymptotics of Generalized Eigenfunctions of the Schrödinger Equation* (Russian), Uspehi Mat. Nauk 16/4, 253 (1961).
- [8] See Reference [2], Chap. VI.
- [9] See Reference [4], Chap. V, § 5.3.