**Zeitschrift:** Helvetica Physica Acta
**Band:** 44 (1971)
**Heft:** 6

**Artikel:** Study of waves at a plasma vacuum boundary

**Autor:** Boulanger, P. / Ashby, N.

**DOI:** https://doi.org/10.5169/seals-114308

# Study of Waves at a Plasma Vacuum Boundary

by **P. Boulanger**[1]) and **N. Ashby**

University of Colorado, Boulder, 80302, USA

(31. III. 71)

*Summary*. We consider in this work the behaviour of waves at a plasma-vacuum interface. Under the specular reflection condition for electrons at the boundary, the dispersion relation for surface waves is calculated. The disappearance of the so-called surface plasmon effect is shown when proper boundary conditions are taken. The generation of transverse waves by longitudinal waves striking the plasma-vacuum transition is obtained as well as the longitudinal waves created by an impinging electromagnetic wave.

## 1. Introduction

In the absence of an external magnetic field a homogeneous plasma can sustain two kind of waves: longitudinal and transverse waves. In analogy with surface transverse wave which can propagate along a plasma vacuum interface many workers [1, 2] have postulated the existence of surface longitudinal waves (or surface plasmons) which would exist at a frequency near $\omega = \omega_p/\sqrt{2}$ where $\omega_p$ is the plasma frequency. Ferrell [3] studied the dispersion relation for such surface plasmons in a plasma slab and the dispersion relation for a semi-infinite media was calculated taking Landau damping into account [1, 2]. They solved the Vlasov-Poisson equations in a semi-infinite plasma assuming specular reflection for the electrons at the plasma vacuum interface.

It is the purpose of the work to show how the spurious resonance corresponding to surface plasmons is identical to surface transverse waves (at least under the specular reflection condition), when one takes into account retardation effects, or the full set of Maxwell's equations coupled with the Vlasov equations, instead of only the Vlasov-Poisson equations.

Furthermore the problem of surface waves is closely related to the study of the generation of waves at a plasma vacuum boundary. If surface plasmons could be excited by transverse waves, the Fresnel equations for the transmission and reflection coefficient would be strongly modified at frequencies close to $\omega = \omega_p/\sqrt{2}$. The problem we are concerned with, the generation of longitudinal waves by transverse waves and conversely the generation of transverse waves by longitudinal waves has been studied for slowly varying density gradients [4] or for a sharp discontinuity but small gradient [5, 6] but we consider here a sharp and large density variation, a plasma vacuum boundary, considered as a perfectly reflecting wall. In the limit of zero temperature,

---

[1]) Adresse actuelle: Centre de Recherches en Physique des Plasmas, Lausanne (Suisse).

when longitudinal waves are not propagating and the plasma not spatially dispersive we shall obtain Fresnel equations as we should.

## 2. Solution of Vlasov Equation in a Semi-Infinite Media; Extension to the Whole Space

We shall consider a plasma in the half space $x > 0$ where $0\,x$ is normal to the plasma vacuum interface. The ions are considered as a positive smeared uniform background, there only to preserve electrical neutrality (electron gas approximation). We shall only consider the dynamic of the electrons and assume specular reflection for the electron at $x = 0$ on the interface. If $f_1$ is the electron density fluctuation in phase space this condition is written

$$f_1\,(x = 0, y, z, v_x, v_y, v_z, t) = f_1\,(x = 0, y, z, -v_x, v_y, v_z, t)\,. \tag{1}$$

The error associated with the specular reflection assumption is difficult to estimate. However Reuther and Sondheimer [7] considering the problem of the anomalous skin effect in metals used both specular and diffuse boundary conditions; in their special case of normal incidence and Fermi Dirac equilibrium distribution function for the electrons, the difference between the results due to the two reflection conditions is small and the experimental results lies in between.

To idealize a plasma vacuum transition which takes place over a Debye length (in the absence of an external magnetic field) by a step discontinuity is valid when the wavelength of the waves considered is larger than the Debye length. This is not a very strong restriction as longitudinal wave propagates in a plasma without being too strongly Landau damped only when their wavelength is bigger than the Debye length.

We shall only consider specular reflection and the dynamics of the plasma is described by the linearized Vlasov equation

$$\frac{\partial f_1}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f_1}{\partial \boldsymbol{r}} - \frac{e}{m}\,\boldsymbol{E}_1 \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} = 0 \tag{2}$$

in which $f_0$ is the equilibrium distribution function for the electrons, taken as Maxwellian in the rest of this work. We shall take a $e^{i(\boldsymbol{k}_{11} \cdot \boldsymbol{R}_{11} - \omega t)}$ dependence for all quantities, as the tangential component of wave vectors are continuous across the boundary. $\boldsymbol{k}_{11}$ is the tangential component of the wave vector and $\boldsymbol{R}_{11}$ the component of the position vector parallel to the plasma vacuum interface. We shall omit this space time dependence from now on and the Vlasov equation is written

$$(-i\,\omega + i\,\boldsymbol{k}_{11} \cdot \boldsymbol{v}_{11})\,f_1 + v_x\,\frac{\partial f_1}{\partial x} = \frac{e}{m}\,\boldsymbol{E}_1 \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} \tag{3}$$

Generalizing to three dimensions a method due to Landau [8], this equation is solved for $f_1$ in terms of the electric field, using (1) and assuming finite fields at $x = +\infty$.

Setting $\alpha = -i\,\omega + i\,\boldsymbol{k}_{11} \cdot \boldsymbol{v}_{11}$ we obtain for $v_x < 0$:

$$f_1 = \frac{e}{m\,v_x}\,e^{-\alpha x/v_x} \int_{-\infty}^{x} e^{\alpha x'/v_x}\,\boldsymbol{E}_1(x') \cdot \frac{\partial f_0}{\partial \boldsymbol{v}}\,dx' \tag{4}$$

and for $v_x > 0$,

$$f_1 = \frac{e}{m\,v_x}\,e^{-\alpha x/v_x}\left[\int_0^x e^{\alpha x'/v_x}\,\boldsymbol{E}_1(x')\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}\,dx' + \int_0^\infty e^{-\alpha x'/v_x}\left(-E_{1x}\frac{\partial f_0}{\partial v_x} + \boldsymbol{E}_{11}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}_{11}}\right)dx'\right]. \tag{5}$$

Knowing $f_1$ we may calculate the charges and currents as functions of the electric field. For the currents:

$$\boldsymbol{j}_1 = -e\int f_1\,\boldsymbol{v}\,d\boldsymbol{v} \tag{6}$$

or for the normal component:

$$j_{1x} = -\frac{e^2}{m}\int d\boldsymbol{v}_{11}\left[\left[\int_{-\infty}^0 dv_x\int_\infty^x dx'\,e^{\alpha(x'-x)/v_x}\,\boldsymbol{E}_1\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} + \int_0^\infty dv_x\int_0^x dx'\,e^{\alpha(x'-x)/v_x}\,\boldsymbol{E}_1\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}\right) + \int_0^\infty dv_x\int_0^\infty dx'\,e^{-\alpha(x'+x)/v_x}\left[-E_{1x}\frac{\partial f_0}{\partial v_x} + E_{11}\frac{\partial f_0}{\partial v_{11}}\right]\right] \tag{7}$$

Defining for $\xi > 0$ the kernels $K_1^i(\xi)$ as:

$$K_1^i(\xi) = \int d\boldsymbol{v}_{11}\int_0^\infty dv_x\,e^{-\alpha\xi/v_x}\,\frac{\partial f_0}{\partial v_i} \tag{8}$$

and introducing the matrix $M_k^l$:

$$M_k^l = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

the current $j_{1x}$ may be written from (7)

$$j_{1x} = -\frac{e^2}{m}\left[\int_\infty^x K_1^i(x'-x)\,M_i^l\,E_{1l}(x')\,dx' + \int_0^x K_1^i(x'-x)\,E_{1i}(x')\,dx' + \int_0^\infty K_1^i(x+x')\,M_i^l\,E_{1l}(x')\,dx'\right]. \tag{10}$$

In the previous formula $j_{1x}$ is defined only for $x > 0$. The specular reflection condition and the form of the matrix $M_k^l$ which corresponds to a symmetry with respect to the $x = 0$ plane, tempt us into using an image method to extend the problem to all space.

We shall define $f_1$ in the non-physical region $x < 0$ in such a way that (10) is still satisfied for $x > 0$; the easiest extension is to take the plane $x = 0$ as a 'mirror' for all phenomena and define $f_1$ for $x < 0$ as

$$f_1(-x, y, z, v_x, v_y, v_z, t) = f_1(x, y, z, -v_x, v_y, v_z, t) .$$

With this definition it is easy to verify that all scalar quantities are symmetrical with respect to the change $x/_{-x}$, the normal component of vectors (like the electric field) antisymmetrical while the parallel components are symmetrical and the normal components of pseudo-vector (like the magnetic field) symmetrical while the parallel components are antisymmetricals.

To satisfy (1) we shall take $f_1$ continuous at $x = 0$, and to take into account the fields that may exist at the boundary we shall take the normal component of the electric field discontinuous at the boundary as well as the tangential components of the magnetic field. They will be related by Maxwell's equations, as we shall see later. We shall define the kernels for $x < 0$ as:

$$K_1^l(-|x|) = -M_i^l K_1^i(|x|) \tag{11}$$

then the expression of the normal component of the polarization current simplifies into:

$$j_{1x}(x) = -\frac{e^2}{m} \int_{-\infty}^{+\infty} K_1^i(x - x')\, E_i(x')\, dx'. \tag{12}$$

The advantage of the extension to the entire plane is now clear as $j_{1x}(x)$ behaves as if the boundary did not exist, and is given by a convolution extended to the whole space which will give an easy to calculate Fourier transform. We shall define the other kernels as

$$K_2^i = \int d\boldsymbol{v}_{11} \int_0^\infty dv_x\, e^{-\alpha \xi/v_x} \frac{\partial f_0}{\partial v_i} \frac{v_y}{v_x}\, dv_x , \tag{13}$$

$$K_3^i = \int d\boldsymbol{v}_{11} \int_0^\infty dv_x\, e^{-\alpha \xi/v_x} \frac{\partial f_0}{\partial v_i} \frac{v_z}{v_x}\, dv_x . \tag{14}$$

and by a similar procedure we obtain

$$j_{1y}(x) = -\frac{e^2}{m} \int_{-\infty}^{+\infty} K_2^i(x - x')\, E_i(x')\, dx , \tag{15}$$

$$j_{1z}(x) = -\frac{e^2}{m} \int_{-\infty}^{+\infty} K_3^i(x - x')\, E_i(x')\, dx . \tag{16}$$

Similarly for the charge density $\varrho(x)$:

$$\varrho(x) = -\frac{e^2}{m} \int_{-\infty}^{+\infty} L^i (x - x') E_i(x') \, dx' \tag{17}$$

in which

$$L^i(\xi) = \int d\boldsymbol{v}_{11} \int_0^{\infty} \frac{e^{-\alpha \xi / v_x}}{v_x} \frac{\partial f_0}{\partial v_i} \, dv_x \,. \tag{18}$$

Defining Fourier transforms as:

$$\tilde{\boldsymbol{A}} = \int_{-\infty}^{+\infty} e^{i k_x x} \boldsymbol{A}(x) \, dx \tag{19}$$

we obtain after some algebra

$$\tilde{\boldsymbol{j}} = i \, \omega \, \varepsilon_0 \left[ (1 - \varepsilon_T) \, \tilde{\boldsymbol{E}} + \boldsymbol{k} \frac{\boldsymbol{k} \cdot \tilde{\boldsymbol{E}}}{k^2} (\varepsilon_T - \varepsilon_L) \right] , \tag{20}$$

$$\tilde{\varrho} = i \, \varepsilon_0 \, (1 - \varepsilon_L) \, \boldsymbol{k} \cdot \tilde{\boldsymbol{E}} \tag{21}$$

in which $\boldsymbol{k} = \boldsymbol{k}_{11} + k_x$, $\varepsilon_T$ is the transverse dielectric constant

$$\varepsilon_T = 1 + \frac{i \, e^2}{m \, \varepsilon_0 \, \omega} \int \frac{f_0 \, d\boldsymbol{v}}{- i \, \omega + i \, \boldsymbol{k} \cdot \boldsymbol{v}} \tag{22}$$

and $\varepsilon_L$ the longitudinal dielectric constant

$$\varepsilon_L = 1 - \frac{i \, e^2}{m \, \varepsilon_0 \, k^2} \int \frac{\boldsymbol{k} \cdot \partial f_0 / \partial \boldsymbol{v}}{- i \, \omega + i \, k \cdot \boldsymbol{v}} \, d\boldsymbol{v} \,. \tag{23}$$

Equations (20) to (23) describe completely the dynamics of the plasma.

## 3. Maxwell's Equations and Their Extension to the Entire Plane

Using the Lorentz gauge and taking retardation into account the potentials are:

$$\boldsymbol{A} = \frac{\mu_0}{4 \pi} \int d\boldsymbol{R}'_{11} \int_0^{\infty} dx' \, \frac{e^{i(\omega/c) \, |\boldsymbol{r} - \boldsymbol{r}'|}}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \boldsymbol{j}(\boldsymbol{R}'_{11}, x') \,, \tag{24}$$

$$\Phi = \frac{1}{4 \pi \, \varepsilon_0} \int d\boldsymbol{R}'_{11} \int_0^{\infty} dx' \, \frac{e^{i(\omega/\ ) \, |\boldsymbol{r} - \boldsymbol{r}'|}}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \varrho(\boldsymbol{R}'_{11}, x') \,. \tag{25}$$

It is to be noted that the integration over $x$ extends only over the half space $x > 0$, $\boldsymbol{j}$ and $\varrho$ being the currents and charge in the 'physical' space. Using the $e^{i \boldsymbol{k}_{11} \cdot \boldsymbol{R}_{11}}$

dependence for $\varrho$ and $\boldsymbol{j}$ we obtain after some simple algebra, for $\omega^2/c^2 > k^2_{11}$

$$\boldsymbol{A}(x) = \frac{\mu_0 \, i}{2} \int_0^\infty dx' \, \frac{e^{i\sqrt{\omega^2/c^2 - k_{11}^2}\,|x-x'|}}{\sqrt{\omega^2/c^2 - k_{11}^2}} \, \boldsymbol{j}(x') \,, \tag{26}$$

$$\Phi(x) = \frac{i}{2\,\varepsilon_0} \int_0^\infty dx' \, \frac{e^{i\sqrt{\omega^2/c^2 - k_{11}^2}\,|x-x'|}}{\sqrt{\omega^2/c^2 - k_{11}^2}} \, \varrho(x') \,. \tag{27}$$

We shall now calculate the fields at $x = 0^+$. The electric field is:

$$\boldsymbol{E} = -\nabla\Phi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{28}$$

we calculate first $E_x|_{0+}$

$$\left.\frac{\partial \phi}{\partial x}\right|_{0+} = \frac{1}{2\,\varepsilon_0} \int_0^\infty e^{i\sqrt{\omega^2/c^2 - k_{11}^2}\,|x'|} \, \varrho(x') dx' \,. \tag{29}$$

Extending as we did before the definition of all quantities to $x < 0$, $\partial\phi/\partial x$ will be odd and discontinuous at $x = 0$. The jump $\Delta(\partial\phi/\partial x)$ due to the charge in the plasma is:

$$\Delta\left(\frac{\partial\phi}{\partial x}\right) = \left.\frac{\partial\phi}{\partial x}\right|_{0+} - \left.\frac{\partial\phi}{\partial x}\right|_{0-} = \frac{1}{\varepsilon_0} \int_0^\infty e^{i\sqrt{\omega^2/c^2 - k_{11}^2}} \, \varrho(x') \, dx' \,, \tag{30}$$

$$\Delta\left(\frac{\partial\phi}{\partial x}\right) = \frac{-\,i\sqrt{\omega^2/c^2 - k_{11}^2}}{2\,\pi\,\varepsilon_0} \int_{-\infty}^{+\infty} \frac{\tilde{\varrho}\,(k_x)\, dk_x}{k^2 - \omega^2/c^2} \,. \tag{31}$$

In the same way for the normal component of the vector potential:

$$\Delta\left(\frac{\partial A_x}{\partial t}\right) \equiv \left.\frac{\partial A_x}{\partial t}\right|_{0+} - \left.\frac{\partial A_x}{\partial t}\right|_{0-} = \frac{i\,\mu_0\,\omega}{2\,\pi\sqrt{\omega^2/c^2 - k_{11}^2}} \int_{-\infty}^{+\infty} \frac{k_x\,\tilde{j}_x(k_x)}{k^2 - \omega^2/c^2} \, dk_x \,. \tag{32}$$

Combining (31) and (32) the jump for the normal component of the electric field due to the currents and charge in the plasma is:

$$\Delta(E_x) = \frac{i}{2\,\pi\,\varepsilon_0\sqrt{\omega^2/c^2 - k_{11}^2}} \int_{-\infty}^{+\infty} \frac{dk_x}{k^2 - \omega^2/c^2} \left[\left(\frac{\omega^2}{c^2} - k_{11}^2\right)\tilde{\varrho}(k_x) - \frac{\omega}{c^2}\,k_x\,\tilde{j}_x(k_x)\right]. \tag{33}$$

In summary the charges and currents in the plasma create an electric field $E_x|_{0+}$ at the boundary; we have added the mirror image so that the Fourier transforms are defined over all space for the extended definition of $\varrho$ and $\boldsymbol{j}$; then the normal component of the electric field is discontinuous and its jump at $x = 0$ due to the charge and current in the plasma is given by (33).

We shall only consider to simplify the algebra, '$p$ polarized' transverse wave such that the magnetic field is parallel to the plane $x = 0$. From symmetry considerations it is easy to show that ,s polarized' transverse wave can not excite any longitudinal wave in the plasma [6], and it is well known that transverse surface waves exist only in the ,p polarization' mode [9]. Thus we have not restrained the generality in any physical sense, the s polarization case not giving rise to any interesting physical phenomena. We take the magnetic field along the 0y axis, and then $k_{11} = k_z$

$$\left.\begin{aligned} B_y \equiv \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = \frac{\mu_0\, i}{2} \int_0^\infty dx'\, e^{i\sqrt{\omega^2/c^2 - k_z^2}\,|x-x'|} \\ \times \left[\frac{i k_z}{\sqrt{\omega^2/c^2 - k_z^2}}\, j_x(x') + i \operatorname{Sign}(x-x')\, j_z(x')\right] \end{aligned}\right\} \quad (34)$$

the jump for the tangential component of the magnetic field is:

$$\left.\begin{aligned} \Delta(B_y) \equiv B_y|_{0+} - B_y|_{0-} = \\ -\frac{i\,\mu_0}{2\pi\sqrt{\omega^2/c^2 - k_z^2}} \int_{-\infty}^{+\infty} dk_x\, \frac{k_x k_z \tilde{j}_x - (\omega^2/c^2 - k_z^2)\, \tilde{j}_z}{k^2 - \omega^2/c^2} \,. \end{aligned}\right\} \quad (35)$$

The jumps of $E_x$ and $B_y$ at the discontinuity are twice the values of the fields at $x = 0^+$, those field being created by the charges and currents in the plasma *in response* to driving field. We now write Poisson's equation

$$\nabla \cdot \boldsymbol{E} = \frac{\varrho}{\varepsilon_0} \quad (36)$$

in the physical space (vacuum for $x < 0$, plasma for $x > 0$). Taking the Fourier transform we have

$$i\,\boldsymbol{k} \cdot \tilde{\boldsymbol{E}} = \frac{\tilde{\varrho}}{\varepsilon_0} + \Delta^T(E_x) \quad (37)$$

in which $\Delta^T(E_x)$ is the jump for the total electric field consisting of the driving electric field $\Delta(E_x)_{\text{driving}}$ and the response of the medium $\Delta(E_x)$

$$\Delta^T(E_x) = \Delta(E_x)_{\text{driving}} + \Delta(E_x)\,. \quad (38)$$

Similarly combining Ampere's law with Faraday's law,

$$\nabla \times \boldsymbol{B} = \mu_0\, \boldsymbol{J} - \frac{i\,\omega}{c^2}\, \boldsymbol{E},$$

$$\nabla \times \boldsymbol{E} = -i\,\omega\, \boldsymbol{B}$$

we obtain after tedious algebra

$$\left(k^2 - \frac{\omega^2}{c^2}\right) \tilde{E}_x = \frac{i\,\mu_0\, c^2 k_x}{\omega} (k_x \tilde{j}_x + k_z \tilde{j}_z) + i\,\omega\,\mu_0\, \tilde{j}_x - \frac{i\, k_x k_z}{\omega}\, c^2\, \Delta^T(B_y)\,, \quad (39)$$

$$\left. \begin{aligned} \left(k^2 - \frac{\omega^2}{c^2}\right) \tilde{E}_z = & - \frac{i \mu_0 c^2 k_x}{\omega} (k_x \tilde{j}_x + k_z \tilde{j}_z) \\ & + i \omega \mu_0 \tilde{j}_z - c^2 \left(\frac{i k_z^2}{\omega} - \frac{i \omega}{c^2}\right) \Delta^T(B_y) . \end{aligned} \right\} \tag{40}$$

As a consistency check we may notice that adding the last two equations the first one multiplied by $k_x$ and the second one by $k_z$, and combining with Poisson's equation, we obtain:

$$\Delta^T(E_x) = k_z \frac{c^2}{\omega} \Delta^T(B_y) \tag{41}$$

which may be written explicitly using (33), (35), (38)

$$\left. \begin{aligned} & \frac{i}{2 \pi \varepsilon_0 \sqrt{\omega^2/c^2 - k_z^2}} \int_{-\infty}^{+\infty} \frac{dk_x}{k^2 - \omega^2/c^2} \left[\left(\frac{\omega^2}{c^2} - k_z^2\right) \left(\tilde{\varrho} - \frac{(k_x \tilde{j}_x + k_z \tilde{j}_z)}{\omega}\right)\right] \\ & \qquad + \Delta(E_x)_{\text{driving}} - \frac{k_z c^2}{\omega} \Delta(B_y)_{\text{driving}} = 0 . \end{aligned} \right\} \tag{42}$$

Maxwell's equations in vacuum gives

$$\Delta(E_x)_{\text{driving}} = \frac{k_z c^2}{\omega} \Delta(B_y)_{\text{driving}} . \tag{43}$$

Thus $\tilde{\varrho} = k_x \tilde{j}_x + k_z \tilde{j}_z/\omega$ which is the Fourier transform of the continuity equation remembering that $j_x|_{0^+} = 0$ due to the specular reflection condition.

Using the dynamics of the plasma described by (20) into (39) and (40) we obtain:

$$\tilde{E}_x = \frac{-i k_x k_z c^2}{\omega k^2} \left\{\frac{1}{\varepsilon_L} + \frac{\omega^2/c^2}{k^2 - \varepsilon_T \omega^2/c^2}\right\} \Delta^T(B_y) , \tag{44}$$

$$\tilde{E}_z = \frac{-i c^2}{\omega k^2} \left\{\frac{k_z^2}{\varepsilon_L} + \frac{k_x^2 \omega^2/c^2}{k^2 - \varepsilon_T \omega^2/c^2}\right\} \Delta^T(B_y) \tag{45}$$

and we may also express $\Delta_T(B_y)$ in terms of $\Delta(B_y)_{\text{driving}}$

$$\left. \begin{aligned} \Delta^T(B_y) = & \frac{\Delta(B_y)_{\text{driving}}}{1 + \frac{i}{2 \pi} \left\{ \frac{k_z^2}{\sqrt{\omega^2/c^2 - k_z^2}} \int_{-\infty}^{+\infty} \frac{dk_x}{k^2} \left(\frac{1}{\varepsilon_L} - 1\right) \right.} \\ & \frac{\Delta(B_y)_{\text{driving}}}{\left. + \frac{\omega^4}{c^4 \sqrt{\omega^2/c^2 - k_z^2}} \int_{-\infty}^{+\infty} \frac{k_x^2 (1 - \varepsilon_T) \, dk_x}{k^2 (k^2 - \varepsilon_T \omega^2/c^2)(k^2 - \omega^2/c^2)} \right\}} . \end{aligned} \right\} \tag{46}$$

## 4. Surface Plasmons

In addition to the poles in the complex $k$ plane which correspond to longitudinal and transverse waves we have an extra resonance when:

$$\left.\begin{aligned} 1 + \frac{i}{2\pi} \Bigg\{ & \frac{k_z^2}{\sqrt{\omega^2/c^2 - k_z^2}} \int_{-\infty}^{+\infty} \frac{dk_x}{k^2} \left( \frac{1}{\varepsilon_L} - 1 \right) \\ & + \frac{\omega^4}{c^4 \sqrt{\omega^2/c^2 - k_z^2}} \int_{-\infty}^{+\infty} \frac{k_z^2 (1 - \varepsilon_T)\, dk_x}{k^2 (k^2 - \varepsilon_T\, \omega^2/c^2)(k^2 - \omega^2/c^2)} \Bigg\} = 0 \,. \end{aligned}\right\} \tag{47}$$

Some authors [1, 2, 3] have found a dispersion relation of this type neglecting retardation effect. Letting the velocity of light being infinite we get:

$$1 - \frac{k_z}{2\pi} \int_{-\infty}^{+\infty} \left( 1 - \frac{1}{\varepsilon_L} \right) \frac{dk_x}{k^2} = 0 \tag{48}$$

which was obtained by [2]. For non spatially dispersive plasma, or neglecting the $k$ dependence of $\varepsilon_L$ we would obtain the dispersion relation:

$$\varepsilon_L = -1 \quad \text{or} \quad \omega = \frac{\omega_p}{\sqrt{2}} \tag{49}$$

which was used in [1, 3].

It is our claim that to neglect retardation is a grave error and leads to non physical results. We shall calculate the second integral inside the bracket in (47), taking the transverse dielectric constant not spatially dispersive or $\varepsilon_T = \varepsilon = 1 - \omega_p^2/\omega^2$ which is correct for non relativistic plasma [10]; we obtain:

$$\left.\begin{aligned} 1 + \frac{i}{2\pi} \Bigg\{ & \frac{k_z^2}{\sqrt{\omega^2/c^2 - k_z^2}} \int_{-\infty}^{+\infty} \frac{dk_x}{k^2} \left( \frac{1}{\varepsilon_L} - 1 \right) \\ & + \frac{i\pi}{\sqrt{\omega^2/c^2 - k_z^2}} \left[ i\, k_z \left( \frac{1}{\varepsilon} - 1 \right) - \frac{\sqrt{\varepsilon\, \omega^2/c^2 - k_z^2}}{\varepsilon} + \sqrt{\omega^2/c^2 - k_z^2} \right] \Bigg\} = 0 \,. \end{aligned}\right\} \tag{50}$$

To obtain the first approximation for the resonance frequency corresponding to the so-called surface plasmons we consider $\varepsilon_L$ as *not* spatially dispersive; then $\varepsilon_L = \varepsilon$ and we can see that the integral in the bracket is *cancelled by the next term due to the retardation effect*. The resonance occurs for

$$1 + \frac{1}{\varepsilon} \sqrt{\frac{\varepsilon\, \omega^2/c^2 - k_z^2}{\omega^2/c^2 - k_z^2}} = 0 \,, \tag{51}$$

or

$$k_z^2 = \frac{\varepsilon\,\omega^2/c^2}{1+\varepsilon}$$

which is the dispersion condition for surface transverse waves.

Taking $\varepsilon_L$ spatially dispersive in the long wavelength approximation modify slightly this result to give

$$k_z^2 = \frac{\varepsilon\,\omega^2/c^2}{1+\varepsilon}\left[1+\sqrt{6}\,\frac{v_0}{c}\,\frac{\varepsilon^{5/2}}{\sqrt{1+\varepsilon}\,(1-\varepsilon^2)}\right]$$

where $\varepsilon_L$ was taken

$$\varepsilon_L = 1 - \frac{\omega_p{}^2}{\omega^2} - \frac{3}{2}\,\frac{k^2 v_0^2}{\omega^2}\,.$$

The so-called surface plasmon is therefore a surface transverse wave treated incorrectly by neglecting retardation; this is substantiated by the result of Stern (Equation 16 of Ferrell's paper [3]) who gets (51) in a slightly different form, when considering the effect of retardation for surface plasmon.

Furthermore, if those surface plasmons could be excited for $\varepsilon = -1$ this would occur even for a usual dielectric as well as a plasma and would greatly modify Fresnel's equation which has not been observed experimentally. Another point of view is to say that in all those papers the proper boundary conditions are not taken into account; the tangential component of the electric field is continuous at the plasma vacuum interface and there is therefore a varying electric field *in vacuum* (for $x = 0^-$) which creates a magnetic field. For nonmagnetic material the magnetic induction $B$ is continuous and there is therefore a magnetic field acting on the particles in the plasma which cannot be neglected. This magnetic field is very important near the boundary because, when the charged particles are reflected, this creates a curl in the current which in turn creates a very large magnetic field near the surface. It is, in fact, the presence of this magnetic field which explains why a longitudinal wave striking a plasma vacuum boundary generates a transverse wave outside through the intermediary of the magnetic field created by the 'bending' of the particle trajectory.

## 5. Generation of Waves at a Plasma Boundary

We shall call $B_0$ the magnetic field at $x = 0^-$ (in vacuum). As the tangential component of the magnetic field is continuous across the boundary we have, $B_y^T|_{0^+}$ being the total magnetic field at $x = 0^+$:

$$\Delta^T(B_y) = 2\,B_y^T|_{0^+} = 2\,B_0$$

then (44) and (45) are written:

$$\tilde{E}_x = \frac{-2\,i\,k_x\,k_z\,c^2}{\omega\,k^2}\left\{\frac{1}{\varepsilon_L} + \frac{\omega^2/c^2}{k^2-\varepsilon_T\,\omega^2/c^2}\right\} B_0\,, \tag{52}$$

$$\tilde{E}_z = \frac{-2\,i\,c^2}{\omega\,k^2}\left\{\frac{k_z^2}{\varepsilon_L} - \frac{\omega^2/c^2\,k_x^2}{k^2-\varepsilon_T\,\omega^2/c^2}\right\} B_0\,. \tag{53}$$

*Landau's case*

Landau [8] treated the case when a longitudinal field strikes a plasma vacuum boundary. This situation does not correspond directly to a physical phenomena as a longitudinal field can not exist in vacuum, but it can be simulated [11]. Considering the one dimensionnal problem when the electric field at $x = 0^+$ $E_0$, is normal to the surface, we must create this field by a surface charge density

$$\sigma = 2\,\varepsilon_0\,E_0 \tag{54}$$

on the boundary plasma vacuum. This corresponds to an extended current

$$g = i\,\omega\,\varepsilon_0\,E_0\,\mathrm{Sign}\,(x)\;. \tag{55}$$

Expressing (39) in terms of $E_0$ instead of $\Delta^T(B_y)$ and adding the Fourier transform of (55) we get after simplification

$$\tilde{E} = -\frac{2\,i\,E_0}{\varepsilon_L}\,P\left(\frac{1}{k}\right)$$

where $P$ stands for principal part. Taking the inverse Fourier transform we obtain in Landau's notation:

$$\left.\begin{aligned}
&\varepsilon_L = -K_k\,,\\
&E = -\frac{i\,E_0}{\pi}\,P\int\limits_{-\infty}^{+\infty}\frac{e^{i\,k\,x}}{k\,(1-K_k)}\,dk = \frac{i\,E_0}{\pi\,\varepsilon}\int\limits_{-\infty}^{+\infty}\frac{K_0-K_k}{k\,(1-K_k)}\,e^{i\,k\,x}\,dk\\
&\qquad\qquad -\frac{i\,E_0}{\pi\,\varepsilon}\,P\int\limits_{-\infty}^{+\infty}\frac{e^{i\,k\,x}}{k}\,dk\,,
\end{aligned}\right\} \tag{56}$$

$$E = \frac{E_0}{\varepsilon} + \frac{i\,E_0}{\pi\,\varepsilon}\int\limits_{-\infty}^{+\infty}\frac{K_0-K_k}{k\,(1-K_k)}\,e^{i\,k\,x}\,dk \tag{57}$$

which is Landau's result.

*Fresnel's equation*

From (53) we obtain the tangential component of the electric field

$$E_z = \frac{-\,i\,c^2\,B_0}{\pi\,\omega}\int\limits_{-\infty}^{+\infty}\frac{e^{i\,k_x\,x}}{k^2}\left\{\frac{k_z^2}{\varepsilon_L} - \frac{\omega^2/c^2\,k_x^2}{k^2-\varepsilon_T\,\omega^2/c^2}\right\}dk_x\,. \tag{58}$$

$\varepsilon_T$ is taken non spatially dispersive, $\varepsilon_T = \varepsilon$. If we take the longitudinal dielectric constant non spatially dispersive, then $\varepsilon_L = \varepsilon$ and the plasma is not different from common dielectric. We may then close the contour of integration in the upper half plane and

$$E_z = \frac{-c^2 B_0}{\varepsilon \omega} \sqrt{\varepsilon \frac{\omega^2}{c^2} - k_z^2}\; e^{i \sqrt{\varepsilon \omega^2/c^2 - k_z^2}\, x} . \tag{59}$$

We take an incoming transverse wave in vacuum as shown in Figure 1.

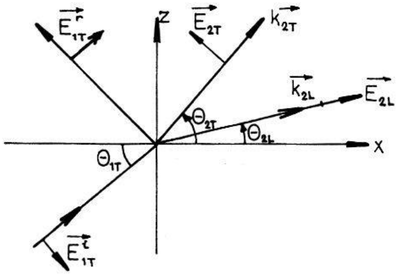


Figure 1
Wave aspect at the boundary.

From Maxwell's equation in vacuum we have

$$B_0 = -\frac{\omega}{k_{1x} c^2} (E_z^i - E_z^r) \tag{60}$$

where

$$k_{1x} = \frac{\omega}{c} \cos\theta_{1T} .$$

Up to now we have used only the continuity of $B_y$. Writing that $E_z$ is continuous across the boundary we obtain

$$Z = \frac{1+r}{1-r} = \frac{1}{k_{1z}\varepsilon} \left[ \sqrt{\epsilon \frac{\omega^2}{c^2} - k_z^2} = \frac{\sin\theta_{2T} \cos\theta_{2T}}{\sin\theta_{1T} \cos\theta_{1T}} \right] \tag{61}$$

where $r = E_z^r/E_z^i = E_{1T}^r/E_{1T}^i$ is the reflection coefficient and $Z$ the surface impedance. Solving for $r$ we get

$$r = \frac{\cos\theta_{1T} \sin\theta_{1T} - \cos\theta_{2T} \sin\theta_{2T}}{\cos\theta_{1T} \sin\theta_{1T} + \cos\theta_{2T} \sin\theta_{2T}} \tag{62}$$

which are the usual Fresnel's formula for the reflection coefficient of a $p$ polarized wave.

*Long Wavelength approximation*

Without considering Landau damping $\omega_p^2$ for frequencies $\omega_p < \omega \lesssim \omega_p/\sqrt{2}$ we may develop $\varepsilon_L$ as:

$$\varepsilon_L = 1 - \frac{\omega\, p^2}{\omega^2} - \frac{3}{2} \frac{k^2 v_0}{\omega^2} . \tag{63}$$

Then $\varepsilon_L$ is spatially dispersive and this introduces a new pole in the upper complex $k_x$ plane. Setting

$$k_T^2 = \varepsilon \frac{\omega^2}{c^2} \quad \text{and} \quad k_L^2 = \frac{2}{3} \varepsilon \frac{\omega^2}{v_0^2} \quad \text{we obtain from (58)}$$

$$E_z = \frac{\omega B_0}{\varepsilon \omega^2/c^2} \left[ k_{T_x} e^{i k_{Tx} x} + \frac{k_z^2}{k_{L_x}} e^{i k_{Lx} x} \right] . \tag{64}$$

From the continuity of $E_z$ across the boundary we now get

$$\left.\begin{aligned} Z = \frac{1+r}{1-r} &= \frac{1}{k_{1x} \varepsilon} \left[ k_{T_x} + \frac{k_z^2}{k_{L_x}} \right] \\ &= \frac{\sin\theta_{2T} \cos\theta_{2T}}{\sin\theta_{1T} \cos\theta_{1T}} + \sqrt{\frac{3}{2}} \frac{v_0}{c} \frac{\sin^2\theta_{1T}}{\varepsilon^{3/2} \cos\theta_{2L} \cos\theta_{1T}} , \end{aligned}\right\} \tag{65}$$

$$r = \frac{1 - \dfrac{\sin\theta_{2T}\cos\theta_{2T}}{\sin\theta_{1T}\cos\theta_{1T}} - \sqrt{\dfrac{3}{2}} \dfrac{v_0}{c} \dfrac{\sin^2\theta_{1T}}{\varepsilon^{3/2}\cos\theta_{2L}\cos\theta_{1T}}}{1 + \dfrac{\sin\theta_{2T}\cos\theta_{2T}}{\sin\theta_{1T}\cos\theta_{1T}} + \sqrt{\dfrac{3}{2}} \dfrac{v_0}{c} \dfrac{\sin^2\theta_{1T}}{\varepsilon^{3/2}\cos\theta_{2L}\cos\theta_{1T}}} . \tag{66}$$

We may calculate a transmission coefficient for transverse waves

$$t_T = \frac{E_L^t}{E_T^i} = \frac{\sin\theta_{1T}\cos\theta_{1T}}{\sin\theta_{2T}\cos\theta_{2T}} (1 - r) \tag{67}$$

and a transmission coefficient for longitudinal waves

$$t_L = \frac{E_L^t}{E_T^i} = \sqrt{\frac{3}{2}} \frac{v_0}{c} \frac{\sin^2\theta_{1T}}{\cos^2\theta_{2L}} \frac{1}{\varepsilon^{3/2}} (1 - r) \tag{68}$$

$v_0/c$ is small but $\varepsilon$ is small too, so that the overall effect might be observable experimentally.

*Discussion of the results*

We suppose $\omega$ fixed so that $\varepsilon = 1 - \omega_p^2/\omega^2$ is given and we shall discuss the meaning of the results obtained previously in functions of the angle $\theta_{1T}$. We take $\varepsilon > 0$; we may write

$$k_{1T_x} = \frac{\omega}{c} \cos\theta_{1T} , \quad k_{2T_x} = \sqrt{\varepsilon} \frac{\omega}{c} \cos\theta_{2T} , \quad k_{2L_x} = \frac{2}{3} \frac{\omega}{v_0} \sqrt{\varepsilon} \cos\theta_{2L} \tag{69}$$

and from the continuity of the tangential component of the wave vectors

$$\sin\theta_{2T} = \frac{\sin\theta_{1T}}{\sqrt{\varepsilon}} , \qquad \sin\theta_{2L} = \sqrt{\frac{3}{2}} \frac{v_0}{c} \frac{1}{\sqrt{\varepsilon}} \sin\theta_{1T} .$$

*Case a:* $0 \leqslant \sin\theta_{1T} \leqslant \sqrt{\varepsilon}$

Then $|\sin\theta_{2T}| \leqslant 1$ and $\cos\theta_{2T}$ is real, as well as $\cos\theta_{2L}$. Then $r_T$ is real and all the waves propagate in the region $x > 0$. We should remember that $\varepsilon$ is small for this theory to be valid.

*Case b:* $\sqrt{\varepsilon} \leqslant \sin\theta_{1T} \leqslant \sqrt{\dfrac{3}{2}} \dfrac{v_0}{c} \dfrac{1}{\sqrt{\varepsilon}}$ .

This corresponds to an internal reflection for the incoming transverse wave in the plasma. We have a transverse surface wave, such that

$$\cos\theta_{2T} = i\sqrt{\sin^2\theta_{2L} - 1}$$

and using (66)

$$r = \frac{1-\beta-i\alpha}{1+\beta+i\alpha}\,, \tag{70}$$

with

$$\beta = \sqrt{\frac{3}{2}}\,\frac{v_0}{c}\,\frac{\sin^2\theta_{1T}}{\varepsilon^{3/2}\cos\theta_{1T}\cos\theta_{2L}}\,,\qquad \alpha = \frac{\sin\theta_{2T}\sqrt{\sin^2\theta_{2T}-1}}{\sin\theta_{1T}\cos\theta_{1T}}\,,$$

$$|r|^2 = 1 - \frac{4\beta}{(1+\beta)^2+\alpha^2}\,,$$

$|r|^2 \neq 1$ as, although the transverse wave is internally reflected, some of the energy is stored in the longitudinal mode.

*Case c:* $\sin\theta_{1T} > \sqrt{\frac{3}{2}}\,\frac{v_0}{c}\,\frac{1}{\sqrt{\varepsilon}}$.

$\cos\theta_{2L}$ is now imaginary as $\sin\theta_{2L} > 1$. The longitudinal wave is a surface wave and $r = 1 - i(\alpha+\beta)/1 + i(\alpha+\beta)$; $|r|^2 = 1$, as it should.

*Transverse wave excited in vacuum by a longitudinal wave in the plasma*

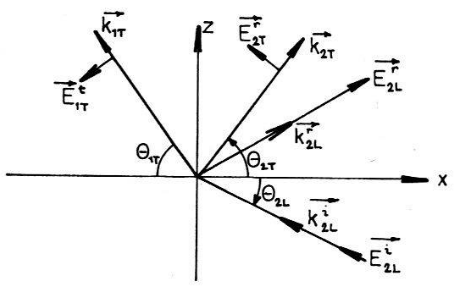


Figure 2
Wave aspect at the boundary.

We shall consider a longitudinal wave arriving on the boundary from $x = +\infty$ at an angle $\theta_{2L}$. The wave vector $k_{2L}$ and the normal to the boundary define the plane of incidence $x = 0$ and the problem is twodimensional. We assume again as a starting hypothesis that we know the fields at $x = 0^-$ in vacuum, equations (52) and (53) are still valid the only difference are the boundary conditions at infinity which corresponds to a different path of integration in the complex $k_x$ plane to obtain the inverse Fourier transform.

The four poles in $k_x$ corresponding to $\varepsilon_L = 0$ (in the approximation (63)) and $k^2 - \varepsilon_T\,\omega^2/c^2 = 0$ lie in the complex $k_x$ plane as shown in Figure 3.

We shall relax one degree of freedom and take into account the pole $-k_{L_x}$ which corresponds to the wave coming from the right by taking as contour of integration the linear combination.

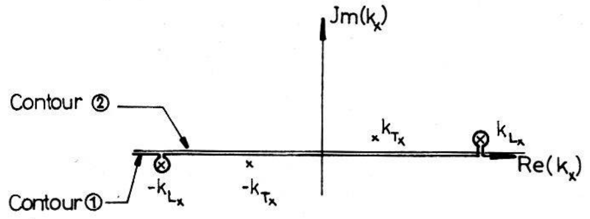


Figure 3
Contours of integration in the complex $k_x$ plane.

$A \times$ contour ① $+ B \times$ contour ②. The pole $+k_{T_x}$ is included in both contours such that $B$ is continuous across the density discontinuity. $A$ and $B$ are subjected to the constraint $A + B = 1$. For both contours we close the contour by a half circle in the upper half of the complex $k_x$ plane, and calculating the residues we obtain in the region $x > 0$:

$$E_{2z} = -\frac{\omega B_1}{\varepsilon\,\omega^2/c^2}\left[k_{2T_x} e^{i k_{2Tx} x} - A \frac{k_z^2}{k_{2L_x}} e^{-i k_{2Lx} x} + B \frac{k_z^2}{k_{2L_x}} e^{i k_{2Lx} x}\right]. \tag{71}$$

Applying Maxwell's equation in the region $x < 0$ and expressing the continuity of $E_z$ we obtain the transmission coefficient:

$$t_T = \frac{E^t_{1T}}{E^i_{2L}} = \frac{\dfrac{2\cos\theta_{2L}\sin\theta_{1T}}{\sin^2\theta_{2L}}}{1 + \sqrt{\dfrac{3}{2}}\,\dfrac{v_0}{c}\left(\dfrac{\sqrt{\varepsilon}\cos\theta_{1T}\cos\theta_{2L} + \cos\theta_{2T}\cos\theta_{2L}}{\sin^2\theta_{2L}}\right)}. \tag{72}$$

*Discussion of the Results*

In order to have a propagating transverse wave in the region $x < 0$, we must have $|\sin\theta_{1T}| \leqslant 1$. As

$$\sin\theta_{1T} = \sqrt{\frac{2\,\varepsilon}{3}}\,\frac{c}{v_0}\sin\theta_{2L}. \tag{73}$$

The angle $\theta_{2L}$ must be inside a small cone aroung the $0_x$ axis. The effect of $c/v_0$ is somewhat attenuated by the factor $\varepsilon$ which is small. For $|\sin\theta_{1T}| \geqslant 1$ we have a surface transverse wave in the region $x < 0$. If $\sin\theta_{2L} < 2/3\; c/v_0$, we have a propagating transverse wave in the region $x > 0$. This condition is much stronger than the previous one as there is no factor of $\varepsilon$ multiplying $c/v_0$. For $\sin\theta_{2L} > 3/2\;\varepsilon\, v_0/c$ no propagating transverse waves are generated – neither in the region $x > 0$ nor in the region $x < 0$ – and the longitudinal wave is totally internally reflected as $\cos\theta_{1T}$ and $\cos\theta_{2T}$ are pure imaginary.

*Reflection coefficient for a transverse wave general case*

The solution given in a previous paragraph was only valid in a certain frequency domain. For $\omega$ arbitrary we must take Landau damping into account and in order to invert Fourier transform (52) and (53) we must discuss more cautiously the properties of $\varepsilon_L$

$$\varepsilon_L = 1 - K_L = 1 - \frac{e^2}{m\,\varepsilon_0\,k^2}\iint \frac{\boldsymbol{k}\cdot\partial f_0/\partial\boldsymbol{v}}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega}\, d\boldsymbol{v}$$

where $K_L$ is the polarization kernel.

The integral is not defined over a line in the $v_x$, $v_z$ plane where

$$k_x v_x + k_z v_z - \omega = 0 .$$

The denominator becomes infinite.

We shall rotate the coordinates such that the axis $u$ is perpendicular to the line $\boldsymbol{k} \cdot \boldsymbol{v} - \omega = 0$. The rotation is written as follows, where $v_0 = 2 K T/m$ and $u$ and $v$ have been normalized:

$$\mu = \frac{1}{k v_0} (k_x v_x + k_z v_z) , \qquad v = \frac{1}{k v_0} (- k_z v_x + k_x v_z) .$$

Setting $\Omega = \omega/k v_0$, the polarization kernel may be written, using

$$\left.\begin{aligned} & k = \sqrt{k_x^2 + k_z^2} , \\ & K_L = \frac{- 2 \omega_p^2}{\pi v_0^2 k^2} \iint \frac{k u e^{-(u^2 + v^2)} du dv}{k u - \Omega} = \frac{- 2 \omega_p^2}{\pi^{1/2} v_0^2 k^2} \int_{-\infty}^{+\infty} \frac{k u e^{-u^2} du}{k u - \Omega} . \end{aligned}\right\} \tag{74}$$

The integral is not defined for $u = \Omega/k$. To give it a sense we suppose, as customary, that $\Omega$ has a small positive imaginary part, $\nu$, ultimately taken equal to zero, but such that for $k$ *positive* the contour of integration will go below the pole in the complex $u$ plane.

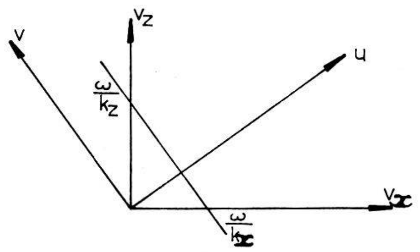


Figure 4
Rotation of the coordinates in the velocity plane.

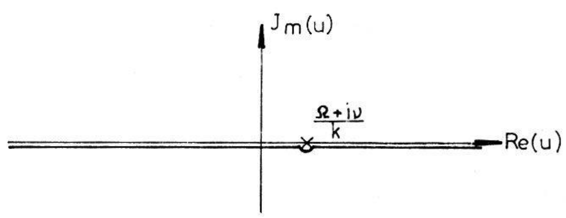


Figure 5
Contour of the integration in the complex $u$ plane for $k$ *positive*.

To take the contour as shown in Figure 5 is equivalent to applying Plejmel's formula

$$\lim_{\varepsilon \to 0} \frac{1}{x - i \varepsilon} = P \frac{1}{x} + i \pi \delta(x) \tag{75}$$

to (74) which gives for $k > 0$ and real,

$$K_L = K_1 = \frac{- 2 \omega_p}{\pi^{1/2} v_0^2 k^2} \left\{ P \int \frac{k u e^{-u^2} du}{k u - \Omega} + \frac{i \pi \Omega}{k} e^{-\Omega^2/k^2} \right\} . \tag{76}$$

For $k < 0$ we shall take the same formula for $K_1$, but in this domain $K_L$ does not reduce to $K_1$.

$$K_1(k) \{k \varepsilon ] 0, \infty [ \} = \frac{2 \omega_p^2}{\pi^{1/2} v_0^2 k^2} \left\{ P \int_{-\infty}^{+\infty} \frac{k u e^{-u^2} du}{k u - \Omega} + \frac{i \pi \Omega}{k} e^{-(\Omega^2/k^2)} \right\} , \tag{77}$$

$$K_1(k)\ \{k\ \varepsilon]-\infty, 0\,[\,\} = \frac{2\,\omega_p^2}{\pi^{1/2}\,v_0^2\,k^2}\left\{P\int_{-\infty}^{+\infty}\frac{k\,u\,e^{-u^2}\,du}{k\,u-\Omega} - \frac{i\,\pi\,\Omega}{|k|}\,e^{-(\Omega^2/k^2)}\right\}. \tag{78}$$

We shall now consider $K_L$ for $k < 0$. The contour of integration will now go above the pole in the complex $u$ plane, such that for $k < 0$, $K_L = K_2$:

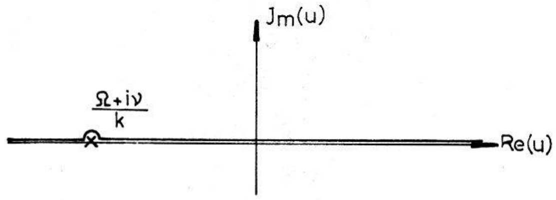


Figure 6
Contour of the integration in the complex $u$ plane for $k$ *potitive.*

$$K_2 = \frac{-\,2\,\omega_p^2}{\pi^{1/2}\,v_0^{1/2}\,k^2}\left\{P\int_{-\infty}^{+\infty}\frac{k\,u\,e^{-u^2}\,du}{k\,u-\Omega} - \frac{i\,\pi\,\Omega}{k}\,e^{-(\Omega^2/k^2)}\right\}$$

and extending the definition of $K_2$ for $k > 0$, using the same formula as above,

$$K_2(k)\ \{k\ \varepsilon]\,0, \infty\,[\,\} = -\frac{2\,\omega_p^2}{\pi^{1/2}\,v_0^{1/2}\,k^2}\left\{P\int_{-\infty}^{+\infty}\frac{k\,u\,e^{-u^2}\,du}{k\,u-\Omega} - \frac{i\,\pi\,\Omega}{k}\,e^{-(\Omega^2/k^2)}\right\}$$

$$K_2(k)\ \{k\ \varepsilon]-\infty, 0\,[\,\} = -\frac{2\,\omega_p^2}{\pi^{1/2}\,v_0^{1/2}\,k^2}\left\{P\int_{-\infty}^{+\infty}\frac{k\,u\,e^{-u^2}\,du}{k\,u-\Omega} + \frac{i\,\pi\,\Omega}{|k|}\,e^{-(\Omega^2/k^2)}\right\}.$$

With those definitions we have the equality

$$K_1(-k) = K_2(k)$$

for $k$ real and different from zero.

We shall now consider the analytic continuation of the function $K_1(k)$ for $\Omega$ real and $k$ complex which reduces to (77) on the real $k$ axis. It may be defined as

$$K_1(k) = -\frac{2\,\omega_p^2}{\pi^{1/2}\,v_0^2\,k^2}\int_{\Gamma_1}\frac{k\,u\,e^{-u^2}}{k\,u-\Omega}\,du\,.$$

The contour $\Gamma_1$ consisting of the real $k$ axis plus a contour passing below the pole $u = \Omega/k$ of $Im(\Omega/k) < 0$. Similarly for the analytic continuation of $K_2$.

$$K_2(k) = -\frac{2\,\omega_p^2}{\pi^{1/2}\,v_0^2\,k^2}\int_{\Gamma_2}\frac{k\,u\,e^{-u^2}}{k\,u-\Omega}\,du\,.$$

The contour $\Gamma_2$ consisting of the real $k$ axis plus a contour passing above the pole $u = \Omega/k$ if $Im(\Omega/k) > 0$.

With those definitions $K_1(k)$ and $K_2(k)$ are analytical functions of $k$ in the entire complex $k$ plane and

$$[K_2(k^*)]^* = K_1(k)\,, \qquad [K_1(k^*)]^* = K_2(k).$$

$K_1$ and $K_2$ have an essential singularity at the point $k = 0$.

Knowing the properties of the polarization kernel we come back to the fields. Taking $\varepsilon_T$ not spatially dispersive we obtain for the tangential component of the electric field:

$$E_z = \frac{i\,B_0\,c^2}{\pi\,\omega} \left\{ k_z^2 \int_{-\infty}^{+\infty} \frac{e^{i\,k_x x}\,(1-(1-K_L)/\varepsilon)}{k^2\,(1-K_L)}\,dk_x + \frac{i\,\pi}{\varepsilon}\,k_{T\,x}\,e^{i\,k_{T\,x} x} \right\}. \tag{79}$$

The first term in the bracket corresponds to a longitudinal field and the second term to a transverse field. To take advantage of our study of $K_L$ as a function of $k$ and not of $k_x$ we change the variable of integration from $k_x$ to $k$ introducing a branch cut in the complex $k$ plane between $-k_z$ and $+k_z$ with the following determinations:

$$k_x < 0, \qquad k_x^2 = -\sqrt{k^2 - k_z^2}\,, \qquad k = \sqrt{k_x^2 + k_z^2}\,,$$

$$k_x > 0, \qquad k_x^2 = +\sqrt{k^2 - k_z^2}\,, \qquad k = \sqrt{k_x^2 + k_z^2}\,.$$

Then

$$\left.\begin{aligned} E_z = \frac{-\,i\,c^2\,B_0}{\pi\,\omega} \Bigg\{ k_z^2 \Bigg[ -\int_{\infty}^{k_z} \frac{e^{-i\sqrt{k^2-k_z^2}\,x}\,(1-(1-K_1(k))/\varepsilon)}{k\sqrt{k^2-k_z^2}\,(1-K_1(k))}\,dk \\ + \int_{k_z}^{\infty} \frac{e^{i\sqrt{k^2-k_z^2}\,x}\,(1-(1-K_1(k))/\varepsilon)}{k\sqrt{k^2-k_z^2}\,(1-K_1(k))}\,dk + \frac{i\,\pi\,k_{T\,x}}{\varepsilon}\,e^{i\,k_{T\,x}\,x} \Bigg\}. \end{aligned}\right\} \tag{80}$$

Using $K_1(-k) = K_2(k)$ it is easy to see that with the contour $\overset{\curvearrowright}{C_1}$ and $\overset{\curvearrowright}{C_2}$ defined as in Figure 7 we have

$$\left.\begin{aligned} E_z = -\frac{i\,c^2\,B_0}{\omega\,\pi} \Bigg\{ k_z^2 \int_{C_1} \frac{e^{i\sqrt{k^2-k_z^2}\,x}\,(1-(1-K_2(k))/\varepsilon)}{k\sqrt{k^2-k_z^2}\,(1-K_2(k))}\,dk \\ + k_z^2 \int_{C_2} \frac{e^{i\sqrt{k^2-k_z^2}\,x}\,(K_1(k)-K_2(k))}{k\sqrt{k^2-k_z^2}\,(1-K_1(k))\,(1-K_2(k))}\,dk + \frac{i\,\pi\,k_{T\,x}}{\varepsilon}\,e^{i\,k_{T\,x}\,x} \Bigg\}. \end{aligned}\right\} \tag{81}$$

The reflection coefficient for the incoming transverse wave $r$ and the surface impedance are determined using the continuity of $E_z$.

$$\left.\begin{aligned} E_z|_{0^+} = -\frac{i\,c^2\,B_1\,k_z^2}{\pi\,\varepsilon\,\omega} \Bigg\{ \int_{C_1} \frac{1-(1-K_2)/\varepsilon}{k\sqrt{k^2-k_z^2}\,(1-K_2)}\,dk \\ + \int_{C_2} \frac{K_1-K_2\,dk}{k\sqrt{k^2-k_z^2}\,(1-K_1)\,(1-K_2)} - \frac{k_{T\,x}\,c^2\,B_1}{\omega\,\varepsilon} \Bigg\}, \end{aligned}\right\} \tag{82}$$

$$B_1 = -\frac{\omega}{k_{1\,x}\,c^2}\,(E_z^i - E_z^r)$$

and

$$Z_S = \frac{1+r}{1-r} = \frac{1}{\varepsilon\, k_{1x}}\, k_{T_x} + \frac{i}{\pi}\, k_z^2 \left\{ \int_{C_1} \frac{1 - ((1-K_1)/\varepsilon)\, dk}{k\sqrt{k^2-k_z^2}\,(1-K_2)} + \int_{C_2} \frac{(K_1-K_2)\, dk}{k\sqrt{k^2-k_z^2}\,(1-K_1)\,(1-K_2)} \right\}\Bigg] \tag{83}$$

and by numerical computation of $Z_S$ we would know $r$.

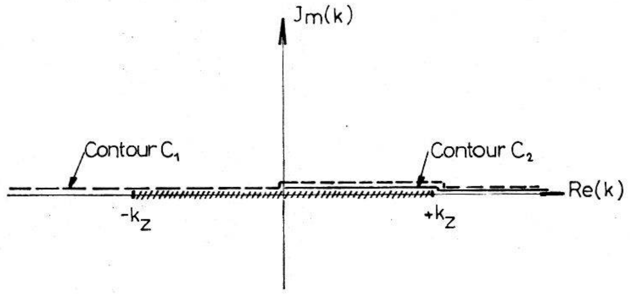


Figure 7
Contour $C_1$ and $C_2$ in the complex $k$ plane.

We shall now stress the fundamental importance of the principal Landau pole $k_1$. Due to the essential singularity at $k = 0$, $1 - K_1 = 0$ has an infinity of roots in the upper half of the complex $k$ plane, we denote them by $k_i$ and we may write:

$$\frac{1}{1-K_1(k)} = -\sum_{i=0}^{\infty} \frac{1}{\partial/\partial k\, K_1(k)\,|_{k=k_i}}\, \frac{1}{k-k_i}\,. \tag{84}$$

Furthermore it is easy to show using Nyquist criteria [6] that for $\varepsilon > 0$, $1 - K_2(k) = 0$ has no solutions in the upper half of the complex $k$ plane and for $\varepsilon < 0$ one root only. In the case $\varepsilon > 0$, the integral over the contour $C_2$ vanishes by pushing the contour of integration to infinity and we obtain

$$Z_S = \frac{1}{\varepsilon\, k_{1x}} \left[ k_{T_x} - \frac{i}{\pi}\, k_z^2 \sum_{i=0}^{\infty} \frac{1}{\partial/\partial k\, K_1(k)\,|_{k=k_i}} \int_{C_2} \frac{dk}{k\sqrt{k^2-k_3^2}\,(k-k_i)} \right.. \tag{85}$$

It is easy to show that if we take only the first Landau pole we reobtain (65) and when $\varepsilon \to 0$ the major contribution comes from the principal Landau pole as $\partial/\partial k\, K_1(k)\,|_{k=k_i}$ vanishes *only* for this value. This does not justify completely our procedure as we have an infinity of other poles and although each contribution may be small the sum may be comparable to the term corresponding to the first pole. All we can hope is to have this theory at best valid asymptotically when $\varepsilon \to 0$. Although some progress has been made in calculating higher order Landau poles [12], no calculations are obtainable for poles of higher order than 4 and this is insufficient to decide over convergence of series of inverse of derivative evaluated at all Landau poles.

In brief we have calculated a formal but exact expression for the surface impedance from which the reflection coefficient may be obtained.

*Remark*

If we have a dielectric of dielectric constant $\varepsilon_1$ outside the plasma, we could imagine that there is a small slab of vacuum between the dielectric and the plasma and relate the field in vacuum to the fields in the dielectric to obtain $E_{x_{ext}}$. This easy generalization could be obtained readily.

## 6. Conclusion

The problem of creation at a density gradient of a longitudinal wave by a transverse wave and of a transverse wave by a longitudinal wave has been studied in this work considering a one-dimensional density gradient, a Maxwell-Boltzmann equilibrium distribution function and no external magnetic field. The various approaches used could be generalized to other types of geometry rather easily (such as cylindrical and spherical boundaries, plasma slabs, etc.). In the case of a solid state plasma it would be more proper to use the Fermi-Dirac equilibrium distribution function instead of the Maxwell-Boltzmann or even better to treat the problem quantum mechanically. In the case of a plasma-vacuum interface it would be interesting also to consider diffuse reflection instead of specular to measure the approximation made assuming the latter condition.

But possibly the most interesting continuation of this work is to consider the identical problem of wave creation in the presence of an external magnetic field; this introduces some complications as longitudinal and transverse waves are no longer pure modes, but it will be very interesting to know how a light wave could excite an Alfvèn wave, for example, and the experimental confirmation would be easier as it would correspond to a much more physical situation in which the plasma confinement would be due to an external magnetic field.

REFERENCES

[1] R. H. Ritchie and A. L. Marusak, *The Surface Plasmon Dispersion Relation for an Electron Gas*, Surface Sci. *4*, 234 (1966).
[2] R. L. Guernsey, *Surface Waves in Hot Plasmas*, Phys. Fluids *12*, 1852 (1969).
[3] R. A. Ferrell, *Predicted Radiation of Plasma Oscillations in Metal Films*, Phys. Rev. *111*, 5, 1214 (1958).
[4] D. A. Tidman and J. M. Boyd, *Radiation by Plasma Oscillations Incident on a Density Discontinuity*, Phys. Fluids *5*, 213 (1962).
[5] A. H. Kritz and D. Mintzer, *Propagation of Plasma Waves Across a Density Discontinuity*, Phys. Rev. *117*, 382 (1960).
[6] P. Boulanger, *Création d'ondes dans un plama à gradient de densité*, J. Phys. *31*, 993 (1970).
[7] G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc *A 195*, 336 (1948).
[8] L. D. Landau, J. Phys. (U.S.S.R.) *10*, 25 (1946).
[9] L. D. Landau and Lifshitz, *Electrodynamics of Continuous Media*.
[10] B. S. Tanenbaum, *Plasma Physics* (McGraw Hill Book Company 1967).
[11] W. E. Drummond, *Theory of Plasma Probes in a Collisionless Plasma*, Rev. Sci. Inst. *34*, 7 (1963).
[12] H. Derfler and T. C. Simonen, *Higher Landau poles* *12*, 269 (1969).