Zeitschrift: Helvetica Physica Acta

Band: 44 (1971)

Heft: 7

Artikel: On the uniqueness of the hamiltonian and of the representation of the CCR for the quartic Boson interaction in three dimensions

Autor: Eckmann, Jean-Pierre / Osterwalder, Konrad

DOI: https://doi.org/10.5169/seals-114317

On the Uniqueness of the Hamiltonian and of the Representation of the CCR for the Quartic Boson Interaction in Three Dimensions

by **Jean-Pierre Eckmann**[1][2]

Brandeis University, Waltham, Mass.,

and **Konrad Osterwalder**[3]

Courant Institute of Mathematical Sciences, New York University, New York

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Abstract. Glimm has constructed a Hamiltonian for the $(:\phi^4:)_{2+1}$ interaction with space cutoff, using a truncated version of the formal wave operator in order to define a domain for this Hamiltonian. For a wide class of such truncations we obtain unitarily equivalent representations of the canonical commutation relations in the sense of Fabrey. We establish unitary equivalence of the closures of the Hamiltonians obtained for many different truncations.

I. Introduction

In the constructive approach to quantum field theory one often encounters complicated limiting procedures involving the choice of some technical parameters. It is natural to ask to what extent the final results depend on such quantities. In [4] Glimm constructed a Hamiltonian for a quartic boson selfinteraction with space cutoff in three dimensional space time. He used a truncated version of the formal wave operator in order to define a domain for this Hamiltonian. In this paper we show that for a wide class of such truncations one obtains unitarily equivalent representations of the canonical commutation relations (CCR) in the sense of Fabrey [3] and even unitary equivalence of the Hamiltonians. It seems that this class of truncations exhausts almost all the possibilities for which Glimm's construction goes through.

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[1]) Supported by the Office of Naval Research, N00014–70–SC–0418.

[2]) On leave from the University of Geneva, Switzerland.

[3]) Supported by the National Science Foundation, Grant GP–24003.

In order to deal rigorously with infinite quantities one introduces an approximate Hamiltonian H_σ with a momentum cutoff σ:

$$H_\sigma = H_0 + \int :\phi_\sigma^4:(x)\, h(x)\, d^2x + M_\sigma + E_\sigma\,, \tag{1.1}$$

where H_0 is the free Hamiltonian, $\phi_\sigma(x)$ is the cutoff free boson field at time zero:

$$\phi_\sigma(x) = \int\limits_{|k| \le \sigma} e^{ikx}\, \mu(k)^{-1/2} \big(a^*(-k) + a(k)\big)\, d^2k\,,$$

$$\mu(k) = (m^2 + k^2)^{1/2}\,, \quad m > 0\,.$$

The expressions M_σ and E_σ are mass and vacuum energy renormalization terms respectively and will be given in section II.

The space cutoff function $h(x)$ will be held fixed throughout this paper and is supposed to be smooth and of compact support.

By $V_{j\sigma}$, $j = 0, 1, \ldots, 4$, we denote that part of the interaction term $\int :\phi_\sigma^4:(x)\, h(x)\, d^2x$ which creates exactly j particles.

Our procedure is to start with a simplified Hamiltonian [5], [6]

$$\hat{H}_\sigma = H_0 + V_{4\sigma} + V_{0\sigma} + \text{counterterms}\,, \tag{1.2}$$

and we shall see later that it exhibits already all the interesting properties of the full Hamiltonian (1.1).

In section III we define a large family of truncations $\hat{T}_\sigma(f, g)$ of the formal wave operator that belongs to (1.2), the truncation depending on two parameters f and g. The operators $\hat{T}_\sigma(f, g)$ are called 'dressing transformations'. Following the ideas of Glimm [4] we show that each of these dressing transformations defines a limit $\hat{H}(f, g)$ of the simplified Hamiltonian as σ goes to infinity, which is a densely defined symmetric operator in a Hilbert space $\hat{\mathscr{I}}(f, g)$ disjoint from the Fock space. Furthermore each of these spaces $\hat{\mathscr{I}}(f, g)$ is a representation space for an non-Fock representation $\hat{W}(f, g \mid y)$ of the Weyl relations, $\hat{W}(f, g \mid y)$ being a certain limit of $e^{i\phi(y)}$, with y an element of a suitably chosen test function space (section VI and [3]). In section IV we construct a natural unitary mapping from $\hat{\mathscr{I}}(f, g)$ to $\hat{\mathscr{I}}(f', g')$, for different (f, g) and (f', g'). This mapping is called natural because it is constructed as a limit of the identity map in Fock space. We show that it yields the unitary equivalence of the representations $\hat{W}(f, g \mid y)$ and $\hat{W}(f', g' \mid y)$ of the Weyl relations.

In section V the construction of section III is repeated for the full Hamiltonian H_σ, equation (1.1), i.e. we construct dressing transformations $T_\sigma(f, g)$ leading to limiting Hamiltonians $H(f, g)$, densely defined in a Hilbert space $\mathscr{I}(f, g)$, which is a representation space for a non-Fock representation $W(f, g \mid y)$ of the Weyl relations. Again by natural unitary mappings we prove the unitary equivalence of $W(f, g \mid y)$ and $\hat{W}(f, g \mid y)$. This is an extension of results of Fabrey [3].

In section VII we use the natural unitary mappings to show that the closures of $H(f, g)$ and of $H(f', g')$ are unitarily equivalent for different (f, g) and (f', g'). These results can be interpreted in the following way:

All Hilbert spaces $\hat{\mathscr{I}}(f, g)$ and $\mathscr{I}(f, g)$ may be identified, using the natural unitary mappings. We call this space $\mathscr{I}_{ren}$. This involves the identification of all the representations $\hat{W}(f, g \mid y)$ and $W(f, g \mid y)$ with a representation $W(y)$ in $\mathscr{I}_{ren}$. Furthermore we can define a Hamiltonian H with domain $\bigcup_{(f,g)} \mathscr{D}(H(f, g)) \subset \mathscr{I}_{ren}$, $\mathscr{D}(H(f, g))$ being the domain of $H(f, g)$, now a dense set in $\mathscr{I}_{ren}$. We set $H|_{\mathscr{D}(H(f,g))} = H(f, g)$.

Thus we end up with a Hilbert space $\mathscr{I}_{ren}$, a representation $W(y)$ of the Weyl relations and a densely defined Hamiltonian H, and the truncation parameters have been eliminated. Furthermore we see that we can obtain $\mathscr{I}_{ren}$ and $W(y)$ already with the very simple dressing transformation $\hat{T}_\sigma(f, g)$ for any (f, g). It would be interesting to find more abstract criteria which characterize $\mathscr{I}_{ren}$ and $W(y)$. If we want to construct in $\mathscr{I}_{ren}$ a dense domain for the Hamiltonian H, then a more complicated dressing transformation $T_\sigma(f, g)$ is needed. Now we could go on, constructing dense domains for powers of H using more and more sophisticated dressing transformations. What we would like best is a dressing transformation which yields a dense set of analytic vectors for H, to prove that the closure of H is a selfadjoint operator. In fact, as recent results of Masson and McClary [8], [10] show, it would even be sufficient to construct a dense set of semianalytic vectors for H. Then if H is shown to be semibounded, the essential selfadjointness of H is a consequence.

II. Notation and Definitions

Let F denote the Fock space of free bosons with mass m; $a(k)$ and $a^*(k)$ the annihilation and creation operators for a particle with momentum k, respectively. An operator of the form

$$W_{mn} = \int \prod_{i=1}^{m} a^*(k_i)\, dk_i \prod_{i=1}^{n} a(l_i)\, dl_i\, w_{mn}(k_1, \ldots, k_m, l_1, \ldots, l_n) \tag{2.1}$$

is called a *Wick monomial* (with numerical kernel w_{mn}). dk stands for d^2k. In modifying Friedrich's perturbation theory, Glimm has introduced the operation Γ, which associates to a Wick monomial W_{mn} the Wick monomial $\Gamma(W_{mn})$. For $m > 0$, $\Gamma(W_{mn})$ is defined by

$$\Gamma(W_{mn}) = \int \prod_{i=1}^{m} a^*(k_i)\, dk_i \prod_{i=1}^{n} a(l_i)\, dl_i \Big(\sum_{i=1}^{m} \mu(k_i)\Big)^{-1} w_{mn}(k_1, \ldots, k_m, l_1, \ldots, l_n)\,.$$

We also define $| W_{mn} |$ to be the Wick monomial (2.1), but $w_{mn}(k_1, \ldots, l_n)$ is replaced by its absolute value $| w_{mn} | (k_1, \ldots, l_n)$.

Let W_{mn} and $W'_{m'n'}$ be two Wick monomials as in (2.1) with numerical kernels w_{mn} and $w'_{m'n'}$ which are symmetric functions of their creation and annihilation variables separately. According to Wick's theorem the product $W_{mn} W'_{m'n'}$ can be expanded as follows.

$$W_{mn} W'_{m'n'} = \sum_{r=0}^{\min(n,m')} \binom{n}{r} \binom{m'}{r} r! \int \prod_{i=1}^{m} a^*(k_i)\, dk_i \prod_{i=r+1}^{m'} a^*(k'_i)\, dk'_i$$

$$\times \prod_{i=r+1}^{n} a(l_i)\, dl_i \prod_{i=1}^{n'} a(l'_i)\, dl'_i \prod_{i=1}^{r} \delta(k'_i - l_i)\, dk'_i\, dl_i$$

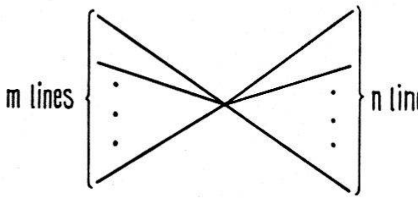
$$w_{mn}(k_1, \ldots, k_m, l_1, \ldots, l_n)\, w'_{m'n'}(k'_1, \ldots, k'_{m'}, l'_1, \ldots, l'_{n'}) \,. \tag{2.2}$$

This expansion is the *Wick expansion*, and a single term in the sum $\sum_{r=0}^{\min(n,m')}$ on the right of (2.2), but without the factor $\binom{n}{r}\binom{m'}{r} r!$ is called a *Wick term*, occurring in $W_{mn} W'_{m'n'}$. The rth term in this sum is denoted by $\underbrace{W_{mn} W'_{m'n'}}_{r}$, it is a sum of $\binom{n}{r}\binom{m'}{r} r!$ identical Wick terms. We also define

$$\underbrace{W_{mn} W'_{m'n'}}_{} = \sum_{r>0} \underbrace{W_{mn} W'_{m'n'}}_{r} = W_{mn} W'_{m'n'} - :W_{mn} W'_{m'n'}: \,.$$

Inductively we extend these definitions to products of more than to Wick monomials.

Wick monomials and Wick terms are usually represented as *graphs*. A Wick monomial W_{mn} is drawn as



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Each Wick term in (2.2) is represented by the graph of W_{mn} to the left of the graph of $W'_{m'n'}$. Those lines whose variables have been identified by $\prod_{i=1}^{r} \delta(k'_i - l_i)$ are connected. They are called internal lines, the other lines are called external lines. Let

$$W = \int \prod_{i=1}^{r} a^*(k_i)\, dk_i \prod_{i=1}^{s} a(l_i)\, dl_i \prod_{i=1}^{t} dp_i\, w(k_1, \ldots, k_r, l_1, \ldots, l_s, p_1, \ldots, p_t)$$

be a Wick term. Then we define a *truncation* of W by replacing $w(k_1, \ldots, p_t)$ by $w(k_1, \ldots, p_t) \cdot \chi(k_1, \ldots, p_t)$, χ being a characteristic function.

A truncation of a sum of Wick terms is defined by truncating each Wick term separately. We now turn to the model under consideration. The expression $\int :\phi^4:(x) \times h(x)\, dx$ has an expansion

$$\sum_{i=0}^{4} V_{i\sigma} = \sum_{i=0}^{4} \int a^*(k_1) \ldots a^*(k_i)\, a(k_{i+1}) \ldots a(k_4)\, v_{i\sigma}(k_1, \ldots, k_4)\, dk_1 \ldots dk_4 \,. \tag{2.2}$$

Here h is in the space $\mathscr{D}$ of Schwartz and

$$v_{i\sigma}(k_1, \ldots, k_4) = \begin{cases} \binom{4}{i} \prod_{j=0}^{4} \mu(k_j)^{-1/2} \tilde{h}(k_1 + \cdots + k_i - k_{i+1} \cdots - k_4) \\ \text{if} \quad |k_j| \leq \sigma \quad \text{for} \quad j = 1, 2, 3, 4 \\ 0 \quad \text{otherwise.} \end{cases}$$

By $\tilde{}$ we denote the Fourier transform; $\mu(k) = (k^2 + m^2)^{1/2}$.

We now give the explicit definitions of the two Hamiltonians which we shall consider later.

$$H_\sigma = H_0 + \sum_{i=0}^{4} V_{i\sigma} + M_\sigma + E_\sigma,$$

$$\hat{H}_\sigma = H_0 + V_{0\sigma} + V_{4\sigma} + \hat{M}_\sigma + \hat{E}_\sigma .$$

M_σ, E_σ, $\hat{M}_\sigma$, $\hat{E}_\sigma$ are the counterterms, whose definition is motivated by perturbation theory, see e.g. [6].

The mass counterterms are defined by

$$\hat{M}_\sigma = 2\, m_\sigma \int\limits_{|k_i| \leq \sigma} a^*(k_1)\, a(k_2)\, \mu(k_1)^{-1/2}\, \mu(k_2)^{-1/2}\, (\tilde{h} * \tilde{h})\, (k_1 - k_2)\, dk_1\, dk_2 \,,$$

$$M_\sigma = m_\sigma \int\limits_{|k_i| \leq \sigma} :\big(a^*(k_1) + a(-k_1)\big)\, \big(a^*(k_2) + a(-k_2)\big):\, \mu(k_1)^{-1/2}\, \mu(k_2)^{-1/2}$$
$$\times\, (\tilde{h} * \tilde{h})\, (k_1 + k_2)\, dk_1\, dk_2 \,,$$

$$m_\sigma = 96 \int\limits_{4\sigma/3} \prod_{i=1}^{3} \frac{d\,p_i}{\mu(p_i)}\, \delta(p_1 + p_2 + p_3)\, \Big(\sum_{i=1}^{3} \mu(p_i)\Big)^{-1} .$$

The scalar counterterms are given by

$$\hat{E}_\sigma = \big(\phi_0,\, V_{0\sigma}\, \Gamma(V_{4\sigma})\, \phi_0\big) \,,$$

$$E_\sigma = \hat{E}_\sigma - \big(\Gamma(V_{4\sigma})\, \phi_0,\, V_{2\sigma}\, \Gamma(V_{4\sigma})\, \phi_0\big) .$$

Here ϕ_0 denotes the Fock vacuum.

We shall also use the notation

$$\Lambda_\sigma = \|\, \Gamma(V_{4\sigma})\, \phi_0\, \|^2 = \underbrace{\big(\Gamma(V_{4\sigma})\big)^*\, \Gamma(V_{4\sigma})}_{4} .$$

The fact that $\Lambda_\sigma = 0(\ln \sigma)$, that is the divergence of Λ_σ as $\sigma \to \infty$, is the reason for a change to a non-Fock representation of the CCR.

The graph representing Λ_σ is  . A graph which does not contain  as a subgraph is called a *skeleton graph*.

We shall often use the following standard domain $\mathscr{D}_0$ in Fock space. $\mathscr{D}_0$ is defined to be the set of all vectors in Fock space which have a finite number of particles (i.e. the n-particle component equals zero for n large) and which have compact support in momentum space.

III. Dressing Transformations for a Simplified Hamiltonian

As was shown [5], [6], the ultraviolet divergencies of $\hat{H}_\sigma$ are formally compensated on the range of a formal dressing transformation

$$\hat{T}_{formal,\sigma} = \exp - \Gamma(V_{4\sigma}) = \exp W_\sigma \, . \tag{3.1}$$

For a rigorous construction of a domain for $\hat{H}_\sigma$ as $\sigma \to \infty$ we use a truncated version of (3.1). The aim of this section is to define a class of such truncations, each of them leading to the definition of a limiting Hamiltonian $\hat{H}_\sigma$.

Let $f: N \to R^+$ be a strictly increasing, nonnegative function on the natural numbers N, and define $f(N) = \{f(i); i \in N\}$. Let $g: N \to N \cup \{0\}$ be a map of the natural numbers into the nonnegative integers.

For $\varrho \in f(N)$, $\sigma \geq 0$ and f^{-1} the inverse function of f,

$$\hat{T}_{\varrho\sigma}(f, g) = \prod_{j \geq f^{-1}(\varrho)} \exp_{g(j)} W_{fj\sigma} \, , \tag{3.2}$$

where

$$\exp_n x = \sum_{k=0}^{n} x^k/k! \quad \text{for} \quad n \geq 0 \, .$$

Furthermore

$$- W_{fj\sigma} = \Gamma(V_{4fj\sigma}) = \int a^*(k_1) \ldots a^*(k_4) \left(\sum_{i=1}^{4} \mu(k_i)\right)^{-1} v_{4fj\sigma}(k_1 \ldots k_4) \, dk_1 \ldots dk_4 \, ,$$

$$v_{4fj\sigma}(k_1, \ldots, k_4) = \begin{cases} v_{4\sigma}(k_1, \ldots, k_4) & \text{if } \max\limits_{i=1,\ldots,4} |k_i| \in [f(j), f(j+1)) \\ 0 & \text{otherwise} \, . \end{cases}$$

In the following we refer to the momentum k_i, $|k_i| \in [f(j), f(j+1))$, as the 'maximal momentum belonging to $W_{fj\sigma}$'.

Note that $\hat{T}_{\varrho\sigma}(f, g)$ is that part of $\hat{T}_{formal,\sigma}$ which contains at most $g(j)$ factors $W_{fj\sigma}$. One can easily verify that $\hat{T}_{\varrho\sigma}(f, g)$ is a truncation of $\hat{T}_{formal,\sigma}$ in the sense of the definition of section II. Namely, for $\sigma \geq f(1)$ we have $-\Gamma(V_{4\sigma}) = -\Gamma(V_{4f(1)}) + \sum_{j=1}^{\infty} W_{fj\sigma}$ and the $W_{fj\sigma}$'s commute one with the other.

By $\hat{T}_{\varrho\sigma n}(f, g)$ we denote the nth order contribution to $\hat{T}_{\varrho\sigma}(f, g)$. The purpose of this chapter is to establish sufficient conditions on the functions f and g such that $\hat{T}_{\varrho\sigma}(f, g)$ is a dressing transformation for $\hat{H}_\sigma$.

Definition. For any $\delta > 0$ we say that f and g satisfy condition C_δ, or $(f, g) \in C_\delta$, if there is a constant $i_0 > 0$ such that for all $i > i_0$:

C1. $f(i) > i^{a_1}$,

C2. $g(i) < f(i)^{a_2}$,

C3. $a_3 \ln f(i) < g(i)$,

C4. $f(i+1) < f(i)^{a_4}$,

with $a_1, \ldots, a_4$ satisfying

A1. $a_1 > \dfrac{2(1+\delta)}{\varepsilon_0}$,

A2. $a_2 < \dfrac{\delta}{1+\delta} \cdot \dfrac{\varepsilon_0}{2}$,

A3. $a_3 > \dfrac{\varepsilon_0}{2(1+\delta)}$,

A4. $a_4 > 1$,

A5. $a_4 < 1 + \min\{1, \varepsilon_0 \lambda^{-1} e^{-56(1+\delta)}\}$.

The positive constants ε_0 and λ are fixed and given by the model. They will be specified below.

The main result of this section is:

Theorem 3.1. *Let (f, g) satisfy C_δ for some $\delta > 0$ and suppose $\varrho, \varrho' \in f(N)$. Then with $\hat{T}_{\varrho\sigma} \equiv \hat{T}_{\varrho\sigma}(f, g)$ as defined in* (3.2) *and $\phi, \psi \in \mathscr{D}_0$ the following holds.*

I. $$\lim_{\sigma \to \infty} (\hat{T}_{\varrho\sigma}\, \phi, \hat{T}_{\varrho'\sigma}\, \psi)\, e^{-A_\sigma} \equiv (\hat{T}_{\varrho\infty}\, \phi, \hat{T}_{\varrho'\infty}\, \psi)_r \tag{3.3}$$

exists.

II. The expression (3.3) *defines a positive definite scalar product $(\cdot, \cdot)_r$ on*

$$\langle \hat{T}_{\varrho\infty}(f, g)\, \phi : \phi \in \mathscr{D}_0, \varrho \in f(N) \rangle \equiv \hat{\mathscr{D}}(f, g), \tag{3.4}$$

$\langle\,\rangle$ denoting the linear hull. $\hat{\mathscr{D}}(f, g)$ together with $(\cdot, \cdot)_r$ is a prehilbert space, whose completion $\hat{\mathscr{I}}(f, g)$ is a separable Hilbert space.

III. $\| \hat{H}_\sigma \hat{T}_{\varrho\sigma}\, \phi \|^2 e^{-A_\sigma}$ is uniformly bounded in $0 \le \sigma \le \infty$, and $\lim_{\sigma \to \infty} (\hat{T}_{\varrho\sigma}\, \phi, \hat{H}_\sigma\, \hat{T}_{\varrho'\sigma}\, \psi)\, e^{-A_\sigma}$ exists and defines a symmetric operator $\hat{H}(f, g)$ with domain $\hat{\mathscr{D}}(f, g) \subset \hat{\mathscr{I}}(f, g)$.

Proof: We have to verify *I*–*III*. The proof follows step by step Glimm's procedure [4].

We sketch only the most important estimates which are needed. Property *I* is a consequence of conditions C1 and C2 on f and g, which guarantee that the truncation omits enough terms from the divergent series of $\hat{T}_{formal,\sigma}$ to make the remaining series convergent. Thus the main tool to prove property *I* is a σ-independent upper bound on

$$| (\hat{T}_{\varrho\sigma}(f, g) \phi, \hat{T}_{\varrho'\sigma}(f, g) \psi) e^{-\Lambda_\sigma} | . \tag{3.5}$$

Glimm's analysis shows that (3.5) is majorized by

$$\sum_{n,m} \sum_{S_{nm}} (n!\, m!)^{-1} (| \phi |, | S_{nm} | | \psi |) , \tag{3.6}$$

where the sum $\sum_{S_{nm}}$ runs over all Wick terms S_{nm} in the expansion of $n!\, m!\, \hat{T}_{\varrho\sigma n}(f, g)^* \hat{T}_{\varrho'\sigma m}(f, g)$ whose graph is a skeleton graph.

By $s_{nm}(p_{ext}, p_{int})$ we denote the numerical kernel of S_{nm}; $p_{ext}(p_{int})$ stands for all the variables belonging to external (internal) lines of the graph of S_{nm}.

Lemma 3.2. (Glimm [4, Theorem 2.2.1]; [2], [9]). *There exist positive constants ε_0' and K, independent of ϱ, σ, f or g, such that for all $\varepsilon < \varepsilon_0'$,*

$$\Big\| \prod_{p_i \in p_{ext}} \mu(p_i)^{-2+\varepsilon/2} \int \prod_{p_i \in p_{int}} \mu(p_i)^{\varepsilon} \, | s_{nm}(p_{ext}, p_{int}) | \, dp_{int} \Big\|_{2,ext} \le K^{n+m} . \tag{3.7}$$

By $\| \cdot \|_{2,ext}$ we denote the L_2 norm with respect to the momenta $p_i \in p_{ext}$.

Remark. The constant ε_0 used in the definition of condition C_δ is chosen to be the maximal possible value of ε_0'; $\varepsilon_0 = 1/6$, by inspection.

Suppose $r \in N$ has been chosen such that for all $s \ge r$ the s-particle components of ϕ and ψ respectively are zero. Then for fixed n and m the number of Wick terms S_{nm} contributing to (3.6) is bounded by

$$\sum_{s_1=0}^{r} \sum_{s_2=0}^{r} \binom{4n}{s_1} \binom{4m}{s_2} \big((4n - s_1)!\, (4m - s_2)!\big)^{1/2} \le 2^{4(n+m)} \big((4n)!\, (4m)!\big)^{1/2} . \tag{3.8}$$

Note that each term in the sum on the left-hand side of (3.8) counts the number of Wick terms which annihilate (create) s_1 (s_2) particles. Furthermore in (3.8) we have used

$$\sum_{s=0}^{4n} \binom{4n}{s} = 2^{4n} .$$

By the Schwarz inequality we get as bound on (3.6)

$$C_{\phi,\psi} \sum_{n,m} 2^{4(n+m)} \big((4n)!\, (4m)!\big)^{1/2} (n!\, m!)^{-1}$$

$$\times \max_{S_{nm}} \Big\| \prod_{p_i \in p_{ext}} \mu(p_i)^{-2} \int | s_{nm}(p_{ext}, p_{int}) | \, dp_{int} \Big\|_{2,ext} , \tag{3.9}$$

where $C_{\phi,\psi}$ is a constant depending on ϕ and ψ only. The definition of the truncation of $\hat{T}_{formal,\sigma}$ implies that $\hat{T}_{\varrho\sigma}(f, g)$ contains $W_{fj\sigma}$ at most $g(j)$ times or, in other words, that $\hat{T}_{\varrho\sigma}(f, g)$ contains at most $\xi(i) = \sum_{j=1}^{i-1} g(j)$ maximal momenta with absolute value smaller than $f(i)$.

Let $t(p_1, \ldots, p_{4n})$ be the numerical kernel of $\hat{T}_{\varrho\sigma n}(f, g)$; $p_1, \ldots, p_n$ the maximal momenta, and suppose that $|p_1| \leq |p_2| \leq \cdots \leq |p_n|$. Then the above statement says that

$$|p_{\xi(i)+1}| \geq f(i)\,. \tag{3.10}$$

Now we use the condition C_δ: for i large,

$$\begin{aligned}\xi(i+1) &= \sum_{j=1}^{i} g(j) < \sum_{j=1}^{i_0} g(j) + \sum_{j=i_0+1}^{i} f(j)^{a_2} \quad \text{by C2}\,,\\ &< i\, f(i)^{a_2} < f(i)^{a_2 + 1/a_1} \quad \text{by C1}\,,\\ &< f(i)^{\varepsilon_1/2}, \text{ for some } \varepsilon_1 < \varepsilon_0, \quad \text{by A1 and A2}\,.\end{aligned} \tag{3.11}$$

Inequalities (3.10, 11) give, for large i,

$$|p_{\xi(i)+1}|^{\varepsilon_1/2} \geq f(i)^{\varepsilon_1/2} > \xi(i+1)\,, \quad \text{or}$$

$$|p_n|^{\varepsilon_1/2} > n \quad \text{for large } n\,.$$

Finally we choose an $\eta > 0$ such that $\varepsilon_1(1+\eta) \equiv \varepsilon < \varepsilon_0$ in order to get

$$|p_n|^{\varepsilon/2} > n^{(1+\eta)}\,, \quad \text{for large } n\,. \tag{3.12}$$

Using Lemma 3.2 and (3.12) we finally get

$$\begin{aligned}\max_{S_{nm}} \Big\| \prod_{p_i \in p_{ext}} \mu(p_i)^{-2} \int |s_{nm}(p_{ext}, p_{int})|\, dp_{int} \Big\|_{2,ext} &\leq \max_{S_{nm}} \Big\| \prod_{p_i \in p_{ext}} \mu(p_i)^{-2+\varepsilon/2} \\ &\times \int \prod_{p_i \in p_{int}} \mu(p_i)^{\varepsilon} |s_{nm}(p_{ext}, p_{int})|\, dp_{int} \Big\|_{2,ext} (n!\, m!)^{-(1+\eta)} K_1^{n+m} \\ &\leq K^{n+m}(n!\, m!)^{-(1+\eta)}\,, \quad \text{for large } n, m,\end{aligned} \tag{3.13}$$

and thus by modifying K, for all $n, m > 0$. Thus (3.9) converges uniformly in ϱ and σ and is therefore the desired finite bound for (3.5).

Remark 3.1. In some later calculations we will know that at least one of the momenta in $s_{nm}(p_{ext}, p_{int})$ is restricted to absolute values larger than some given number τ. Then we can improve (3.13):

$$\begin{aligned}&\max_{S_{nm}} \Big\| \prod_{p_i \in p_{ext}} \mu(p_i)^{-2} \int |s_{nm}(p_{ext}, p_{int})|\, dp_{int} \Big\|_{2,\,ext} \\ &\quad \leq K^{n+m}(n!\, m!)^{-(1+\eta/2)}\, \tau^{-(\varepsilon\eta/4)} \quad \text{for all } n, m > 0\,.\end{aligned} \tag{3.14}$$

Property *II*, the positive definiteness of the 'renormalized' scalar product, is essentially a consequence of conditions C3 and C4 on f and g. They guarantee that the truncation of $\hat{T}_{formal,\sigma}$ is not too strong, whereas C1 and C2 were designed to make the truncation strong enough.

Lemma 3.3. *Let* $\Lambda_{fj\sigma} = \| W_{fj\sigma}\, \phi_0 \|^2 = 4!\, \| w_{fj\sigma} \|_2^2$.

Then there exists a constant λ, *independent of* f, j *and* σ *such that*

$$\Lambda_{fj\sigma} \le \lambda \ln \frac{f(j+1)}{f(j)} .$$

The proof of this lemma is straightforward. The above constant λ is used in the formulation of condition C_δ.

Lemma 3.4. *If* (f, g) *satisfy condition* C_δ *for some* $\delta > 0$, *then*

$$\lim_{j_0\to\infty} \lim_{\sigma\to\infty} \prod_{j \ge j_0} \exp_{g(j)} \Lambda_{fj\sigma} \exp - \Lambda_{fj\sigma} = 1 . \tag{3.15}$$

Proof: If we set $b_j = \exp_{g(j)} \Lambda_{fj\sigma} \exp - \Lambda_{fj\sigma}$, then

$$0 < b_i < 1 \quad \text{and} \quad (1 - b_j) \le \frac{(\Lambda_{fj\sigma})^{g(j)+1}}{(g(j)+1)!} . \tag{3.16}$$

Thus

$$\begin{aligned} 1 - \prod_{j \ge j_0} b_j &\le \sum_{j \ge j_0} (1 - b_j) \\ &\le \sum_{j \ge j_0} \frac{(\Lambda_{fj\sigma})^{g(j)+1}}{(g(j)+1)!}, \quad \text{using (3.16)} \\ &\le \sum_{j \ge j_0} \left(\frac{e\, \lambda [\ln f(j+1) - \ln f(j)]}{g(j)} \right)^{g(j)+1} , \end{aligned}$$

by Lemma 3.3 and by Stirling's formula

$$\begin{aligned} &\le \sum_{j \ge j_0} \left[\frac{e\, \lambda}{a_3} (a_4 - 1) \right]^{g(j)+1} , \quad \text{by C3 and C4} \\ &\le \sum_{j \ge j_0} e^{-[g(j)+1]} , \quad \text{by A5 [see inequality (3.30) below]} \\ &\le \sum_{j \ge j_0} e^{-a_1 a_3 \ln j} = \sum_{j \ge j_0} j^{-a_1 a_3} , \quad \text{by C1 and C3} \end{aligned}$$

$< \infty$, for any $j_0 \ge 1$, by A1 and A3.

This bound combined with the fact that $\lim_{\sigma\to\infty} \Lambda_{fj\sigma}$ exists establishes Lemma 3.4.

The remainder of the proof of property *II* follows from Glimm's arguments, see [4, p. 35] or [3].

The essential tools in the proof of property *III* are bounds, uniform in σ, on

$$\| (H_0 + V_{4\sigma})\, \hat{T}_{\varrho\sigma}(f, g)\, \phi \|^2\, e^{-\Lambda_\sigma} \tag{3.17}$$

and on

$$\| (V_{0\sigma} + \hat{M}_\sigma + \hat{E}_\sigma)\, \hat{T}_{\varrho\sigma}(f, g)\, \phi \|^2\, e^{-\Lambda_\sigma} . \tag{3.18}$$

We only indicate how one gets the bound on (3.17). The methods for the bound on (3.18) are identical. For the remaining steps in the proof of ***III*** we refer again to [4]. Using the definitions (3.2) we obtain for $\varrho \in f(N)$, $\varrho < \sigma$,

$$V_{4\sigma} = V_{4\varrho} + \sum_{j \geq f^{-1}(\varrho)} V_{4fj\sigma},$$

and

$$\begin{aligned} \|(H_0 + V_{4\sigma})\, \hat{T}_{\varrho\sigma}(f,g)\, \phi\|\, e^{-\Lambda_\sigma/2} &\leq \|\hat{T}_{\varrho\sigma}(f,g)\, H_0\, \phi\|\, e^{-\Lambda_\sigma/2} \\ &+ \|V_{4\varrho}\, \hat{T}_{\varrho\sigma}(f,g)\, \phi\|\, e^{-\Lambda_\sigma/2} \\ &+ \sum_{j \geq f^{-1}(\varrho)} \Big\| V_{4fj\sigma} \frac{(W_{fj\sigma})^{g(j)}}{g(j)!} \prod_{\substack{i \geq f^{-1}(\sigma) \\ i \neq j}} \exp_{g(i)} W_{fi\sigma}\, \phi \Big\|\, e^{-\Lambda_\sigma/2}. \end{aligned} \tag{3.19}$$

The first two terms on the right-hand side of (3.19) are bounded uniformly in σ, by the arguments used in the proof of property ***I***. The remaining terms come from the sum of $[H_0, \hat{T}_{\varrho\sigma}(f,g)]\, \phi$ and $\sum_{j \geq f^{-1}(\varrho)} V_{4fj\sigma}\, \hat{T}_{\varrho\sigma}(f,g)\, \phi$ and have to be estimated. First we observe that due to the definition (3.2) the following inequality holds:

$$\begin{aligned} |V_{4fj\sigma}| &\leq \mathrm{const} f(j+1)\, |\Gamma\, V_{4fj\sigma}| \\ &= \mathrm{const} f(j+1)\, |W_{fj\sigma}|, \end{aligned} \tag{3.20}$$

with a constant that does not depend on f, j or σ. Therefore it suffices to look for a bound on

$$(g(j)!)^{-2}\, (f(j+1))^2\, \Big\|\, |W_{fj\sigma}|^{g(j)+1} \prod_{i \neq j} \exp_{g(i)} |W_{fi\sigma}|\, \phi \Big\|^2 e^{-\Lambda_\sigma}, \tag{3.21}$$

and to show that $\sum_j \sqrt{(3.21)}$ converges.

As in the proof of property ***I*** we find that (3.21) is majorized by

$$\begin{aligned} &(g(j)!)^{-2}\, (f(j+1))^2 \sum_{n,m} \sum_{t=0}^{g(j)+1} (n!\, m!)^{-1} \binom{g(j)+1}{t}^2 \\ &\times (g(j)+1-t)!\, \Lambda_{fj\sigma}^{g(j)+1-t}\, e^{-\Lambda_{fj\sigma}} \sum_{S_{nmt}} (|\phi|, |S_{nmt}|\, |\phi|), \end{aligned} \tag{3.22}$$

where the sum $\sum_{S_{nmt}}$ runs over all Wick terms S_{nmt} in the expansion of

$$n!\, m!\, \Big(\prod_{i \neq j} \exp_{g(i)} |W_{fi\sigma}|\Big)^{*}_{n\text{th order}}\, |W^{*}_{fj\sigma}|^t\, |W_{fj\sigma}|^t\, \Big(\prod_{i \neq j} \exp_{g(i)} |W_{fi\sigma}|\Big)_{m\text{th order}},$$

whose graph is a skeleton graph. As in (3.8) we find that the number of such Wick terms is smaller than

$$2^{4(n+m+2t)}\, \big((4(n+t))!\, (4(m+t))!\big)^{1/2}. \tag{3.23}$$

Using Lemma 3.1 and inequality (3.14) we get with the aid of (3.23) the following bound on (3.22):

$$\begin{aligned} C_\phi(g(j)!)^{-2}\,(f(j+1))^2 &\sum_{n,m}\sum_{t=0}^{g(j)+1}(n!\,m!)^{-1}\binom{g(j)+1}{t}^2(g(j)+1-t)! \\ &\times \Lambda_{fj\sigma}^{g(j)+1-t}\,e^{-\Lambda_{fj\sigma}}\,K^{(n+m+2t)}\,\big((4(n+t))!\,(4(m+t))!\big)^{1/2} \\ &\times \big((n+t)!\,(m+t)!\big)^{-(1+\eta)}\,f(j)^{-t\eta} \end{aligned} \tag{3.24}$$

for some $\eta > 0$ and a constant C_ϕ that depends on ϕ only. Here and in the sequel K denotes a finite constant which, however, may change its value from one inequality to the other.

We use the inequality

$$(a+b)! \le 2^{a+b}\,a!\,b! \tag{3.25}$$

to decouple n and m from t:

$$(4(n+t))! \le K^n(n!)^4\,K^t(t!)^4$$

for some fixed K.

After the summation over n and m which obviously converges, we are left with the following bound on (3.24):

$$\mathrm{const}(g(j)+1)^2\,(f(j+1))^2\sum_{t=0}^{g(j)+1}K^t(f(j))^{-t\eta}\,\frac{\Lambda_{fj\sigma}^{g(j)+1-t}}{(g(j)+1-t)!}\,e^{-\Lambda_{fj\sigma}}\,. \tag{3.26}$$

By C2 and C4,

$$(g(j)+1)^2\,(f(j+1))^2 \le 2\,f(j)^{2a_4(1+a_2)}\,, \quad \text{for large } j\,. \tag{3.27}$$

Let t_0 be the smallest integer such that $f(j)^{2a_4(1+a_2)-t_0\eta} \le \mathrm{const}\,j^{-3}$. Such a t_0 always exists due to C1. Then (3.26) is majorized by

$$\mathrm{const}\sum_{t=0}^{t_0}f(j)^{2a_4(1+a_2)}\,\frac{\Lambda_{fj\sigma}^{g(j)+1-t}}{(g(j)+1-t)!} + \mathrm{const}\,j^{-3}\sum_{t=t_0+1}^{g(j)+1}(K\,f(j)^{-\eta})^{t-t_0}\,. \tag{3.28}$$

The first term in (3.28) is smaller than

$$\mathrm{const}\sum_{t=0}^{t_0}e^{2a_4(1+a_2)a_3^{-1}g(j)}\left(\frac{e\,\lambda(\ln f(j+1)-\ln f(j))}{g(j)+1-t}\right)^{g(j)+1-t},$$

by C3, Lemma 3.3 and Stirling's formula, and thus smaller than

$$\mathrm{const}\sum_{t=0}^{t_0}(e^{2a_4(1+a_2)a_3^{-1}}\,e\,\lambda(a_4-1)\,a_3^{-1})^{g(j)+1-t}\left(\frac{g(j)}{g(j)-t_0}\right)^{g(j)}, \tag{3.29}$$

by C3 and C4. By A5, A3, A2, and using the value $\varepsilon_0 = 1/6$ of Lemma 3.2, we get

$$\begin{aligned} e^{2a_4(1+a_2)a_3^{-1}}\,e\,\lambda(a_4-1)\,a_3^{-1} &< e^{2\cdot 2[1+\delta\varepsilon_0/(2(1+\delta))]\,2(1+\delta)/\varepsilon_0}\,e\lambda(a_4-1)\left(\frac{2(1+\delta)}{\varepsilon_0}\right) \\ &< e^{54(1+\delta)}\,\lambda\,\varepsilon_0^{-1}(a_4-1) < e^{-2}\,. \end{aligned} \tag{3.30}$$

Therefore (3.29) can be bounded by

$$\text{const} \sum_{t=0}^{t_0} e^{-2[g(j)+1-t]} \le \text{const}\, e^{-2g(j)} \le \text{const}\, \big(f(i)\big)^{-2a_3} \le \text{const}\, j^{-2a_1a_3} \le \text{const}\, j^{-(2+\varepsilon)},$$

for some $\varepsilon > 0$, by C3, C1, A1 and A3.

As the second term in (3.28) is obviously bounded by constj^{-3} for large j we finally conclude that (3.28) and thus (3.21) is bounded by const$j^{-(2+\varepsilon)}$, too. The convergence of $\sum_j \sqrt{(3.21)}$ is then obvious and establishes the uniform boundedness of (3.17). This concludes the proof of property *III* and thus the proof of Theorem 3.1.

IV. Natural unitary mappings

In the preceding section we showed that any $(f, g) \in C_\delta$ define a Hilbert space $\hat{\mathscr{I}}(f, g)$ and a limiting Hamiltonian $\hat{H}(f, g)$ which is densely defined in $\hat{\mathscr{I}}(f, g)$. Now we proceed to construct unitary operators which map one of these Hilbert spaces onto another (Theorem 4.4). We call them 'natural' because they emerge in a natural way from the identity map in Fock space. They also give the connection between the different limiting Hamiltonians as will be seen in section VII.

Let $(f_\alpha, g_\alpha) \in C_\delta$, $\alpha = 1, 2$ for some $\delta > 0$. Then we define the mapping $U = U(f_2\, g_2, f_1\, g_1)$ from $\hat{\mathscr{D}}(f_1, g_1) \subset \hat{\mathscr{I}}(f_1, g_1)$ into $\hat{\mathscr{I}}(f_2, g_2)$ by

$$U\, \hat{T}_{\varrho\infty}(f_1, g_1)\, \phi = \lim_{n\to\infty} \hat{T}_{f_2(n)\,\infty}(f_2, g_2)\, \hat{T}_{\varrho f_2(n)}(f_1, g_1)\, \phi \tag{4.1}$$

for $\phi \in \mathscr{D}_0$, $\varrho \in f_1(N)$. Note that $T_{\varrho f_2(n)}(f_1, g_1)\, \phi = \theta_m \in \mathscr{D}_0$ whenever $\phi \in \mathscr{D}_0$ and $n < \infty$. We shall prove that the closure of U, which we denote again by U, is unitary, using the following

Lemma 4.1. (Fabrey [3], Lemma 4.2). *Suppose that for each $\phi \in \mathscr{D}_0$ and each $\varrho_1 \in f_1(N)$, $\varrho_2 \in f_2(N)$ there exist θ_{1n} and $\theta_{2n} \in \mathscr{D}_0$ and $\tau_1(n), \tau_2(n) \ge 0$ such that*

$$\lim_{n\to\infty} \lim_{\sigma\to\infty} \| \hat{T}_{\varrho_1\sigma}(f_1, g_1)\, \phi - \hat{T}_{\tau_2(n)\sigma}(f_2, g_2)\, \theta_{2n} \|^2\, e^{-\Lambda\sigma} = 0 \tag{4.2}$$

and

$$\lim_{n\to\infty} \lim_{\sigma\to\infty} \| \hat{T}_{\varrho_2\sigma}(f_2, g_2)\, \phi - \hat{T}_{\tau_1(n)\sigma}(f_1, g_1)\, \theta_{1n} \|^2\, e^{-\Lambda\sigma} = 0\,. \tag{4.3}$$

Then the closure of $U(f_2\, g_2, f_1\, g_1)$ as defined in (4.1) *is unitary.*

The following lemma establishes the assumptions of Lemma 4.1 for a special case.

Lemma 4.2. *Let $(f_\alpha, g_\alpha) \in C_\delta$, $\alpha = 1, 2$; $\delta > 0$, and suppose that $f_1(N) \supset f_2(N)$. Then for every $\phi \in \mathscr{D}_0$, and each $\varrho_\alpha \in f_\alpha(N)$ the equations* (4.2) *and* (4.3) *hold if we set*

$$\tau_1(n) = \tau_2(n) = f_2(n)\,, \tag{4.4}$$

$$\theta_{1n} = \hat{T}_{\varrho_2 f_2(n)}(f_2, g_2)\, \phi;\ \theta_{2n} = \hat{T}_{\varrho_1 f_2(n)}(f_1, g_1)\, \phi\,. \tag{4.5}$$

Proof: We will prove (4.3) only; the proof of (4.2) is analogous. We have to show that for any $\varepsilon > 0$ there is an $N(\varepsilon)$, such that for all $n > N(\varepsilon)$, uniformly in σ

$$\Big\| \prod_{\substack{j \geq f_2^{-1}(\varrho) \\ g_2(j)}} \exp W_{f_2 j\sigma}\, \phi - \prod_{\substack{j \geq f_1^{-1}[f_2(n)] \\ g_1(j)}} \exp W_{f_1 j\sigma} \cdot \prod_{\substack{i = f_2^{-1}(\varrho) \\ g_2(j)}}^{n-1} \exp W_{f_2 j\sigma}\, \phi \Big\|^2 e^{-A_\sigma} \tag{4.6}$$

is smaller than ε.

By $[\ ]_\nu$ we denote the 'νth order term'. With $M = M(\sigma) = \max\{j \mid W_{f_2 j\sigma} \neq 0\}$, $W_{f_\alpha j\sigma} = W_{\alpha j}$ and $J(n) = f_1^{-1}(f_2(n))$ we have

$$\begin{aligned} L_\sigma &\equiv \prod_{\substack{j \geq n \\ g_2(j)}} \exp W_{2j} - \prod_{\substack{i \geq J(n) \\ g_1(i)}} \exp W_{1i} \\ &= \prod_{\substack{j \geq n \\ g_2(j)}} \exp \Big(\sum_{i = J(j)}^{J(j+1)-1} W_{1i} \Big) - \prod_{j \geq n} \Big(\prod_{\substack{i = J(j) \\ g_1(i)}}^{J(j+1)-1} \exp W_{1i} \Big) \\ &= \sum_{\nu_n, \ldots, \nu_M \geq 0} \Big\{ \prod_{j=n}^{M} \Big[\exp_{g_2(j)} \Big(\sum_{i=J(j)}^{J(j+1)-1} W_{1i} \Big) \Big]_{\nu_j} - \prod_{j=n}^{M} \Big[\prod_{\substack{i = J(j) \\ g_1(i)}}^{J(j+1)-1} \exp W_{1i} \Big]_{\nu_j} \Big\}. \end{aligned} \tag{4.7}$$

It is now important to note that for

$\nu_j \leq \gamma(j) \equiv \min\{g_2(j), g_1(J(j)), \ldots, g_1(J(j+1)-1)\}$ one has

$$\Big[\exp_{g_2(j)} \sum_{i=J(j)}^{J(j+1)-1} W_{1i} \Big]_{\nu_j} = \Big[\prod_{\substack{i = J(j) \\ g_1(i)}}^{J(j+1)-1} \exp W_{1i} \Big]_{\nu_j} . \tag{4.8}$$

Therefore in each nonvanishing term of the expression (4.7) for L_σ there is at least one $j \geq n$ with $\nu_j > \gamma(j)$.

Now the expression (4.6) can be rewritten:

$$\begin{aligned} (4.6) &= \Big\| L_\sigma \prod_{\substack{j = f_2^{-1}(\varrho) \\ g_2(j)}}^{n-1} \exp W_{2j}\, \phi \Big\|^2 e^{-A_\sigma} \\ &= \big(\phi, \hat{T}^*_{\varrho f_2(n)}(f_2, g_2)\, L_\sigma^*\, L_\sigma\, \hat{T}_{\varrho f_2(n)}(f_2, g_2)\, \phi\big)\, e^{-A_\sigma} = I_1 + I_2 . \end{aligned} \tag{4.9}$$

We describe the decomposition $I_1 + I_2$, which is obtained by partitioning the space of variables of the numerical kernel of each Wick term of $\hat{T}^* L^* L \hat{T}$ in the following way. In I_1, the momenta larger than $f_2(n)$ occur in the Λ-components of the Wick term only; the complementary region applies for I_2. Note that in I_1, all contributions from L_σ occur in the Λ-components only.

By the methods of the previous section, we bound I_2 by

$$\sum_S (|\phi|, |S|\,|\phi|),$$

where $\sum_S$ runs over all Wick terms S in the Wick expansion of $\hat{T}^* L_\sigma^* L_\sigma \hat{T}$ whose graph is a skeleton graph and whose kernel $s(q_1, \ldots, q_t)$ vanishes unless at least one of the variables has absolute value larger than $f_2(n)$. Using Remark 3.1 and inequality (3.14) we conclude that there is some $\eta > 0$, such that uniformly in σ

$$I_2 \leq \text{const} f_2(n)^{-\eta} . \tag{4.10}$$

In order to estimate I_1 we have to use the representation (4.7) of L_σ and equation (4.8), i.e. we have to take advantage of cancellations in L_σ. We can write

$$I_1 = \| \hat{T}_{\varrho f_2(n)}(f_2, g_2)\, \phi \|^2\, e^{-\Lambda_{f_2(n)}} \cdot C_n\,, \tag{4.11}$$

and $\| \hat{T}_{\varrho f_2(n)}(f_2, g_2)\, \phi \|^2\, e^{-\Lambda_{f_2(n)}}$ is uniformly bounded in n by Theorem 3.1 (Property *I*). Furthermore

$$C_n = e^{-[\Lambda_\sigma - \Lambda_{f_2(n)}]}(\phi_0, L_\sigma^* L_\sigma \phi_0)_\Lambda\,, \tag{4.12}$$

where the subscript Λ indicates that one has to take only those contributions to $L_\sigma^* L_\sigma$ whose graph consists of Λ-components only, by construction of I_1.

We certainly increase the value of (4.12) if we replace L_σ by

$$2 \sum_{j=n}^{M} (\exp|W_{2j}| - \underset{\gamma(j)}{\exp}|W_{2j}|) \prod_{\substack{i=n \\ i \neq j}}^{M} \exp|W_{2i}|\,, \tag{4.13}$$

which means that with $\Lambda_{2j} \equiv \Lambda_{f_2 j\sigma}$,

$$\begin{aligned} C_n &\leq 4\, e^{-[\Lambda_\sigma - \Lambda_{f_2(n)}]} \sum_{j=n}^{M} (\exp \Lambda_{2j} - \underset{\gamma(j)}{\exp} \Lambda_{2j}) \prod_{\substack{i=n \\ i \neq j}}^{M} \exp \Lambda_{2i} \\ &= 4 \sum_{j=n}^{M} (\exp - \Lambda_{2j})(\exp \Lambda_{2j} - \underset{\gamma(j)}{\exp} \Lambda_{2j}) \leq 4 \sum_{j=n}^{\infty} \frac{(\Lambda_{2j})^{\gamma(j)+1}}{(\gamma(j)+1)!}\,. \end{aligned} \tag{4.14}$$

By definition, we have

$$\begin{aligned} \gamma(j) &= \min\{g_2(j), g_1(J(j)), \ldots, g_1(J(j+1)-1)\} \quad \text{and thus} \\ \gamma(j) &> \min\{a_{3,2} \ln f_2(j), a_{3,1} \ln f_1(J(j))\}\,, \\ &\qquad \text{for some} \quad a_{3,1}, a_{3,2} > \frac{\varepsilon_0}{2(1+\delta)}\,, \quad \text{by C3}\,, \\ &= \min\{a_{3,1}, a_{3,2}\} \ln f_2(j), \quad \text{by the definition of } J\,. \end{aligned} \tag{4.15}$$

Inequality (4.15) and the assumption $(f_\alpha, g_\alpha) \in C_\delta$, $\alpha = 1, 2$, ensure that (f_2, γ) satisfy conditions C_δ with a possible exception of C2. But as was shown in the proof of Lemma 3.4, C1, C3, C4 and A1–A5 are sufficient to guarantee that for fixed n the sum in (4.14) is finite and thus goes to zero as n tends to infinity.

Therefore $I_1 + I_2 =$ (4.6) tends to zero as n goes to infinity, uniformly in σ. This ends the proof of (4.3) and thus of Lemma 4.2.

A next step will be to eliminate the assumption $f_1(N) \supset f_2(N)$ in Lemma 4.2. Let $(f_\alpha, g_\alpha) \in C_\delta$, $\alpha = 1, 2$ for some $\delta > 0$. Then we define (uniquely) a function f: $N \to R^+$, strictly increasing, such that $f(N) = f_1(N) \cup f_2(N)$. Let for $\alpha = 1, 2$, $j_\alpha(i) = \min\{j \mid f_\alpha(j) \geq f(i)\}$. We define g: $N \to N \cup \{0\}$ by

$$g(i) = \min\{g_1(j_1(i)), g_2(j_2(i))\}\,. \tag{4.16}$$

We write $(f, g) = \phi\,((f_1\, g_1), (f_2\, g_2))$.

Lemma 4.3. *Let* $(f_\alpha, g_\alpha) \in C_\delta$, $\alpha = 1, 2$, $\delta > 0$. *Then* $(f, g) = \phi\big((f_1\, g_1), (f_2\, g_2)\big)$ *satisfy condition* C_δ, *too.*

We postpone the proof of Lemma 4.3 and discuss its consequences. It follows immediately from the definition of (f, g) that $f_1(N) \subset f(N)$ and $f_2(N) \subset f(N)$. Therefore by Lemma 4.3 we can apply the Lemmata 4.1 and 4.2 to conclude that

$$U(f_2\, g_2, f_1\, g_1) = U(f_2\, g_2, f\, g)\; U(f\, g, f_1\, g_1) \tag{4.17}$$

is a unitary map from $\hat{\mathscr{I}}(f_1, g_1)$ to $\hat{\mathscr{I}}(f_2, g_2)$. Finally, we consider the most general case and we suppose that $(f_\alpha, g_\alpha) \in C_{\delta_\alpha}$, $\alpha = 1, 2$, for some $\delta_1, \delta_2 > 0$. Note that for $f_0(j) = 2^j$, $g_0(j) = j^2$, we get $(f_0, g_0) \in C_\delta$ for all $\delta > 0$. Let $(f_{\alpha 0}, g_{\alpha 0}) = \phi\big((f_\alpha\, g_\alpha), (f_0\, g_0)\big)$, $\alpha = 1, 2$. Then the equality

$$\begin{aligned} &U(f_2\, g_2, f_1\, g_1) \\ &\quad = U(f_2\, g_2, f_{20}\, g_{20})\; U(f_{20}\, g_{20}, f_0\, g_0)\; U(f_0\, g_0, f_{10}\, g_{10})\; U(f_{10}\, g_{10}, f_1\, g_1) \end{aligned} \tag{4.18}$$

shows that $U(f_2\, g_2, f_1\, g_1)$ is unitary since each of the factors on the right of (4.18) is. We state this result as

Theorem 4.4. *Let* $(f_\alpha, g_\alpha) \in C_{\delta_\alpha}$, $\delta_\alpha > 0$, $\alpha = 1, 2$. *Then* $U(f_2\, g_2, f_1\, g_1)$ *is a unitary operator on* $\hat{\mathscr{I}}(f_1, g_1)$ *to* $\hat{\mathscr{I}}(f_2, g_2)$.

We now prove Lemma 4.3.

Let $a_{1\alpha}, \ldots, a_{4\alpha}$ be the constants for which (f_α, g_α) satisfy C1–C4, $\alpha = 1, 2$. We check C1–C4, A1–A5 for $(f, g) = \phi\big((f_1\, g_1), (f_2\, g_2)\big)$;

$$\begin{aligned} f(i) &> \max\{f_1(j_1(i) - 1), f_2(j_2(i) - 1)\} \\ &> \max\{(j_1(i) - 1)^{a_{11}}, (j_2(i) - 1)^{a_{12}}\}, \quad \text{by C1,} \\ &> \left(\frac{i}{2} - 1\right)^{\min\{a_{11}, a_{12}\}} > i^{a_1}, \quad \text{for large } i\,. \end{aligned}$$

We have used $j_1(i) + j_2(i) \geq i$ and we define a_1 such that $2(1 + \delta)\, \varepsilon_0^{-1} < a_1 < \min\{a_{11}, a_{12}\}$. Then C1 and A1 are verified for (f, g).

$$\begin{aligned} g(i) &= \min\{g_1(j_1(i)), g_2(j_2(i))\} \\ &\leq \min\{f_1(j_1(i))^{a_{21}}, f_2(j_2(i))^{a_{22}}\}, \quad \text{by C2} \\ &< \big(\min\{f_1(j_1(i)), f_2(j_2(i))\}\big)^{a_2} \\ &= f(i)^{a_2}, \end{aligned}$$

where $a_2 = \max\{a_{21}, a_{22}\}$. This proves C2 and A2.

$$\begin{aligned} \ln f(i) &= \min\{\ln f_1(j_1(i)), \ln f_2(j_2(i))\} \\ &< \min\{g_1(j_1(i))\, a_{31}^{-1}, g_2(j_2(i))\, a_{32}^{-1}\}, \quad \text{by C3,} \\ &\leqslant g(i) \cdot a_3^{-1} \end{aligned}$$

where $a_3 = \min\{a_{31}, a_{32}\}$. This proves C3 and A3. Finally, since $f(N) = f_1(N) \cup f_2(N)$, by C4 $f(i + 1) < f(i)^{a_4}$, where $a_4 = \min\{a_{41}, a_{42}\}$. This establishes C4 and A4, A5 for (f, g) and ends the proof of Lemma 4.3.

V. The Complete Interaction

In this section we give the modifications which are needed to carry over the results of the previous sections to the case of the full Hamiltonian (1.1). First we define a family of dressing transformations $T(f, g)$ which lead to Hilbert spaces $\mathscr{I}(f, g)$ in which the limiting Hamiltonians $H(f, g)$ are densely defined. We then construct the natural unitary operators on one of the $\mathscr{I}(f, g)$ to another in the sense of section IV.

We need some definitions: $V'_{2\sigma}$ is the truncation of $V_{2\sigma}$ in which $v_{2\sigma}(k_1, \ldots, k_4)$ is replaced by zero if the momenta k_1, k_2 belonging to the creation operator satisfy $|k_1| + |k_2| \le 2\,(|k_3| + |k_4|)$. We set $\Delta^{14}_\sigma = M_{2\sigma} - \underbrace{V_{1\sigma}\,\Gamma(V_{4\sigma})}_{3}$, and furthermore for $j \in N$, $V_{3j\sigma}$, $V_{2j\sigma}$, $\Delta^{14}_{j\sigma}$ are defined by restricting the momentum of largest magnitude $|k|$ created by $V_{3\sigma}$, $V_{2\sigma}$ and Δ^{14}_σ respectively to the region where $|k| \in [2^j, 2^{j+1})$. For the following definitions, see also [9]. Let Y be the set of all functions x which map N into Z.

Then we set

$$X_{\varrho\sigma}(f, g) = X_{\varrho\sigma} = \{x : x \in Y, \quad \text{for all } i \in N, \quad 0 \le x(i) \le g(i), \quad \text{and}$$
$$x(i) = 0 \quad \text{if} \quad f(i) < \varrho \quad \text{or if} \quad f(i) \ge \sigma\}\,. \tag{5.1}$$

For $x \in Y$ we define $x_k \in Y$ by $x_k(i) = x(i) - \delta_{ik}$ and $\bar{x} \in Y$ by $\bar{x}(i) = g(i) - x(i)$.

Let furthermore $(j_1, \ldots, j_m) = j^m \in (N)^m$; $j^0 = \emptyset$, and let

$$J_\varrho = J_{\varrho\xi} = \{j^m : 2^{j_1} \ge \varrho, \Big(\sum_{i=1}^{k-1} j_i\Big)^\xi < j_k,\ k = 2, 3, \ldots\}; \quad \text{with } \xi < 1\,.$$

We shall keep $\xi = 3/4$ fixed and omit it in the following.

We now define $S_{\tau\sigma}(f, g \mid j^m, x) = S_{\tau\sigma}(j^m, x)$ by

$$S_{\tau\sigma}(j^0, x) = \begin{cases} 1 & \text{if} \quad x \equiv 0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

$$\begin{aligned} S_{\tau\sigma}(j^m, x) = & - \Gamma\big((V_{3j_m\sigma} + V'_{2j_m\sigma} - \Delta^{14}_{j_m\sigma})\, S_{\tau\sigma}(j^{m-1}, x)\big) \\ & - \sum_{i=0}^{\infty} \Gamma\big((V_{3j_m\sigma} + \underbrace{V_{2j_m\sigma})\, W_{fi\tau}}\, S_{\tau\sigma}(j^{m-1}, x_i)\big) \\ & - \sum_{i_1, i_2 = 0}^{\infty} \Gamma\left(\frac{1}{2}\, \underbrace{\underbrace{V_{2j_m\sigma}\, W_{fi_1\tau}}\, W_{fi_2\tau}}\, S_{\tau\sigma}(j^{m-1}, x_{i_1 i_2})\right), \end{aligned} \tag{5.2}$$

and finally for $\varrho_1 \in f(N)$, $\varrho_2 \ge 0$,

$$T_{\varrho_1\varrho_2\sigma}(f, g) = \sum_{x \in X_{\varrho_1\sigma}} \prod_{i = f^{-1}(\varrho_1)}^{\infty} (\exp W_{fi\sigma})_{\bar{x}(i)} \sum_{j^m \in J_{\varrho_2}} S_{\sigma\sigma}(j^m, x)\,. \tag{5.3}$$

This is the definition of the dressing transformation for the full Hamiltonian (1.1). This definition insures that $W_{fi\sigma}$ occurs at most $g(i)$ times in $T_{\varrho_1\varrho_2\sigma}(f, g)$. We note that for $\varrho_1 = \varrho_2 = 0$ and $f(i) = 2^i$, $g(i) = i$, $\xi = 3/4$, Equation (5.3) is an explicit definition of Glimm's original dressing transformation [4, p. 26].

By modifying Glimm's proofs [4] in the sense of our Theorem 3.1, we arrive at the

Theorem 5.1. *Let* $(f, g) \in C_\delta$ *for some* $\delta > 0$, $\varrho_1 \in f(N)$, $\varrho_2 \geq 0$, $T_{\varrho_1 \varrho_2 \sigma}(f, g) = T_\sigma$ *defined as above. Then* T_σ *satisfies for all* $\phi, \psi \in \mathscr{D}_0$:

I. $$\lim_{\sigma \to \infty} (T_\sigma \phi, T_\sigma \psi)\, e^{-\Lambda_\sigma} = (T_\infty \phi, T_\infty \psi)_r \tag{5.4}$$

exists.

II. *The expression* (5.4) *defines a positive definite scalar product on*

$$\mathscr{D}(f, g) = \langle T_{\varrho_1 \varrho_2 \infty}(f, g)\, \phi : \phi \in \mathscr{D}_0,\, \varrho_1 \in f(N),\, \varrho_2 \geq 0 \rangle \tag{5.5}$$

III. $\| H_\sigma T_\sigma \phi \|^2 e^{-\Lambda_\sigma}$ *is uniformly bounded in* $\sigma \leq \infty$ *and* $\lim_{\sigma \to \infty} (T_\sigma \phi, H_\sigma T_\sigma \psi)\, e^{-\Lambda_\sigma}$ *defines a symmetric operator* $H(f, g)$ *on* $\mathscr{D}(f, g)$.

$\mathscr{D}(f, g)$ *together with* $(\cdot, \cdot)_r$ *is a prehilbert space, whose completion we denote by* $\mathscr{I}(f, g)$.

Next we want to compare $\mathscr{I}(f, g)$ with $\hat{\mathscr{I}}(f, g)$. We need the following extension of Lemma 4.2:

Lemma 5.2. *Let* $(f, g) \in C_\delta$, $\delta > 0$. *For each* $\phi \in \mathscr{D}_0$, *each* $\varrho_1 \in f(N)$, $\varrho_2 \geq 0$, *there exists* $\theta_{1n}, \theta_{2n} \in \mathscr{D}_0$ *such that*

$$\lim_{n \to \infty} \lim_{\sigma \to \infty} \| T_{\varrho_1 \varrho_2 \sigma}(f, g)\, \phi - \hat{T}_{f(n)\sigma}(f, g)\, \theta_{1n} \|^2 e^{-\Lambda_\sigma} = 0 \tag{5.6}$$

and

$$\lim_{n \to \infty} \lim_{\sigma \to \infty} \| \hat{T}_{\varrho\sigma}(f, g)\, \phi - T_{f(n) f(n) \sigma}(f, g)\, \theta_{2n} \|^2 e^{-\Lambda_\sigma} = 0\,. \tag{5.7}$$

Proof: The proof is completely analogous to the one of Lemma 4.2. We only give θ_{1n} and θ_{2n}.

$$\theta_{1n} = \sum_{x \in X_{\varrho_1 f(n)}} \prod_{i = f^{-1}(\varrho_1)}^{n-1} (\exp W_{fi\sigma})_{\bar{x}(i)} \sum_{j^m \in J_{\varrho_2}} S_{f(n) f(n)}(j^m, x)\, \phi\,, \tag{5.8}$$

$$\theta_{2n} = \prod_{i = f^{-1}(\varrho_1)}^{n-1} \exp W_{fi\sigma}{}_{g(i)}\, \phi\,. \tag{5.9}$$

Lemma 5.2 suffices to obtain the natural unitary operator mapping $\mathscr{I}(f, g)$ onto $\hat{\mathscr{I}}(f, g)$.

This combined with the natural unitary mapping $U(f_2 g_2, f_1 g_1)$: $\hat{\mathscr{I}}(f_1, g_1) \to \hat{\mathscr{I}}(f_2, g_2)$, established in Theorem 4.4, defines a natural unitary mapping of $\mathscr{I}(f_1, g_1)$ onto $\mathscr{I}(f_2, g_2)$ for $(f_\alpha, g_\alpha) \in C_{\delta_\alpha}$, $\delta_\alpha > 0$, $\alpha = 1, 2$.

An explicit definition of this unitary map $V(f_2 g_2, f_1 g_1)$: $\mathscr{I}(f_1, g_1) \to \mathscr{I}(f_2, g_2)$ can be given for $f_{\alpha_1}(N) \supset f_{\alpha_2}(N)$ by

$$V(f_2 g_2, f_1 g_1)\, T_{\varrho_1 \varrho_2 \infty}(f_1, g_1)\, \phi = \lim_{n \to \infty} T_{\tau\tau^2 \infty}(f_2, g_2)\, \theta_{\varrho_1 \varrho_2 \tau}\,, \tag{5.10}$$

where $\phi \in \mathscr{D}_0$, $\varrho_1 \in f_1(N)$, $\tau = \tau(n) = f_{\alpha_2}(n)$ and

$$\theta_{\varrho_1 \varrho_2 \tau} = \sum_{x \in X_{\varrho_1 \tau}(f_1, g_1)} \prod_{i = f_1^{-1}(\varrho_1)}^{n-1} (\exp W_{f_1 i \sigma})_{\bar{x}(i)} \sum_{j^m \in J_{\varrho_2}} S_{\tau\tau^2}(j^m, x)\, \phi \,. \tag{5.11}$$

The proof of this statement is again a straightforward application of the arguments introduced in section IV. Of course the choice of $T_{\tau\tau^2\infty}(f_2, g_2)\, \theta_{\varrho_1 \varrho_2 \tau}$ to approximate the vector $V(f_2\, g_2, f_1\, g_1)\, T_{\varrho_1 \varrho_2 \infty}(f_1\, g_1)\, \phi$ is very special, but it will be helpful in section VII, where we shall compare two Hamiltonians $H(f_1, g_1)$ and $H(f_2, g_2)$.

VI. Equivalence of (Non Fock) Weyl Systems

It was shown by Fabrey [3] that on the spaces $\hat{\mathscr{I}}(f, g)$, $f(i) = \alpha^i$, $\alpha > 1$, $g(i)$ strictly increasing but polynomially bounded, one gets representations of the CCR which are not unitarily equivalent to a direct sum of Fock representations. Furthermore, he proved that two such representations, given in the form of exponential Weyl systems, are unitarily equivalent provided the f's are equal and the g's are different in only finitely many points.

A slightly different discussion of the representations of the CCR in the $(\phi^4)_3$ model is given by Hepp in [6], [7]. He starts from Glimm's original 'renormalized' Hilbert space $\mathscr{I}_{ren}$, which in our notation is the closure of $\langle T_{\varrho\infty}(f, g)\, \mathscr{D}_0$, $\varrho = 0$, $f(i) = 2^i$, $g(i) = i\rangle$ in the $\big(\lim (\cdot, \cdot)\, e^{-A_\sigma}\big)^{1/2}$ norm. This space may be too small for the desired representation of the CCR; $e^{i\phi}$ is not known to be a unitary operator on $\mathscr{I}_{ren}$. Hepp constructs a larger space $\mathscr{H} \supset \mathscr{I}_{ren}$, using the Gelfand Neumark Segal construction and then obtains a non-Fock representation of the CCR on $\mathscr{H}$. It is straightforward that $\mathscr{H}$ can be identified with a subspace of $\mathscr{I}(f, g)$ and that Hepp's representation of the CCR is a subrepresentation of the one constructed by Fabrey. An easy calculation should show that $\mathscr{H} = \mathscr{I}(f, g)$ and that the two representations are the same.

It is the purpose of this section to establish the existence of exponential Weyl systems on all spaces $\mathscr{I}(f, g)$ and $\hat{\mathscr{I}}(f, g)$ for $(f, g) \in C_\delta$, $\delta > 0$, and to show that all of them are unitarily equivalent.

Definition. (Weyl [11], Chaiken [1, def. 1.1]). A Weyl system is a map $y \to W(y)$ from a complex inner product space $\mathscr{K}$ to unitary operators on a complex Hilbert space $\mathscr{H}$ such that

$$W(y_1)\, W(y_2) = \exp[2^{-1}\, i\, \mathrm{Im}\langle y_1, y_2\rangle]\, W(y_1 + y_2) \tag{6.1}$$

and for each $y \in \mathscr{K}$, $W(t\, y)$ considered as a function of the real variable t is weakly continuous at $t = 0$. Our inner product space is given by

$$\mathscr{K} = \{y \in L^2(R^2) : \| \mu^\vartheta\, y \|_2 < \infty\}\,, \tag{6.2}$$

and the inner product is $\langle y_1, y_2\rangle = \int \bar{y}_1(k)\, \big(2\, \mu(k)\big)^{-1}\, y_2(k)\, dk$. We choose $\vartheta = 2$, in view of a later application of Lemma 3.2. We remark that we also could take any $\vartheta > \varepsilon_0/2$, but this would require a slight modification of Lemma 3.2., which we do not want to give here.

For $y \in \mathscr{K}$ let

$$\phi(y) = \int \mu(k)^{-1/2} [\bar{y}(k)\, a^*(k) + y(k)\, a(k)]\, dk\,, \tag{6.3}$$

then by setting $W(y) = e^{i\phi(y)}$, $\mathscr{H} = \mathscr{I}$, the Fock space, we get the Fock representation.

The (non-Fock) representations $W(f, g \mid y)$, $y \in \mathscr{K}$ on the Hilbert space $\mathscr{I}(f, g)$ will be defined by

$$\begin{aligned}(T_{\varrho_1\varrho_2\infty}(f, g)\, \phi, W(f, g \mid y)\, T_{\varrho_1'\varrho_2'\infty}(f, g)\, \psi)_r \\ = \lim_{\sigma\to\infty}\bigl(T_{\varrho_1\varrho_2\sigma}(f, g)\, \phi, W(y)\, T_{\varrho_1'\varrho_2'\sigma}(f, g)\, \psi\bigr)\, e^{-\Lambda_\sigma}\,,\end{aligned} \tag{6.4}$$

and correspondingly we define the representations $\widehat{W}(f, g \mid y)$ on $\widehat{\mathscr{I}}(f, g)$.

The results of this section are summarized in the following two theorems.

Theorem 6.1. *Let* $(f, g) \in C_\delta$ *for some* $\delta > 0$. *Then* $W(f, g \mid y)$ *and* $\widehat{W}(f, g \mid y)$, *as defined in* (6.4), *are Weyl systems.*

Theorem 6.2. *Let* $(f_\alpha, g_\alpha) \in C_{\delta_\alpha}$, $\delta_\alpha > 0$, $\alpha = 1, 2$. *Then* $W(f_1, g_1 \mid y)$, $W(f_2, g_2 \mid y)$, $\widehat{W}(f_1, g_1 \mid y)$ *and* $\widehat{W}(f_2, g_2 \mid y)$ *are all unitarily equivalent.*

Remark. The unitary equivalence will be established using the natural unitary maps introduced in sections IV and V. First we prove a technical lemma.

Lemma 6.3. *Let* $(f, g) \in C_\delta$ *for some* $\delta > 0$, $y \in \mathscr{K}$. *Then the limits*

$$\lim_{\sigma\to\infty}\bigl(T_{\varrho_1\varrho_2\sigma}\varphi, \phi(y)^m\, T_{\varrho_1'\varrho_2'\sigma}\psi\bigr)\, e^{-\Lambda_\sigma}\,, \tag{6.5}$$

$$\lim_{\sigma\to\infty} \| \phi(y)^m\, T_{\varrho_1\varrho_2\sigma}(f, g)\, \varphi \|\, e^{-\Lambda_\sigma/2} \tag{6.6}$$

exist and (6.6) *is bounded by*

$$K^m (m!)^{1/2} \| \mu^2\, y \|_2^m \tag{6.7}$$

for a constant K which depends on ϕ *only. The limit* (6.5) *defines an operator* $\phi(f, g \mid y)^m$ *on* $\mathscr{D}(f, g)$.

The same statements hold if we replace $T_{\varrho_1\varrho_2\sigma}$ by $\widehat{T}_{\varrho\sigma}$ and $\mathscr{D}(f, g)$ by $\widehat{\mathscr{D}}(f, g)$, and we denote the operators thus obtained by $\widehat{\phi}(f, g \mid y)^m$.

Proof: The existence of the limits (6.5) and (6.6) is proved by the methods of sections III and V. To obtain the bound (6.7) we conclude as in the proof of Theorem 3.1 that

$$\| \phi(y)^m\, T_{\varrho_1\varrho_2\sigma}(f, g)\, \varphi \|^2\, e^{-\Lambda_\sigma} \le \sum_{n_1 n_2} \sum_{S_{n_1 n_2}} (| \varphi |, | S_{n_1 n_2} |\, | \varphi |)\,, \tag{6.8}$$

where the sum $\sum_{S_{n_1 n_2}}$ runs over all Wick terms $S_{n_1 n_2}$ in the expansion of $(T^*_{\varrho_1\varrho_2\sigma})_{n_1}(f, g)\, (\phi(y)^m)^*\, (\phi(y)^m)\, (T_{\varrho_1\varrho_2\sigma})_{n_2}(f, g)$, whose graph is a skeleton graph.

Using Lemma 3.2 and the Schwarz inequality in the contracted variables between $T^* \ldots T \ldots$ and $(\phi(y)^m)^*\, (\phi(y)^m)$ we obtain

$$(| \varphi |, | S_{n_1 n_2} |\, | \varphi |) \le C_\varphi\, K^{n_1 + n_2 + m}\, \| \mu^2\, y \|_2^{2m} (n_1!\, n_2!)^{-(2+\eta)} \tag{6.10}$$

for some $\eta > 0$.

The number of Wick terms $S_{n_1 n_2}$ is bounded by

$$K^{n_1+n_2+m}(n_1!\, n_2!)^2\, m!\,, \tag{6.11}$$

and (6.10, 11) combined with (6.8) give the desired bound (6.7). Now Theorem 6.1 is proved as Theorem 2 in [3]. To establish Theorem 6.2 we prove first the unitary equivalence of $\hat{W}(f_1, g_1 \mid y)$ and $\hat{W}(f_2, g_2 \mid y)$ for $(f_\alpha, g_\alpha) \in C_\delta$, $\delta > 0$ and $f_1(N) \supset f_2(N)$. We assert that

$$\hat{W}(f_1, g_1 \mid y) = U^{-1}(f_2\, g_2, f_1\, g_1)\, \hat{W}(f_2, g_2 \mid y)\, U(f_2\, g_2, f_1\, g_1) \tag{6.12}$$

with the unitary operator $U(f_2\, g_2, f_1\, g_1)$ as defined in (4.1). We use the following abbreviations:

$$U = U(f_2\, g_2, f_1\, g_2)\,;\; \hat{W}(f_\alpha\, g_\alpha \mid y) = \hat{W}(\alpha \mid y)\,;$$

$$\hat{T}_{\varrho\sigma}(f_\alpha, g_\alpha) = \hat{T}_{\varrho\sigma}(\alpha),\; \hat{\mathscr{I}}(f_\alpha, g_\alpha) = \hat{\mathscr{I}}(\alpha),\; \alpha = 1, 2\,.$$

Choose a sequence of vectors $\hat{T}_{\varrho(n)\infty}(1)\, \phi_n \in \hat{\mathscr{D}}(f_1, g_1)$ such that

$$\|\, \hat{W}(1 \mid y)\, \hat{T}_{\varrho\infty}(1)\, \varphi - \hat{T}_{\varrho(n)\infty}(1)\, \varphi_n \,\|_r \to 0 \quad \text{as} \quad n \to \infty\,. \tag{6.13}$$

$$\|\cdot\|_r^2 = (\,\cdot, \cdot\,)_r = \lim_{\sigma \to \infty} (\,\cdot, \cdot\,)\, e^{-\Lambda_\sigma}\,.$$

We now write

$$\begin{aligned}
&\big|\big(\hat{T}_{\varrho'\infty}(2)\, \psi, U\, \hat{W}(1 \mid y)\, \hat{T}_{\varrho\infty}(1)\, \varphi\big)_r - \big(\hat{T}_{\varrho'\infty}(2)\, \psi, \hat{W}(2 \mid y)\, U\, \hat{T}_{\varrho\infty}(1)\, \varphi\big)_r\big| \\
&\quad \le \lim_{\sigma \to \infty} \big|\big(\hat{T}_{\varrho'\infty}(2)\, \psi, U\, \hat{W}(1 \mid y)\, \hat{T}_{\varrho\infty}(1)\, \varphi\big)_r - \big(\hat{T}_{\varrho'\sigma}(2)\, \psi, e^{i\phi(y)}\, \hat{T}_{\varrho\sigma}(1)\, \varphi\big)\, e^{-\Lambda_\sigma}\big| \quad (6.14) \\
&\quad + \lim_{\sigma \to \infty} \big|\big(\hat{T}_{\varrho'\infty}(2)\, \psi, \hat{W}(2 \mid y)\, U\, \hat{T}_{\varrho\infty}(1)\, \varphi\big)_r - \big(\hat{T}_{\varrho'\sigma}(2)\, \psi, e^{i\phi(y)}\, \hat{T}_{\varrho\sigma}(1)\, \varphi\big)\, e^{-\Lambda_\sigma}\big|\,.
\end{aligned}$$

We give an upper bound on the first term on the right of (6.14),

$$\begin{aligned}
&\lim_{n \to \infty} \lim_{\sigma \to \infty} \big|\big(\hat{T}_{\varrho'\infty}(2)\, \psi, U\, \hat{W}(1 \mid y)\, \hat{T}_{\varrho\infty}(1)\, \varphi\big)_r - \big(\hat{T}_{\varrho'\sigma}(2)\, \psi, \hat{T}_{\varrho(n)\sigma}(1)\, \varphi_n\big)\, e^{-\Lambda_\sigma}\big| \\
&+ \lim_{n \to \infty} \lim_{\sigma \to \infty} \big|\big(\hat{T}_{\varrho'\sigma}(2)\, \psi, \big(e^{i\phi(y)}\, \hat{T}_{\varrho\sigma}(1)\, \varphi - \hat{T}_{\varrho(n)\sigma}(1)\, \varphi_n\big)\big)\, e^{-\Lambda_\sigma}\big| \le \quad (6.15) \\
&0 + \lim_{n \to \infty} \lim_{\sigma \to \infty} \|\, \hat{T}_{\varrho'\sigma}(2)\, \psi \,\|\, e^{-\Lambda_\sigma/2}\, \|\, e^{i\phi(y)}\, \hat{T}_{\varrho\sigma}(1)\, \varphi - \hat{T}_{\varrho(n)\sigma}(1)\, \varphi_n \,\|\, e^{-\Lambda_\sigma/2} = 0\,.
\end{aligned}$$

In the same way we show that the second term on the right of (6.14) is equal to zero, and thus the left side of (6.14) vanishes for all ϱ, ϱ'; $\varphi, \psi \in \mathscr{D}_0$.

As $\hat{\mathscr{D}}(f_2, g_2)$, the set of vectors of the form $\hat{T}_{\varrho'\infty}(2)\, \psi$, $\psi \in \mathscr{D}_0$, and finite linear combinations of them, is dense in $\hat{\mathscr{I}}(f_2, g_2)$, assertion (6.12) is proved.

For all other cases the unitary equivalence is established in the same way if in (6.12) we use the natural unitary operator which maps the Hilbert space of one Weyl system onto the Hilbert space of the other. All these unitary operators have been constructed in sections IV and V. This ends the proof of Theorem 6.2.

VII. Unitary Equivalence of the Hamiltonians

As we have stated in section V, each $(f, g) \in C_\delta$ can be used to define a Hamiltonian $H(f, g)$ as a symmetric operator in $\mathscr{I}(f, g)$. In this section, our main result is the following connection between these Hamiltonians.

Theorem 7.1. *For any* $(f_\alpha, g_\alpha) \in C_{\delta_\alpha}$, $\delta_\alpha > 0$, $\alpha = 1, 2$, *the closures*

$$\bar{H}(f_\alpha, g_\alpha) \quad \text{of} \quad H(f_\alpha, g_\alpha) \quad \text{in} \quad \mathscr{I}(f_\alpha, g_\alpha),\ \alpha = 1, 2$$

are unitarily equivalent.

Proof: As in the proof of Theorem 6.2 it is sufficient to prove the assertion for $(f_\alpha, g_\alpha) \in C_\delta$, $\delta > 0$, $f_1(N) \supset f_2(N)$ or $f_1(N) \subset f_2(N)$.

Our main ingredient is the following lemma, in which we make use of the approximation (5.10, 11) in the construction of the unitary operator $V(f_2\, g_2, f_1\, g_1)$ on $\mathscr{I}(f_1, g_1)$ to $\mathscr{I}(f_2, g_2)$.

Lemma 7.2. *Let* $(f_\alpha, g_\alpha) \in C_\delta$, $\delta > 0$, $\alpha = 1, 2$ *and suppose* $f_1(N) \subset f_2(N)$ [or $f_2(N) \subset f_1(N)$]. *Let* $\phi, \psi \in \mathscr{D}_0$, $\varrho_{\alpha 1} \in f_1(N) \cap f_2(N)$, $\varrho_{\alpha 2} \in R^+$. *Finally let* $\theta_{\varrho_{11}\varrho_{12}\tau}$ *be defined as in* (5.11). *Then with* $\tau = \tau(n) = f_2(n)$ [or $\tau(n) = f_1(n)$], *one has*

$$\lim_{\sigma \to \infty} \Big(T_{\varrho_{21}\varrho_{22}\sigma}(f_2, g_2)\, \psi, H_\sigma\big(T_{\tau\tau^2\sigma}(f_2, g_2)\, \theta_{\varrho_{11}\varrho_{12}\tau} - T_{\varrho_{11}\varrho_{12}\sigma}(f_1, g_1)\, \phi\big)\Big)\, e^{-A_\sigma} \to 0\,,$$

$$\text{as} \quad n \to \infty\,. \tag{7.1}$$

Furthermore

$$\limsup_{\sigma \to \infty} \| H_\sigma\, T_{\tau\tau^2\sigma}(f_2, g_2)\, \theta_{\varrho_{11}\varrho_{12}\tau} \|^2\, e^{-A_\sigma}\,, \tag{7.2}$$

is bounded uniformly in τ, *and*

$$\limsup_{\sigma \to \infty} \| H_\sigma\, T_{\varrho_{11}\varrho_{12}\sigma}(f_1, g_1)\, \phi \|^2\, e^{-A_\sigma} \tag{7.3}$$

is finite.

We postpone the proof of Lemma 7.2 and prove now Theorem 7.1. Since $H(f_\alpha, g_\alpha)$, $\alpha = 1, 2$ is symmetric, it is closable. Now (7.1)–(7.3) say that for $\phi \in \mathscr{D}_0$, $V_{21}\, T_{\varrho_{11}\varrho_{12}\infty}\, \phi$ is in the domain $\mathscr{D}(\bar{H}(f_2, g_2))$. We have written $V_{21} \equiv V(f_2\, g_2, f_1\, g_1)$. (7.1)–(7.3) imply also

$$\bar{H}(f_2, g_2)\, |_{V_{21}\mathscr{D}[H(f_1, g_1)]} = V_{21}\, H(f_1, g_1)\, V_{21}^{-1}\,.$$

In other terms,

$$V_{21}\, H(f_1, g_1)\, V_{21}^{-1} \subset \bar{H}(f_2, g_2)\,,$$

or

$$H(f_1, g_1) \subset V_{21}^{-1}\, \bar{H}(f_2, g_2)\, V_{21}$$

and therefore

$$\bar{H}(f_1, g_1) \subset V_{21}^{-1}\, \bar{H}(f_2, g_2)\, V_{21}\,. \tag{7.4}$$

Exchanging (f_1, g_1) and (f_2, g_2) we get

$$\bar{H}(f_2, g_2) \subset V_{21}\, \bar{H}(f_1, g_1)\, V_{21}^{-1} \tag{7.5}$$

and combining the two inclusions (7.4), (7.5) we complete the proof of Theorem 7.1.

Proof of Lemma 7.2: We first note that (7.1) and (7.2) do not follow from our earlier estimates for the following reason: In $T_{\tau\tau^2\sigma}(f_2, g_2)\, \theta_{\varrho_{11}\varrho_{12}\tau}$ we approximate $T_{\varrho_{11}\varrho_{12}\sigma}(f_1, g_1)\, \phi$, but since T and θ are of the form $\exp W_j$'s followed by an S factor, we find that in $T_{\tau\tau^2\sigma}(f_2, g_2)\, \theta_{\varrho_{11}\varrho_{12}\tau}$ the $S_{\sigma\sigma}$ of $T_{\tau\tau^2\sigma}(f_2, g_2)$ and the $\exp V_j$'s of $\theta_{\varrho_{11}\varrho_{12}\tau}$ are not in the same order as in $T_{\varrho_{11}\varrho_{12}\sigma}(f_1, g_1)\, \phi$.

Therefore the cancellations of the infinities of H_σ on $T_{\tau\tau^2\sigma}(f_2, g_2)\, \theta_{\varrho_{11}\varrho_{12}\tau}$ will not be as good as they are one on $T_{\varrho_{11}\varrho_{12}\sigma}(f_1, g_1)\, \phi$. The particular choice of $\theta_{\varrho_{11}\varrho_{12}\tau}$ (see Equation (5.11)) ensures that the additional uncancelled Wick terms give rise to convergent kernels whose contributions go to zero as $\tau \to \infty$. It will be crucial that the W_j's in the S part of $\theta_{\varrho_{11}\varrho_{12}\tau}$ are more strongly truncated than the V_3's, V_2''s etc.

We finally remark that (7.3) is known from Glimm's analysis [4], see also Theorem 5.1.

The calculations leading to (7.1) and (7.2) are long. We present here as an example only the calculations in connection with the cancellation of the $V_{3\sigma}$-part of H_σ in (7.1), and of this term only those contributions in which $V_{3\sigma}$ is not contracted with the $\exp W_j$ part of $T_{\varrho_{11}\varrho_{12}\sigma}(f_1, g_1)$, $T_{\varrho_{21}\varrho_{22}\sigma}(f_2, g_2)$ or $\theta_{\varrho_{11}\varrho_{12}\tau}$. In this example, however, the reader will find all the interesting cancellation and convergence arguments which are necessary for the complete calculation. For the terms arising from $V_{4\sigma}$, see also the calculations in the proof of Theorem 3.1, property *III*.

We define $V_{3(\varrho\tau)}$ as that truncation of $V_{3\sigma}$ in which the maximal magnitude of the created momenta lies in the interval $[\varrho, \tau)$.

Let $\varrho_1 = \varrho_{11}$, $\varrho_2 = \varrho_{12}$, let $l = f_1^{-1}(\varrho_1)$, and let $X_{\varrho\sigma}^{(\alpha)} = X_{\varrho\sigma}(f_\alpha, g_\alpha)$, $\alpha = 1, 2$ (see also (5.1)); $S_{\tau\sigma}^{(\alpha)}(j^k, x) = S_{\tau\sigma}(f_\alpha, g_\alpha \mid j^k, x)$. We start by writing down $V_{3\sigma}\, T_{\varrho_1\varrho_2\sigma}(f_1, g_1)\, \phi$:

$$V_{3\sigma}\, T_{\varrho_1\varrho_2\sigma}(f_1, g_1)\, \phi = \sum_{x \in X_{\varrho_1\sigma}^{(1)}} \prod_{i=l}^{\infty} \exp W_{\substack{f_1 i\sigma \\ \bar{x}(i)}} \sum_{j^k \in J_{\varrho_2}} V_{3(\tau^2\sigma)}\, S_{\sigma\sigma}^{(1)}(j^k, x)\, \phi \tag{7.6.1}$$

$$+ \sum_{x \in X_{\varrho_1\sigma}^{(1)}} \prod_{i=l}^{\infty} \exp W_{\substack{f_1 i\sigma \\ \bar{x}(i)}} \sum_{j^k \in J_{\varrho_2}} V_{3(0\tau^2)}\, S_{\sigma\sigma}^{(1)}(j^k, x)\, \phi \tag{7.6.2}$$

$+$ terms where $V_{3\sigma}$ is contracted with $W_{f_1 i\sigma}$ of $\exp W_{\substack{f_1 i\sigma \\ \bar{x}(i)}}$.

Our next term is $V_{3\sigma}\, T_{\tau\tau^2\sigma}(f_2, g_2)\, \theta_{\varrho_1\varrho_2\tau}$. Define m_α by $\tau = f_\alpha(m_\alpha)$, $\alpha = 1, 2$.

$$V_{3\sigma}\, T_{\tau\tau^2\sigma}(f_2, g_2)\, \theta_{\varrho_1\varrho_2\tau} = \sum_{x \in X_{\tau\sigma}^{(2)}} \prod_{i=m_2}^{\infty} \exp W_{\substack{f_2 i\sigma \\ \bar{x}(i)}} \sum_{j^k \in J_{\tau^2}} V_{3(\tau^2\sigma)}\, S_{\sigma\sigma}^{(2)}(j^k, x)\, \theta_{\varrho_1\varrho_2\tau} \tag{7.7.1}$$

$$+ T_{\tau\tau^2\sigma}(f_2, g_2) \sum_{x \in X_{\varrho_1\tau}^{(1)}} \prod_{i=l}^{m_1-1} \exp W_{\substack{f_1 i\sigma \\ \bar{x}(i)}} \sum_{j^k \in J_{\varrho_2}} V_{3(0\tau^2)}\, S_{\tau\tau^2}^{(1)}(j^k, x)\, \phi \tag{7.7.2}$$

$$+ \sum_{x \in X_{\tau\sigma}^{(2)}} \prod_{i=m_2}^{\infty} \exp W_{\substack{f_2 i\sigma \\ \bar{x}(i)}} \sum_{j^k \in J_{\tau^2}} [V_{3(0\tau^2)}, S_{\sigma\sigma}^{(2)}(j^k, x)]\, \theta_{\varrho_1\varrho_2\tau} \tag{7.7.3}$$

$+$ terms where $V_{3\sigma}$ is contracted with $W_{f_2 i\sigma}$ of $\exp W_{\substack{f_2 i\sigma \\ \bar{x}(i)}}$, or with $W_{f_1 i\sigma}$ of $\exp W_{\substack{f_1 i\sigma \\ \bar{x}(i)}}$.

We now need the following parts of $H_0\, T_{\varrho_1\varrho_2\sigma}(f_1, g_1)\, \phi$ to cancel the infinities in (7.6.1) and (7.7.2).

$$-\sum_{x\in X^{(1)}_{\varrho_1\sigma}} \prod_{i=l}^{\infty} \exp W_{f_1 i\sigma} \sum_{\bar{x}(i)} \sum_{\substack{j^k\in J_{\varrho_2}\\ 2^{j_k}\geq\tau^2}} V_{3j_k\sigma}\, S^{(1)}_{\sigma\sigma}(j^{k-1}, x)\, \phi \tag{7.8.1}$$

$$-\sum_{x\in X^{(1)}_{\varrho_1\sigma}} \prod_{i=l}^{\infty} \exp W_{f_1 i\sigma} \sum_{\bar{x}(i)} \sum_{j^k\in J_{\varrho_2}} V_{3j_k\tau^2}\, S^{(1)}_{\sigma\sigma}(j^{k-1}, x)\, \phi\,. \tag{7.8.2}$$

The corresponding quantity from $H_0\, T_{\tau\tau^2\sigma}(f_2, g_2)\, \theta_{\varrho_1\varrho_2\tau}$ is

$$-\sum_{x\in X^{(2)}_{\tau\sigma}} \prod_{i=m_2}^{\infty} \exp W_{f_2 i\sigma} \sum_{\bar{x}(i)} \sum_{j^k\in J_{\tau^2}} V_{3j_k\sigma}\, S^{(2)}_{\sigma\sigma}(j^{k-1}, x)\, \theta_{\varrho_1\varrho_2\tau} \tag{7.9.1}$$

$$-\, T_{\tau\tau^2\sigma}(f_2, g_2) \sum_{x\in X^{(1)}_{\varrho_1\tau}} \prod_{i=l}^{m_1-1} \exp W_{f_1 i\sigma} \sum_{\bar{x}(i)} \sum_{j^k\in J_{\varrho_2}} V_{3j_k\tau^2}\, S^{(1)}_{\tau\tau^2}(j^{k-1}, x)\, \phi\,. \tag{7.9.2}$$

We now combine (7.6.α) with (7.8.α), (7.7.α) with (7.9.α), $\alpha = 1, 2$ and find cancellations in $\sum_{j^k\in J_{\varrho_2}}$. We obtain

$$(7.6.1) + (7.8.1) = \sum_{x\in X^{(1)}_{\varrho_1\sigma}} \prod_{i=l}^{\infty} \exp W_{f_1 i\sigma} \sum_{x(i)} {\sum}' V_{3j\sigma}\, S^{(1)}_{\sigma\sigma}(j^k, x)\, \phi\,, \tag{7.10}$$

where $\sum'$ runs over the set $\{j^k, j\colon j^k\in J_{\varrho_2},\, 2^j\geq\tau^2,\, j\leq(\sum_{i=1}^k j_i)^{\xi}\}$.

First we observe that $\|(7.10)\|^2\, e^{-A_\sigma}$ is uniformly bounded in σ. This follows by a standard argument by Glimm [4]: one has to use the fact that in $\sum'$, $j\leq(\sum_{i=1}^k j_i)^{\xi}$. Furthermore as $2^j\geq\tau^2 = (\tau(n))^2 = (f_2(n))^2$, all terms in (7.10) contain at least one momentum larger than $(\tau(n))^2$. Thus by Remark 3.1, inequality (3.14), the contribution of (7.10) to (7.1) is $0(\tau(n)^{-\eta})$ for some $\eta > 0$. Next we compute

$$(7.6.2) + (7.8.2) = \sum_{x\in X^{(1)}_{\varrho_1\sigma}} \prod_{i=l}^{\infty} \exp W_{f_1 i\sigma} \sum_{\bar{x}(i)} \sum_{j^k\in J_{\varrho_2}} V_{3(0\varrho_2)}\, S^{(1)}_{\sigma\sigma}(j^k, x)\, \phi \tag{7.11.1}$$

$$+ \sum_{x\in X^{(1)}_{\varrho_1\sigma}} \prod_{i=l}^{\infty} \exp W_{f_1 i\sigma} \sum_{\bar{x}(i)} {\sum}'' V_{3j\tau^2}\, S^{(1)}_{\sigma\sigma}(j^k, x)\, \phi\,, \tag{7.11.2}$$

and $\sum''$ runs now over the set $\{j^k, j\colon j^k\in J_{\varrho_2};\, \varrho_2\leq j\leq(\sum_{i=1}^k j_i)^{\xi}\}$. The sum (7.7.1) + (7.9.1) contributes $0(\tau(n)^{-\eta})$ to (7.1) by the same argument as above.

$$(7.7.2) + (7.9.2) = T_{\tau\tau^2\sigma}(f_2, g_2) \sum_{x\in X^{(1)}_{\varrho_1\tau}} \prod_{i=l}^{m_1-1} \exp W_{f_1 i\sigma} \sum_{\bar{x}(i)} \sum_{j^k\in J_{\varrho_2}} V_{3(0\varrho_2)}\, S^{(1)}_{\tau\tau^2}(j^k, x)\, \phi \tag{7.12.1}$$

$$+\, T_{\tau\tau^2\sigma}(f_2, g_2) \sum_{x\in X^{(1)}_{\varrho_1\tau}} \prod_{i=l}^{m_1-1} \exp W_{f_1 i\sigma} \sum_{\bar{x}(i)} {\sum}'' V_{3j\tau^2}\, S^{(1)}_{\tau\tau^2}(j^k, x)\, \phi\,. \tag{7.12.2}$$

The terms (7.11.1, 2), (7.12.1, 2) give finite contributions to (7.1), but they are not necessarily small. Nevertheless we assert that the contributions of (7.11.1)—(7.12.1) and of (7.11.2) — (7.12.2) go to zero as n approaches infinity. For a proof we note that those Wick terms in (7.11.1, 2), (7.12.1, 2) which have large momenta $(> \tau(n))$ in the skeleton contribute individually $0(\tau(n)^{-\eta})$ to (7.1). To Wick terms in (7.11.1) and (7.11.2) who have only small momenta in the skeleton there is always a term in (7.12.1) and (7.12.2) respectively with exactly the same skeleton (and the same skeleton part of the numerical kernel). Thus in the difference of two such corresponding terms we have to consider the difference in the high momentum part of the Λ-factors only. Differences of this kind have been discussed in the proof of Lemma 4.2. It has been shown there that these expressions go to zero as n goes to infinity.

The remaining term is (7.7.3). There are no other terms for cancellation. Again we observe first that the contribution of a single Wick term in (7.7.3) to (7.1) is finite (for all $\sigma \leq \infty$), because with $V_{3(0\tau^2)}$ there is always at least one other vertex in the skeleton graph (because ot the commutator), and creating at least one momentum larger in magnitude than all momenta created by $V_{3(0\tau^2)}$. Then by 'power gymnastics' [see Equation (7.14) below] the assertion follows. Secondly, we have to make sure that the summation over all contributions of (7.7.3) to (7.1) converges uniformly in σ. This is shown with the arguments we have used in the proof of Theorem 3.1, property *I*.

Finally, as each skeleton occurring in the contribution of (7.7.3) to (7.1) contains at least one line with momentum larger in magnitude than τ^2, inequality (3.14) ensures that the contribution of (7.7.3) is in fact $0(\tau(n)^{-\eta})$. This concludes the discussion of the terms (7.6 — 7.9). For all other contributions to (7.1) or to (7.2) we can use the same arguments, with one exception:

The exceptional term appears in the discussion of $V_{4\sigma}\, T_{\tau\tau^2\sigma}(f_2, g_2)\, \theta_{\varrho_1\varrho_2\tau}$, where one will get expressions containing commutators:

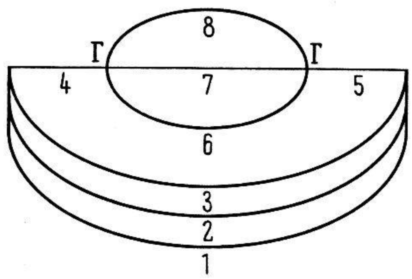
$$\sum_{j^k \in J_{\tau^2}} [V_{4(0\tau)}, S^{(2)}_{\sigma\sigma}(j^k, x)] \,. \tag{7.13}$$

They show up in terms analogous to (7.7.3). It is for these expressions we have made the very special choice of $T_{\tau\tau^2\infty}\, \theta_{\varrho_1\varrho_2\tau}$ in (5.10, 11). By definition $V_{4(0\tau)}$ creates only momenta smaller in magnitude than τ, while there is always at least one vertex in $\sum_{j^k \in J_{\tau^2}} S^{(2)}_{\sigma\sigma}(j^k, x)$ which in (7.13) is contracted with $V_{4(0\tau)}$ and which creates a momentum larger in magnitude than $\tau^{3/2} = \tau^{2\xi}$ (see definition of $J_{\varrho\xi}$ in section V).

Consider for example

$$[V_{4(0\tau)}, \sum_{2^j \geq \tau^2} \Gamma\, V_{3j\sigma}] = Q_\tau \,.$$

The expression $\|\, Q_\tau\, \phi_0 \,\|^2$ has as graph



The momenta belonging to the lines 1–5 have absolute value smaller than τ, while one of the lines 6, 7, 8 has momentum k with $|k| > \tau^2$. Thus we can multiply the numerical kernel of $Q_\tau^* Q_\tau$ by

$$\frac{\mu(k)^{2/3}}{\tau} > 1 \tag{7.14}$$

('power gymnastics') and then convince ourselves (and possibly the reader) by power counting that $\| Q_\tau \phi_0 \|^2$ is finite and in fact goes to zero for τ going to infinity.

This ends our discussion of (7.1)–(7.3).

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