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Causality in S-Matrix Theory, II

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Abstract. Two scattering processes are discussed for which there exist points in the physical region of the mass shell at which the analytic S-matrix cannot be represented as the boundary value of a single analytic function. At such points the S-matrix must instead be represented as a sum of at least two such boundary value terms.

I. Introduction

A basic assumption of analytic S-matrix theory is that, apart from energy-momentum conservation delta functions, the connected parts of the momentum space S-matrix are boundary values of functions holomorphic in the complex mass shell [1]. The question studied here is whether a *single* boundary value term suffices for each scattering function T_c at each point of its physical region.

The answer to this question is no, at least within the framework of an earlier paper [2] in which the holomorphy assumption was justified on the basis of a causality requirement. In that paper the question was formulated in the following way. Consider a scattering process involving a total of n initial and final particles, and let $\overline{K} = (\overline{k}_1, \dots, \overline{k}_n)$ be the set of mathematical energy-momentum vectors of the particles. The vectors $\overline{k}_i$ are related to the physical energy-momentum vectors $\overline{p}_j$ of the particles by $\overline{k}_i = \sigma_j \, \overline{p}_j$, where

$$\begin{aligned} \sigma_j &= +1 \text{ for final particles,} \\ &= -1 \text{ for initial particles.} \end{aligned} \tag{1.1}$$

The point $\overline{K}$ belongs to the n-particle mass shell

$$\mathcal{M}_n = \{K \mid K = (k_1, \dots, k_n),\ k_j^2 = (k_{j0})^2 - \boldsymbol{k}^2 = m_j^2,\ \sigma_j \, k_{j0} > 0,\ \textstyle\sum k_j = 0\}\,, \tag{1.2}$$

where the m_j are the particle[1]). Suppose, in addition, that $\overline{K}$ does not belong to $\mathcal{M}_{n,0}$, the set of points of $\mathcal{M}_n$ at which at least two initial particle energy-momentum vectors are parallel or at least two final particle energy-momentum vectors are parallel[2]). Then only a finite number of different leading positive-α Landau surfaces L_g^+, where g belongs to a finite index set I, can pass through $\overline{K}$ and each has a local representation

$$L_g^+ = \{K \mid K \in \mathcal{M}_n,\ \Lambda_g(K) = 0\}\,. \tag{1.3}$$

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[1]) Spin and other quantum numbers are not important here and are suppressed.

[2]) Points of $\mathcal{M}_{n,0}$ are excluded because of the well known difficulty that at such points the particles with parallel energy-momentum vectors interact over an infinite time span.

The functions Λ_g are real analytic functions in some 4 n-dimensional neighborhood of $\overline{K}$, and the gradients

$$\nabla \Lambda_g(K) = (\gamma_1, \dots, \gamma_n) , \tag{1.4}$$

where

$$\gamma_j^\mu = \left(\frac{\partial \Lambda_g}{\partial k_{j\mu}}\right) (\overline{K}) , \tag{1.5}$$

are nonzero. In addition, the gradients $\nabla \Lambda_g$ are not of the form

$$U_0(\overline{K}) = (a + t_1 \overline{k}_1, \dots, a + t_n \overline{k}_n) \tag{1.6}$$

where a is any four-vector and the t_i are real. The problem is then to find a point $\overline{K}$ through which pass surfaces L_g^+ that can be labeled by two distinct index sets I_1 and I_2. The Landau diagrams D_g corresponding to the surfaces L_g^+, $g \in I_1$, are all contractions of a single diagram $\overline{D}_1$, and those corresponding to L_g^+, $g \in I_2$, are all contractions of $\overline{D}_2$. If at such a point a relation[3])

$$\sum_{I_1} \lambda_g \nabla \Lambda_g(\overline{K}) = - \sum_{I_2} \lambda_g \nabla \Lambda_g(\overline{K}) + U_0(K) \tag{1.7}$$

is possible with non-negative λ_g (but some nonzero λ_g on each side), then at least two boundary-value terms are needed to represent T_c at $\overline{K}$.

In Section II an example of such a point is given. This point is an example of the type I points of Ref. [2]. Then in Section III an example is given of a point with the additional property that the two index sets I_1 and I_2 are not disjoint. This is an example of a type II point.

No new fundamental problems appear to arise from the existence of these points. They pose no obstacle to holomorphic continuation of scattering functions, for example. This is because the union of all such points is a subset $\mathfrak{L}_1^+$ of relative measure zero of $\mathfrak{L}^+$, the union of all positive-α Landau surfaces [2]. Paths of continuation can simply go around $\mathfrak{L}_1^+$. Even in general arguments involving arbitrary paths of continuation only a small amount of additional algebra is needed to allow for points in $\mathfrak{L}_1^+$ [3]. This paper merely confirms that this extra effort is more than just a precaution and is actually necessary.

II. Type I Point

The first example involves the elastic scattering of four particles, two of mass M and two of mass m, $M > m$. The mathematical energy-momentum vectors of the particles are given by

$$\overline{k}_j = (\overline{k}_{j0}, \overline{\boldsymbol{k}}_j) = \sigma_j\, m_j (\cosh \theta_j, \sinh \theta_j\, \hat{e}) , \tag{2.1}$$

where the σ_i are definied by (1.1) and $\hat{e}$ is an arbitrary (3-dimensional) unit vector. The initial particles are specified by $1 \leqslant j \leqslant 4$ and the final particles by $5 \leqslant j \leqslant 8$. The indices are chosen so that

$$M = m_1 = m_2 = m_5 = m_6 , \qquad m = m_3 = m_4 = m_7 = m_8 . \tag{2.2}$$

[3]) The symbol $U_0(\overline{K})$ always means a quantity of the form (1.6).

Finally the angles θ_j are chosen so that

$$\theta_j = -\theta_{j+1}\,,\ j \text{ odd}\,, \tag{2.3}$$

$$\theta_1 > \theta_5\,, \tag{2.4}$$

$$\theta_3 = -\frac{1}{2}(\theta_1 - \theta_5) + \sinh^{-1}\left[(M/m)\sinh\frac{1}{2}(\theta_1 + \theta_5)\right], \tag{2.5}$$

$$\theta_7 = \frac{1}{2}(\theta_1 - \theta_5) + \sinh^{-1}\left[(M/m)\sinh\frac{1}{2}(\theta_1 + \theta_5)\right]. \tag{2.6}$$

It is easily verified that the point $\overline{K}$ so defined belongs to $\mathfrak{M}_8$. The conditions $\bar{k}_j^2 = m_j^2$ and $\sigma_j\,\bar{k}_{j0} > 0$ are trivial consequences of (2.1). Momentum conservation is a trivial consequence of (2.3), and energy conservation,

$$0 = \Sigma\,\sigma_j\,m_j\cosh\theta_j = 2\,M\,(\cosh\theta_5 - \cosh\theta_1) + 2\,m\,(\cosh\theta_7 - \cosh\theta_3) \tag{2.7}$$

follows immediately from (2.5) and (2.6).

It is also easy to show that θ_1 and θ_5 can be chosen so that $\overline{K}$ does not belong to $\mathfrak{M}_{8,0}$. The condition that any two of the vectors (2.1), say $\bar{k}_i$ and $\bar{k}_j$, $i \neq j$, are collinear is that $\theta_i = \theta_j$. Enumeration of the various possibilities shows that $\theta_1 = \theta_3$ is the only possible equality under the conditions (2.4) and $M > m$. Even this equality is impossible in a small neighborhood of $\theta_1 = \theta_5$, as can be seen from (2.5). Thus, for θ_1–θ_5 sufficiently small, the point $\overline{K}$ not only does not lie on $\mathfrak{M}_{8,0}$ but in fact has the property that *no* two of the vectors $\bar{k}_j$, initial *or* final, are collinear[4]).

The important property of the point $\overline{K}$ is that it lies on the intersection

$$J = \bigcap_{g=1}^{4} L_g^+ \tag{2.8}$$

of the leading positive-α Landau surfaces L_g^+, $1 \leqslant g \leqslant 4$, of the triangle diagrams of Figures 1 and 2. To show, for example, that $\overline{K}$ belongs to L_1^+, one must first compute

$$\begin{aligned} x_1 &= (2\,M^2)^{-1}\,[(\bar{k}_1 + \bar{k}_2 + \bar{k}_4 + \bar{k}_7)^2 - 2\,M^2] = \cosh(\theta_1 - \theta_5)\,,\\ x_2 &= (2\,M^2)^{-1}\,[(\bar{k}_3 + \bar{k}_8)^2 - 2\,M^2] = -\cosh(\theta_1 + \theta_5)\,,\\ x_3 &= (2\,M^2)^{-1}\,[(\bar{k}_5 + \bar{k}_6)^2 - 2\,M^2] = \cosh 2\theta_5\,. \end{aligned} \tag{2.9}$$

Direct substitution of (2.9) into the Landau equation

$$\Lambda_1 = 1 - x_1^2 - x_2^2 - x_3^2 - 2\,x_1\,x_2\,x_3 = 0 \tag{2.10}$$

shows that $\overline{K}$ belongs to the (general-α) Landau surfaces L_1 [4]. Equation (2.4) insures that

$$1 < x_1,\ x_3 < -x_2\,, \tag{2.11}$$

so that $\overline{K}$ is in fact on L_1^+ (see Fig. 3). The proof that $\overline{K}$ lies on each of the other surfaces L_g^+ is similar.

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[4]) Thus, the point $\overline{K}$ does not lie on the boundary of the physical region in the space of Lorentz invariant variables.

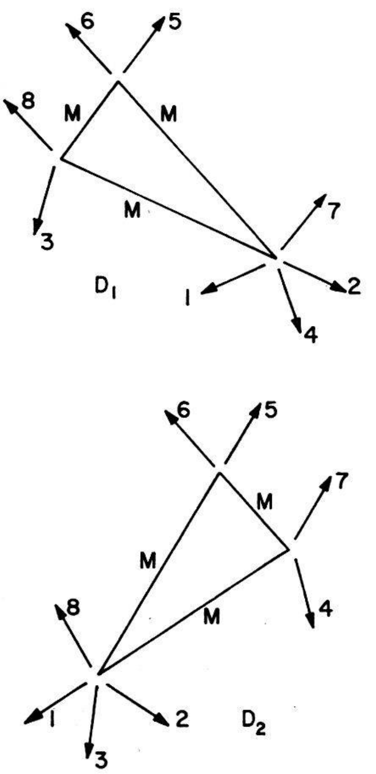

Figure 1

Triangle diagrams D_1 and D_2 with heavy intermediate particles.

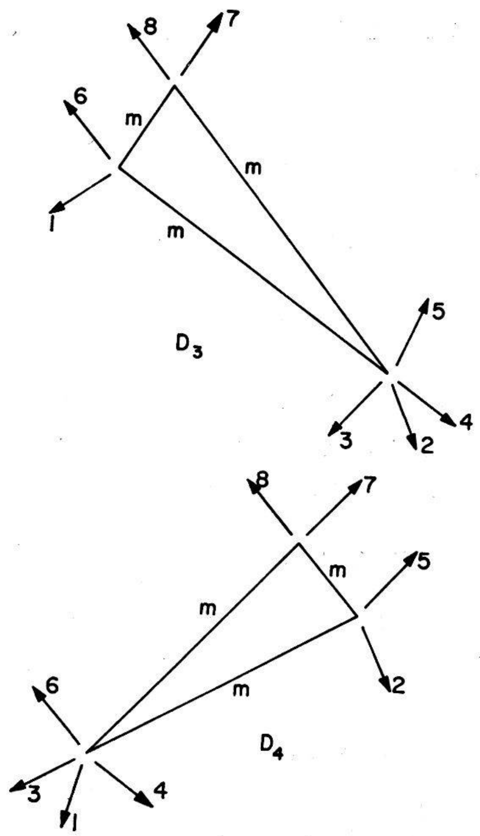

Figure 2

Triangle diagrams D_3 and D_4 with light intermediate particles.

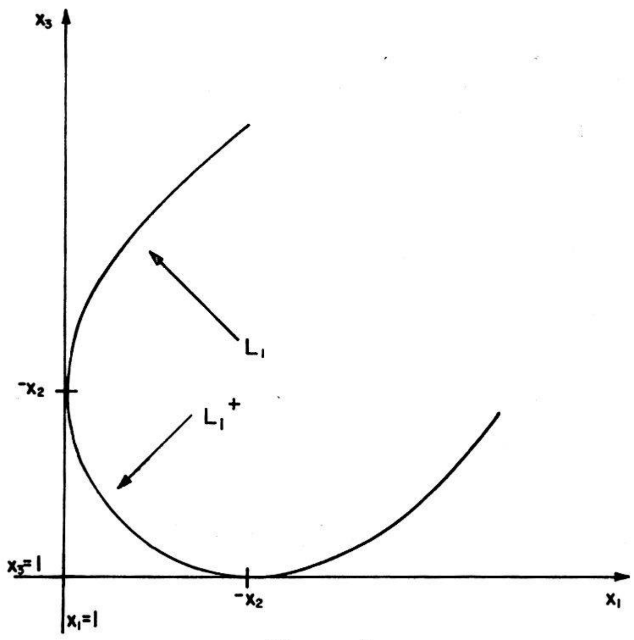

Figure 3

Real section of the Landau surface for D_1.

Consider now the quantity

$$U = \nabla A_1(\overline{K}) + \nabla A_2(\overline{K}) = (u_1, \ldots, u_8) . \tag{2.12}$$

The values of the vectors u_j are computed by substitution of (2.1) into the definitions (1.4) and (1.5):

$$\begin{aligned} u_1 &= u_2 = u , \\ u_3 &= u_8 = u - \alpha \sinh\theta_5 \bar{k}_2 , \\ u_4 &= u_7 = u - \alpha \sinh\theta_5 \bar{k}_1 , \\ u_5 &= u_6 = u - \alpha \sinh\theta_5 k_1 + \alpha \sinh\theta_1 \bar{k}_6 , \\ \alpha &= 8 M^{-2} \cosh^2\theta_5 [\sinh\theta_1 + \sinh\theta_5] . \end{aligned} \tag{2.13}$$

The value of u is unimportant.

Consider next the quantity

$$V = \nabla A_3(\overline{K}) + \nabla A_4(\overline{K}) = (v_1, \ldots, v_8) . \tag{2.14}$$

The values of the vectors v_j are also computed by substitution of (2.1) into (1.4) and (1.5):

$$\begin{aligned} v_1 &= v_6 = v - \beta \sinh\theta_7 \bar{k}_4 , \\ v_2 &= v_5 = v - \beta \sinh\theta_7 \bar{k}_3 , \\ v_3 &= v_4 = v , \\ v_7 &= v_8 = v - \beta \sinh\theta_7 \bar{k}_3 + \beta \sinh\theta_3 \bar{k}_8 , \\ \beta &= 8 m^{-2} \cosh^2\theta_7 [\sinh\theta_3 + \sinh\theta_7] . \end{aligned} \tag{2.15}$$

The value of v is of no consequence.

The remarkable thing about these two quantities is that they satisfy

$$U = -\lambda V + U_0(\overline{K}) , \tag{2.16}$$

where

$$\lambda = [\alpha M \sinh\theta_1 \sinh\theta_5] [\beta m \sinh\theta_3 \sinh\theta_7]^{-1} . \tag{2.17}$$

The quantity $U_0(\overline{K})$ is of the form (1.6) with

$$\begin{aligned} a &= u + \lambda v - t_1 \bar{k}_1 - t_3 \bar{k}_3 , \\ t_1 &= t_2 = \alpha \sinh\theta_5 , \\ t_3 &= t_4 = \lambda \beta \sinh\theta_7 , \\ t_5 &= t_6 = \alpha \sinh\theta_1 , \\ t_7 &= t_8 = \lambda \beta \sinh\theta_3 . \end{aligned} \tag{2.18}$$

Equation (2.16) is, of course, just (1.7) rewritten. It follows that the scattering function T_c for this process must be the sum of at least two boundary value terms. A similar conclusion must hold for a continuum of nearby points K. This follows from the fact that the properties of $\overline{K}$ do not depend on the exact values of θ_1, θ_5 and $\hat{e}$, but only on the provision that θ_1–θ_5 be small enough. Hence the continuum has dimension of at least four.

Finally, the point $\bar{K}$ defined by (2.1) is a type I point. This means that one of the boundary value terms in the representation of T_c at $\bar{K}$ is singular only on $L_1^+ \cup L_2^+$, and the other is singular only on $L_3^+ \cup L_4^+$. Such a canonical choice of boundary value terms is unknown for type II points, and it is of interest to know if such points exist outside of the set $\mathfrak{M}_{n,0}$[5]).

III. Type II Point

There is a type II point for the elastic scattering of seven particles. Let eight of the fourteen initial and final particles have energy-momentum vectors given by (2.1), and let the remaining energy-momentum vectors $(\bar{k}_9, \dots, \bar{k}_{14})$ be *any* point on the positive-α surface of a triangle diagram. For simplicity assume that the additional particles all have mass M. The initial particles are labeled $9 \leqslant j \leqslant 11$, and the final particles $12 \leqslant j \leqslant 14$. The additional energy-momentum vectors are, of course, chosen so that no two of the vectors $\bar{k}_i$, $1 \leqslant j \leqslant 14$, are collinear.

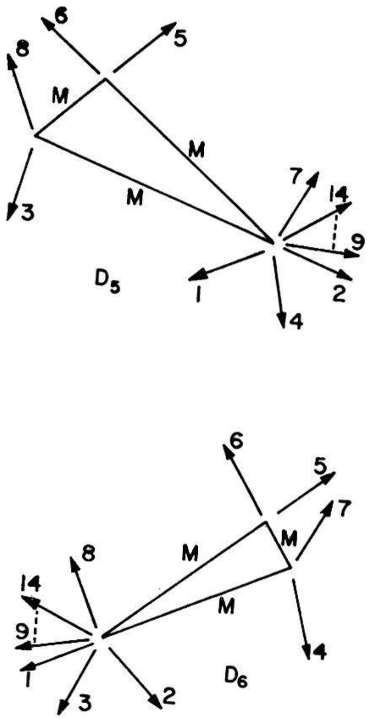

Figure 4

Triangle diagrams D_5 and D_6 with heavy intermediate particles.

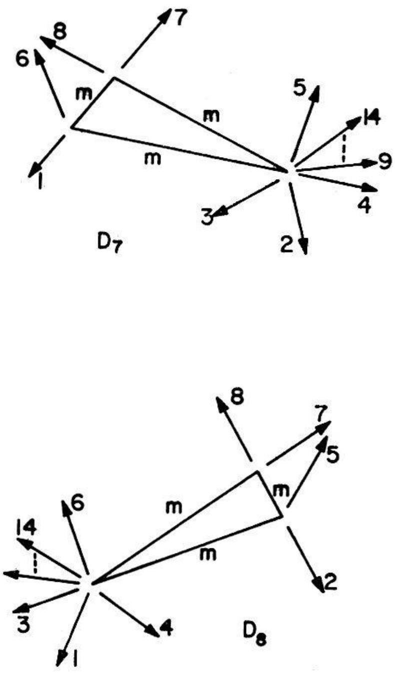

Figure 5

Triangle diagrams D_7 and D_8 with light intermediate particles.

The point $\bar{K}$ so defined lies on the intersection

$$J' = \bigcap_{g=5}^{9} L_g^+ \tag{3.1}$$

[5]) It was shown in Reference 2 that all points of $\mathfrak{M}_{n,0}$ are of type II.

of the positive-α Landau surfaces L_g^+ of the diagrams of Figures 4–6. Inspection of the diagrams shows that the two index sets I_1 and I_2 can be chosen to be

$$I_1 = \{5, 6, 9\} , \qquad I_2 = \{7, 8, 9\} . \tag{3.2}$$

The conclusion that

$$U = \nabla A_5(\overline{K}) + \nabla A_6(\overline{K}) , \qquad V = \nabla A_7(\overline{K}) + \nabla A_8(\overline{K}) \tag{3.3}$$

satisfy (2.16) is valid just as before. The additional quirk is that I_1 and I_2 overlap. This means that the point is of type II and that, unlike the situation of Section II, there is no canonical way to choose the boundary value terms for the representation of the scattering amplitude T_c.

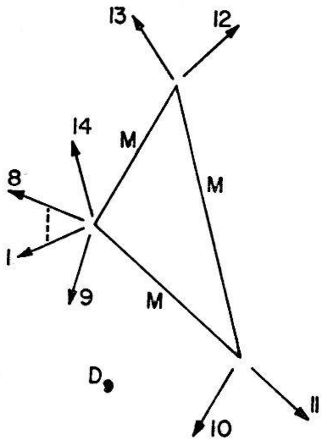

Figure 6

Triangle diagram D_9 with heavy intermediate particles.

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[1] See, for example, G. F. Chew, *The Analytic S-Matrix* (W. A. Benjamin, Inc., New York 1966).

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