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Boundary Curves of the Double Spectral Functions in the Mandelstam Representation[1])

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Summary. The boundary curves of the double spectral functions in the Mandelstam representation for the invariant amplitudes of a two-particle → two-particle collision process are evaluated for a number of hadronic processes. Use is made only of elastic unitarity and of 'extended' unitarity and a general formula is given which applies to any case where an anomalous threshold is absent. It is shown that subtractions in the Mandelstam representation do not alter the boundary curves.

1. Introduction

It is well known that the double spectral functions in the Mandelstam representation for the invariant scattering amplitudes of a two-particle→two-particle collision process do not begin to differ from zero at the square of the total mass of the two-particle state, of lowest possible total mass, with the internal quantum numbers of the appropriate channel. The region in which a double spectral function is non-zero is not rectangular; in general, unitarity restricts it to a smaller region bounded by curves asymptotic to the squares of the lowest masses just mentioned. These boundary curves have been calculated for $\pi\pi \to \pi\pi$ using elastic unitarity [1] and for $\pi N \to \pi N$ using 'extended' unitarity as well [2]. In these cases the possibility of subtractions being needed in the Mandelstam representation was not considered.

The boundary curves of the double spectral functions for $N N \to N N$ have been calculated by using the obvious box diagram of fourth order [3]. From a practical point of view, this is sufficient. However, we believe that it is desirable to see how such boundary curves (and indeed those for any binary collision process) can be obtained by using only elastic unitarity and 'extended' unitarity, without recourse to diagrams. The purpose of this paper is to obtain in just this way a general formula for boundary curves of double spectral functions which applies to any binary collision when no anomalous threshold is present. We shall verify that boundary curves are unaltered if subtractions are required in the Mandelstam representation and we shall use our general formula to obtain boundary curves for several important hadronic

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processes. These boundary curves are often needed in the study of strong interaction processes by means of partial wave dispersion relations, to determine when the absorptive part on an unphysical cut can be calculated by means of a partial wave expansion. The boundary curves also need to be known if one wishes to obtain the double spectral functions from the absorptive parts of the scattering amplitude. Exploratory work in this direction has been done, for example, by Martin [4].

Throughout the paper we do not consider the inclusion of spin or isospin. This simplification will not affect the calculation of the boundary curves. We know [5] that for binary collisions involving non-zero mass particles with spin it is possible to obtain invariant amplitudes free from kinematic singularities, for which the Mandelstam representation may be expected to hold. The unitarity condition for each amplitude of definite isospin will then be much more complicated than in the spinless case, even for the simple case of (spin 0 + spin $^1/_2$ → spin 0 + spin $^1/_2$). For this case, the necessary algebra is given explicitly by Mandelstam [1]. What emerges, even for the general case, is that in each term on the right side of the unitarity relation, there is an integral to be evaluated of the type

$$\iint d\Omega_{\bar{\boldsymbol{n}}} \frac{[\text{polynomial function of } (\bar{\boldsymbol{n}} \cdot \boldsymbol{n}') \text{ and } (\bar{\boldsymbol{n}} \cdot \boldsymbol{n})]}{(\tau_2' - \bar{\boldsymbol{n}} \cdot \boldsymbol{n}')\,(\tau_2 - \bar{\boldsymbol{n}} \cdot \boldsymbol{n})},$$

whereas in the spinless case without subtractions the polynomial function is just unity. This means that the complexities introduced by spin can be handled exactly like those which come from subtractions, which we shall consider in Section 3. As we shall see, the boundary curves are not altered.

Though isospin is not included, we shall require it to be conserved. This means, for example, that Λ is not possible as an intermediate state for the process $\pi \Lambda \to \pi \Lambda$. We shall use other conservation laws as well. Thus an intermediate $(3\,\pi)$ state is not possible for the process $\pi\pi \to \pi\pi$ because of G-parity conservation and Σ is not a possible intermediate state for $K N \to K N$ because of strangeness conservation.

The plan of the paper is as follows. There are sufficient preliminaries concerning the Mandelstam representation, fixed variable dispersion relations and unitarity for it to be desirable to collect them together in Section 2. In Section 3 we consider the process $\pi\pi \to \pi\pi$. Although this can be found in [1, 2], we consider in detail the effect of subtractions in the fixed variable dispersion relations which are fed into the unitarity relation.

In Section 4 we discuss what is meant by 'extended' unitarity and give a general formula for the boundary curves. It is not difficult to explain what 'extended' unitarity is; it is much more difficult to give the conditions under which it may be expected to hold true. These are precisely the conditions for the absence of anomalous thresholds. We are not going to discuss anomalous thresholds in this paper; this requires detailed analysis of fourth order 'box' diagrams [6], or delicate arguments involving analytic continuation (an outline of this method of approach is given by Barut [7]). We shall derive our general formula in Section 4 on the assumption that no anomalous thresholds are present. This will hold true for the special processes to be discussed in Section 5, namely, $\pi N \to \pi N$, $\pi K \to \pi K$, $\pi \Lambda \to \pi \Lambda$, $K N \to K N$ and $N N \to N N$. An appendix shows how to evaluate, using real variable methods only, the basic integral involved in calculating the boundary curves.

2. Mandelstam representation. Unitarity

Consider the four spinless particles A, B, C, D with antiparticles $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$. To avoid annoying subscripts, without producing any confusion, we shall also denote the masses of these particles by A, B, C, D. In the special processes of Section 5 we shall also use the same letter for the particle and its mass, except for the pion, whose mass will be denoted by μ. We label the processes below as follows:

1. $AB \to CD$,
2. $A\bar{C} \to D\bar{B}$,
3. $A\bar{D} \to \bar{B}C$.

To avoid the very slight complication of identical particles in the initial or final state of any of these processes, we assume that A, B, $\bar{C}$ and $\bar{D}$ are all different.

For process 1, let the four-momenta of the particles in some inertial frame be p_A, p_B, p_C, p_D and define the invariants

$$s_1 = -(p_A + p_B)^2 = -(p_C + p_D)^2 ,$$
$$t_1 = -(p_A - p_C)^2 = -(p_D - p_B)^2 ,$$
$$u_1 = -(p_A - p_D)^2 = -(p_C - p_B)^2 .$$

Then $(s_1 + t_1 + u_1) = (A^2 + B^2 + C^2 + D^2) = \Sigma$, say. The process may also be characterised by its total energy W_1 and scattering angle θ_1 in the centre-of-momentum system (CMS). θ_1 is the angle between $\boldsymbol{p}_A$ and $\boldsymbol{p}_C$ which satisfies $0 \leqslant \theta_1 \leqslant \pi$. If q_1 is the magnitude of the three-momentum of either A or B and q_1' is the magnitude of the three-momentum of either C or D in the CMS, then

$$W_1 = \sqrt{A^2 + q_1^2} + \sqrt{B^2 + q_1^2} = \sqrt{C^2 + q_1'^2} + \sqrt{D^2 + q_1'^2},$$
$$s_1 = W_1^2,$$
$$\begin{aligned} t_1 &= A^2 + C^2 - 2\sqrt{A^2 + q_1^2}\sqrt{C^2 + q_1'^2} + 2\, q_1\, q_1' \cos\theta_1 \\ &= \frac{1}{2}\Sigma - \frac{1}{2} s_1 - \frac{1}{2}(A^2 - B^2)(C^2 - D^2)\, s_1^{-1} + 2\, q_1\, q_1' \cos\theta_1 , \end{aligned} \tag{1}$$

$$\begin{aligned} u_1 &= A^2 + D^2 - 2\sqrt{A^2 + q_1^2}\sqrt{D^2 + q_1'^2} - 2\, q_1\, q_1' \cos\theta_1 \\ &= \frac{1}{2}\Sigma - \frac{1}{2} s_1 + \frac{1}{2}(A^2 - B^2)(C^2 - D^2)\, s_1^{-1} - 2\, q_1\, q_1' \cos\theta_1 . \end{aligned} \tag{2}$$

The physical process corresponds to $s_1 \geqslant \max\{(A + B)^2, (C + D)^2\}$, $|\cos\theta_1| \leqslant 1$.

In the same way we may define kinematical invariants for the other processes as follows. For process 2,

$$s_2 = -(p_A + p_{\bar{C}})^2 = -(p_{\bar{B}} + p_D)^2 ,$$
$$t_2 = -(p_A - p_D)^2 = -(p_{\bar{B}} - p_{\bar{C}})^2 ,$$
$$u_2 = -(p_A - p_{\bar{B}})^2 = -(p_D - p_{\bar{C}})^2 ;$$

for process 3,

$$\begin{aligned} s_3 &= -(p_A + p_{\overline{D}})^2 = -(p_{\overline{B}} + p_C)^2 , \\ t_3 &= -(p_A - p_{\overline{B}})^2 = -(p_C - p_{\overline{D}})^2 , \\ u_3 &= -(p_A - p_C)^2 = -(p_{\overline{B}} - p_{\overline{D}})^2 . \end{aligned}$$

As before, $(s_2 + t_2 + u_2) = (s_3 + t_3 + u_3) = \Sigma$. We may define quantities W, θ, q, q' for each process and write equations exactly like (1), (2) with 1 replaced by 2 or 3.

With plane wave states normalised so that $\langle \boldsymbol{p}' | \boldsymbol{p} \rangle = \delta^{(3)} (\boldsymbol{p}' - \boldsymbol{p})$ we define the Lorentz invariant scattering amplitude $T_1(s_1, t_1)$ in terms of the S-operator by

$$\begin{aligned} &\langle \boldsymbol{p}_C \, \boldsymbol{p}_D | (S - 1) | \boldsymbol{p}_A \, \boldsymbol{p}_B \rangle \\ &\quad = -i(2\pi)^{-2} \, \delta^{(4)} (p_C + p_D - p_A - p_B) \, \frac{1}{4} \, (E_A \, E_B \, E_C \, E_D)^{-1/2} \, T_1(s_1, t_1) \, . \end{aligned} \tag{3}$$

Exactly similar definitions may be given of $T_2(s_2, t_2)$ and $T_3(s_3, t_3)$.

If $\boldsymbol{n}_1$, $\boldsymbol{n}_1'$ are unit vectors in the directions of $\boldsymbol{p}_A$, $\boldsymbol{p}_C$ respectively in the CMS, then the differential cross-section for scattering into the differential of solid angle $d\Omega_{\boldsymbol{n}_1'}$ is

$$\frac{d\sigma}{d\Omega_{\boldsymbol{n}_1'}} = \frac{q_1'}{q_1} \, \frac{| T_1(s_1, t_1(s_1, \boldsymbol{n}_1 \cdot \boldsymbol{n}_1')) |^2}{64 \, \pi^2 \, s_1} \, . \tag{4}$$

For the other channels, simply replace 1 by 2 or 3.

Suppose that for process 1 a two-particle intermediate state (EF) is possible. Then, dropping the subscript 1, the contribution of this state to $Im \, T_{AB \to CD}(s, t(s, \boldsymbol{n} \cdot \boldsymbol{n}'))$ in the unitarity relation is

$$\begin{aligned} &Im \, T^{(EF)}_{AB \to CD}(s, t(s, \boldsymbol{n} \cdot \boldsymbol{n}')) \\ &\quad = - \frac{q_{EF}}{32 \, \pi^2 \, W} \iint d\Omega_{\overline{\boldsymbol{n}}} \, T^*_{CD \to EF}(s, t(s, \boldsymbol{n}' \cdot \overline{\boldsymbol{n}})) \, T_{AB \to EF}(s, \boldsymbol{n} \cdot \overline{\boldsymbol{n}}) \, . \end{aligned} \tag{5}$$

If the states (AB) and (CD) are different, then time-reversal invariance is needed to write

$$\begin{aligned} &\frac{1}{2i} \, [T_{AB \to CD}(s, t(s, \boldsymbol{n} \cdot \boldsymbol{n}')) - T^*_{CD \to AB}(s, t(s, \boldsymbol{n} \cdot \boldsymbol{n}'))] \\ &\qquad = Im \, T_{AB \to CD}(s, t(s, \boldsymbol{n} \cdot \boldsymbol{n}')) \, . \end{aligned}$$

q_{EF} is the magnitude of the three-momentum of either E or F in the CMS, corresponding to total energy $W = s^{1/2}$, and $\overline{\boldsymbol{n}}$ is an arbitrary unit vector. The integration is thus over the whole surface of the unit sphere.

Writing subscripts again we see that we have introduced three invariant amplitudes $T_i(s_i, t_i)$ for $i = 1, 2, 3$ and given in equation (4) the relation between these amplitudes and experimental differential cross-sections. We now formulate the Mandelstam hypothesis by postulating an analytic function $F(z_1, z_2, z_3)$ of three complex variables z_1, z_2, z_3, which is defined (and regular) except when any one of the

z_i is real and satisfies $\Sigma_i \leqslant z_i < \infty$ and except for isolated simple poles which will appear below. The first assumption about F is that it has the representation

$$\begin{aligned}
F(z_1, z_2, z_3) &= F(Z_1, Z_2, Z_3) \\
&+ \sum_{i=1}^{3} \sum_{m=0}^{N-1} (m!)^{-1} D_i^m F(Z_1, Z_2, Z_3) (z_i - Z_i)^m \\
&+ \sum_{\substack{(ij) \\ =(12,23,31)}} \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} (m!\, n!)^{-1} D_i^m D_j^n F(Z_1, Z_2, Z_3) (z_i - Z_i)^m (z_j - Z_j)^n \\
&+ \sum_{i=1}^{3} \sum_{m=1}^{n_i} \frac{(z_i - Z_i)^N}{(z_{im} - Z_i)^N} \frac{R_{im}}{(z_i - z_{im})} \\
&+ \sum_{i=1}^{3} \frac{(z_i - Z_i)^N}{\pi} \int_{\Sigma_i}^{\infty} \frac{d\sigma_i}{(\sigma_i - z_i)(\sigma_i - Z_i)^N} \\
&\times \sum_{m=0}^{N-1} [f_{im}(\sigma_i)(z_j - Z_j)^m + g_{im}(\sigma_i)(z_k - Z_k)^m] \\
&+ \sum_{\substack{(ij) \\ =(12,23,31)}} \frac{(z_i - Z_i)^N (z_j - Z_j)^N}{\pi^2} \\
&\times \int_{\Sigma_i}^{\infty} \int_{\Sigma_j}^{\infty} \frac{d\sigma_i\, d\sigma_j\, \varrho_k(\sigma_i, \sigma_j)}{(\sigma_i - z_i)(\sigma_j - z_j)(\sigma_i - Z_i)^N (\sigma_j - Z_j)^N} .
\end{aligned} \tag{6}$$

In (6), $(i\, j\, k)$ is always a cyclic permutation of (1, 2, 3). N is a non-negative integer; if $N = 0$ only the pole terms and the three double integral terms appear. The functions $f_{im}(\sigma_i)$ and $g_{im}(\sigma_i)$ ($i = 1, 2, 3$ and $m = 0, 1, \dots, (N-1)$) and the three double spectral functions $\varrho_k(\sigma_i, \sigma_j)$ are assumed *real*-valued. The real numbers z_{im} are the squares of the masses of single particles with the same internal quantum numbers as the initial and final states of process i; the residues R_{im} at these simple poles of F are assumed real. The real numbers Σ_i are the squares of the total masses of the two-particle states, of lowest possible total mass, with the same internal quantum numbers as the initial and final states of process i. For all hadronic processes, $(\Sigma_1 + \Sigma_2 + \Sigma_3) > \Sigma$ and it is convenient to take the coordinates of the subtraction point as

$$Z_i = \frac{1}{3} \Sigma + \frac{2}{3} \Sigma_i - \frac{1}{3} \Sigma_j - \frac{1}{3} \Sigma_k .$$

The Z_i will then be real and $(Z_1 + Z_2 + Z_3) = \Sigma$, $(\Sigma_1 - Z_1) = (\Sigma_2 - Z_2) = (\Sigma_3 - Z_3)$. With the further assumptions that F and its partial derivatives which appear in equation (6) are all real at (Z_1, Z_2, Z_3), it follows that F has the reflection property

$$F(z_1^*, z_2^*, z_3^*) = F^*(z_1, z_2, z_3) .$$

Our subtracted form of the Mandelstam representation in (6) agrees with that of Cheung [8], except for his omission of the third term involving the mixed partial derivatives of F; this term is clearly necessary.

The second assumption about F is that the physical invariant amplitudes $T_i(s_i, t_i(s_i, \cos\theta_i))$ are the *boundary values* of F according to the following exact prescription:

$$T_1(s_1, t_1(s_1, \cos\theta_1)) = \lim_{\substack{\zeta\to 0\\ Im\,\zeta > 0}} F(s_1 + \zeta, t_1(s_1, \cos\theta_1), u_1(s_1, \cos\theta_1)) ,$$

$$T_2(s_2, t_2(s_2, \cos\theta_2)) = \lim_{\substack{\zeta\to 0\\ Im\,\zeta > 0}} F(u_2(s_2, \cos\theta_2), s_2 + \zeta, t_2(s_2, \cos\theta_2)) ,$$

$$T_3(s_3, t_3(s_3, \cos\theta_3)) = \lim_{\substack{\zeta\to 0\\ Im\,\zeta > 0}} F(t_3(s_3, \cos\theta_3), u_3(s_3, \cos\theta_3), s_3 + \zeta) . \tag{7}$$

For this prescription to be meaningful, we must require that

$$\max\{t_1(s_1, \cos\theta_1) \mid (s_1, \theta_1) \text{ physical}\}$$

be less than the square of the mass of the state, of lowest possible total mass, with the same internal quantum numbers as $A\overline{C}$, together with five similar conditions which can readily be written down. These conditions are satisfied for the hadronic processes of interest to us.

We have assumed that the limits in (7) exist; for this it is sufficient to assume that the functions f_{im}, g_{im} and the double spectral functions satisfy Lipschitz conditions. In the work of this section and the next we shall carry out a number of such mathematical procedures and we shall assume that conditions are imposed which justify them. For example, the assumption that the integrals in (6) are absolutely convergent allows the use of Fubini's theorem to invert the order of integration in several places. But we shall not write out a set of conditions in full detail as it would be tedious and of little value.

For s_1 real and $\Sigma_1 < s_1 < \infty$, define the function $T_1(s_1, z_2, z_3)$ by the limiting process

$$T_1(s_1, z_2, z_3) = \lim_{\substack{\zeta\to 0\\ Im\,\zeta > 0}} F(s_1 + \zeta, z_2, z_3) \tag{8}$$

and similarly for the functions $T_2(z_1, s_2, z_3)$ and $T_3(z_1, z_2, s_3)$. We shall write the limit on the right side of (8) as $F(s_1+, z_2, z_3)$ and the limit from below as $F(s_1-, z_2, z_3)$. Whenever one of the variables approaches a cut, we shall use this notation. Define further

$$\begin{aligned} A_1(s_1, z_2, z_3) &= \frac{1}{2i}\,[F(s_1+, z_2, z_3) - F(s_1-, z_2, z_3)] \\ &= \sum_{m=0}^{N-1} [f_{1m}(s_1)\,(z_2 - Z_2)^m + g_{1m}(s_1)\,(z_3 - Z_3)^m] \\ &+ \frac{(z_2 - Z_2)^N}{\pi} \int_{\Sigma_2}^{\infty} \frac{d\sigma_2\,\varrho_3(s_1, \sigma_2)}{(\sigma_2 - z_2)\,(\sigma_2 - Z_2)^N} \\ &+ \frac{(z_3 - Z_3)^N}{\pi} \int_{\Sigma_3}^{\infty} \frac{d\sigma_3\,\varrho_2(\sigma_3, s_1)}{(\sigma_3 - z_3)\,(\sigma_3 - Z_3)^N} . \end{aligned} \tag{9}$$

Similar definitions may be given for $A_2(z_1, s_2, z_3)$ and $A_3(z_1, z_2, s_3)$.

After a substantial amount of manipulation one can deduce from the Mandelstam representation (6) three fixed variable dispersion relations, of which we write one:

$$
\begin{aligned}
F(z_1, z_2, \Sigma - z_1 - z_2) = &\sum_{m=0}^{2N-2} \phi_m(z_1)\,(z_2 - Z_2)^m \\
&+ \sum_{p=1}^{n_2} \frac{(z_2 - Z_2)^{2N-1}}{(z_{2p} - Z_2)^{2N-1}}\,\frac{R_{2p}}{(z_2 - z_{2p})} \\
&+ \sum_{p=1}^{n_3} \frac{(-1)^{2N-1}\,(z_1 + z_2 - Z_1 - Z_2)^{2N-1}}{(z_{3p} - Z_3)^{2N-1}}\,\frac{R_{3p}}{(\Sigma - z_1 - z_2 - z_{3p})} \\
&+ \frac{(z_2 - Z_2)^{2N-1}}{\pi} \int_{\Sigma_2}^{\infty} \frac{d\sigma_2\, A_2(z_1, \sigma_2, \Sigma - z_1 - \sigma_2)}{(\sigma_2 - z_2)\,(\sigma_2 - Z_2)^{2N-1}} \\
&+ \frac{(-1)^{2N-1}\,(z_1 + z_2 - Z_1 - Z_2)^{2N-1}}{\pi} \\
&\times \int_{\Sigma_3}^{\infty} \frac{d\sigma_3\, A_3(z_1, \Sigma - \sigma_3 - z_1, \sigma_3)}{(\sigma_3 + z_1 + z_2 - \Sigma)\,(\sigma_3 - Z_3)^{2N-1}}
\end{aligned}
\qquad (10)
$$

The functions $\phi_m(z_1)$ have a cut along the real axis from Σ_1 to $+\infty$, and $\phi_0(z_1)$ includes the pole terms in (6) for $i = 1$. Now take z_1 onto the cut $[\Sigma_1, \infty)$ from above to obtain

$$
\begin{aligned}
T_1(s_1, z_2, \Sigma - s_1 - z_2) = &\sum_{m=0}^{2N-2} \phi_m(s_1 +)\,(z_2 - Z_2)^m + (\text{pole terms}) \\
&+ \frac{(z_2 - Z_2)^{2N-1}}{\pi} \int_{\Sigma_2}^{\infty} \frac{d\sigma_2\, A_2(s_1 +, \sigma_2, \Sigma - s_1 - \sigma_2)}{(\sigma_2 - z_2)\,(\sigma_2 - Z_2)^{2N-1}} \\
&+ \frac{(-1)^{2N-1}\,(s_1 + z_2 - Z_1 - Z_2)^{2N-1}}{\pi} \\
&\times \int_{\Sigma_3}^{\infty} \frac{d\sigma_3\, A_3(s_1 +, \Sigma - s_1 - \sigma_3, \sigma_3)}{(\sigma_3 + s_1 + z_2 - \Sigma)\,(\sigma_3 - Z_3)^{2N-1}}\,.
\end{aligned}
\qquad (11)
$$

Equation (11) is valid except when z_2 is real and $\Sigma_2 \leqslant z_2 < \infty$, $-\infty < z_2 \leqslant (\Sigma - \Sigma_3 - s_1)$, $z_2 = z_{2p}$ $(p = 1, 2, \dots, n_2)$ or $z_2 = (\Sigma - s_1 - z_{3p})$ $(p = 1, 2, \dots, n_3)$. Finally, taking z_2 *real*, $z_2 = x_2$, and using (11), we have

$$
\begin{aligned}
Im\, T_1(s_1, x_2, \Sigma - s_1 - x_2) = &\sum_{m=0}^{2N-2} Im\, \phi_m(s_1 +)\,(x_2 - Z_2)^m \\
&+ \frac{(x_2 - Z_2)^{2N-1}}{\pi} \int_{\Sigma_2}^{\infty} \frac{d\sigma_2\, \varrho_3(s_1, \sigma_2)}{(\sigma_2 - x_2)\,(\sigma_2 - Z_2)^{2N-1}} \\
&+ \frac{(-1)^{2N-1}\,(s_1 + x_2 - Z_1 - Z_2)^{2N-1}}{\pi} \\
&\times \int_{\Sigma_3}^{\infty} \frac{d\sigma_3\, \varrho_2(\sigma_3, s_1)}{(\sigma_3 + s_1 + x_2 - \Sigma)\,(\sigma_3 - Z_3)^{2N-1}}\,.
\end{aligned}
\qquad (12)
$$

Here we have assumed that x_2 is not a singular point of T_1 and we have used the relations

$$Im\, A_2\,(s_1 +, \sigma_2, \Sigma - s_1 - \sigma_2) = \varrho_3(s_1, \sigma_2)\,,$$
$$Im\, A_3\,(s_1 +, \Sigma - s_1 - \sigma_3, \sigma_3) = \varrho_2(\sigma_3, s_1)\,;$$

these follow from the definitions of A_2 and A_3 which are analogous to equation (9). It is equations (11) and (12), together with the unitarity relation (5), which we shall use in order to obtain the boundaries of the double spectral functions.

3. The Process $\pi\,\pi \to \pi\,\pi$. Subtractions

We assume that $A = C$ and $B = D$ and that all four particles have the same mass μ. We work exclusively in channel 1 and omit the subscript 1 for convenience. For the process $\pi\,\pi \to \pi\,\pi$ the only possible intermediate state for $4\,\mu^2 \leqslant s < 16\,\mu^2$ is the state (AB). Writing T for $T_{AB\to AB}$, the elastic unitarity relation is, from (5),

$$Im\, T(s, t(s, \boldsymbol{n}\cdot\boldsymbol{n}')) = -\frac{q}{32\,\pi^2\,W}\iint d\Omega_{\overline{\boldsymbol{n}}}\; T^*(s, t(\boldsymbol{n}'\cdot\overline{\boldsymbol{n}}))\; T(s, t(\boldsymbol{n}\cdot\overline{\boldsymbol{n}}))\,. \tag{13}$$

If the particles A and B are identical, the factor 32 is replaced by 64. The kinematic relations are

$$W = 2\sqrt{\mu^2 + q^2},\; s = W^2\,,$$
$$t(s, \boldsymbol{n}\cdot\boldsymbol{n}') = -2\,q^2\,(1 - \boldsymbol{n}\cdot\boldsymbol{n}')\,,$$
$$u(s, \boldsymbol{n}\cdot\boldsymbol{n}') = -2\,q^2\,(1 + \boldsymbol{n}\cdot\boldsymbol{n}')\,.$$

Inserting equations (11) and (12), in their *unsubtracted* form, into (13), we have

$$\begin{aligned}
&\frac{1}{\pi}\int\limits_{4\mu^2}^{\infty}\frac{d\sigma_2\,\varrho_3(s,\sigma_2)}{(\sigma_2 + 2\,q^2 - 2\,q^2\,\boldsymbol{n}\cdot\boldsymbol{n}')} + \frac{1}{\pi}\int\limits_{4\mu^2}^{\infty}\frac{d\sigma_3\,\varrho_2(\sigma_3, s)}{(\sigma_3 + 2\,q^2 + 2\,q^2\,\boldsymbol{n}\cdot\boldsymbol{n}')}\\
&\quad= -\frac{q}{32\,\pi^2\,W}\iint d\Omega_{\overline{\boldsymbol{n}}}\left[\frac{1}{\pi}\int\limits_{4\mu^2}^{\infty}\frac{d\sigma_2'\,A_2^*(s+, \sigma_2', 4\,\mu^2 - s - \sigma_2')}{(\sigma_2' + 2\,q^2 - 2\,q^2\,\boldsymbol{n}'\cdot\overline{\boldsymbol{n}})}\right.\\
&\quad\left. + \frac{1}{\pi}\int\limits_{4\mu^2}^{\infty}\frac{d\sigma_3'\,A_3^*(s+, 4\,\mu^2 - s - \sigma_3', \sigma_3')}{(\sigma_3' + 2\,q^2 + 2\,q^2\,\boldsymbol{n}'\cdot\overline{\boldsymbol{n}})}\right]\\
&\quad\times\left[\frac{1}{\pi}\int\limits_{4\mu^2}^{\infty}\frac{d\sigma_2''\,A_2(s+, \sigma_2'', 4\,\mu^2 - s - \sigma_2'')}{(\sigma_2'' + 2\,q^2 - 2\,q^2\,\boldsymbol{n}\cdot\overline{\boldsymbol{n}})}\right.\\
&\quad\left. + \frac{1}{\pi}\int\limits_{4\mu^2}^{\infty}\frac{d\sigma_3''\,A_3(s+, 4\,\mu^2 - s - \sigma_3'', \sigma_3'')}{(\sigma_3'' + 2\,q^2 + 2\,q^2\,\boldsymbol{n}\cdot\overline{\boldsymbol{n}})}\right]\,.
\end{aligned} \tag{14}$$

The right side of (14) is the sum of four integrals. We assume that repeated integrals may be written as double integrals and that the order of integration can be inverted to give, for the first of the four terms on the right side, the expression

$$-\frac{1}{128\,\pi^4\,q^3\,W}\int\limits_{4\mu^2}^{\infty}\int\limits_{4\mu^2}^{\infty} d\sigma_2'\,d\sigma_2''\,A_2^*(s+,\sigma_2',4\,\mu^2-s-\sigma_2')$$
$$\times\,A_2(s+,\sigma_2'',4\,\mu^2-s-\sigma_2'')\iint\frac{d\Omega_{\bar{\boldsymbol{n}}}}{(\tau_2'-\bar{\boldsymbol{n}}\cdot\boldsymbol{n}')\,(\tau_2''-\bar{\boldsymbol{n}}\cdot\boldsymbol{n})}\,,$$

where $\tau_2''=1+\sigma_2''/2\,q^2$, $\tau_2'=1+\sigma_2'/2\,q^2$, so that $\tau_2''>1$, $\tau_2'>1$. This brings us to the standard integral

$$I=\iint\frac{d\Omega_{\bar{\boldsymbol{n}}}}{(\tau_2'-\bar{\boldsymbol{n}}\cdot\boldsymbol{n}')\,(\tau_2''-\bar{\boldsymbol{n}}\cdot\boldsymbol{n})}\,,\quad\tau_2''>1,\,\tau_2'>1\,, \tag{15}$$

the integration being over the whole surface of the unit sphere.

The integral I is shown in the Appendix to be

$$I=4\,\pi\int\limits_{u_0}^{\infty}\frac{du}{(u-\boldsymbol{n}\cdot\boldsymbol{n}')\,[(u-\tau_2'\,\tau_2'')^2-(\tau_2'^{\,2}-1)\,(\tau_2''^{\,2}-1)]^{1/2}}\,,$$

where

$$u_0=\tau_2'\,\tau_2''+(\tau_2'^{\,2}-1)^{1/2}\,(\tau_2''^{\,2}-1)^{1/2}.$$

Now put

$$2\,q^2\,u=(\sigma_2+2\,q^2)$$

and express τ_2', τ_2'' in terms of σ_2', σ_2'' to give

$$I=16\,\pi\,q^3\int\limits_{f(s,\sigma_2',\sigma_2'')}^{\infty}\frac{d\sigma_2}{(\sigma_2+2\,q^2-2\,q^2\,\boldsymbol{n}\cdot\boldsymbol{n}')\,[g(s,\sigma_2,\sigma_2',\sigma_2'')]^{1/2}}\,,$$

where

$$g(s,\sigma_2,\sigma_2',\sigma_2'')=(s-4\,\mu^2)$$
$$\times\,(\sigma_2^2+\sigma_2'^{\,2}+\sigma_2''^{\,2}-2\,\sigma_2\,\sigma_2'-2\,\sigma_2\,\sigma_2''-2\,\sigma_2'\,\sigma_2'')-4\,\sigma_2\,\sigma_2'\,\sigma_2''\,,$$

$$f(s,\sigma_2',\sigma_2'')=(\sigma_2'+\sigma_2'')+\frac{\sigma_2'\,\sigma_2''}{2\,q^2}+\left[\sigma_2'\,\sigma_2''\left(2+\frac{\sigma_2'}{2\,q^2}\right)\left(2+\frac{\sigma_2''}{2\,q^2}\right)\right]^{1/2}.$$

The first of the four terms on the right side of (14) has now been expressed in the form

$$\frac{-1}{8\,\pi^3\,s^{1/2}}\int\limits_{4\mu^2}^{\infty}\int\limits_{4\mu^2}^{\infty} d\sigma_2'\,d\sigma_2''\,A_2^*(s+,\sigma_2',4\,\mu^2-s-\sigma_2')\,A_2(s+,\sigma_2'',4\,\mu^2-s-\sigma_2'')$$
$$\times\int\limits_{f(s,\sigma_2',\sigma_2'')}^{\infty}\frac{d\sigma_2}{(\sigma_2+2\,q^2-2\,q^2\,\boldsymbol{n}\cdot\boldsymbol{n}')\,[g(s,\sigma_2,\sigma_2',\sigma_2'')]^{1/2}}\,.$$

Now for fixed σ_2' (resp. σ_2'') it is clear that $f(s, \sigma_2', \sigma_2'')$ is a monotonic increasing function of σ_2'' (resp. σ_2'). Thus the smallest value of σ_2 appearing in the above integral is

$$\phi(s) = 8\,\mu^2 + \frac{8\,\mu^4}{q^2} + 4\,\mu^2 \left(2 + \frac{2\,\mu^2}{q^2}\right) = \frac{16\,\mu^2\,s}{(s - 4\,\mu^2)} \,.$$

Thus, reversing the order of integration, we obtain finally the integral

$$\frac{1}{\pi} \int\limits_{\phi(s)}^{\infty} \frac{d\sigma_2\, F(s, \sigma_2)}{(\sigma_2 + 2\,q^2 - 2\,q^2\, \boldsymbol{n} \cdot \boldsymbol{n}')} \,,$$

with

$$F(s, \sigma_2) = \frac{-1}{8\,\pi^2\, s^{1/2}} \times \iint\limits_{D(s,\sigma_2)} \frac{d\sigma_2'\, d\sigma_2''\, A_2^*\,(s+, \sigma_2', 4\,\mu^2 - s - \sigma_2')\, A_2\,(s+, \sigma_2'', 4\,\mu^2 - s - \sigma_2'')}{[g(s, \sigma_2, \sigma_2', \sigma_2'')]^{1/2}} \,,$$

where

$$D(s, \sigma_2) = \{(\sigma_2', \sigma_2'') \mid f(s, \sigma_2', \sigma_2'') \leqslant \sigma_2\} \,.$$

The evaluation of the other three integrals on the right side of (14) proceeds in exactly the same way. One replaces $\boldsymbol{n}$ by $-\boldsymbol{n}$, or $\boldsymbol{n}'$ by $-\boldsymbol{n}'$, or both. The result is then

$$\int\limits_{4\mu^2}^{\infty} \frac{d\sigma_2\, \varrho_3(s, \sigma_2)}{(\sigma_2 + 2\,q^2 - 2\,q^2\, \boldsymbol{n} \cdot \boldsymbol{n}')} + \int\limits_{4\mu^2}^{\infty} \frac{d\sigma_3\, \varrho_2(\sigma_3, s)}{(\sigma_3 + 2\,q^2 + 2\,q^2\, \boldsymbol{n} \cdot \boldsymbol{n}')}$$
$$= \int\limits_{\phi(s)}^{\infty} \frac{d\sigma_2\, F_3(s, \sigma_2)}{(\sigma_2 + 2\,q^2 - 2\,q^2\, \boldsymbol{n} \cdot \boldsymbol{n}')} + \int\limits_{\phi(s)}^{\infty} \frac{d\sigma_3\, F_2(\sigma_3, s)}{(\sigma_3 + 2\,q^2 + 2\,q^2\, \boldsymbol{n} \cdot \boldsymbol{n}')} \,, \qquad (16)$$

where

$$F_3(s, \sigma_2) = \frac{-1}{8\,\pi^2\, s^{1/2}} \times \iint\limits_{D(s,\sigma_2)} d\xi\, d\eta\, \frac{[A_2^*\,(s+, \xi, 4\,\mu^2 - s - \xi)\, A_2\,(s+, \eta, 4\,\mu^2 - s - \eta) + A_3^*\,(s+, 4\,\mu^2 - s - \xi, \xi)\, A_3\,(s+, 4\,\mu^2 - s - \eta, \eta)]}{[g(s, \sigma_2, \xi, \eta)]^{1/2}} \,,$$

$$F_2(\sigma_3, s) = \frac{-1}{8\,\pi^2\, s^{1/2}} \times \iint\limits_{D(s,\sigma_3)} d\xi\, d\eta\, \frac{[A_2^*\,(s+, \xi, 4\,\mu^2 - s - \xi)\, A_3\,(s+, 4\,\mu^2 - s - \eta, \eta) + A_3^*\,(s+, 4\,\mu^2 - s - \xi, \xi)\, A_2\,(s+, \eta, 4\,\mu^2 - s - \eta)]}{[g(s, \sigma_3, \xi, \eta)]^{1/2}} \,, \qquad (17)$$

and

$$D(s, \sigma) = \{(\xi, \eta) \mid f(s, \xi, \eta) \leqslant \sigma\}\,.$$

Equation (16) will hold for $4\,\mu^2 \leqslant s < 16\,\mu^2$, and for $-1 \leqslant (\boldsymbol{n}\cdot\boldsymbol{n}') \leqslant 1$. But it is clear that each side of (16) defines an analytic function of $z = \boldsymbol{n}\cdot\boldsymbol{n}'$ which is regular except for the obvious cuts on the real axis. From the principle of analytic continuation (see, for example, (9.4.3) of Dieudonné [9]) it follows that these functions are equal at all regular points z. Then a slight extension of a theorem in Widder [10] (see Theorem 5b, Chapter VIII) shows that, almost everywhere (and thus everywhere under the mild assumption of continuity of the ϱ's and F's),

$$\varrho_3(s_1, s_2) = F_3(s_1, s_2)\,, \quad s_2 \geqslant 4\,\mu^2\,, \qquad \varrho_2(s_3, s_1) = F_2(s_3, s_1)\,, \quad s_3 \geqslant 4\,\mu^2\,, \tag{18}$$

in each case for $4\,\mu^2 \leqslant s_1 < 16\,\mu^2$. But since, by (17), $F_2(s_3, s_1)$ (resp. $F_3(s_1, s_2)$) is zero for $s_3 < \phi(s_1)$ (resp. $s_2 < \phi(s_1)$), we have obtained the boundary curves of $\varrho_2(s_3, s_1)$ and $\varrho_3(s_1, s_2)$ for $4\,\mu^2 < s_1 < 16\,\mu^2$, namely

$$s_2\ (\text{or } s_3) = \frac{16\,\mu^2\, s_1}{(s_1 - 4\,\mu^2)} = 16\,\mu^2 + \frac{64\,\mu^4}{(s_1 - 4\,\mu^2)}\,. \tag{19}$$

The other boundaries for $\pi\pi \to \pi\pi$ are obvious; all three processes are identical.

Our second job in this section is to demonstrate that substitution of the subtracted relations (11) and (12) (that is, with $N \geqslant 1$) into (13) does not alter the boundary curves (19). We show, too, that the relations given by equations (17), (18) are not changed. Write $m = (2\,N - 1)$. It is convenient to shift the subtraction point in each integral in (11). Noting that

$$\frac{(\alpha+\gamma)^m}{(\beta+\gamma)^m\,(\beta-\gamma)} = \frac{\alpha^m}{\beta^m\,(\beta-\gamma)} + \gamma \sum_{p=0}^{m-1} \frac{(\alpha+\gamma)^p\,\alpha^{m-p-1}}{(\beta+\gamma)^{p+1}\,\beta^{m-p}}$$

and putting $\alpha = 2\,q^2\,\boldsymbol{n}\cdot\boldsymbol{n}'$, $\beta = (\sigma_2 + 2\,q^2)$, $\gamma = -2\,q^2 - 4\,\mu^2/3$, we have

$$\begin{aligned} T(s, t(s, \boldsymbol{n}\cdot\boldsymbol{n}')) &= \sum_{p=0}^{m-1} \psi_p(s)\,(\boldsymbol{n}\cdot\boldsymbol{n}')^p \\ &+ \frac{(2\,q^2)^m\,(\boldsymbol{n}\cdot\boldsymbol{n}')^m}{\pi} \int_{4\mu^2}^{\infty} \frac{d\sigma_2\,A_2\,(s+, \sigma_2, 4\,\mu^2 - s - \sigma_2)}{(\sigma_2 + 2\,q^2 - 2\,q^2\,\boldsymbol{n}\cdot\boldsymbol{n}')\,(\sigma_2 + 2\,q^2)^m} \\ &+ \frac{(-2\,q^2)^m\,(\boldsymbol{n}\cdot\boldsymbol{n}')^m}{\pi} \int_{4\mu^2}^{\infty} \frac{d\sigma_3\,A_3\,(s+, 4\,\mu^2 - s - \sigma_3, \sigma_3)}{(\sigma_3 + 2\,q^2 + 2\,q^2\,\boldsymbol{n}\cdot\boldsymbol{n}')\,(\sigma_3 + 2\,q^2)^m}\,. \end{aligned}$$

The modified version of equation (12) is clear. These modified equations are then to be substituted into (13). On the right side there are four double integrals of which the first is

$$\begin{aligned} &\frac{-1}{128\,\pi^4\,q^3\,W} \int_{4\mu^2}^{\infty}\int_{4\mu^2}^{\infty} d\sigma_2'\,d\sigma_2''\,A_2^*\,(s+, \sigma_2', 4\,\mu^2 - s - \sigma_2')\,A_2\,(s+, \sigma_2'', 4\,\mu^2 - s - \sigma_2'') \\ &\times \iint \frac{d\Omega_{\overline{n}}(\overline{\boldsymbol{n}}\cdot\boldsymbol{n}')^m\,(\overline{\boldsymbol{n}}\cdot\boldsymbol{n})^m}{(\tau_2' - \overline{\boldsymbol{n}}\cdot\boldsymbol{n}')\,(\tau_2'' - \overline{\boldsymbol{n}}\cdot\boldsymbol{n})\,\tau_2'^{\,m}\,\tau_2''^{\,m}}\,. \end{aligned}$$

Instead of the standard integral (15) we have a subtracted form which we evaluate as follows.

Note first that

$$\frac{(\boldsymbol{n}\cdot\boldsymbol{n}')^m}{(\tau_2'-\overline{\boldsymbol{n}}\cdot\boldsymbol{n}')\,\tau_2'^{\,m}}=\frac{1}{(\tau_2'-\overline{\boldsymbol{n}}\cdot\boldsymbol{n}')}-\sum_{p=0}^{m-1}\frac{(\overline{\boldsymbol{n}}\cdot\boldsymbol{n}')^p}{\tau_2'^{\,p+1}}\,. \tag{20}$$

Then there are various terms to consider:

(i) The standard integral I of equation (15), which we subtract again to give

$$I=16\,\pi\,q^3(2\,q^2)^m\,(\boldsymbol{n}\cdot\boldsymbol{n}')^m$$
$$\times\int\limits_{f(s,\sigma_2',\sigma_2'')}^{\infty}\frac{d\sigma_2}{(\sigma_2+2\,q^2-2\,q^2\,\boldsymbol{n}\cdot\boldsymbol{n}')\,(\sigma_2+2\,q^2)^m\,[g(s,\sigma_2,\sigma_2',\sigma_2'')]^{1/2}}$$
$$+\,16\,\pi\,q^3\sum_{p=0}^{m-1}(2\,q^2)^p\,(\boldsymbol{n}\cdot\boldsymbol{n}')^p\int\limits_{f(s,\sigma_2',\sigma_2'')}^{\infty}\frac{d\sigma_2}{(\sigma_2+2\,q^2)^{p+1}\,[g(s,\sigma_2,\sigma_2',\sigma_2'')]^{1/2}}\,.$$

(ii) Terms of the form

$$I_p=\tau_2'^{\,-p-1}\iint\frac{d\Omega_{\overline{\boldsymbol{n}}}(\overline{\boldsymbol{n}}\cdot\boldsymbol{n}')^p}{(\tau_2''-\overline{\boldsymbol{n}}\cdot\boldsymbol{n})}\,,\quad p=0,\ldots,(m-1)\,.$$

Taking $\boldsymbol{n}$ as polar axis and putting $\boldsymbol{n}\cdot\boldsymbol{n}'=x$, $\boldsymbol{n}\cdot\overline{\boldsymbol{n}}=\mu$, we have

$$I_p=\tau_2'^{\,-p-1}\int\limits_{-1}^{1}d\mu\,(\tau_2''-\mu)^{-1}\int\limits_{0}^{2\pi}d\phi\,[(1-x^2)^{1/2}\,(1-\mu^2)^{1/2}\cos\phi+x\,\mu]^p\,.$$

The binomial expansion gives a polynomial of degree p in $\cos\phi$; only even powers of $\cos\phi$ contribute on integration, so that

$$I_p=\sum_{q=0}^{(p/2)}c(p,q)\,\tau_2'^{\,-p-1}\,(1-x^2)^q\,x^{p-2q}\int\limits_{-1}^{1}d\mu\,(1-\mu^2)^q\,\mu^{p-2q}\,(\tau_2''-\mu)^{-1}\,,$$

where $(p/2)=p/2$ for even p, $(p-1)/2$ for odd p. Thus I_p is a polynomial in x of degree p whose coefficients are functions of τ_2' and τ_2''. The same is true of the integral

$$J_p=\tau_2''^{\,-p-1}\iint\frac{d\Omega_{\overline{\boldsymbol{n}}}(\overline{\boldsymbol{n}}\cdot\boldsymbol{n})^p}{(\tau_2'-\overline{\boldsymbol{n}}\cdot\boldsymbol{n}')}\,,\quad p=0,\ldots,(m-1)\,.$$

(iii) Terms of the form

$$I_{p,q}=\tau_2''^{\,-p-1}\,\tau_2'^{\,-q-1}\iint d\Omega_{\overline{\boldsymbol{n}}}\,(\overline{\boldsymbol{n}}\cdot\boldsymbol{n})^p\,(\overline{\boldsymbol{n}}\cdot\boldsymbol{n}')^q\,,$$

with $p=0,1,\ldots,(m-1)$, $q=0,1,\ldots,(m-1)$. Expand each power in a series of Legendre polynomials and use the relation

$$\iint d\Omega_{\overline{\boldsymbol{n}}}\,P_l(\overline{\boldsymbol{n}}\cdot\boldsymbol{n})\,P_{l'}(\overline{\boldsymbol{n}}\cdot\boldsymbol{n}')=\frac{4\,\pi}{(2\,l+1)}\,\delta_{l\,l'}\,P_l(\boldsymbol{n}\cdot\boldsymbol{n}')\,.$$

We see then that $I_{p,q}=0$ for $(p-q)$ odd, while $I_{p,q}$ is a polynomial in $(\boldsymbol{n}\cdot\boldsymbol{n}')$ of degree $\min\{p,q\}$ for $(p-q)$ even. The coefficients are clearly functions of τ_2', τ_2''.

The final result we have obtained is

$$\iint \frac{d\Omega_{\bar{n}}(\bar{\boldsymbol{n}}\cdot\boldsymbol{n}')^m\,(\bar{\boldsymbol{n}}\cdot\boldsymbol{n})^m}{(\tau_2'-\bar{\boldsymbol{n}}\cdot\boldsymbol{n}')\,(\tau_2''-\bar{\boldsymbol{n}}\cdot\boldsymbol{n})\,\tau_2'^{\,m}\,\tau_2''^{\,m}} = \sum_{p=0}^{m-1} g_p(s,\sigma_2',\sigma_2'')\,(\boldsymbol{n}\cdot\boldsymbol{n}')^p$$

$$+\,16\,\pi\,q^3(2\,q)^m\,(\boldsymbol{n}\cdot\boldsymbol{n}')^m$$

$$\times \int\limits_{f(s,\sigma_2',\sigma_2'')}^{\infty} \frac{d\sigma_2}{(\sigma_2+2\,q^2-2\,q^2\,\boldsymbol{n}\cdot\boldsymbol{n}')\,(\sigma_2+2\,q^2)^m\,[g(s,\sigma_2,\sigma_2',\sigma_2'')]^{1/2}}\,.$$

Now the integral in the second term is, apart from a factor,

$$I(s,\sigma_2',\sigma_2'',\boldsymbol{n}\cdot\boldsymbol{n}') = \int\limits_{u_0}^{\infty} \frac{du}{(u-\boldsymbol{n}\cdot\boldsymbol{n}')\,(u-u_0)^{1/2}\,(u-u_1)^{1/2}\,u^m}$$

$$= u_0^{-m-1} \int\limits_{1}^{\infty} \frac{dv}{(v-\boldsymbol{n}\cdot\boldsymbol{n}'/u_0)\,(v-u_1/u_0)^{1/2}\,(v-1)^{1/2}\,v^m}\,,$$

where

$$\begin{matrix} u_0 \\ u_1 \end{matrix} = \tau_2'\,\tau_2'' \pm (\tau_2'^{\,2}-1)^{1/2}\,(\tau_2''^{\,2}-1)^{1/2}\,.$$

Hence for u_0 large the integral behaves like u_0^{-m-1}. It thus has exactly the correct behaviour for the integral

$$\int\limits_{4\mu^2}^{\infty}\int\limits_{4\mu^2}^{\infty} d\sigma_2'\,d\sigma_2''\,A_2^*\,(s+,\sigma_2',4\,\mu^2-s-\sigma_2')\,A_2\,(s+,\sigma_2'',4\,\mu^2-s-\sigma_2'')\;I(s,\sigma_2',\sigma_2'',\boldsymbol{n}\cdot\boldsymbol{n}')$$

to converge and for the order of integration to be reversed again. This means, too, that the integral

$$\int\limits_{4\mu^2}^{\infty}\int\limits_{4\mu^2}^{\infty} d\sigma_2'\,d\sigma_2''\,A_2^*\,(s+,\sigma_2',4\,\mu^2-s-\sigma_2')\,A_2\,(s+,\sigma_2'',4\,\mu^2-s-\sigma_2'')$$

$$\times\left(\sum_{p=0}^{m-1} g_p(s,\sigma_2',\sigma_2'')\,(\boldsymbol{n}\cdot\boldsymbol{n}')^p\right)$$

converges for $-1 \leqslant (\boldsymbol{n}\cdot\boldsymbol{n}') \leqslant +1$, and thus that each of the integrals

$$\int\limits_{4\mu^2}^{\infty}\int\limits_{4\mu^2}^{\infty} d\sigma_2'\,d\sigma_2''\,A_2^*\,(s+,\sigma_2',4\,\mu^2-s-\sigma_2')\,A_2\,(s+,\sigma_2'',4\,\mu^2-s-\sigma_2'')\;g_p(s,\sigma_2',\sigma_2'')\,,$$

with $p = 0, 1, \ldots, (m-1)$, is convergent.

Evaluation of each of the other three double integrals proceeds in the same way. There are other terms on the right side of the unitarity relation, but they involve integrals already considered. There are terms of the form

$$\iint d\Omega_{\bar{n}}\,\psi_p^*(s)\,\psi_q(s)\,(\bar{\boldsymbol{n}}\cdot\boldsymbol{n}')^p\,(\bar{\boldsymbol{n}}\cdot\boldsymbol{n})^q,\ p,q = 0, 1, \ldots, (m-1)\,.$$

From (iii) above, these terms, when summed over p and q, give a polynomial in $(\boldsymbol{n} \cdot \boldsymbol{n}')$ of degree $(m-1)$, whose coefficients are functions of s. Finally there are the terms (omitting a factor)

$$\psi_p^*(s) \int_{4\mu^2}^{\infty} d\sigma_2' \, A_2\,(s+, \sigma_2', 4\,\mu^2 - s - \sigma_2') \iint d\Omega_{\overline{\boldsymbol{n}}} \, \frac{(\overline{\boldsymbol{n}} \cdot \boldsymbol{n}')^p \, (\overline{\boldsymbol{n}} \cdot \boldsymbol{n})^m}{(\tau_2' - \overline{\boldsymbol{n}} \cdot \boldsymbol{n}) \, \tau_2'^{\,m}}$$

and

$$\psi_p(s) \int_{4\mu^2}^{\infty} d\sigma_2' \, A_2^*(s+, \sigma_2', 4\,\mu^2 - s - \sigma_2') \iint d\Omega_{\overline{\boldsymbol{n}}} \, \frac{(\overline{\boldsymbol{n}} \cdot \boldsymbol{n}')^m \, (\overline{\boldsymbol{n}} \cdot \boldsymbol{n})^p}{(\tau_2' - \overline{\boldsymbol{n}} \cdot \boldsymbol{n}') \, \tau_2'^{\,m}}\,,$$

$p = 0, 1, \ldots, (m-1)$. Use of equation (20) gives terms already dealt with in (ii) and (iii) above. Each of the above terms is thus a polynomial in $(\boldsymbol{n} \cdot \boldsymbol{n}')$ of degree p, whose coefficients are functions of s.

This completes the discussion of the right side of the unitarity relation and it is clear that the expressions for $\varrho_2(s_3, s_1)$ and $\varrho_3(s_1, s_2)$ in terms of A_2 and A_3, as given in (17) and (18), are unchanged by the presence of subtractions in the Mandelstam representation. In particular, the boundary curves of ϱ_2 and ϱ_3 given in (19) are unaltered.

4. General Formula for Boundary Curves. Extended Unitarity

We turn now to obtaining a general formula for the boundary curves of the double spectral functions in the Mandelstam representation for the processes 1, 2, 3 of Section 2. For this we look again at equation (5), namely,

$$\begin{aligned} Im\; & T^{(EF)}_{AB \to CD}(s, t(s, \boldsymbol{n} \cdot \boldsymbol{n}')) \\ & = \frac{-q_{EF}}{32\,\pi^2\,s^{1/2}} \iint d\Omega_{\overline{\boldsymbol{n}}} \; T^*_{CD \to EF}(s, t(s, \overline{\boldsymbol{n}} \cdot \boldsymbol{n}')) \; T_{AB \to EF}(s, t(s, \boldsymbol{n} \cdot \overline{\boldsymbol{n}}))\,. \end{aligned}$$

All the scattering amplitudes appearing in this equation are accessible to actual experimental measurement if and only if

$$s > \max\{(A+B)^2, (C+D)^2, (E+F)^2\};$$

in this case we have the usual unitarity relation, which has a physical interpretation in terms of probability conservation.

Suppose, however, that $(E+F)$ is less than at least one of the numbers $(A+B)$, $(C+D)$ and that

$$(E+F)^2 < s < \max\{(A+B)^2, (C+D)^2\}\,.$$

Then at least one of the processes $AB \to EF$, $CD \to EF$ is no longer physical. However, equipped with the Mandelstam hypothesis, a formal extension of equation (5) to this case can be written in the following way:

$$\begin{aligned} \frac{1}{2\,i} & [F_{(ABCD)}\,(s+, t(s, \boldsymbol{n} \cdot \boldsymbol{n}'), u(s, \boldsymbol{n} \cdot \boldsymbol{n}')) - F_{(ABCD)}\,(s-, t(s, \boldsymbol{n} \cdot \boldsymbol{n}'), u(s, \boldsymbol{n} \cdot \boldsymbol{n}'))]^{(EF)} \\ & = \frac{-q_{EF}}{32\,\pi^2\,s^{1/2}} \iint d\Omega_{\overline{\boldsymbol{n}}} \; F^*_{(CDEF)}\,(s+, t(s, \overline{\boldsymbol{n}} \cdot \boldsymbol{n}'), u(s, \overline{\boldsymbol{n}} \cdot \boldsymbol{n}')) \\ & \times F_{(ABEF)}\,(s+, t(s, \overline{\boldsymbol{n}} \cdot \boldsymbol{n}), u(s, \overline{\boldsymbol{n}} \cdot \boldsymbol{n}))\,. \end{aligned} \tag{21}$$

The function $F_{(ABCD)}$ is the analytic function which describes the process $AB \to CD$ and its related processes; similarly for $F_{(CDEF)}$ and $F_{(ABEF)}$. All three functions have $\Sigma_1 \leqslant (E+F)^2$, and the notation $(s\,\pm)$ has the meaning ascribed to it in Section 2. Each of the quantities appearing in (21) has a precise meaning, except for the fact that $q_{AB}(s)$ and $q_{CD}(s)$ (which appear in the definitions of the t's and u's, by equations (1), (2)) are ambiguous up to a sign; for example,

$$q_{AB}(s) = \frac{[s-(A+B)^2]^{1/2}\,[s-(A-B)^2]^{1/2}}{2\,s^{1/2}}\,.$$

This ambiguity does not affect (21); changing the sign of q_{AB} (resp. q_{CD}) on each side of (21) is exactly equivalent to changing the sign of $\boldsymbol{n}$ (resp. $\boldsymbol{n}'$). The awkward notation on the left side of (21) is intended to denote the contribution of the two-particle state (EF) only to the quantity written there.

Equation (21) might be called an 'extended' unitarity relation. It only has a meaning through the analyticity assumptions which give a meaning to the unphysical quantities contained in it. There are some grounds for the hope that it is true when anomalous thresholds are absent. An argument in support of it has been given by Mandelstam [11] and it is certainly true that the use of 'extended' unitarity has been fruitful in the study of the nucleon form factors and of pion-nucleon scattering by dispersion relation techniques (the original papers are those of Frazer and Fulco [12] and Hamilton and Spearman [13]). We assume the validity of (21) from now on, and apply it to obtaining the boundary curves of the double spectral functions. Thus (EF) will be the two-particle state, of *lowest* total mass, with the same internal quantum numbers as the states (AB) and (CD). Then, for some range of values of s above $(E+F)^2$, say $(E+F)^2 \leqslant s < M^2$, the *only* contribution to the left side of (21) will be from the state (EF), so that we may drop the superscript (EF).

We may avoid troublesome but unimportant complications due to kinematics below threshold if we use (21) *as if* s were greater than $\max\{(A+B)^2, (C+D)^2\}$. As we shall see, $q_{AB}(s)$ and $q_{CD}(s)$ disappear from the final results for the boundaries, so the results we obtain are not affected by our having treated $q_{AB}(s)$ and $q_{CD}(s)$ as real and positive. Denote by $M_{A\bar{E}}$ the total mass of the one- or two-particle state, of lowest total mass, with the same internal quantum numbers as the states $(A\bar{E})$ and $(F\bar{B})$. The quantities $M_{C\bar{E}}$, $M_{A\bar{F}}$ and $M_{C\bar{F}}$ will have exactly similar meanings. Then the right side of (21) leads to an integral

$$\iint d\Omega_{\bar{n}}\,[M^2_{A\bar{E}} - t_{AB\to EF}(s,\boldsymbol{n}\cdot\bar{\boldsymbol{n}})]^{-1}\,[M^2_{C\bar{E}} - t_{CD\to EF}(s,\boldsymbol{n}'\cdot\bar{\boldsymbol{n}})]^{-1}$$

$$= \iint d\Omega_{\bar{n}}\left[M^2_{A\bar{E}} - \frac{1}{2}\,(A^2+B^2+E^2+F^2) + \frac{1}{2}\,s + \frac{1}{2}\,(A^2-B^2)\right.$$

$$\left.\times\,(E^2-F^2)\,s^{-1} - 2\,q_{AB}\,q_{EF}\,\boldsymbol{n}\cdot\bar{\boldsymbol{n}}\right]^{-1}\left[M^2_{C\bar{E}} - \frac{1}{2}\,(C^2+D^2+E^2+F^2)\right.$$

$$\left. + \frac{1}{2}\,s + \frac{1}{2}\,(C^2-D^2)\,(E^2-F^2)\,s^{-1} - 2\,q_{CD}\,q_{EF}\,\boldsymbol{n}'\cdot\bar{\boldsymbol{n}}\right]^{-1}$$

$$= (4\,q_{AB}\,q_{CD}\,q^2_{EF})^{-1}\iint d\Omega_{\bar{n}}\,(\tau''_2 - \boldsymbol{n}\cdot\bar{\boldsymbol{n}})^{-1}\,(\tau'_2 - \boldsymbol{n}'\cdot\bar{\boldsymbol{n}})^{-1}\,,$$

where

$$\tau_2'' = (2\, q_{AB}\, q_{EF})^{-1} \Big[M^2_{A\bar{E}} - \frac{1}{2} (A^2 + B^2 + E^2 + F^2) + \frac{1}{2} s + \frac{1}{2} (A^2 - B^2)(E^2 - F^2)\, s^{-1} \Big] ,$$

$$\tau_2' = (2\, q_{CD}\, q_{EF})^{-1} \Big[M^2_{C\bar{E}} - \frac{1}{2} (C^2 + D^2 + E^2 + F^2) + \frac{1}{2} s + \frac{1}{2} (C^2 - D^2)(E^2 - F^2)\, s^{-1} \Big] .$$

As in Section 3, we have the representation

$$\iint \frac{d\Omega_{\bar{\boldsymbol{n}}}}{(\tau_2'' - \boldsymbol{n} \cdot \bar{\boldsymbol{n}})(\tau_2' - \boldsymbol{n}' \cdot \bar{\boldsymbol{n}})} = 4\pi \int_{u_0}^{\infty} \frac{du}{(u - \boldsymbol{n} \cdot \boldsymbol{n}')\, [(u - \tau_2' \tau_2'')^2 - (\tau_2'^2 - 1)(\tau_2''^2 - 1)]^{1/2}}$$

with

$$u_0 = \tau_2' \tau_2'' + (\tau_2'^2 - 1)^{1/2} (\tau_2''^2 - 1)^{1/2} .$$

The integral over u may be written as an integral over a new variable σ_2, with the factor $(\sigma_2 - t_{AB \to CD}(s, \boldsymbol{n} \cdot \boldsymbol{n}'))$ in the denominator, and lower limit of integration

$$\phi(s) = 2\, q_{AB}\, q_{CD}\, u_0 + \frac{1}{2} (A^2 + B^2 + C^2 + D^2) - \frac{1}{2} s - \frac{1}{2} (A^2 - B^2)(C^2 - D^2)\, s^{-1} .$$

Now

$$\begin{aligned} 4\, q_{AB}\, q_{CD}\, q^2_{EF}\, [\tau_2' \tau_2'' &+ (\tau_2'^2 - 1)^{1/2} (\tau_2''^2 - 1)^{1/2}] \\ &= \Big[M^2_{A\bar{E}} - \frac{1}{2} (A^2 + B^2 + E^2 + F^2) + \frac{1}{2} s + \frac{1}{2} (A^2 - B^2)(E^2 - F^2)\, s^{-1} \Big] \\ &\times \Big[M^2_{C\bar{E}} - \frac{1}{2} (C^2 + D^2 + E^2 + F^2) + \frac{1}{2} s + \frac{1}{2} (C^2 - D^2)(E^2 - F^2)\, s^{-1} \Big] \\ &+ [s\, M^2_{A\bar{E}} + M^4_{A\bar{E}} - M^2_{A\bar{E}} (A^2 + B^2 + E^2 + F^2) + (A^2 - E^2)(B^2 - F^2) \\ &+ s^{-1} \{ M^2_{A\bar{E}} (A^2 - B^2)(E^2 - F^2) + (A^2 F^2 - B^2 E^2)(A^2 - B^2 - E^2 + F^2) \}]^{1/2} \\ &\times [s\, M^2_{C\bar{E}} + M^4_{C\bar{E}} - M^2_{C\bar{E}} (C^2 + D^2 + E^2 + F^2) + (C^2 - E^2)(D^2 - F^2) \\ &+ s^{-1} \{ M^2_{C\bar{E}} (C^2 - D^2)(E^2 - F^2) + (C^2 F^2 - D^2 E^2)(C^2 - D^2 - E^2 + F^2) \}]^{1/2} \end{aligned}$$

and

$$4\, q^2_{EF} = s - 2 (E^2 + F^2) + (E^2 - F^2)^2 s^{-1} .$$

After further manipulation, we arrive at the following expression for $\phi(s)$:

$$\phi(s) = M^2_{A\bar{E}} + M^2_{C\bar{E}} + [s - 2 (E^2 + F^2) + (E^2 - F^2)^2 s^{-1}]^{-1} F(s) , \tag{22a}$$

where

$$\begin{aligned}
F(s) &= 2\, M_{A\bar{E}}^2 M_{C\bar{E}}^2 + M_{A\bar{E}}^2 (E^2 + F^2 - C^2 - D^2) + M_{C\bar{E}}^2 (E^2 + F^2 - A^2 - B^2) \\
&+ 2 \left[E^2 - \frac{1}{2} (A^2 + C^2)\right] \left[F^2 - \frac{1}{2} (B^2 + D^2)\right] - \frac{1}{2} (A^2 - C^2) (B^2 - D^2) \\
&+ s^{-1} (E^2 - F^2) \left[(A^2 - B^2) \left\{M_{C\bar{E}}^2 - \frac{1}{2} (C^2 + D^2 + E^2 + F^2)\right\}\right. \\
&+ (C^2 - D^2) \left\{M_{A\bar{E}}^2 - \frac{1}{2} (A^2 + B^2 + E^2 + F^2)\right\} \\
&\left. - (E^2 - F^2) \left\{M_{A\bar{E}}^2 + M_{C\bar{E}}^2 - \frac{1}{2} (A^2 + B^2 + C^2 + D^2)\right\}\right] \\
&+ s^{-1} (E^2 + F^2) (A^2 - B^2) (C^2 - D^2) \\
&+ 2\, [s\, M_{A\bar{E}}^2 + M_{A\bar{E}}^4 - M_{A\bar{E}}^2 (A^2 + B^2 + E^2 + F^2) + (A^2 - E^2) (B^2 - F^2) \\
&+ s^{-1} \{M_{A\bar{E}}^2 (A^2 - B^2) (E^2 - F^2) + (A^2 F^2 - B^2 E^2) (A^2 - B^2 - E^2 + F^2)\}]^{1/2} \\
&\times [s\, M_{C\bar{E}}^2 + M_{C\bar{E}}^4 - M_{C\bar{E}}^2 (C^2 + D^2 + E^2 + F^2) + (C^2 - E^2) (D^2 - F^2) \\
&+ s^{-1} \{M_{C\bar{E}}^2 (C^2 - D^2) (E^2 - F^2) + (C^2 F^2 - D^2 E^2) \\
&\times (C^2 - D^2 - E^2 + F^2)\}]^{1/2} .
\end{aligned} \tag{22b}$$

Note that, as $s \to \infty$, $\phi(s) \to (M_{A\bar{E}} + M_{C\bar{E}})^2$.

Equation (22) is the general formula which we require. Replacing s by s_1, $\phi(s_1)$ is one of the two possible boundary curves of $\varrho_3(s_1, s_2)$ for $(E + F)^2 < s_1 < M^2$. The other possible boundary can be obtained from (22) by making the replacements

$$M_{A\bar{E}}^2 \to M_{A\bar{F}}^2 - (E^2 - F^2) (A^2 - B^2) s_1^{-1} ,$$
$$M_{C\bar{E}}^2 \to M_{C\bar{F}}^2 - (E^2 - F^2) (C^2 - D^2) s_1^{-1} ,$$

or, alternatively, by the replacements

$$M_{A\bar{E}} \to M_{A\bar{F}} ,\quad M_{C\bar{E}} \to M_{C\bar{F}} ,$$
$$E \leftrightarrow F \text{ (or } A \leftrightarrow B \text{ and } C \leftrightarrow D) .$$

It is necessary to check in each individual case which of these two possible boundary curves is the actual boundary by seeing which one gives the smaller value of $\phi(s_1)$.

Our general formula is also able to give the boundary curve of $\varrho_2(s_3, s_1)$ for $(E + F)^2 < s_1 < M^2$. Again there are two possible boundary curves; one is obtained from (22) by replacing $M_{A\bar{E}}$ by $M_{A\bar{F}}$ and by interchanging A and B, the other by replacing $M_{C\bar{E}}$ by $M_{C\bar{F}}$ and interchanging C and D. Again we must check in each case which of these two possible curves is the actual boundary.

For each process there are *four* further boundary curves to be obtained. Again we do not need to write any further formulae. It suffices to write s_2 or s_3 for s in (22), and to replace A, B, C, D by the particles which correspond to them in processes 2 or 3 (see Section 5). The particles E and F will also alter, of course, according to the process being considered and so will the various masses $M_{A\bar{E}}, \ldots$. It remains in the final section to compute these boundary curves for a number of interesting hadronic processes.

5. Boundary Curves for Special Processes

In each case the process we shall write down will be treated as process 1.

$\pi K \to \pi K$

For process 1, we have

$$A = C = E = \pi , \quad B = D = F = K ,$$

$$M_{A\bar{E}} = M_{C\bar{E}} = 2\mu , \quad M_{A\bar{F}} = M_{C\bar{F}} = (\mu + K) .$$

We use μ for the pion mass, but otherwise denote the particle and its mass by the same symbol.

For the boundary of $\varrho_3(s_1, s_2)$ we note that, for $s_1 \geqslant (\mu + K)^2$, $(\mu + K)^2 - (K^2 - \mu^2)^2 s_1^{-1} \geqslant 4\mu K > 4\mu^2$. Hence the boundary curve of $\varrho_3(s_1, s_2)$, for $(\mu + K)^2 < s_1 < (3\mu + K)^2$, is given by inserting $M_{A\bar{E}}$ and $M_{C\bar{E}}$ above into (22). On simplification this gives

$$s_2 = 16\mu^2 + 64\mu^4 [s_1 - 2(\mu^2 + K^2) + (K^2 - \mu^2)^2 s_1^{-1}]^{-1} .$$

The two possible boundary curves of $\varrho_2(s_3, s_1)$ are identical in this case; for $(\mu + K)^2 < s_1 < (3\mu + K)^2$, we have the boundary curve

$$s_3 = (3\mu + K)^2 + 16\mu^2 (\mu + K)^2 [s_1 - (K + \mu)^2]^{-1} .$$

For process 2, we may use (22) with the following identification of the particles:

$$A = B = E = F = \pi , \quad C = K, \quad D = \bar{K};$$

further,

$$M_{A\bar{E}} = M_{A\bar{F}} = 2\mu, \quad M_{C\bar{E}} = M_{C\bar{F}} = (\mu + K) .$$

The two possible boundary curves of $\varrho_1(s_2, s_3)$ and of $\varrho_3(s_1, s_2)$ are all the same; for $4\mu^2 < s_2 < 16\mu^2$, we have

$$s_3 \text{ (or } s_1) = (3\mu + K)^2 + 32\mu^3 (\mu + K)(s_2 - 4\mu^2)^{-1} .$$

Process 3 gives curves identical to those of process 1, with s_1 replaced by s_3. Thus, for $(\mu + K)^2 < s_3 < (3\mu + K)^2$, the boundary of $\varrho_2(s_3, s_1)$ is

$$s_1 = (3\mu + K)^2 + 16\mu^2 (\mu + K)^2 [s_3 - (K + \mu)^2]^{-1} ,$$

while that of $\varrho_1(s_2, s_3)$ is

$$s_2 = 16\mu^2 + 64\mu^4 [s_3 - 2(\mu^2 + K^2) + (K^2 - \mu^2)^2 s_3^{-1}]^{-1} .$$

$\pi N \to \pi N$

Now, for process 1, we have

$$A = C = E = \pi, \quad B = D = F = N,$$

$$M_{A\bar{E}} = M_{C\bar{E}} = 2\mu, \quad M_{A\bar{F}} = M_{C\bar{F}} = N.$$

For the boundary of $\varrho_3(s_1, s_2)$ we again use $M_{A\bar{E}}$ and $M_{C\bar{E}}$ in (22), since, for $s_1 \geqslant (\mu + N)^2$, $N^2 - (N^2 - \mu^2)^2 s_1^{-1} \geqslant \mu(2N - \mu) > 4\mu^2$. Thus, as for $\pi K \to \pi K$, the boundary curve of $\varrho_3(s_1, s_2)$ for $(\mu + N)^2 < s_1 < (2\mu + N)^2$, is

$$s_2 = 16\mu^2 + 64\mu^4 [s_1 - 2(\mu^2 + N^2) + (N^2 - \mu^2)^2 s_1^{-1}]^{-1}.$$

With s_1 replaced by s_3 on the right side, this is also the boundary of $\varrho_1(s_2, s_3)$ for $(\mu + N)^2 < s_3 < (2\mu + N)^2$.

The two possible boundary curves of $\varrho_2(s_3, s_1)$ are again identical, but now they are much more complicated. After some simplifications the result for this boundary is

$$s_3 = 4\mu^2 + N^2 + [s_1 - 2(\mu^2 + N^2) + (N^2 - \mu^2)^2 s_1^{-1}]^{-1} F(s_1),$$

where

$$\begin{aligned} F(s_1) &= 8\mu^2 N^2 - 8\mu^2 (N^2 - \mu^2)^2 s_1^{-1} \\ &\quad + 4\mu N s_1 \{1 - (N^2 - \mu^2) s_1^{-1}\} \{1 - (N^2 + 2\mu^2) s_1^{-1}\}^{1/2} \\ &\quad \times \{1 - (N - \mu^2 N^{-1})^2 s_1^{-1}\}^{1/2}; \end{aligned}$$

this result holds for $(\mu + N)^2 < s_1 < (2\mu + N)^2$. With s_1 and s_3 interchanged, this also gives the boundary of $\varrho_2(s_3, s_1)$ for $(\mu + N)^2 < s_3 < (2\mu + N)^2$. After some pages of algebra, this explicit result can be cast into the implicit form given by Frazer and Fulco [13].

To use (22) for process 2 we take

$$\begin{aligned} &A = B = E = F = \pi, \; C = N, \; D = \overline{N}, \\ &M_{A\overline{E}} = M_{A\overline{F}} = 2\mu, \; M_{C\overline{E}} = M_{C\overline{F}} = N. \end{aligned}$$

Again, the two possible boundary curves of $\varrho_1(s_2, s_3)$ and of $\varrho_3(s_1, s_2)$ are all identical; for $4\mu^2 < s_2 < 16\mu^2$, we have

$$\begin{aligned} s_3 \text{ (or } s_1) &= 4\mu^2 + N^2 + (s_2 - 4\mu^2)^{-1} \\ &\quad \times \left[8\mu^4 + 4\mu N s_2 \left\{ 1 - 4\mu^2 \left(1 - \frac{1}{4}\mu^2 N^{-2} \right) s_2^{-1} \right\}^{1/2} \right]. \end{aligned}$$

This result can be quickly changed into that of Frazer and Fulco [13].

$N N \to N N$

For process 1,

$$\begin{aligned} &A = B = C = D = E = F = N, \\ &M_{A\overline{E}} = M_{C\overline{E}} = M_{A\overline{F}} = M_{C\overline{F}} = \mu. \end{aligned}$$

The boundary curves of $\varrho_3(s_1, s_2)$ and $\varrho_2(s_3, s_1)$ are the same; for $4N^2 < s_1 < (2N + \mu)^2$ we have the simple result

$$s_2 \text{ (or } s_3) = 4\mu^2 + 4\mu^4 (s_1 - 4N^2)^{-1}.$$

Processes 2 and 3 are the same in this case. For process 2 we take

$$\begin{aligned} &A = C = N, \; B = D = \overline{N}, \; E = F = \pi, \\ &M_{A\overline{E}} = M_{C\overline{E}} = M_{A\overline{F}} = M_{C\overline{F}} = N. \end{aligned}$$

Again, the boundary curves of $\varrho_1(s_2, s_3)$ and $\varrho_3(s_1, s_2)$ are the same; for $4\mu^2 < s_2 < 9\mu^2$, we have

$$s_3 \text{ (or } s_1) = 4N^2 + 4\mu^4 (s_2 - 4\mu^2)^{-1}.$$

Further, for $4\,\mu^2 < s_3 < 9\,\mu^2$, we have

$$s_1 \text{ (or } s_2) = 4\,N^2 + 4\,\mu^4\,(s_3 - 4\,\mu^2)^{-1}$$

for the boundaries of $\varrho_2(s_3, s_1)$ and $\varrho_1(s_2, s_3)$.

Note that, in this case, all the boundary curves have the same functional form. In particular, for $\varrho_3(s_1, s_2)$ and $\varrho_2(s_3, s_1)$ we have a single bounding curve, with asymptotes $s_1 = 4\,N^2$ and s_2 (or s_3) $= 4\,\mu^2$. The two pieces we determined above (for $4\,N^2 < s_1 < (2\,N + \mu)^2$ and for $4\,\mu^2 < s_2$ (or s_3) $< 9\,\mu^2$) have an arc in common.

$\pi\,\Lambda \to \pi\,\Lambda$

For process 1,

$$A = C = E = \pi,\quad B = D = F = \Lambda,$$

while

$$M_{A\bar{E}} = M_{C\bar{E}} = 2\,\mu,\quad M_{A\bar{F}} = M_{C\bar{F}} = \Sigma,$$

since a $(\pi\,\Lambda)$ state has total isospin 1. Now, for $s_1 \geqslant (\mu + \Lambda)^2$,

$$\Sigma^2 - (\Lambda^2 - \mu^2)^2\,s_1^{-1} \geqslant \Sigma^2 - (\Lambda - \mu)^2 > 4\,\mu^2$$

and so the boundary curve of $\varrho_3(s_1, s_2)$, for $(\mu + \Lambda)^2 < s_1 < (2\,\mu + \Lambda)^2$, is given by inserting $M_{A\bar{E}}$ and $M_{C\bar{E}}$ above into (22). This gives

$$s_2 = 16\,\mu^2 + 64\,\mu^4\,[s_1 - 2\,(\mu^2 + \Lambda^2) + (\Lambda^2 - \mu^2)^2\,s_1^{-1}]^{-1}\,.$$

With s_1 replaced by s_3 on the right side, this is also the boundary of $\varrho_1(s_2, s_3)$ for $(\mu + \Lambda)^2 < s_3 < (2\,\mu + \Lambda)^2$.

The two possible boundary curves of $\varrho_2(s_3, s_1)$ are identical; for $(\mu + \Lambda)^2 < s_1 < (2\,\mu + \Lambda)^2$ we have the boundary

$$s_3 = 4\,\mu^2 + \Sigma^2 + [s_1 - 2\,(\mu^2 + \Lambda^2) + (\Lambda^2 - \mu^2)^2\,s_1^{-1}]^{-1}\,F(s_1),$$

where

$$F(s_1) = 8\,\mu^2\,\{\Sigma^2 - (\Lambda^2 - \mu^2)^2\,s_1^{-1}\} + 4\,\mu\,\Sigma\,s_1\,\{1 - (\Lambda^2 - \mu^2)\,s_1^{-1}\}$$
$$\times\,\{1 - (\Lambda^2 - \mu^2)^2\,\Sigma^{-2}\,s_1^{-1}\}^{1/2}\,\{1 - (2\,\mu^2 + 2\,\Lambda^2 - \Sigma^2)\,s_1^{-1}\}^{1/2}.$$

With s_1 and s_3 interchanged, this also gives the boundary of $\varrho_2(s_3, s_1)$ for $(\mu + \Lambda)^2 < s_3 < (2\,\mu + \Lambda)^2$.

It will be seen that we have claimed that these boundaries hold up to $(2\,\mu + \Lambda)^2$. However, a $(\pi\,\Sigma)$ intermediate state is possible in this case, and $(\mu + \Sigma) < (2\,\mu + \Lambda)$. But, being a *two* particle intermediate state, we can calculate the boundaries arising from it; they will be obtained by replacing Λ by Σ throughout the expressions above. Then we see that the value of s_2 or s_3 arising from the $(\pi\,\Sigma)$ intermediate state is greater than that arising from $(\pi\,\Lambda)$, for each value of s_1 in the range $(\mu + \Sigma)^2 < s_1 < (2\,\mu + \Lambda)^2$. This is obvious for the case of s_2. For s_3, numerical evaluation leaves no doubt that both curves are monotonic decreasing in the range under consideration; we have not constructed a rigorous proof. The curve arising from $(\pi\,\Sigma)$ is always above that from $(\pi\,\Lambda)$; indeed, s_3 for the former, at $s_1 = (2\,\mu + \Lambda)^2$, is greater than s_3 for the latter, at $s_1 = (\mu + \Sigma)^2$.

For process 2, we insert

$$A = B = E = F = \pi, \ C = \Lambda, \ D = \overline{\Lambda},$$

$$M_{A\overline{E}} = M_{A\overline{F}} = 2\,\mu, \ M_{C\overline{E}} = M_{C\overline{F}} = \Sigma\,.$$

The two possible boundary curves of $\varrho_1(s_2, s_3)$ and of $\varrho_3(s_1, s_2)$ are yet again all the same; using (22), we have, for $4\,\mu^2 < s_2 < 16\,\mu^2$,

$$s_3 \text{ (or } s_1) = 4\,\mu^2 + \Sigma^2 + (s_2 - 4\,\mu^2)^{-1}\,[8\,\mu^2\,(\Sigma^2 - \Lambda^2 + \mu^2) + 4\,\mu\,\Sigma\,s_2\{1 - (\Sigma - \Sigma^{-1}\,(\Lambda - \mu)^2)\,(\Sigma^{-1}\,(\Lambda + \mu)^2 - \Sigma)\,s_2^{-1}\}^{1/2}]\,.$$

$K\,N \to K\,N$

For process 1,

$$A = C = E = K, \ B = D = F = N,$$

$$M_{A\overline{E}} = M_{C\overline{E}} = 2\,\mu, \ M_{A\overline{F}} = M_{C\overline{F}} = \Lambda\,.$$

Now, for $s_1 \geqslant (K + N)^2, \Lambda^2 - (N^2 - K^2)^2\,s_1^{-1} \geqslant \Lambda^2 - (N - K)^2 > 4\,\mu^2$. Hence, for the boundary of $\varrho_3(s_1, s_2)$ we use (22) directly to obtain, for $(K + N)^2 < s_1 < (\mu + K + N)^2$,

$$s_2 = 16\,\mu^2 + 64\,\mu^4\,[s_1 - 2\,(K^2 + N^2) + (N^2 - K^2)^2\,s_1^{-1}]^{-1}\,.$$

The two possible boundary curves of $\varrho_2(s_3, s_1)$ are identical once more. For $(K + N)^2 < s_1 < (\mu + K + N)^2$, the boundary is

$$s_3 = 4\,\mu^2 + \Lambda^2 + [s_1 - 2\,(K^2 + N^2) + (N^2 - K^2)^2\,s_1^{-1}]^{-1}\,F(s_1),$$

where

$$F(s_1) = 8\,\mu^2\{\Lambda^2 - (N^2 - K^2)^2\,s_1^{-1}\} + 4\,\mu\,\Lambda\,s_1\{1 - 2\,(K^2 + N^2 - 2\,\mu^2)\,s_1^{-1} + (N^2 - K^2)^2\,s_1^{-2}\}^{1/2}\,\{1 - (2\,K^2 + 2\,N^2 - \Lambda^2)\,s_1^{-1}\}^{1/2} \times \{1 - (N^2 - K^2)^2\,\Lambda^{-2}\,s_1^{-1}\}^{1/2}\,.$$

For process 2 the particles are

$$A = K, \ B = \overline{K}, \ C = N, \ D = \overline{N}, \ E = F = \pi$$

and the required masses are

$$M_{A\overline{E}} = M_{A\overline{F}} = (\mu + K), \ M_{C\overline{E}} = M_{C\overline{F}} = N\,.$$

As usual, there are four identical boundary curves. For $4\,\mu^2 < s_2 < 16\,\mu^2$ the boundaries of $\varrho_1(s_2, s_3)$ and $\varrho_3(s_1, s_2)$ are given by

$$s_3 \text{ (or } s_1) = (\mu + K)^2 + N^2 + (s_2 - 4\,\mu^2)^{-1} \times \left[4\,\mu^3\,(\mu + K) + 2\,(\mu + K)\,N\,s_2\left\{1 - 4\,\mu^2\left(1 - \frac{1}{4}\,\mu^2\,N^{-2}\right)s_2^{-1}\right\}^{1/2}\right].$$

Process 3 for this case is the most complicated of all the situations we have had to consider. We have

$$A = K, \ B = \overline{N}, \ C = \overline{N}, \ D = K, \ E = \pi, \ F = \overline{\Lambda},$$

$$M_{A\overline{E}} = M_{C\overline{F}} = (\mu + K), \ M_{A\overline{F}} = M_{C\overline{E}} = N\,.$$

The two possible boundaries of $\varrho_2(s_3, s_1)$ are identical; for $(\mu + \Lambda)^2 < s_3 < (2\mu + \Lambda)^2$ the boundary is given by

$$s_1 = (\mu + K)^2 + N^2 + [s_3 - 2(\mu^2 + \Lambda^2) + (\Lambda^2 - \mu^2)^2 s_3^{-1}]^{-1} G(s_3),$$

where

$$\begin{aligned} G(s_3) = {} & \mu(\mu + 2K)(\mu^2 + \Lambda^2 + N^2 - K^2) + 2\mu^2 \Lambda^2 - s_3^{-1} \mu \{(\mu + 2K) \\ & \times (\Lambda^2 - \mu^2)^2 + (\mu + 2K)(N^2 - K^2)(\Lambda^2 - \mu^2) + 2\mu(N^2 - K^2)^2\} \\ & + 2 s_3 (\mu + K) N [1 - \{(K\Lambda^2 + \mu N^2)(K + \mu)^{-1} - \mu K\} s_3^{-1}] \\ & \times [1 - (2\Lambda^2 + \mu^2 + \mu^2 K^2 N^{-2} - \mu^2 \Lambda^2 N^{-2}) s_3^{-1} \\ & + \{(\Lambda^2 - \mu^2)(\Lambda^2 - \mu^2 K^2 N^{-2}) + \mu^2 (N - K^2 N^{-1})^2\} s_3^{-2}]^{1/2}. \end{aligned}$$

For the boundary of $\varrho_1(s_2, s_3)$ there are two possible curves. One curve corresponds to using $M_{A\bar{F}}$ and $M_{C\bar{E}}$; the other corresponds to using $M_{A\bar{E}}$ and $M_{C\bar{F}}$ and may be found by substituting $(\mu + K)^2 + (N^2 - K^2)(\Lambda^2 - \mu^2) s_3^{-1}$ for N^2 in the equation of the first curve. The two curves intersect for that value of s_3 which makes

$$N^2 = (\mu + K)^2 + (N^2 - K^2)(\Lambda^2 - \mu^2) s_3^{-1}.$$

Call the value of s_3 at this crossover point s_{3c}; then

$$s_{3c} = \frac{(N^2 - K^2)(\Lambda^2 - \mu^2)}{N^2 - (\mu + K)^2}.$$

Numerical evaluation shows that

$$(\mu + \Lambda)^2 < s_{3c} < (\mu + \Sigma)^2.$$

Thus the crossover point occurs in the range of values of s_3 for which we expect the 'extended' unitarity relation to hold. For $(\mu + \Lambda)^2 < s_3 \leqslant s_{3c}$, the boundary curve of $\varrho_1(s_2, s_3)$ is obtained by using $M_{A\bar{F}}$ and $M_{C\bar{E}}$; the result is

$$\begin{aligned} s_2 = {} & 4N^2 + [s_3 - 2(\mu^2 + \Lambda^2) + (\Lambda^2 - \mu^2)^2 s_3^{-1}]^{-1} \\ & \times 4\mu^2 [(N^2 + \Lambda^2 - K^2) + (N^2 - K^2)(\Lambda^2 + N^2 - K^2 - \mu^2) s_3^{-1}]. \end{aligned}$$

For $s_{3c} \leqslant s_3 < (2\mu + \Lambda)^2$ we use $M_{A\bar{E}}$ and $M_{C\bar{F}}$ to obtain the boundary

$$\begin{aligned} s_2 = {} & 4(\mu + K)^2 + [s_3 - 2(\mu^2 + \Lambda^2) + (\Lambda^2 - \mu^2)^2 s_3^{-1}]^{-1} \\ & \times 4\mu [2(\mu + K)\{(\mu + K)^2 + \Lambda^2 - N^2\} \\ & - (\Lambda^2 - \mu^2 - N^2 + K^2)(2K\Lambda^2 + \mu N^2 + \mu\Lambda^2 - 2\mu^2 K - \mu K^2 - \mu^3) s_3^{-1}]. \end{aligned}$$

As for the process $\pi\Lambda \to \pi\Lambda$, we have claimed that the above boundaries hold in a range of s_3 which extends, not just to $(\mu + \Sigma)^2$, but to $(2\mu + \Lambda)^2$. This is because the curve arising from $(\pi\bar{\Sigma})$ is always above that from $(\pi\bar{\Lambda})$, in the range $(\mu + \Sigma)^2 \leqslant s_3 < (2\mu + \Lambda)^2$. For the boundary of $\varrho_1(s_2, s_3)$ it is not difficult to prove that both curves are monotonic decreasing in this range; moreover s_2, for the $(\pi\bar{\Sigma})$ curve, at $s_3 = (2\mu + \Lambda)^2$, is greater than s_2, for the $(\pi\bar{\Lambda})$ curve, at $s_3 = (\mu + \Sigma)^2$. By numerical evaluation it is clear that exactly similar statements can be made for the boundary of $\varrho_2(s_3, s_1)$.

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Appendix

We give an evaluation of the integral I of equation (15) which uses real analysis only. The first steps are completely standard. Since

$$\int_0^1 \frac{d\alpha}{[a\,\alpha + b\,(1-\alpha)]^2} = \frac{1}{a\,b} \text{ if } a\,b > 0\,,$$

we have

$$I = \int_0^1 d\alpha \iint \frac{d\Omega_{\bar{n}}}{[\alpha\,\tau_2'' + (1-\alpha)\,\tau_2' - (\alpha\,\boldsymbol{n} + (1-\alpha)\,\boldsymbol{n}')\cdot\bar{\boldsymbol{n}}]^2}$$

on reversing the order of integration; the integrand is always positive, so this is justified. Now take the vector $(\alpha\,\boldsymbol{n} + (1-\alpha)\,\boldsymbol{n}')$ as the axis of a system of spherical polar coordinates to obtain

$$I = 2\pi \int_0^1 d\alpha \int_{-1}^{+1} \frac{dx}{[\alpha\,\tau_2'' + (1-\alpha)\,\tau_2' - |\alpha\,\boldsymbol{n} + (1-\alpha)\,\boldsymbol{n}'|\,x]^2}\,.$$

But

$$\int_{-1}^{+1} \frac{dx}{(a + b\,x)^2} = \frac{2}{(a^2 - b^2)} \text{ if } |b| < |a|\,,$$

and so

$$I = 2\pi \int_0^1 \frac{d\alpha}{\alpha\,(1-\alpha)} \left[\frac{\alpha^2\,(\tau_2''^2 - 1) + (1-\alpha)^2\,(\tau_2'^2 - 1)}{2\,\alpha\,(1-\alpha)} + \tau_2'\,\tau_2'' - \boldsymbol{n}\cdot\boldsymbol{n}'\right]^{-1}.$$

Now change the variable of integration to

$$u = \frac{\alpha^2\,(\tau_2''^2 - 1) + (1-\alpha)^2\,(\tau_2'^2 - 1)}{2\,\alpha\,(1-\alpha)} + \tau_2'\,\tau_2''\,.$$

Then $du/d\alpha$ vanishes just once in (0, 1), when

$$\alpha_0\,(1-\alpha_0)^{-1} = (\tau_2'^2 - 1)^{1/2}\,(\tau_2''^2 - 1)^{-1/2}\,.$$

As α increases from 0 to α_0, u decreases monotonically from $+\infty$ to u_0; as α increases from α_0 to 1, u increases monotonically from u_0 to $+\infty$. The minimum value of u is

$$u_0 = \tau_2'\,\tau_2'' + (\tau_2'^2 - 1)^{1/2}\,(\tau_2''^2 - 1)^{1/2}\,.$$

The equation connecting u and α is a quadratic in α of the form

$$0 = Q(\alpha, u) = p(u)\,\alpha^2 + 2\,q(u)\,\alpha + r(u)\,,$$

where

$$Q(\alpha, u) = \phi(\alpha) - (u - \tau_2' \tau_2'')\,\psi(\alpha)\,,$$

$$\phi(\alpha) = \frac{1}{2}\,\alpha^2\,(\tau_2'^2 - 1) + \frac{1}{2}\,(1 - \alpha)^2\,(\tau_2''^2 - 1)\,,$$

$$\psi(\alpha) = \alpha\,(1 - \alpha)\,.$$

Then

$$\begin{aligned}\psi(\alpha)\,du/d\alpha &= \phi'(\alpha) - (u - \tau_2' \tau_2'')\,\psi'(\alpha) = \partial Q(\alpha, u)/\partial\alpha \\ &= 2(p(u)\,\alpha + q(u)) = \pm\,2\,[q^2(u) - p(u)\,r(u)]^{1/2}\,.\end{aligned}$$

Thus, for the two values of α which give the same value of u, $\psi(\alpha)\,du/d\alpha$ has values which are equal in magnitude but opposite in sign. Since

$$4\,(q^2(u) - p(u)\,r(u)) = (u - \tau_2' \tau_2'')^2 - (\tau_2'^2 - 1)\,(\tau_2''^2 - 1)$$

the integral now becomes

$$I = 4\,\pi \int_{u_0}^{\infty} \frac{du}{(u - \boldsymbol{n} \cdot \boldsymbol{n}')\,[(u - \tau_2' \tau_2'')^2 - (\tau_2'^2 - 1)\,(\tau_2''^2 - 1)]^{1/2}}\,.$$

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