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Singular Domains of Space

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Abstract. A generalization of a definition of cosmological singularity is proposed, which allows to formulate singularity theorems so that they refer only to a finite domain of space-time. In this way, two theorems due to Hawking are sharpened, by means of what it can be shown that our Universe cannot be singularity free, unless the causal loops violating the strong causality required by Hawking entirely lie in an explicitly indicated compact region of our past.

1. Introduction

In the papers [1–6] on singularities of the solutions of Einstein's equations, a number of criteria have been found for a space-time to be null and time-like geodesically incomplete (i.e. there exists at least one time-like or null geodesic which cannot be extended to arbitrary length within the space-time). The incompleteness implies a singularity (of a certain kind [7]) only under some additional conditions. Usually, the so-called Postulate of Inextendability of the space-time is mentioned [5]. There are ways, in principle, to define the inextendability, to establish that a given space-time is extendable, and to extend it [8]. But the calculations cannot be carried out except in cases of highly symmetric spaces. Inextendability is, therefore, difficult enough to check mathematically. Moreover, the data we are able to measure always refer to a limited domain of the Universes; so, if we wish to draw experimentally meaningful conclusions, we must use the properties of a finite, and accordingly extendable, space-time patch only. The following is an attempt in this line.

2. Notation

The following symbole and conventions will be used: $\overline{N}^{\mathfrak{M}}$ is a closure of the set N in the space $\mathfrak{M}$. $\gg$, $>$, $<$, $\langle p, q\rangle$ etc. characterize the chronologic and causal relations between points and sets as they have been introduced in [9], see also [5].

$T\,\mathfrak{M}_p$ is the tangent space at p to the manifold $\mathfrak{M}$.

Lorentz metric g on a manifold $\mathfrak{M}$ is a symmetric bilinear form of the signature $+---$, and of class at least C^2 given in tangent space of every point of $\mathfrak{M}$. Covariant differentiation compatible with the metric is indicated by a semi-colon. For the corresponding Riemann and Ricci tensors, the equations

$$v_{a;bc} = v_{a;cb} + R^p_{abc}\, v_p\,, \qquad R_{ab} = R^c_{abc}$$

hold, where v_a is a vector field of class C^2.

3. Singular Domain of Space

Definition 1. A four-dimensional connected differentiable manifold of class at least C^3 with a Lorentz metric g is called domain of space, if for each $p \in \mathcal{M}$ and each time-like vector $v^a \in T\mathcal{M}_p$, the following inequality is satisfied

$$R_{ij}\, v^i\, v^j \geqslant 0 \,.$$

Definition 2. Two domains of space $\mathcal{M}$ and $\mathcal{M}'$ are equivalent, if there exists a homeomorphism ϕ of $\mathcal{M}$ on $\mathcal{M}'$ which is an isometry with respect to g and g'.

Definition 3. Extension of a domain of space $\mathcal{M}$ is a domain of space $\mathcal{M}'$ such that there is an open subset of $\mathcal{M}'$ which is equivalent to $\mathcal{M}$.

Definition 4. Given any set $\mathcal{N} \subset \mathcal{M}$ which has a property in $\mathcal{M}$, say, $P^{\mathcal{M}}$, then, if $\mathcal{N}$ has the property $P^{\mathcal{M}'}$ in every extension $\mathcal{M}'$ of $\mathcal{M}$, $P^{\mathcal{M}}$ is called extendable.

Examples:

1. Every local property is extendable.
2. The property of being compact in $\mathcal{M}$ is extendable.
3. The property of being a Cauchy surface in $\mathcal{M}$ (for the definition see e.g. [3]) is not, in general, extendable: Define the submanifold L of Minkovski space with coordinates t, x, y, z and metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

by inequalities

$$t > 0,\ t^2 - x^2 - y^2 - z^2 > 0 \,.$$

The hypersurface given by

$$t^2 - x^2 - y^2 - z^2 = \text{const}$$

is a Cauchy surface in L, but it is a partial Cauchy surface only in Minkovski space. (The domain of space L satisfies all conditions of Hawking's theorem 1 in [3], is obviously not null and time-like geodesically complete, but it is regular.)

4. The property of a point $p \in \mathcal{M}$ that every causal curve in $\mathcal{M}$ passes through p not more than once is called $\mathcal{M}$-causality at p. $\mathcal{M}$-causality at a point p is not, in general, extendable.
5. The property of a point p that every neighbourhood of p open in $\mathcal{M}$ contains an open neighbourhood which no causal curve in $\mathcal{M}$ passes more through than once is called $\mathcal{M}$-strong causality at p. $\mathcal{M}$-strong causality is not extendable.

Definition 5. A domain of space $\mathcal{M}$ is singular, if it has no null or time-like geodesically (or bounded acceleration) complete extension.

There are domains of space having both singular and regular extension. As an instance, we take the submanifold $\mathcal{V}$ of Minkovski space defined by the inequalities

$$t > 0\,,\ -k < \frac{x}{t} < k\,,\ k < 1\,, \qquad -\infty < y < \infty\,,\quad -\infty < z < \infty\,.$$

The Minkovski space is one of its regular extensions. Its singular extension $\mathcal{V}'$ is obtained by adding the set to $\mathcal{V}$ whose elements are the pairs of points $(t, -k\,t, y, z)$, $(t, k\,t, y, z)$, where $0 < t$, $-\infty < y, z < \infty$. The segments $t = \text{const}$, $y = \text{const}$,

$z = \text{const}$, $-k\,t \leqq x \leqq k\,t$ are then closed curves. The following mapping of $\mathcal{V}'$ into the five-dimensional space $\mathcal{A}$ of coordinates T, X, Y, Z, U and metric

$$dS^2 = dT^2 - dX^2 - dY^2 - dZ^2 - dU^2$$

given by

$$T = \frac{v}{\sqrt{1-v^2}} \sqrt{t^2 - x^2}\,,$$

$$X = \frac{v}{\sqrt{1-v^2}} \sqrt{t^2 - x^2} \cos\left(\frac{\sqrt{1-v^2}}{2\,v} \lg \frac{t+x}{t-x}\right),$$

$$Y = \frac{v}{\sqrt{1-v^2}} \sqrt{t^2 - x^2} \sin\left(\frac{\sqrt{1-v^2}}{2\,v} \lg \frac{t+x}{t-x}\right),$$

$$Z = y\,, \quad U = z\,,$$

where

$$v = \frac{\operatorname{arctangh} k}{\sqrt{\pi^2 + \operatorname{arctangh}^2 k}}\,,$$

has, as can be proved, the following properties:

a) it is one-to-one,

b) it is continuous on $\mathcal{V}$, and has a continuous inverse there,

c) it is an isometry on $\mathcal{V}$, if the metric on the image of $\mathcal{V}$ is induced by that of $\mathcal{A}$,

d) the image of $\mathcal{V}'$ is the hypersurface K in $\mathcal{A}$ given by the relations:

$$v^2\,T^2 - X^2 - Y^2 = 0\,, \quad T > 0\,.$$

The properties b) and c) may be, per definition, extended on the set $\mathcal{V}'$, which is then topologically and metrically equivalent to K, i.e. $\mathcal{V}'$ is an extension of $\mathcal{V}$ and is clearly singular.

Theorem 1. A domain of space $\mathcal{M}$ is singular, if it fulfils the conditions:

1) There is a compact, space-like, three-dimensional, imbedded submanifold $\mathcal{H}$.

2) The contraction of the second fundamental form of $\mathcal{H}$ is either everywhere positive or everywhere negative. (Cf. [5], Theorem 1.)

Proof. $\mathcal{H}$ is a compact, three-dimensional, imbedded submanifold of $\mathcal{M}$, if there exists a compact, three-dimensional manifold, $\mathcal{H}_0$, and a mapping ϕ: $\mathcal{H}_0 \to \mathcal{H}$ with the properties:

a) $\phi(\mathcal{H}_0) = \mathcal{H}$,

b) ϕ is of rank 3 everywhere on $\mathcal{H}_0$,

c) if the topology of $\mathcal{H}$ is that induced by the topology of $\mathcal{M}$, then ϕ is a homeomorphism $\mathcal{H}_0$ on $\mathcal{H}$. (Cf. [11], p. 42.)

Let $\mathcal{M}'$ be any extension of $\mathcal{M}$. Call $i: \mathcal{M} \to \mathcal{M}'$ the identity injection of $\mathcal{M}$ into $\mathcal{M}'$. i is a homeomorphism of $\mathcal{M}$ onto $i(\mathcal{M})$, where the topology of $i(\mathcal{M})$ is induced by that of $\mathcal{M}'$. This implies that the manifold $\mathcal{H}_0$ and the mapping $i_0\,\phi$ satisfy conditions a), b) and c) in $\mathcal{M}'$. The property of being a three-dimensional imbedded submanifold of $\mathcal{M}$ is, therefore, extendable.

To be space-like and to have a second fundamental form S_{ij} with $S^i_i > 0$ or $S^i_i < 0$, are both extendable properties (they are local).

Now, it is clear that the set $\mathcal{H}$ fulfils both conditions of the Theorem 1 in every extension of $\mathcal{M}$. Hence, every extension of $\mathcal{M}$ has all properties required by Theorem 1 in [5] and cannot, therefore, be null or time-like geodesically complete. Thus, $\mathcal{M}$ is singular.

Theorem 2. There is no domain of space $\mathcal{M}$ with the following properties:

1) A point $p \in \mathcal{M}$, a past-directed time-like unit vector $\omega^a \in T\,\mathcal{M}_p$, and a positive constant b exist such that on every past-directed time-like geodesic through p either $\theta = v^a_{;a}$ becomes less than $-3\,c\,b^{-1}$ within the distance $b\,c^{-1}$ from p or

$$\int_0^{b/c} R_{ab}\, v^a\, v^b \sin^2 \frac{\pi\, c\, s}{2\, b}\, ds > \frac{3\,\pi^2\, c}{4\, b},$$

where $c = \omega^a\, v_a\, p$ and v_a is the unit tangent vector to the geodesic.

2) Let $\mathcal{N} \subset T\,\mathcal{M}_p$ be defined in ortho-normal co-ordinates x^0, x^1, x^2, x^3, where $\omega^0 = 1$, $\omega^1 = \omega^2 = \omega^3 = 0$ by

$$x^0 \leqq 0\,,\ (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \geqq 0\,,$$
$$(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 < 2\, b\, \sqrt{2}\,.$$

If the mapping $\exp: \{w^a\} \to \mathcal{M}$, $w^a \in T\,\mathcal{M}_p$, is well-defined everywhere on $\mathcal{N}$, then the $\mathcal{M}$-strong causality holds on $\overline{(p\rangle}^{\mathcal{M}}$. (Cf. [5], Theorem 2.)

Proof. Let $\exp \mathcal{N} \subset \mathcal{M}$; as exp is continuous and $\mathcal{N}$ is compact, $\exp \mathcal{N}$ will be compact. Raychaudhuri's expansion equation [12] together with condition 1) imply that, on every past-directed time-like geodesic through p, there is a conjugate point within $\exp \mathcal{N}$ [3]. Then, the past-directed time-like geodesic joining p with any point outside of $\exp \mathcal{N}$ cannot be of locally extremal length [3].

Seyfert's theorem [13] states that the geodesic joining two given points p and q is extremal, if $\langle q, p\rangle^{\mathcal{M}}$ is non-empty, compact, and $\mathcal{M}$-strong causality holds in all its points. Let $\mathcal{C}$ be the set of all points q with the property that $\langle q, p\rangle^{\mathcal{M}}$ satisfies all three conditions of Seyfert's theorem and $q \ll p$. Then $\mathcal{C} \subset \exp \mathcal{N}$. As $\mathcal{M}$ is Hausdorf, the compact sets are closed, so that $\overline{\mathcal{C}}^{\mathcal{M}} \subset \exp \mathcal{N}$ and is compact.

At each point of $\overline{\mathcal{C}}^{\mathcal{M}}$, the $\mathcal{M}$-strong causality holds, because $\mathcal{C} \subset (p\rangle^{\mathcal{M}}$. Thus, each point of $\overline{\mathcal{C}}^{\mathcal{M}}$ has a local causality neighbourhood whose closure is compact [5]. The neighbourhoods yield an open covering of $\overline{\mathcal{C}}^{\mathcal{M}}$ with a finite subcovering, say, $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_n$.

Let $p \in \mathcal{U}_0$. Choose an arbitrary point $q \in \mathcal{U}_0 \cap (p\rangle\rangle^{\mathcal{M}}$. Every causal curve in $\mathcal{M}$ from q to p must lie in $\mathcal{U}_0$, otherwise $\mathcal{U}_0$ would not be a local causality neighbourhood. This implies $\overline{\langle q, p\rangle}^{\mathcal{M}} \in \overline{\mathcal{U}}_0^{\mathcal{M}}$, so that $\overline{\langle q, p\rangle}^{\mathcal{M}}$ is compact and thus $\mathcal{U}_0 \cap (p\rangle\rangle^{\mathcal{M}} \subset \mathcal{C}$. Then, since $(p\rangle^{\mathcal{M}} \subset \overline{(p\rangle\rangle}^{\mathcal{M}}$, we have $\mathcal{U}_0 \cap (p\rangle^{\mathcal{M}} \subset \mathcal{U}_0 \cap \overline{(p\rangle\rangle}^{\mathcal{M}} \subset \overline{\mathcal{U}_0 \cap (p\rangle\rangle}^{\mathcal{M}}$ so that

$$\mathcal{U}_0 \cap (p\rangle^{\mathcal{M}} \subset \overline{\mathcal{C}}^{\mathcal{M}}\,. \tag{1}$$

Now, choose a piecewise smooth past-directed causal curve $\gamma_1(t)$ through p, $\gamma_1(0) = p$. Because of (1), there is $t_1 < 0$ such that $\gamma_1(t) \in \overline{\mathcal{C}}^{\mathcal{M}}$ for all t satisfying $t_1 \leqq t \leqq 0$. The set of all such t_1 has a finite negative minimum, T_1, say, since $\overline{\mathcal{C}}^{\mathcal{M}}$ is compact and

$\mathcal{M}$-strong causality holds on it. Let $\gamma_1(T_1) \in \mathcal{U}_1$. $\mathcal{U}_1$ is open; there must be, therefore, $t < T_1$ such that $p_1 = \gamma_1(t) \in \mathcal{U}_1$, $p_1 \notin \overline{C}^{\mathcal{M}}$. All causal curves in $\mathcal{M}$ from p_1 to p cannot lie in $\bigcup_{i=0}^{n} \mathcal{U}_i$, because then $\langle p_1, p\rangle^{\mathcal{M}}$ should be compact and $p_1 \in \overline{C}^{\mathcal{M}}$. Let $\gamma_2(t), \gamma_2(0) = p, \gamma_2(-1) = p_1$, denote a piecewise smooth causal curve in $\mathcal{M}$ not lying entirely in $\bigcup_{i=0}^{n} \mathcal{U}_i$. On γ_2, we can perform a construction similar to that on γ_1 and find a point $p_2 \in \mathcal{M}$, $p_2 \in (p\rangle$, $p_2 \notin \overline{C}^{\mathcal{M}}$, $p_2 \in \mathcal{U}_2$, $(\mathcal{U}_2 \neq \mathcal{U}_1)$. Then, there is a curve from p_2 to p not entirely lying in $\bigcup_{i=0}^{n} \mathcal{U}_i$ etc. In this way, a causal curve γ in $\mathcal{M}$ through points $p_1, p_2, \dots$ is constructed so that, between p_i and p_{i+1}, it is identical to γ_i and lies, therefore, in $(p\rangle^{\mathcal{M}}$. The succession $\{p_i\}$ cannot be finite, but, for every i, p_i lies in $\mathcal{U}_i$ and $\{\mathcal{U}_i\}$ is finite. We have a contradiction, which proves the theorem.

4. Comment

By comparing HAWKING's Theorem 2 in [5], which has been used in [14] to show that the Universe contains a singularity with our Theorem 2, we see that they both do not exclude that a breakdown of strong causality might save the Universe from the singularity. However, whereas the former restricts in no way the kind of the breakdown, the latter requires the closed causal curves (or partially closed causal curves) to lie entirely in the compact region exp of our past. Thus, HAWKING's Theorem would let open a physically interesting possibility, namely, that there might be a great unknown portion of the Universe over which some causal curve could be closed (what, by no means, would hurt the causality of our experience. Unfortunately, it can not be the case, as we have proved.

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