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N/D Equations with Marginally Singular Kernels[1])[2])

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Summary. The singular N/D equations which arise in the case of partial-wave dispersion relations with an asymptotically constant left-hand discontinuity $\phi(z)$ are investigated. It is proved that the resolvent (considered as an analytic function of the 'coupling constant' $\lambda = \lim_{z\to\infty} \phi(z)$) has a non polar singularity in the λ-plane. The location of the singular point is controlled by the rate of inelasticity at infinite energy. This singularity gives rise to a multiplicity of solutions.

I. Introduction

When partial-wave dispersion relations are solved by the N/D method, it is generally assumed that the distant part of the left-hand discontinuity does not affect appreciably the scattering amplitude for not too high energies. This 'nearby singularities hypothesis' is supported by the idea that the short range forces are of little weight in the low energy region. Actually, this is known to be true if the asymptotic left-hand discontinuity tends to zero rapidly enough. Then the N/D equations are equivalent to a Fredholm equation with a Hilbert-Schmidt kernel. More precisely, if $T_l(z)$ is a partial-wave amplitude for the elastic scattering of equal mass, scalar, particles ($z = q^2$, the squared center-of-mass momentum) and if its left-hand discontinuity $\phi_l(z)$ satisfies:

$$\phi_l(z) \underset{z\to-\infty}{\simeq} \text{const.}\, |z|^{-\alpha} (\text{Log}\, |z|)^{-\beta} ,$$

the Hilbert-Schmidt condition requires [1]:

$$\alpha > 0 \quad \text{or} \quad \alpha = 0\,, \quad \beta > \frac{1}{2}\,.$$

Then we can compare the N/D solution $T_l(z)$ with the perturbed amplitude $T_{l,Z}(z)$ resulting from the truncated left-hand discontinuity $\phi_l(z)\, \theta\, (z + Z)$. As the cut-off Z goes to infinity, one obtains for $l \geqslant 1$:

$$T_l(z) - T_{l,Z}(z) \underset{Z\to\infty}{=} t(z) \begin{cases} 0\, [Z^{-\alpha} (\text{Log}\, Z)^{-\beta}] & (\alpha > 0) \\ 0\, [(\text{Log}\, Z)^{-\beta+1/2}] & \left(\alpha = 0,\, \beta > \frac{1}{2}\right), \end{cases}$$

where $t(z)$ is bounded over any compact domain of the z-plane whose distance from the left-hand cut is positive.

Entirely new features appear when $\phi_l(z)$ does not decrease asymptotically. This happens for instance if the left-hand discontinuity is approximated by the Born contribution of a vector meson exchange [2, 3]. Already in the so-called 'marginal

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case' ($\phi_l(z) = \lambda\, \Phi_l(z), \lim_{z \to -\infty} \Phi_l(z) = 1$), an incompatibility with the unitarity condition (via the Phragmén-Lindelöf theorem [4]) reflects in some pathology of the N/D solutions. The resolvent of the Fredholm kernel (meromorphic in λ when the Hilbert-Schmidt condition holds) exhibits now non polar singularities in the λ-plane. As shown on solvable (but unrealistic) examples [1], these singularities appear to be generally branch points, producing a multiplicity of inequivalent solutions.

The purpose of this work is to prove a similar result in the general marginal case. This problem has been treated in a different way by Atkinson and Contogouris [5]. Their method is more constructive than ours, but their assumptions are more restrictive.

Our problem and our results are stated precisely in Section II. In Section III some inequalities are derived. They are used in Section IV for the proof of the singular behaviour of the N/D solutions.

II. Statement of the Problem and Results

We restrict ourself to the P-wave elastic scattering of two spinless, equal mass, particles. The higher waves can be treated in a quite analogous way and the results are the same (only the S-wave scattering requires some modifications).

The P-wave amplitude $T(z)$ with a given left-hand discontinuity $\phi(x)$ and a given inelastic factor $R(x)$ has the usual representation (for details, we refer to [6], Section II):

$$\begin{cases} T(z) = z\, \dfrac{N(z)}{D(z)}\,, \\ N(z) = \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{-a} dx\, \frac{\phi(x)\, D(x)}{x\,(x-z)}\,, \\ D(z) = 1 - \dfrac{z}{\pi} \displaystyle\int_0^{\infty} dx\, \varrho(x)\, \frac{N(x)}{x-z}\,. \end{cases}$$

where:

$$\begin{cases} \varrho(x) = \sqrt{\dfrac{x}{x+1}}\, R(x) \\ 0 < \varrho(x) < I \quad \text{for} \quad x > 0\,, \\ \varrho(x) \underset{x\to 0}{\simeq} \sqrt{x}\,. \end{cases} \tag{II.1}$$

This leads to the integral equation for N:

$$N(z) = B(z) + \int_0^{\infty} dz'\, K(z, z')\, N(z') \quad (N = B + K\,N \text{ in operator form})\,, \tag{II.2}$$

where:

$$B(z) = \frac{1}{\pi} \int_a^{\infty} dx\, \frac{\phi(-x)}{x\,(x+z)}\,, \tag{II.3}$$

$$K(z, z') = \frac{\varrho(z')}{\pi^2} \int_a^{\infty} dx\, \frac{\phi(-x)}{(x+z)\,(x+z')}\,. \tag{II.4}$$

As we consider the marginal case:

$$\lim_{x\to\infty} \phi(-x) = \lambda . \tag{II.5}$$

We can regard λ as a multiplicative 'coupling constant' of the left-hand discontinuity and extend it onto the complex plane.

Another basic assumption we need is the existence of $\lim_{X\to\infty} R(x)$. For definiteness, we take:

$$R(x) \underset{x\to\infty}{=} R_\infty + 0\left(\frac{1}{x}\right) \qquad (R_\infty > 1) , \tag{II.6}$$

we wish to show that the equation (II.2) has a solution $N(\lambda)$ which is meromorphic in the circle $|\lambda| < 1/R_\infty$, with a non polar singularity at the point $\lambda = 1/R_\infty$ (whatever the rapidity of the convergence $\phi(-x) \to \lambda$ may be).

To do this, we use a 'variable cut-off' technique which consists in splitting the left-hand discontinuity into a 'long range' part Φ_L and a 'short range' part Φ_S:

$$\begin{cases} \phi(-x) = \lambda\, \Phi_L(-x) + \lambda\, \Phi_S(-x) , \\ \lambda\, \Phi_S(-x) = \theta\,(x - Z)\, \phi(-x) . \end{cases} \tag{II.7}$$

The abscissa Z of the cut-off is variable and can be chosen arbitrarily large (generally this variable will be implicit in the notation). From (II.5) and (II.7):

$$\lim_{x\to\infty} \Phi_S(-x) = 1 .$$

Hence, for Z large enough:

$$A' \leqslant \Phi_S(-x) \leqslant A'' \quad (x \in [Z, \infty]) , \tag{II.8}$$

with:

$$\begin{cases} 0 < A' \leqslant 1 , \; A'' \geqslant 1 , \\ \lim_{z\to\infty} A' = \lim_{z\to\infty} A'' = 1 . \end{cases} \tag{II.9}$$

The decomposition (II.7) involves, via equation (II.4), a similar decomposition of the kernel K:

$$K(z, z') = \lambda\, K_L(z, z') + \lambda\, K_S(z, z') , \tag{II.10}$$

with obvious definitions of K_L and K_S.

One deduces easily from (II.1), (II.3–5) and (II.10) that:

$$\begin{cases} B \in \mathfrak{L}^2(0, \infty) , \quad K_L \in \mathfrak{L}^2[(0, \infty) \times (0, \infty)] , \\ K_S \notin \mathfrak{L}^2[(0, \infty) \times (0, \infty)] . \end{cases}$$

Therefore, K is not a Hilbert-Schmidt kernel. Instead of (II.10), we shall use a decomposition where the more singular part K_2 does not contain the factor $\varrho(x)$ (this is possible by (II.6)):

$$K(z, z') = \lambda\, K_1(z, z') + \lambda\, K_2(z, z') ,$$

$$\begin{cases} K_1(z, z') = K_L(z, z') + \left[\dfrac{\varrho(z')}{R_\infty} - 1\right] K_2(z, z') , \\ K_2(z, z') = \dfrac{R_\infty}{\pi^2} \displaystyle\int_Z^\infty dx \, \dfrac{\Phi_S(-x)}{(x+z)\,(x+z')} . \end{cases} \tag{II.11}$$

Now equation (II.2) is equivalent to the set of coupled integral equations:

$$\mathfrak{n} = B + \lambda K_1 N , \qquad \text{(II.12)}$$

$$N = \mathfrak{n} + \lambda K_2 N . \qquad \text{(II.13)}$$

In order to establish the announced singular behaviour of $N(\lambda)$ (and quoted as a theorem in Section IV), we prove that the singular equation (II.13) ($K_2 \notin \mathfrak{L}^2$) has a perturbative solution of the form:

$$N = \mathfrak{n} + R_2(\lambda)\, \mathfrak{n} .$$

The resolvent $R_2(\lambda)$ presents the same singular behaviour. Furthermore, the remaining equation (II.12), now equivalent to:

$$N = [B + R_2(\lambda)\, B] + \lambda\, [K_1 + R_2(\lambda)\, K_1]\, N , \qquad \text{(II.14)}$$

has a Hilbert-Schmidt kernel.

Once a non polar singularity of the solution $N(\lambda)$ (and hence of the amplitude itself) has been exhibited, the next problem is to determine the type of this singularity. Unfortunately, it seems quite hard to do this. Actually, our method does not allow to decide between an essential singularity, a branch point, or a limit point of isolated singularities. Probably more informations are needed, as the answer appears to depend on the asymptotic behaviour of $\phi(-x) - \lambda$. However, assuming that the convergence $\phi(-x) \to \lambda$ is fast enough and using elastic unitarity, Atkinson and Contogouris have been able to show that $\lambda = 1$ is a branch point of $N(\lambda)$, this function being meromorphic anywhere else. As a matter of fact, the analytic continuation of the resolvent $R_2(\lambda)$ outside the circle $|\lambda| = 1/R_\infty$ is an open question in the general case.

Furthermore, the existence of $R_2(\lambda)$ for some $\lambda > 1/R_\infty$ does not imply directly the existence of a solution $N(\lambda)$, as nothing prevents the kernel of equation (II.14) from being singular when $|\lambda| > 1/R_\infty$. Nevertheless, the assumptions of [5] produce essentially the same analyticity properties in λ as a constant left-hand discontinuity $\phi(-x) = \lambda$. In such a case, the continuation of the solution $N(\lambda)$ onto the second sheet of the branch point $\lambda = 1$ is feasible and allows the construction of two independent solutions for $\lambda > 0$. On the other hand, one has, at least in this example, solutions which are not continuations of a solution holomorphic in the neighbourhood of $\lambda = 0$ [3]. It seems that the various solutions may be distinguished according to their asymptotic behaviour in the z-plane. Their physical relevance is still ambiguous.

III. Proof of Some Inequalities

Let us introduce the iterated kernels:

$$K_2^{(n)} = \underbrace{K_2\, K_2 \ldots K_2}_{n} .$$

Our method is based on the following inequalities:

i) $$K_2^{(n)}(z, z') < \left(\frac{A''\, R_\infty}{\sin^2 \pi\, \mu}\right)^{n-1} \frac{c_\mu\, A''\, R_\infty}{\pi^2\, (z+Z)^\mu\, (z'+Z)^{1-\mu}} \qquad (0 < \mu < 1)$$

where

$$c_\mu = \left(\frac{1}{\mu} - 1\right)^{1-2\mu} .$$

ii)
$$K_2^{(n)}(z, z') \geqslant \frac{(A' R_\infty)^n}{\sqrt{(z+nZ)(z'+nZ)}} \frac{1}{\pi} \int_0^\infty \frac{dt}{ch^{2n} \pi t} \cos\left(t \log \frac{z'+nZ}{z+nZ}\right).$$

iii)
$$\frac{1}{Z} \left(\frac{A' R_\infty}{\pi^2}\right)^n G_n < K_2^{(n)}(0, 0) < \frac{1}{Z} \left(\frac{A'' R_\infty}{\pi^2}\right)^n G_n$$

where:

$$G_n = \int_0^\infty \frac{dy_1\, dy_2 \ldots dy_{2n-1}}{(1+y_1)(y_1+y_2+1) \ldots (y_{2n-2}+y_{2n-1}+1)(y_{2n-1}+1)} \qquad (n = 1, 2, \ldots) ,$$

iv)
$$G_n > \sum_{k=1}^{n} G_{k-1} G_{n-k} \qquad (n = 1, 2, \ldots; G_0 = 1) .$$

Proofs

i) From (II.8) and (II.11):

$$K_2(z, z') \leqslant \frac{A'' R_\infty}{\pi^2} \int_0^\infty \frac{dy}{[y+(z+Z)]\,[y+(z'+Z)]} .$$

Using Hölder's inequality [7]:

$$\int_0^\infty \frac{dy}{(y+\xi)(y+\xi')} \leqslant \left[\int_0^\infty \frac{dy}{(y+\xi)^p}\right]^{\frac{1}{p}} \left[\int_0^\infty \frac{dy}{(y+\xi)^q}\right]^{\frac{1}{q}} \quad \left(\xi, \xi' > 0; p, q > 1; \frac{1}{p} + \frac{1}{q} = 1\right) ,$$

one gets ($1/q \to \mu$):

$$K_2(z, z') < \frac{A'' R_\infty c_\mu}{\pi^2 (z+Z)^\mu (z'+Z)^{1-\mu}} .$$

which is the inequality i) for $n = 1$.

Now, supposing that i) is true for some n:

$$K_2^{(n+1)}(z, z') < \frac{R_\infty}{\pi^2} \int_0^\infty dz'' \int_Z^\infty dx \frac{\Phi_S(-x)}{(x+z)(x+z'')} \left(\frac{A'' R_\infty}{\sin^2 \pi \mu}\right)^{n-1} \frac{c_\mu A'' R_\infty}{\pi^2 (z''+Z)^\mu (z'+Z)^{1-\mu}}$$

$$< \frac{(A'' R_\infty)^n c_\mu A'' R_\infty}{\pi^4 \sin^{2n-2} \pi \mu (z'+Z)^{1-\mu}} \int_Z^\infty \frac{dx}{x+z} \int_0^\infty \frac{dz''}{(z''+x)(z''+Z)^\mu} .$$

But:

$$\int_0^\infty \frac{dz''}{(z''+x)(z''+Z)^\mu} = \int_Z^\infty \frac{dy}{y+(x-Z)\, y^\mu} < \int_0^\infty \frac{dy}{[y+(x-Z)]\, y^\mu} = \frac{\pi}{(x-Z)^\mu \sin \pi \mu} .$$

so that:

$$K_2^{(n+1)}(z, z') < \frac{(A'' R_\infty)^n c_\mu A'' R_\infty}{\pi^3 \sin^{2n-1} \pi \mu (z'+Z)^{1-\mu}} \int_Z^\infty \frac{dx}{(x+z)(x-Z)^\mu}$$

$$= \left(\frac{A' R_\infty}{\sin^2 \pi \mu}\right)^n \frac{c_\mu A'' R_\infty}{\pi^2 (z+Z)^\mu (z'+Z)^{1-\mu}} . \qquad \text{q.e.d.}$$

ii) From:

$$K_2(z, z') \geqslant \frac{A' R_\infty}{\pi^2} \int_Z^\infty \frac{dx}{(x+z)(x+z')} ,$$

we deduce:

$$K_2^{(n)}(z, z') \geqslant \left(\frac{A' R_\infty}{\pi^2}\right)^n \int_Z^\infty \frac{dx_1\, dx_2 \ldots dx_{2n-1}}{(z+x_1)(x_1+x_2)\ldots(x_{2n-2}+x_{2n-1})(x_{2n-1}+z')}$$

$$= \frac{1}{Z}\left(\frac{A' R_\infty}{\pi^2}\right)^n F_{2n-1}(y, y'),$$

where:

$$\begin{cases} F_{2n-1}(y, y') = \displaystyle\int_0^\infty \frac{dy_1\, dy_2 \ldots dy_{2n-1}}{(y+y_1+1)(y_1+y_2+1)\ldots(y_{2n-2}+y_{2n-1}+1)(y_{2n-1}+y'+1)} \\ y = \dfrac{z}{Z},\ y' = \dfrac{z'}{Z} \quad \left(y_i = \dfrac{x_i}{Z} - 1\right). \end{cases}$$

In order to reduce $F_{2n-1}(y)$ (we forget the variable y' for a moment), we shall use the obvious recurrence formula:

$$F_{2n-1}(y-1) = \int_0^\infty \frac{dy_1}{y+y_1} F_{2n-2}(y_1)$$

written in the form:

$$F_{2n-1}(y-1) = \int_0^\infty \frac{dt}{t}\,\frac{1}{t+1}\,F_{2n-2}\left(\frac{y}{t}\right). \tag{III.1}$$

The right-hand side of this relation can be factorized by performing a Mellin transformation [8]. The Mellin transform of a function f of the real variable y is the function $\mathcal{M}[f]$ of the complex variable s defined by:

$$\mathcal{M}[f(y)](s) = \int_0^\infty dy\, y^{s-1} f(y).$$

Then the right-hand side of (III.1) appears as the analogue of the Fourier convolution of $1/(y+1)$ and $F_{2n-2}(y)$. Hence:

$$\mathcal{M}[F_{2n-1}(y-1)](s) = \mathcal{M}\left[\frac{1}{y+1}\right](s)\,\mathcal{M}[F_{2n-2}(y)](s) = \frac{\pi}{\sin\pi s}\,\mathcal{M}[F_{2n-2}(y)](s), \tag{III.2}$$

where $0 < R_l\, s < 1$, in order that the integral representing $\mathcal{M}[1/y+1](s) = \pi/\sin\pi s$ converges.

By using (III.2) repeatedly, one obtains:

$$\mathcal{M}[F_{2n-1}(y-n)](s) = \left(\frac{\pi}{\sin\pi s}\right)^{2n} (y'+n)^{s-1},$$

which gives, when using the inverse Mellin transformation:

$$F_{2n-1}(y, y') = \frac{\pi^{2n}}{y+n}\,\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \frac{ds}{\sin^{2n}\pi s}\left(\frac{y'+n}{y+n}\right)^{s-1} \quad (0 < c < 1).$$

Finally, by choosing $c = 1/2$ $(s = 1/2 + i\,t)$:

$$F_{2n-1}(y, y') = \frac{\pi^{2n}}{\sqrt{(y+n)(y'+n)}}\,\frac{1}{\pi}\int_0^\infty \frac{dt}{c\,h^{2n}\,\pi t}\cos\left(t\,\mathrm{Log}\,\frac{y'+n}{y+n}\right) \qquad \text{q.e.d.}$$

iii) These inequalities follow directly from (II.8) and from:

$$K_2^{(n)}(0,0) = \left(\frac{R\infty}{\pi^2}\right)^n \int_Z^\infty dx_1\, dx_2 \ldots dx_{2n-1} \frac{\Phi_S(-x_1)\,\Phi_S(-x_3)\ldots\Phi_S(-x_{2n-1})}{x_1(x_1+x_2)\ldots(x_{2n-2}+x_{2n-1})\,x_{2n-1}} = \frac{1}{Z}\left(\frac{R\infty}{\pi^2}\right)^n$$

$$\times \int_0^\infty dy_1\, dy_2 \ldots dy_{2n-1} \frac{\Phi_S[-Z(y_1+1)]\,\Phi_S[-Z(y_3+1)]\ldots\Phi_S[-Z(y_{2n-1}+1)]}{(y_1+1)(y_1+y_2+1)\ldots(y_{2n-2}+y_{2n-1}+1)(y_{2n-1}+1)} \quad \text{q.e.d.}$$

iv) We begin by performing $(n-1)$ integrations in G_n:

$$G_n = \int_0^\infty \frac{dy_1\, dy_3 \ldots dy_{2n-1}}{(y_1+1)(y_{2n-1}+1)} \int_0^\infty \frac{dy_2}{(y_1+y_2+1)(y_2+y_3+1)} \cdots \int_0^\infty \frac{dy_{2n-2}}{(y_{2n-3}+y_{2n-2}+1)(y_{2n-2}+y_{2n-1}+1)}$$

$$= \int_0^\infty \frac{dy_1\, dy_2 \ldots dy_{2n-1}}{(y_1+1)(y_{2n-1}+1)} \frac{\operatorname{Log}\dfrac{y_1+1}{y_3+1}}{y_1-y_3} \cdots \frac{\operatorname{Log}\dfrac{y_{2n-3}+1}{y_{2n-1}+1}}{y_{2n-3}-y_{2n-1}}\,.$$

which becomes, with $y_j = 1/(t_i - 1)$:

$$G_n = \int_0^1 dt_1\, dt_2 \ldots dt_n \frac{\operatorname{Log}\dfrac{t_1}{t_2}}{t_1-t_2} \frac{\operatorname{Log}\dfrac{t_2}{t_3}}{t_2-t_3} \cdots \frac{\operatorname{Log}\dfrac{t_{n-1}}{t_n}}{t_{n-1}-t_n}\,.$$

An equivalent form of G_n is obtained if we put $t_i = (1/\alpha)\, u_i$, where α is a new real variable:

$$G_n = \frac{1}{\alpha} I_n(\alpha)\,,$$

with:

$$I_n(\alpha) = \int_0^\alpha du_1 \ldots du_n \frac{\operatorname{Log}\dfrac{u_1}{u_2}}{u_1-u_2} \frac{\operatorname{Log}\dfrac{u_2}{u_3}}{u_2-u_3} \cdots \frac{\operatorname{Log}\dfrac{u_{n-1}}{u_n}}{u_{n-1}-u_n}\,.$$

Then (in a shortened notation):

$$G_n = \left.\frac{dI_n}{d\alpha}\right|_{\alpha=1} = \int_0^1 (u_1 = 1) + \int_0^1 (u_2 = 1) + \ldots + \int_0^1 (u_n = 1)$$

$$= \int_0^1 du_2 \ldots du_n \frac{\operatorname{Log} u_2}{u_2-1} \frac{\operatorname{Log}\dfrac{u_2}{u_3}}{u_2-u_3} \cdots \frac{\operatorname{Log}\dfrac{u_{n-1}}{u_n}}{u_{n-1}-u_n}$$

$$+ \ldots + \int_0^1 du_1 \ldots du_{k-1}\, du_{k+1} \ldots du_n \frac{\operatorname{Log}\dfrac{u_1}{u_2}}{u_1-u_2} \cdots \frac{\operatorname{Log}\dfrac{u_{k-2}}{u_{k-1}}}{u_{k-2}-u_{k-1}}$$

$$\times \left(\frac{\operatorname{Log} u_{k-1}}{u_{k-1}-1} \frac{\operatorname{Log} u_{k+1}}{u_{k+1}-1}\right) \frac{\operatorname{Log}\dfrac{u_{k+1}}{u_{k+2}}}{u_{k+1}-u_{k+2}} \cdots \frac{\operatorname{Log}\dfrac{u_{n-1}}{u_n}}{u_{n-1}-u_n}$$

$$+ \ldots + \int_0^1 du_1 \ldots du_{n-1} \frac{\operatorname{Log}\dfrac{u_1}{u_2}}{u_1-u_2} \cdots \frac{\operatorname{Log}\dfrac{u_{n-2}}{u_{n-1}}}{u_{n-2}-u_{n-1}} \frac{\operatorname{Log} u_{n-1}}{u_{n-1}-1}\,.$$

so that, as $\mathrm{Log}\, u/(u-1) \geqslant 1$ for $0 \leqslant u \leqslant 1$:

$$G_n > \sum_{k=1}^{n} G_{k-1}\, G_{n-k} \qquad (G_0 = 1) \qquad \text{q.e.d.}$$

IV. Proof of the Singular Behaviour in λ

We proceed step by step and establish three lemmas.

Lemma 1: Assuming $\mathcal{N}$ to be a known (and well-behaved) function, the singular equation (II.13) has a solution which is holomorphic in a neighbourhood of $\lambda = 0$. This solution is given by:

$$N = \mathcal{N} + R_2(\lambda)\, \mathcal{N}\,, \tag{IV.1}$$

where the resolvent $R_2(\lambda)$ is the power series with positive coefficients:

$$R_2(z, z'; \lambda) = \sum_{n=1}^{\infty} K_2^{(n)}(z, z')\, \lambda^n \qquad (z \geqslant 0,\, z' \geqslant 0)\,. \tag{IV.2}$$

The radius of convergence $\lambda_0 = \lambda_0(z, z')$ of this series is such that:

$$\frac{1}{A''\, R_\infty} \leqslant \lambda_0(z, z') \leqslant \frac{1}{A'\, R_\infty} \tag{IV.3}$$

Proof

The expression (IV.1) is the iterative solution of equation (II.13). It is well defined as long as the series (IV.2) converges. The radius of convergence λ_0 is given by:

$$\frac{1}{\lambda_0(z, z')} = \limsup_{n \to \infty} \sqrt[n]{K_2^{(n)}(z, z')}\,.$$

Introducing the bound i) with $\mu = 1/2$, we obtain the first of the inequalities (IV.3).

Besides, for $|\lambda| < (\sin^2 \pi\, \mu)/A''\, R_\infty$, one deduces from i) and (IV.2) the bound:

$$|R_2(z, z'; \lambda)| \leqslant \frac{c_\mu\, A''\, R_\infty\, |\lambda|}{\pi^2 \left(1 - \dfrac{A''\, R_\infty\, |\lambda|}{\sin^2 \pi\, \mu}\right)}\, \frac{1}{(z+Z)^\mu\, (z'+Z)^{1-\mu}} \qquad (0 < \mu < 1) \tag{IV.4}$$

which will be used later on.

Next, according to the inequality ii):

$$\frac{1}{\lambda_0(z, z')} \geqslant A'\, R_\infty \lim_{n \to \infty} \left[\frac{1}{\pi \sqrt{(z+n\, Z)\, (z'+n\, Z)}} \int_0^\infty \frac{dt}{c\, h^{2n}\, \pi\, t} \cos\left(t\, \mathrm{Log}\, \frac{z'+n\, Z}{z+n\, Z}\right)\right]^{\frac{1}{n}}$$

$$= A'\, R_\infty \lim_{n \to \infty} \left[\int_0^\infty \frac{dt}{c\, h^{2n}\, \pi\, t}\right]^{\frac{1}{n}}.$$

where we have used the strong decrease of $(\mathrm{ch}\, \pi\, t)^{-2n}$ with t for large n, and $\lim_{n \to \infty} \cos[t\, \mathrm{Log}[(z' + n\, Z)/(z + n\, Z)]] = 1$. The second part of (IV.3) then follows from the property [9]:

$$\lim_{u \to \infty} \left[\int_0^\infty \frac{dt}{c\, h^{2n}\, \pi\, t}\right]^{\frac{1}{n}} = \operatorname*{Sup}_{0 \leqslant t < \infty} \frac{1}{c\, h^2\, \pi\, t} = 1 \qquad \text{q.e.d.}$$

As a consequence of (IV.3), the series (IV.2) is uniformly convergent with respect to z and z' over every circle of the λ-plane with radius $|\lambda| < 1/A''\, R_\infty$.

Lemma 2: Equation (II.14) is a Fredholm equation with a Hilbert-Schmidt kernel when $|\lambda| < 1/A'' R_\infty$.

Proof

$$\alpha)\ K_1 \in \mathfrak{L}^2 .$$

From (II.11):

$$|K_1(z, z')| < |K_L(z, z')| + \left| \frac{\varrho(z')}{R_\infty} - 1 \right| |K_2(z, z')| .$$

Obviously, K_L is a Hilbert-Schmidt kernel. More precisely:

$$|K_L(z, z')| = \left| \frac{1}{\pi^2} \int_a^Z dx \frac{\Phi_L(-x)}{(x+z)(x+z')} \right| < \text{const.} \frac{z}{(z+a)(z'+a)} \quad (z \geqslant 0,\ z' \geqslant 0) .$$

On the other hand, by assumption (II.6):

$$\left| \frac{\varrho(z')}{R_\infty} - 1 \right| < \text{const.} \frac{Z}{z'+Z} .$$

Thus, taking into account the inequality i) for $n = 1$:

$$|K_1(z, z')| < \frac{\text{const.}\, Z}{(z+a)(z'+a)} + \frac{\text{const.}\, c_\nu A'' Z}{(z+Z)^\nu (z'+Z)^{2-\nu}} \quad (0 < \nu < 1) . \tag{IV.5}$$

Hence $K_1 \in \mathfrak{L}^2$, as we can choose for example $\nu = 2/3$.

$$\beta)\ R_2(\lambda)\, B \in \mathfrak{L}^2 .$$

According to equations (II.3) and (IV.4):

$$|R_2(\lambda)\, B(z)| \leqslant \frac{\text{const.}\, c_\mu A'' |\lambda|}{1 - \dfrac{A'' R_\infty |\lambda|}{\sin^2 \pi\mu}} \frac{1}{(z+Z)^\mu} \int_0^\infty \frac{dz'}{z'+Z)^{1-\mu}} \int_a^\infty \frac{dx}{x(x+z')} .$$

Now:

$$\int_0^\infty \frac{dz'}{(z'+Z)^{1-\mu}} \int_a^\infty \frac{dx}{x(x+z')} < \int_a^\infty \frac{dx}{x} \int_0^\infty \frac{dz'}{(z'+x)(z'+a)^{1-\mu}}$$

$$< \int_a^\infty \frac{dx}{x} \int_0^\infty \frac{dy}{[y+(x-a)]\, y^{1-\mu}} = \left(\frac{\pi}{\sin \pi\mu} \right)^2 a^{\mu-1} .$$

Thus:

$$|R_2(\lambda)\, B(z)| < \frac{c_\mu A'' |\lambda|}{\sin^2\pi\mu - A'' R_\infty |\lambda|} \cdot \frac{\text{const.}}{(z+Z)^\mu} \quad \text{for} \quad |\lambda| < \frac{\sin^2\pi\mu}{A'' R_\infty} . \tag{IV.6}$$

Once $|\lambda| < 1/A'' R_\infty$ is given, a positive ε can be found such that $|\lambda| < \sin^2[\pi(1/2 + \varepsilon)]/A'' R_\infty$. Then it suffices to choose $\mu = 1/2 + \varepsilon$ to conclude from (IV.6) that $R_2(\lambda)\, B \in \mathfrak{L}^2$.

$$\gamma)\ R_2(\lambda)\, K_1 \in \mathfrak{L}^2 .$$

q.e.d.

This property can be deduced in the same way from the bounds (IV.4) and (IV.5) by putting $\mu = 1/2 + \varepsilon$, $\nu = 1 - \varepsilon$.

Collecting α), β) and γ), we see that:

$$[B + R_2(\lambda)\, B] \in \mathfrak{L}^2 , \quad [K_1 + R_2(\lambda)\, K_1] \in \mathfrak{L}^2 .$$

At this stage, the radius of convergence λ_0 depends on z, z' and Z. The coefficients $K_2^{(n)}(z, z')$ of the series (IV.2) being positive (for $z \geqslant 0$, $z' \geqslant 0$), $R_2(z, z'; \lambda)$ is necessarily singular at $\lambda = \lambda_0(z, z')$.

Before we show that λ_0 does not depend on z, z' and Z, we establish that the point $\lambda = \lambda_0(z, z')$ cannot be a pole of $R_2(z, z'; \lambda)$ (as a function of λ only).

Lemma 3: The resolvent $R_2(z, z'; \lambda)$ presents a non polar singularity at $\lambda = \lambda_0(z, z')$.

Proof

As the generalization makes no difficulty, we consider the resolvent for $z = z' = 0$ only. We want to show that the presence of a pole of $R_2(0, 0; \lambda)$ at $\lambda = \lambda_0(0, 0)$ is incompatible with the inequalities iii) and iv).

Starting from the coefficients G_n defined in iii), let us construct a function G of λ (and of Z) by means of the power series:

$$G(\lambda) = \sum_{n=0}^{\infty} G_n \left(\frac{A' R_\infty \lambda}{\pi^2}\right)^n \quad (G_0 = 1) . \tag{IV.7}$$

According to (IV.2) and iii):

$$G(\lambda) < 1 + Z R_2(0, 0; \lambda) , \tag{IV.8}$$

$$G(\lambda) > 1 + Z R_2(0, 0; \frac{A'}{A''} \lambda) , \tag{IV.9}$$

for real positive λ (not too large).

On the other hand, by multiplying both sides of iv) with $(A' R_\infty \lambda/\pi^2)^n$ and summing over n, one gets:

$$G(\lambda) - 1 > \frac{A' R_\infty \lambda}{\pi^2} G^2(\lambda) ,$$

which gives:

$$G(\lambda) < \frac{\pi^2}{A' R_\infty \lambda} \tag{IV.10}$$

The bound (IV.8) insures the convergence of the series (IV.7) in the circle $|\lambda| < \lambda_0(0, 0)$. Thus the inequalities (IV.9) and (IV.10) hold when $0 < \lambda < \lambda_0(0, 0)$. They imply:

$$0 < Z R_2\left(0, 0; \frac{A'}{A''} \lambda\right) < \frac{\pi^2}{A' R_\infty \lambda} - 1 \quad \forall \lambda \begin{cases} > 0 \\ < \lambda_0(0, 0) \end{cases} \tag{IV.11}$$

A resolvent presenting a pole at $\lambda = \lambda_0(0, 0)$ could be written in the form:

$$Z R_2(0, 0; \lambda) = \sum_{q=1}^{Q} \frac{r_q}{[\lambda_0(0, 0) - \lambda]^q} + r(\lambda) , \tag{IV.12}$$

where Q is some positive (but finite) integer and $r(\lambda)$ a function holomorphic in a neighbourhood of $\lambda = \lambda_0(0, 0)$.

Comparing (IV.11) and (IV.12) in the limit $Z \to \infty$ and taking (II.9) into account, we obtain:

$$0 < \sum_{q=1}^{Q} \frac{R_q}{(\lambda_0 - \lambda)^q} + \lim_{Z \to \infty} r(\lambda) < \frac{\pi^2}{R_\infty \lambda} - 1 \qquad (0 < \lambda < \lambda_0) , \tag{IV.13}$$

where $\lambda_0 = \lim_{Z \to \infty} \lambda_0(0,0)$, $R_q = \lim_{Z \to \infty} r_q$.

Now the R_q are not all vanishing, since $\lim_{Z \to \infty} Z\, R_2(0, 0; \lambda)$ cannot be regular at $\lambda = \lambda_0$ (this would contradict the inequalities ii)). Thus (IV.12) is impossible, and the proof is complete. q.e.d.

As a consequence of Lemma 2 and of the Fredholm theorem, the only singularities of the resolvent of equation (II.14) lying in the circle $|\lambda| < \lambda_0(z, z')$ are real poles without limit points (their meaning has been given in [6]). This result, when associated to Lemmas 1 and 3, establishes the existence of a solution $N(z; \lambda)$ meromorphic in the circle $|\lambda| < \lambda_0(z) \equiv \operatorname{Inf}_{z' \geqslant 0} \lambda_0(z, z')$ with a non polar singularity at $\lambda = \lambda_0(z)$ (we do not discuss the possibility for this singularity to be cancelled by the integrations over z' which equation (II.14) contains). Now, since the function $N(z; \lambda)$ does not depend on Z, it must be so for $\lambda_0(z)$. Furthermore, the parameter Z being arbitrarily large, (II.3) and (IV.3) imply:

$$\lambda_0(z) = \frac{1}{R_\infty} \quad \forall\, z \geqslant 0 .$$

To summarize, we can assert the:

Theorem: When $\lim_{X \to \infty} \phi(-x) = \lambda$, the integral equation for N has a solution $N(z; \lambda)$ which is meromorphic in the circle $|\lambda| < 1/R_\infty$. This solution presents a non polar singularity at $\lambda = 1/R_\infty$.

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