Zeitschrift: Helvetica Physica Acta
Band: 42 (1969)
Heft: 7-8

Artikel: Formal scattering treatment of the neutral K meson system
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DOI: https://doi.org/10.5169/seals-114107

Formal Scattering Treatment of the Neutral K Meson System[1][2])

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(26. VII. 69)

Abstract. Resonance and decay phenomena associated with the neutral K mesons are discussed in the context of a modified formal scattering system. The master equation for the time evolution of the K mesons under weak interaction is obtained and the resonant structure of the scattering cross sections is established. Within the pole approximation the generalized Wigner-Weisskopf formalism and simple Breit-Wigner forms for the scattering resonances may be valid in the case of CP conservation. Deviations in the case of CP violation are discussed in detail for an explicitly soluble model. The regeneration problem is discussed in an Appendix. It is concluded that the validity of the generalized Wigner-Weisskopf formulation should properly be a subject for experimental investigation and not a basic assumption for the description of neutral K meson decay.

I. Introduction

The neutral K meson system can be described in terms of a scattering system [1] [2] with Hamiltonian $H = H_0 + V$, where H_0 contains the strong interactions under which the K mesons appear as stable particles with mass λ_0 and where V induces their decay into the continuous spectrum of H_0. The spectrum of H_0 therefore consists of the discrete eigenvalue λ_0 and the continuum of final states. We denote by P and $\overline{P}$ the complementary and orthogonal projection operators on the discrete and continuous parts of the Hilbert space $\mathcal{H}$ with respect to H_0; the subspace[3]) $P\,\mathcal{H}$ of $\mathcal{H}$ is spanned by K_1 and K_2 (or, equivalently, K^0, $\overline{K}^0$).

In order that V may induce the decay of the K mesons, it suffices that $\overline{P}\,V\,P \neq 0$. In this case the total Hamiltonian H is not reduced by P and therefore the total evolution $U(t) = e^{-iHt}$ does not conserve the subspace $P\,\mathcal{H}$. We further require that (H, H_0) forms a scattering system in which the K mesons emerge as resonances. This may be assured, for example, by assuming [2] V to be of trace class. We shall treat in a later section an algebraically soluble model for a V of finite rank. We define a decay system as a scattering system in which H_0 and V have the structure specified above [3].

[1]) Supported in part by a Summer Research Grant, University of Denver Research Institute.

[2]) Part of this work has been reported at the Boulder Conference on High Energy Physics, Aug. 18–22, 1969.

[3]) In this paper we study the decay laws assuming that the initial K meson state is pure but arbitrary within $P\,\mathcal{H}$. The formalism can easily be generalized to take into account mixing due to incoherence in the production of K^0 and $\overline{K}^0$.

Our principal results are based on the properties of the reduced resolvent $R'(z) = P\,R(z)\,P$. In fact, these properties determine both the time evolution of the K meson subspace and the structure of the scattering amplitude.

We first discuss in Section II the analytic properties of the reduced resolvent (the 'propagator' in the K meson space) and the properties of its pole residues on the second sheet. We find that, in general, the pole residues are non-orthogonal rank one operators. The projection associated with the range of each of these is a pure state; these are identified with K_S and K_L. In the case of CP conservation the residues would be themselves orthogonal projections corresponding to the CP eigenstates K_1 and K_2; the problem would decompose into a direct sum of independent decay systems corresponding to the CP eigenspaces.

In Section III we discuss the time evolution of the K meson subspace, i.e. the reduced motion. A master equation is derived which describes this motion. It is clear from this result that the evolution of the system may differ significantly from that which can be described by a phenomenological theory with an effective time-independent non-Hermitian Hamiltonian (the generalized Wigner-Weisskopf formalism). For example, the transition probability has zero slope at $t = 0$ and the approximate exponential behavior builds up gradually. The process is analogous to phenomena in statistical mechanics where the true evolution of a subspace differs from the exponential form which could be obtained through use of the repeated random phase approximation. In pole approximation, the time evolution of the subspace is represented by the poles of $R'(z)$; the non-orthogonality of the pole residues is another reflection of the lack of semi-group property. The unitary sum rule of BELL and STEINBERGER [4], however, is consistent with our results in pole approximation.

To obtain more detailed results in the case of small symmetry breaking (CP violation) we introduce in Section IV a model [5] [6] which admits an explicit solution. It consists essentially of omitting the part $\overline{P}\,V\,\overline{P}$ of V, i.e., of neglecting weak direct continuum interactions. An explicit form for the reduced resolvent is obtained and a detailed correspondence with the unitary sum rule is established.

In Section V we show in this soluble model that the pole residues are non-orthogonal to first order in CP violation (amplitude).

We conclude in Section VI with a discussion of the two-channel scattering theory in the soluble model. Explicit formulas for the total cross-sections are obtained. If CP were conserved, the pole approximation could yield a superposition of Breit-Wigner resonance forms. However, in the case of CP violation, a deviation from Breit-Wigner form (in the restricted sense of a Lorentz distribution) occurs even within the pole approximation as a consequence of the non-Hermitian structure of the residue operators.

In the Appendix we formulate the regeneration problem. The results differ from previous formulations which assumed the semi-group property in that there are additional parameters to be experimentally determined.

It appears, therefore, that the validity of the generalized Wigner-Weisskopf formulation should properly be a subject for experimental investigation and not a basic assumption for the description of neutral K meson decay.

II. Properties of the Reduced Resolvent

The resolvent $R(z)$ of the total Hamiltonian H is defined as

$$R(z) = \frac{1}{z-H} \tag{2.1}$$

and is analytic on the entire complex plane except for the spectrum of H. Since the discontinuity of $R(z)$ is unbounded across the cut along the spectrum of H, the cut forms a natural boundary and the resolvent cannot be analytically continued across.

For a suitable V, however, the discontinuity of the reduced resolvent $R'(z) = P\,R(z)\,P$ can be regular on some open set belonging to the spectrum Λ of H; there may then exist an analytic continuation of $R'(z)$. We give an example in Section IV where this condition is satisfied; in the following we shall assume it [7].

For sufficiently weak coupling we may assume that the degenerate discrete eigenvalues in the spectrum of H_0 appear as two poles in the second sheet of the reduced total resolvent which are non-degenerate due to the different strength of coupling (and phase space) to the decay channels. We further assume that the rank of $R'(z)$ remains two for small enough couplings, in its domain of regularity, and hence $R'(z)$ admits an inverse. To investigate the properties of the residues, we note that the identity

$$R'(z)\,h(z) = h(z)\,R'(z) = P\,, \tag{2.2}$$

where $h(z)$ is the inverse of $R'(z)$ in $P\,\mathcal{H}$, is also continuable to the second sheet. Integrating (2.2) around the pole in the second sheet, we obtain

$$g_p\,h(z_p)^{\mathrm{II}} = h(z_p)^{\mathrm{II}}\,g_p = 0\,, \tag{2.3}$$

where g_p are the residues of $R'(z)^{\mathrm{II}}$ at the pole positions z_p. If $g_p \neq 0$, then $\det h(z_p)^{\mathrm{II}} = 0$, i.e., $h(z_p)^{\mathrm{II}}$ has rank one if it is not the zero operator; conversely, g_p is also of rank one, and for any $\varphi \in P\,\mathcal{H}$, $g_p\,\varphi$ is in the null space of $h(z_p)^{\mathrm{II}}$.

In general, $g_p\,g_{p'} \neq 0$ for $p \neq p'$; we estimate this product in a later section using a specific explicitly soluble model. To understand this non-orthogonality, we consider here a point of view somewhat analogous to the introduction of a phenomenological Hamiltonian, i.e., we define a two dimensional matrix $W(z)$ by [8]

$$R'(z) = \frac{1}{z-W(z)}\,. \tag{2.4}$$

If the eigenvectors of $W(z)$ are linearly independent, we can construct a unique decomposition of $R'(z)$ in the form

$$R'(z) = \sum_p \frac{1}{z-w_p(z)}\,Q_p(z)\,, \tag{2.5}$$

where $w_p(z)$ are the eigenvalues of $W(z)$, and

$$Q_p(z) = |\,p(z)\rangle\langle\tilde{p}(z)\,|\,, \tag{2.6}$$

$$\langle\tilde{p}(z)\,|\,p'(z)\rangle = \delta_{pp'}\,. \tag{2.7}$$

The operators Q_p are therefore orthogonal idempotents. However, the equations

$$z - w_p(z) = 0 \tag{2.8}$$

have roots at, in general, $z_p \neq z_{p'}$. In the absence of a symmetry which enforces orthogonality, $\langle \tilde{p}(z_p) \mid p'(z_{p'}) \rangle \neq 0$, for $p \neq p'$, and hence $Q_p(z_p)\, Q_{p'}(z_{p'}) \neq 0$. In fact as we shall show in our model, this product is formally of the order of the amplitude of CP violation.

As is clear from the result (2.6), the state $|p(z_p)\rangle$ spans the one-dimensional range of g_p if the zero of (2.8) is simple. Let P_p be the corresponding projectors. They represent pure states which we identify with K_L and K_S, the resonant states of the K meson subspace.

It follows from equation (2.3) that the resonance energies and widths are determined as expectation values of the Hermitian and anti-Hermitian parts of the operators $W(z)^{\mathrm{II}}$ at the corresponding poles z_p in the states P_p, i.e.,

$$h(z_p)^{\mathrm{II}} = z_p - W(z_p)^{\mathrm{II}} \tag{2.9}$$

and hence

$$[z_p - W(z_p)^{\mathrm{II}}]\, g_p = 0\,. \tag{2.10}$$

The residue g_p is proportional to $Q_p(z_p) = |p(z_p)\rangle \langle \tilde{p}(z_p)|$.

Calling

$$z_p = \lambda_p - i\, \Gamma_p/2\,, \tag{2.11}$$

$$W(z_p)^{\mathrm{II}} = W(z_p)^{\mathrm{II}}_{+} + i\, W(z_p)^{\mathrm{II}}_{-}\,, \tag{2.12}$$

where the subscripts $\pm$ refer to Hermitian and skew-Hermitian parts, we find from (2.10) that

$$\lambda_p = Tr\big(W(z_p)^{\mathrm{II}}_{+}\, P_p\big) = \langle W(z_p)^{\mathrm{II}}_{+} \rangle_p \tag{2.13}$$

and

$$-\Gamma_p/2 = Tr\big(W(z_p)^{\mathrm{II}}_{-}\, P_p\big) = \langle W(z_p)^{\mathrm{II}}_{-} \rangle_p\,. \tag{2.14}$$

We may identify, from this result, the matrices W^{II}_{+} and W^{II}_{-} as the mass matrix and the decay matrix, respectively.

In case CP were not violated, then the operator $W(z)$ would be automatically in diagonal form for every z in the basis in which CP is diagonal. Hence the eigenstates $|p(z)\rangle$ would be orthogonal, and the pole residues would have this property also. The non-orthogonality of the residues may therefore be considered a reflection of symmetry breaking (this phenomenon should not be confused with the well-known lack of orthogonality between $|K_L\rangle$ and $|K_S\rangle$; the structure of (2.6) has already taken into account the fact that $W(z)$ has non-orthogonal eigenfunctions).

III. Time Evolution of the K Meson Subspace

The transition probability for a state φ belonging to the K meson subspace $P\,\mathcal{H}$ to decay into the continuum of H_0, $\bar{P}\,\mathcal{H}$, is the integrated sum of absolute squares of the spectral amplitude ${}_0\langle \lambda, c | \varphi)$ of φ with respect to H_0. The labels c here correspond to a decomposition of the continuum states of H_0 into CP $= \pm 1$ subspaces and into specific channels such as $\pi\,\pi$, $\pi\,\pi\,\pi$, $\pi\, l\, \bar{\nu}$. At any given time, a state which is initially $\varphi \in P\,\mathcal{H}$ decays into channel c with energy λ with probability (we absorb phase space factors into the amplitude)

$$|\, {}_0\langle \lambda, c |\, U(t)\, \varphi)\,|^2\,, \tag{3.1}$$

and the total decay probability is

$$\sum_c \int d\lambda \, |{}_0\langle \lambda, c | \, U(t)\, \varphi) \, |^2 = 1 - \sum_i |\, (K_i, \, U(t)\, \varphi) \, |^2 \, . \tag{3.2}$$

To determine the decay rate as a function of time, it is therefore of interest to investigate the evolution operator

$$U'(t) = P \, U(t) \, P \tag{3.3}$$

inducing the reduced motion of the K meson subspace.

While $U(t) = e^{-iHt}$ satisfies a differential equation (Schrödinger equation with total constant Hamiltonian H), the reduced motion $U'(t)$ is a solution of an integro-differential equation (master equation). To obtain this result, we observe that the identity

$$z \, \frac{1}{z-H} - 1 = H \, \frac{1}{z-H} \tag{3.4}$$

implies that

$$\begin{aligned} z \, R'(z) - P &= P \, H \, R'(z) + P \, H \, \bar{P} \, R(z) \, P \, , \\ z \, \bar{P} \, R(z) \, P &= \bar{P} \, H \, R'(z) + \bar{P} \, H \, \bar{P} \, R(z) \, P \, . \end{aligned} \tag{3.5}$$

Eliminating $\bar{P} \, R(z) \, P$ from this system, one obtains

$$z \, R'(z) - P = P \, H \, P \, R'(z) + P \, H \, \bar{P} \, \frac{1}{z - \bar{P} \, H \, \bar{P}} \, \bar{P} \, H \, P \, R'(z) \, . \tag{3.6}$$

The unitary time development of the complete system is given by

$$U(t) = \frac{1}{2\pi i} \oint R(z) \, e^{-izt} \, dz \, , \tag{3.7}$$

where the path of integration encloses the spectrum of H. The evolution of the reduced motion (3.3) is therefore expressed in terms of the reduced resolvent as

$$U'(t) = \frac{1}{2\pi i} \oint R'(z) \, e^{-izt} \, dz \, . \tag{3.8}$$

Carrying out this inverse Laplace transform on equation (3.6), we obtain the master equation for the time evolution of the K meson subspace

$$i \, \frac{d}{dt} \, U'(t) = P \, H \, P \, U'(t) - i \int_0^t d\tau \, P \, H \, \bar{P} \, e^{-i\bar{P}H\bar{P}\tau} \, \bar{P} \, H \, P \, U' \, (t-\tau) \, . \tag{3.9}$$

The solution of this integro-differential equation does not satisfy the semi-group property and can therefore not be generated by a generalized Wigner-Weisskopf equation with non-Hermitian constant Hamiltonian. For example, in the neighborhood of $t = 0$, the contribution of the second term on the right hand side is negligible. The subsystem therefore develops according to the projected complete Hamiltonian and therefore there is intially no decay. That the decay rate vanishes at $t = 0$ can be seen directly by examining the time derivative of equation (3.2). The decay rate is given by

$$\frac{d}{dt} \sum_c \int d\lambda \, |{}_0\langle \lambda, c | \, U(t)\, \varphi) \, |^2 = - \, \frac{d}{dt} \sum_i |\, (K_i, \, U(t)\, \varphi) \, |^2 \, . \tag{3.10}$$

The right hand side contains the norm in the subspace $P\,\mathcal{H}$ of $U'(t)\,\varphi$, and we therefore evaluate

$$\frac{d}{dt}\,\|\,U'(t)\,\varphi\|^2 = \frac{d}{dt}\,(U'(t)\,\varphi,\,U'(t)\,\varphi)$$

at $t = 0$. In this limit one obtains

$$\frac{d}{dt}\,\|\,U'(t)\,\varphi\|^2\,|_{t=0} = 2\,Im(\varphi,\,P\,H\,P\,\varphi) = 0\,. \tag{3.11}$$

Furthermore, if a measurement (such as detection or regeneration) is made at any time $t > 0$ which destroys the coherence between the K meson subspace $P\,\mathcal{H}$ and its decay product space $\overline{P}\,\mathcal{H}$, then the balance established between these by equation (3.9) and the other equations similarly derived from equation (3.5) is destroyed and the evolution must build up again to a new equilibrium.

Let us now turn to the large time behavior. After sufficient time has elapsed, the behavior of $U'(t)$ may become exponential [9], in which case the reduced motion can be approximately described by a Schrödinger equation with non-Hermitian effective Hamiltonian. Initial conditions at $t = 0$, however, are not applicable to its solutions. In order to see this, and to develop a useful phenomenological formalism, we may, instead of using the master equation, start from the equation (3.8).

Equation (3.8) may be approximated by deforming the contour of integration into the second sheet [6]. The pole contributions dominate the long time behaviour (up to the order of lifetime of the long lived state). We obtain in this pole approximation

$$U'(t) = g_S\,e^{-iz_S t} + g_L\,e^{-iz_L t}\,, \tag{3.12}$$

where g_S and g_L are the rank one residua introduced in Section II. As remarked there, in general (contrary to the conclusion one would reach from a solution of the generalized Wigner-Weisskopf formalism)

$$g_S\,g_L \neq 0\,, \tag{3.13}$$

and

$$U'(0) = g_S + g_L \neq P\,. \tag{3.14}$$

The *initial states* which formally exhibit exponential behavior at finite times are not the resonant states P_S and P_L, but rather $P - P_L$ and $P - P_S$.

To illustrate the deviation from semi-group property, consider the effect of a measurement at time $t_1 > 0$ which destroys the coherence between $P\,\mathcal{H}$ and $\overline{P}\,\mathcal{H}$. For initial state $\varphi \in P\,\mathcal{H}$, the state at time t_1 is given by the configuration

$$\begin{aligned} &U'(t_1)\,\varphi \text{ in } P\,\mathcal{H}\,,\\ &\overline{P}\,U(t_1)\,\varphi \text{ in } \overline{P}\,\mathcal{H}\,. \end{aligned}$$

After the measurement, the first part develops as follows:

$$U'(t_1)\,\varphi \to \left\{\begin{array}{l} U'\,(t-t_1)\,U'(t_1)\,\varphi \text{ in } P\,\mathcal{H}\\ \overline{P}\,U\,(t-t_1)\,U'(t_1)\,\varphi \text{ in } \overline{P}\,\mathcal{H}\end{array}\right\},$$

while the second part develops incoherently with these. However, $U'(t-t_1)\,U'(t_1) \neq U'(t)$ (cf. equation (3.3)). In the pole approximation, we obtain from equation (3.12)

$$U'(t-t_1)\,U'(t_1) = g_S^2\, e^{-iz_S t} + g_L^2\, e^{-iz_L t} + g_S\, g_L\, e^{-i(z_S(t-t_1)+z_L t_1)} + g_L\, g_S\, e^{-i(z_L(t-t_1)+z_S t_1)}\,.$$

As an example of this phenomenon, we treat the regeneration process briefly in an Appendix.

For times large compared to $|Im\, z_S|^{-1}$, an arbitrary initial state φ in the K meson subspace goes over into $g_L\, e^{-iz_L t}\,\varphi$, i.e., an element in $P_L\, P\, \mathcal{H} = P_L\, \mathcal{H}$.

We finally turn to the unitary sum rule of BELL and STEINBERGER [4]. What we shall show is that if a transition operator T exists, then the unitary sum rule in the usual form is not inconsistent with the pole approximation and the lack of orthogonality of the idempotent residua. From

$$-\frac{d}{dt}\,\|\,U'(t)\,\varphi\|^2 = \sum_c \int d\lambda\;|\,{}_0\langle\lambda, c\,|\;T\,U'(t)\,\varphi)\,|^2 \tag{3.15}$$

for arbitrary $\varphi \in P\,\mathcal{H}$, it follows that [10]

$$-\frac{d}{dt}\,U'(t)^\dagger\,U'(t) = U'(t)^\dagger\,\gamma\,U'(t)\,, \tag{3.16}$$

where

$$\gamma = P\,T^\dagger\,\bar{P}\,T\,P\,. \tag{3.17}$$

In the pole approximation equation (3.12), the independence of the exponential time factors implies that

$$-i\,(z_p^* - z_{p'})\,g_p^\dagger\,g_{p'} = g_p^\dagger\,\gamma\,g_{p'}\,. \tag{3.18}$$

Since, as pointed out above, the g_p's are proportional to idempotents of the form (2.6), we may write, for example,

$$-i\,(z_L^* - z_S)\,\langle L(z_L)\;|\;S(z_S)\rangle = \langle L(z_L)\;|\;\gamma\;|\;S(z_S)\rangle \tag{3.19}$$

and

$$\Gamma_p = \langle p(z_p)\;|\;\gamma\;|\;p(z_p)\rangle\,, \tag{3.20}$$

the usual statement of the unitary sum rule.

IV. An Algebraically Soluble Model

Explicit solutions for the reduced resolvent $R'(z)$ can be obtained in a model for a decay system in which[4])

$$\bar{P}\,V\,\bar{P} = P\,V\,P = 0\,. \tag{4.1}$$

The second of (4.1) is introduced for mathematical convenience in the present exposition; it is not essential to the solubility of the model, although its consideration

[4]) Note that all strong interactions, including strong final state interactions, are included in H_0. V is only the weak interaction part of the Hamiltonian.

would be interesting for the study of a 'superweak' theory [11] in which it is assumed that *only* $P\,V\,P$ is CP violating. The first condition of (4.1) corresponds to the deletion of weak continuum interactions; they could in fact be included in H_0 without altering the general structure of the theory [12] [13].

What remains of V is $P\,V\,\bar{P} + \bar{P}\,V\,P$; for $P\,\mathcal{H}$ of dimension 2, V is therefore of rank 4. KATO [2] has shown that a potential of finite rank generates a scattering system, and hence this model (H_0, H) corresponds to a decay system as defined in Section I.

The second resolvent equation

$$R(z) = R_0(z) + R_0(z)\,V\,R(z) \tag{4.2}$$

together with (4.1) implies

$$R'(z) = P\,R_0(z)\,P + P\,R_0(z)\,P\,V\,\bar{P}\,R(z)\,P$$

and

$$\bar{P}\,R(z)\,P = \bar{P}\,R_0(z)\,\bar{P}\,V\,R'(z)\,. \tag{4.3}$$

Eliminating $\bar{P}\,R(z)\,P$ from the equation (4.3) yields

$$[z - H_0 - P\,V\,\bar{P}\,R_0(z)\,\bar{P}\,V\,P]\,R'(z) = P\,. \tag{4.4}$$

Comparing with (2.2), and assuming that $H_0\,P = \lambda_0\,P$, i.e., the discrete spectrum of H_0 is degenerate (equal masses for K_1, K_2), we obtain

$$h(z) = (z - \lambda_0)\,P - P\,V\,\bar{P}\,R_0(z)\,\bar{P}\,V\,P\,. \tag{4.5}$$

In the CP basis for $P\,\mathcal{H}$,

$$h_{ij}(z) = (K_i, h(z)\,K_j) = (z - \lambda_0)\,\delta_{ij} - \int \frac{X_{ij}(\lambda)\,d\lambda}{z-\lambda}\,, \tag{4.6}$$

where

$$X_{ij}(\lambda) = \sum_c \overline{{}_0\langle\lambda, c \mid V\,K_i\rangle}\;{}_0\langle\lambda, c \mid V\,K_j\rangle\,. \tag{4.7}$$

It is clear from this result that $R'(z)$ can be continued into its second sheet if we assume, for example, that $X(\lambda)$ is an entire function. In view of (4.4), the continuation of $R'(z)$ into its second sheet is equivalent to the explicit continuation of $h(z)$ into its second sheet. To see this, we note that

$$R'\,(\lambda + i\,\varepsilon)^{\mathrm{I}} = [h\,(\lambda + i\,\varepsilon)^{\mathrm{I}}]^{-1} = R'\,(\lambda - i\,\varepsilon)^{\mathrm{II}} = [h\,(\lambda - i\,\varepsilon)^{\hat{\mathrm{II}}}]^{-1} \tag{4.8}$$

where the designation $h^{\hat{\mathrm{II}}}$ is specified by definition through the continuation of $R'(z)$. It follows from (4.8) that

$$h\,(\lambda + i\,\varepsilon)^{\mathrm{I}} = h\,(\lambda - i\,\varepsilon)^{\hat{\mathrm{II}}}\,. \tag{4.9}$$

Hence $h\,(\lambda - i\,\varepsilon)^{\hat{\mathrm{II}}}$ is $h\,(\lambda - i\,\varepsilon)^{\mathrm{II}}$. From (4.6) it follows that

$$\begin{aligned} h\,(\lambda - i\,\varepsilon)^{\mathrm{II}} &= h\,(\lambda + i\,\varepsilon)^{\mathrm{I}} - h\,(\lambda - i\,\varepsilon)^{\mathrm{I}} + h\,(\lambda - i\,\varepsilon)^{\mathrm{I}} \\ &= (\lambda - \lambda_0)\,P - \int \frac{X(\lambda')\,d\lambda'}{\lambda - i\,\varepsilon - \lambda'} + 2\,\pi\,i\,X(\lambda)\,. \end{aligned} \tag{4.10}$$

Continuing (4.10) we find that

$$h(z)^{\mathrm{II}} = (z-\lambda_0)\,P - \int \frac{X(\lambda')\,d\lambda'}{z-\lambda'} + 2\pi i\,X(z)\,. \tag{4.11}$$

We note that comparison with (2.10) yields

$$W(z_p)^{\mathrm{II}} = \lambda_0\,P + \int \frac{X(\lambda')\,d\lambda'}{z_p-\lambda'} - 2\pi i\,X(z_p)\,. \tag{4.12}$$

The operator $X(z)$ is at most $O(g^2)$ where $g = |||\,V\,|||$ is the weak coupling constant; assuming $|\,Im\,z_p| = O(g^2)$ also, we may approximate (4.12) by

$$W(z_p)^{\mathrm{II}} \cong \lambda_0\,P + \fint \frac{X(\lambda')\,d\lambda'}{\lambda_p-\lambda'} - \pi i\,X(\lambda_p)\,. \tag{4.13}$$

Hence it follows from (2.14) that

$$\frac{\Gamma_p}{2} = \pi\,Tr\big(X(\lambda_p)\,P_p\big)\,. \tag{4.14}$$

The assumption that $|\,Im\,z_p| = O(g^2)$ is therefore consistent, and (4.14) provides an explicit connection with the decay matrix X. We note in passing that

$$\lambda_p = Tr\left\{\left(\lambda_0\,P + \fint \frac{X(\lambda')\,d\lambda'}{\lambda_p-\lambda'}\right)P_p\right\}. \tag{4.15}$$

We conclude this section with a discussion of the unitary sum rule. We find that the matrix γ of equation (3.17) is proportional to the decay matrix X in the limit of weak interaction when the discrete spectrum of H_0 is degenerate to $O(g^2)$ (no γ would exist if these conditions were not satisfied). To derive the unitary sum rule in our model, we note that equations (2.3) and (4.11) imply that

$$\left[(z_p-\lambda_0) - \fint \frac{X(\lambda)\,d\lambda}{\lambda_p-\lambda} + \pi i\,X(\lambda_p)\right] g_p = 0 \tag{4.16}$$

in the weak coupling limit. Taking the adjoint of (4.16) for the pole at $z_{p'}$, and multiplying on the left and right respectively by $g_{p'}^{\dagger}$ and g_p, then subtracting, one obtains

$$\begin{aligned} &-i\,g_p^{\dagger}\left(z_p^{*} - z_{p'} - \fint X(\lambda)\left[\frac{1}{\lambda_p-\lambda} - \frac{1}{\lambda_{p'}-\lambda}\right]d\lambda\right) g_{p'} \\ &\qquad = \pi\,g_p^{\dagger}\,[X(\lambda_p) + X(\lambda_{p'})]\,g_{p'}\,. \end{aligned} \tag{4.17}$$

If the discrete spectrum of H_0 is degenerate to $O(g^2)$, then λ_p and $\lambda_{p'}$ differ by at most $O(g^2)$; in this case, neglecting terms of $O(g^4)$ in (4.17), we obtain (3.18) with

$$\gamma = 2\pi\,X(\lambda_0)\,.$$

V. CP Violation

In this section we investigate the structure of the residues of $R'(z)$ for CP nonconservation. We have emphasized that in case there is no symmetry enforcing the orthogonality of eigenvectors of $R'(z)$ at the distinct poles, the residue idempotents may not be orthogonal. In the context of the explicitly soluble model discussed in the previous section we can calculate these residues explicitly and, within order of

magnitude, show that their structure qualitatively does not depend on the value of the weak coupling constant in the limit of weak interaction. It depends only on the relative strength of symmetry breaking, and it is to this order that the residues are non-orthogonal.

To estimate the effect of weak symmetry breaking, we assume that

$$\begin{aligned} {}_0\langle\lambda, + |V K_1\rangle &= O(g) , \\ {}_0\langle\lambda, - |V K_1\rangle &= O(g\,\alpha) , \\ {}_0\langle\lambda, + |V K_2\rangle &= O(g\,\alpha) , \\ {}_0\langle\lambda, - |V K_2\rangle &= O(g) , \end{aligned} \tag{5.1}$$

where α is the relative strength of the CP violation ($\alpha^2 \sim 10^{-3}$). It follows that $X(\lambda)$ is of the form

$$X = g^2 \begin{pmatrix} O(1) & O(\alpha) \\ O(\alpha) & O(1) \end{pmatrix}, \tag{5.2}$$

where we denote matrix elements in order of magnitude only.

To find the form of the resolvent residues, we note that

$$R'(z)^{\mathrm{II}} = [h(z)^{\mathrm{II}}]^{-1} = \frac{1}{\det h(z)^{\mathrm{II}}} \begin{vmatrix} h_{22}(z)^{\mathrm{II}} & -h_{21}(z)^{\mathrm{II}} \\ -h_{12}(z)^{11} & h_{11}(z)^{\mathrm{II}} \end{vmatrix}. \tag{5.3}$$

In the neighborhood of $z = z_p$,

$$\det h(z)^{\mathrm{II}} = (z - z_p)\,\{h_{11}^{\mathrm{II}'}\, h_{22}^{\mathrm{II}} + h_{11}^{\mathrm{II}}\, h_{22}^{\mathrm{II}'} - h_{12}^{\mathrm{II}'}\, h_{21}^{\mathrm{II}} - h_{12}^{\mathrm{II}}\, h_{21}^{\mathrm{II}'}\}_{z=z_p} . \tag{5.4}$$

To estimate the size of the elements $h_{ij}(z)^{\mathrm{II}}$, we consider the pole closest to the zero of $h_{11}(z)^{\mathrm{II}}$ and identify this pole as z_S. From

$$\det h(z_S)^{\mathrm{II}} = (h_{11}^{\mathrm{II}}\, h_{22}^{\mathrm{II}} - h_{12}^{\mathrm{II}}\, h_{21}^{\mathrm{II}})_{z=z_S} = 0 ,$$

we have

$$h_{11}(z_S)^{\mathrm{II}} = \left.\frac{h_{12}^{\mathrm{II}}\, h_{21}^{\mathrm{II}}}{h_{22}^{\mathrm{II}}}\right|_{z=z_S} . \tag{5.5}$$

According to equation (4.13),

$$h_{21}(z_p)^{\mathrm{II}} \text{ and } h_{12}(z_p)^{\mathrm{II}} = O(g^2\,\alpha) . \tag{5.6}$$

Since for small symmetry breaking, $h_{22}(z_S)^{\mathrm{II}} = O(g^2)$, it follows from (5.5) and (5.6) that

$$h_{11}(z_S)^{\mathrm{II}} = O(g^2\,\alpha^2) . \tag{5.7}$$

Finally,

$$\begin{aligned} h_{11}^{\mathrm{II}'} \text{ and } h_{22}^{\mathrm{II}'} &= O(1) , \\ h_{12}^{\mathrm{II}'}\, h_{21}^{\mathrm{II}} \text{ and } h_{12}^{\mathrm{II}}\, h_{21}^{\mathrm{II}'} &= O(g^2\,\alpha^2) , \end{aligned} \tag{5.8}$$

and hence in the neighborhood of $z = z_S$, (5.4) yields

$$\det h(z)^{\mathrm{II}} = (z - z_S)\; O(g^2) . \tag{5.9}$$

These results, together with (5.3) imply that the coefficient of $(z - z_S)^{-1}$ is

$$g_S = \begin{pmatrix} O(1) & O(\alpha) \\ O(\alpha) & O(\alpha^2) \end{pmatrix}. \tag{5.10}$$

Similarly, we obtain

$$g_L = \begin{pmatrix} O(\alpha^2) & O(\alpha) \\ O(\alpha) & O(1) \end{pmatrix}. \tag{5.11}$$

For small α, it therefore follows that

$$g_S\, g_L = O(\alpha)\,. \tag{5.12}$$

VI. Scattering Theory

The mass shell scattering amplitude ($T = S - 1$) can be expressed in terms of the resolvent as follows:

$$\begin{aligned} {}_0\langle \lambda, c \mid T \mid \lambda', c' \rangle_0 &= \delta(\lambda - \lambda')\, T_{cc'}(\lambda)\,, \\ T_{cc'}(\lambda) &= -2\pi i \lim_{\epsilon \to 0_+} {}_0\langle \lambda, c \mid V + V\, R(\lambda + i\varepsilon)\, V \mid \lambda\, c' \rangle_0\,. \end{aligned} \tag{6.1}$$

We restrict ourselves in the following to a scattering theory in the model of Section IV, where the scattering amplitude is expressed entirely in terms of the reduced resolvent. In fact, for $\overline{P}\, V\, \overline{P} = P\, V\, P = 0$, equation (6.1) becomes

$$\begin{aligned} T_{cc'}(\lambda) &= -2\pi i \lim_{\epsilon \to 0_+} {}_0\langle \lambda, c \mid V\, R'(\lambda + i\varepsilon)\, V \mid \lambda, c' \rangle_0 \\ &= -2\pi i \lim_{\epsilon \to 0_+} Tr_P\, [X^{cc'}(\lambda)\, R'(\lambda + i\varepsilon)]\,, \end{aligned} \tag{6.2}$$

where

$$X_{ij}^{cc'}(\lambda) = \overline{{}_0\langle \lambda, c \mid V\, K_i \rangle}\; {}_0\langle \lambda, c' \mid V\, K_j \rangle\,, \tag{6.3}$$

and Tr_P is carried out over the indices (i, j). Let us define

$$h^{(\pm)}(\lambda) = \lim_{\epsilon \to 0_+} h(\lambda \pm i\varepsilon)^{\mathrm{I}}\,. \tag{6.4}$$

Then, the total cross-section, up to kinematical factors, is

$$\sigma(\lambda) = \frac{1}{4} \sum_{c,\, c'} |\, T_{cc'}(\lambda)\,|^2\,. \tag{6.5}$$

It therefore follows from (6.2) and (6.3) that

$$\sigma(\lambda) = \pi^2\, Tr_P\, [X(\lambda)\, h^{(+)}(\lambda)^{-1}\, X(\lambda)\, h^{(-)}(\lambda)^{-1}]\,, \tag{6.6}$$

where $X(\lambda)$ is defined by (4.7). Furthermore

$$2\pi i\, X(\lambda) = h^{(+)}(\lambda) - h^{(-)}(\lambda) \tag{6.7}$$

and therefore

$$\sigma(\lambda) = \frac{i\pi}{2}\, Tr_P\, [X(\lambda)\, (h^{(+)}(\lambda)^{-1} - h^{(-)}(\lambda)^{-1})]\,. \tag{6.8}$$

In pole approximation, we may replace $h^{(+)}(\lambda)^{-1}$ by

$$h^{(+)}(\lambda)^{-1} = \sum_p \frac{g_p}{\lambda - z_p}, \tag{6.9}$$

and hence (for λ near the resonance energies λ_p)

$$\sigma(\lambda) \cong \frac{i\pi}{2} \sum_p \left\{ \frac{Tr_P\,(X(\lambda_p)\, g_p)}{\lambda - z_p} - \frac{Tr_P\,(X(\lambda_p)\, g_p^\dagger)}{\lambda - z_p^*} \right\}, \tag{6.10}$$

i.e.,

$$\sigma(\lambda) \cong \frac{i\pi}{2} \sum_p \frac{1}{(\lambda - \lambda_p)^2 + (\Gamma_p^2/4)} \times \left\{ (\lambda - \lambda_p)\, Tr_P\,[X(\lambda_p)\,(g_p - g_p^\dagger)] - i\frac{\Gamma_p}{2}\, Tr_P[X(\lambda_p)\,(g_p + g_p^\dagger)] \right\}. \tag{6.11}$$

In case CP is conserved, the $Q_p(z_p)$ are self-adjoint. If we assume that, furthermore, $g_p = g_p^\dagger$ (i.e., $w_p'(z_p)$ is either real or negligible compared to unity; cf. equations (2.5) and (A.12)), we may use the relations (2.14) and (4.13) to obtain

$$Tr_P(X(\lambda_p)\, g_p) = \frac{\Gamma_p}{2\pi} \frac{1}{1 - w_p'(z_p)} \tag{6.12}$$

and hence

$$\sigma(\lambda) \cong \sum_p \frac{(\Gamma_{p/2})^2}{(\lambda - \lambda_p)^2 + (\Gamma_{p/2})^2} \frac{1}{1 - w_p'(z_p)}. \tag{6.13}$$

The last factor ('inelasticity') would be unity if, for example, $g_p\, g_{p'} = \delta_{pp'}$ so that the propagation law (3.12) could have the semi-group property.

In the absence of CP conservation, the $Q_p(z_p)$ are not, in general self-adjoint. Equation (6.12) is then not valid, and the first term of (6.11) will contribute. Hence $\sigma(\lambda)$, in this case, is not the sum of simple Breit-Wigner forms. This is analogous to the situation in the time dependent theory in which the reduced evolution (3.12) is compatible with the generalized Wigner-Weisskopf formulation only if both the pole approximation (with no 'inelasticity') and CP invariance hold.

Appendix: Regeneration

The phenomenon of regeneration is useful in determining the mass difference $\Delta m = \lambda_L - \lambda_S$ and the phase of, for example,

$$\eta_{+-} = \frac{\text{amplitude}\,(K_L \to \pi^+\pi^-)}{\text{amplitude}\,(K_S \to \pi^+\pi^-)}. \tag{A.1}$$

It is generally induced by placing a slab of material, such as copper, in the K meson beam. Since the K^0 and $\bar{K}^0$ interact differently with nuclei, a K_L state passing into the slab will emerge as a linear combination [14] of K_L and K_S. The efficiency of regeneration of the K_S component is usually very small, and therefore one may observe the interference between the two pi meson decay from the regenerated K_S and that of the relatively unperturbed K_L component. We shall show in the following how the analysis of experiments of this kind can be carried out in the context of the theory discussed in the main part of this paper.

Let us assume for simplicity that the beam is initially a pure state entirely in $P\,\mathcal{H}$. Then, at time $t \leqslant t_R$, where t_R corresponds to the time required for the beam to reach the first surface of the regenerator, the wave function for the system is

$$\varphi(t) = U(t)\,\varphi(0)\,. \tag{A.2}$$

We assume, as in the usual treatment of regeneration [14], that the relation between $\varphi(t_R)$ and $\varphi(t'_R)$, where t'_R is the time required for the beam to reach the second surface of the regenerator, is described by a linear transformation A determined by the characteristics of the regenerator relevant to strong interactions. Hence, for $t \geqslant t'_R$

$$\varphi(t) = U\,(t - t'_R)\,A\,U(t_R)\,\varphi(0)\,. \tag{A.3}$$

Since the strong interactions do not connect the subspaces $P\,\mathcal{H}$ and $\overline{P}\,\mathcal{H}$, A is reduced by P and $\overline{P}$. The part of $\varphi(t)$ lying in $P\,\mathcal{H}$ therefore may be expressed as

$$P\,\varphi(t) = U'\,(t - t'_R)\,A\,U'(t_R)\,\varphi(0) + P\,U\,(t - t'_R)\,\overline{P}\,A\,\overline{P}\,U(t_R)\,\varphi(0)\,. \tag{A.4}$$

We shall ignore the second term of (A.4) in what follows.

For simplicity we treat here the case of $t_R \gg \Gamma_S^{-1}$; in this case, the long-lived 2π mode interferes with regenerated short-lived 2π mode. Differences from previous formulations similar to those which we shall find in this case also occur in the analysis of experiments [15] in which $t_R \sim 5\,\Gamma_S^{-1}$, where the regenerated short-lived 2π mode interferes with the short-lived 2π mode in the original beam.

Under these conditions, in the pole approximation (3.12),

$$P\,U(t_R)\,\varphi(0) \cong g_L\,\varphi(0)\,e^{-iz_L t_R}\,, \tag{A.5}$$

since $\Gamma_S \gg \Gamma_L$. We normalize this wave function and call it K_L:

$$K_L = \frac{g_L\,\varphi(0)}{\|g_L\,\varphi(0)\|} \in P_L\,\mathcal{H}\,. \tag{A.6}$$

The phase of K_L is therefore specified only up to the phase of $\varphi(0)$; however, its phase relative to the short-lived component is determined if we define

$$K_S = \frac{g_S\,\varphi(0)}{\|g_S\,\varphi(0)\|} \in P_S\,\mathcal{H}\,. \tag{A.7}$$

A knowledge of the strong interaction characteristics of the regenerator then suffices to determine the coefficients $\hat{\alpha}$, $\hat{\beta}$ in

$$A\,K_L = \hat{\alpha}\,K_L + \hat{\beta}\,K_S\,; \tag{A.8}$$

here $\hat{\alpha} \approx 1$ and $\hat{\beta}$ is essentially the usual regeneration parameter. At time $t > t'_R$ the wave function of the K meson system is given, in pole approximation, by

$$\begin{aligned} P\,\varphi(t) &\cong \left(g_S\,e^{-iz_S\,(t-t'_R)} + g_L\,e^{-iz_L\,(t-t'_R)}\right)(\hat{\alpha}\,K_L + \hat{\beta}\,K_S) \\ &\cong \varrho_S\,K_S\,e^{-iz_S\,(t-t'_R)} + \varrho_L\,K_L\,e^{-iz_L\,(t-t'_R)}\,, \end{aligned} \tag{A.9}$$

where

$$\varrho_S = \hat{\alpha}\, a_{SL} + \hat{\beta}\, a_{SS}\,, \qquad \varrho_L = \hat{\alpha}\, a_{LL} + \hat{\beta}\, a_{LS} \tag{A.10}$$

and the overlap coefficients $a_{pp'}$ are defined by

$$g_p\, K_{p'} = a_{pp'}\, K_p\,. \tag{A.11}$$

In the generalized Wigner-Weisskopf formulation, $a_{SL} = a_{LS} = 0$ and $a_{SS} = a_{LL} = 1$, so that $\varrho_S = \hat{\beta}$ and $\varrho_L \approx 1$. If, as suggested in Section V, a_{SL} is of the order of the CP violation (amplitude), then the first term of ϱ_S in (A.10) could make a significant contribution. The quantity ϱ_L may also differ from unity, since the pole residues (see (2.5)) have the form

$$g_p = \frac{Q_p(z_p)}{1 - w'_p(z_p)}\,, \tag{A.12}$$

if the zero of (2.8) is simple. It is possible (see (4.11) and (2.4)) that $w'_p(z_p)$ is 0(1), and hence $a_{pp} = [1 - w'_p(z_p)]^{-1} \neq 1$. In this case, the resonances described by (6.10) will reflect 'inelasticity' (e.g. (6.13)).

Proceeding in the usual way [14], we find that

$$\frac{dN_{+-}}{dt} \cong \Gamma_{S,+-} \Big\{ |\varrho_L|^2\, |\eta_{+-}|^2\, e^{-\Gamma_L (t - t'_R)} + |\varrho_S|^2\, e^{-\Gamma_S (t - t'_R)}$$
$$+ 2\, |\varrho_L| \cdot |\varrho_S| \cdot |\eta_{+-}| \cos(\Delta m\, (t - t'_R) - \varphi_{\eta_{+-}} - \varphi_{\varrho_L} + \varphi_{\varrho_S})\, e^{-\Gamma_S/2\, (t - t'_R)} \Big\}, \tag{A.13}$$

where $\Gamma_{S,+-}$ is the rate for $K_S \to \pi^+ \pi^-$. Note that, in case the first term of the expression (A.10) for ϱ_S makes a significant contribution, φ_{ϱ_S} would not be entirely determined by the strong interactions.

Acknowledgments

The point of view we have taken in the algebraic treatment of the solutions of the resolvent equations is due in large part to C. PIRON. In the early stages of this work, we profited from discussions with G. MOHAN, and we are grateful to M. GOURDIN for discussions and very helpful suggestions.

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