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Representation of External Fields by Means of Coherent States

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Abstract. The artificial introduction of classical ('external') electromagnetic fields, interacting with matter, in quantum electrodynamics can be avoided. To all orders of the perturbation expansion in powers of e complete equivalence is shown between the conventional external field method and a description in terms of quantum fields alone, where suitably chosen photon states simulate the effects of the external field. These states are uniquely determined, they are the coherent states of GLAUBER.

1. Introduction

For certain investigations in quantum field theory, e.g. the calculation of anomalous magnetic moments or of vacuum polarization, it is customary to introduce a so-called external electromagnetic field by writing

$$A_\mu = A_\mu^q + A_\mu^{\text{ex}} . \tag{1.1}$$

A_μ is the total electromagnetic potential; it is split into a dynamical (quantized) part A_μ^q containing photon creation and annihilation operators and an external part A_μ^{ex} which is treated as a prescribed classical (c-number) field. To justify this it is argued that the external part A_μ^{ex} of the field may be considered as due to charged sources of large mass whose motion is little affected by the reactions of the dynamical system. However, since the 'correct' theory is a pure quantum theory the a priori introduction of a classical potential in (1.1) (or a similar split for the material part of the system) seems rather artificial and is not very satisfactory. The following question therefore arises: Is it possible to work with a pure quantum-dynamical system where radiation and matter are described in terms of quantum operators (without introducing c-number fields or sources into the theory) but to choose a suitable state of the photons that reproduces the physical effects of an external field? It will be shown in this paper that this is indeed the case in all situations where an arbitrary number of particles are present together with a source-free external field. The photon state in question will be explicitly constructed and complete equivalence with the conventional external field description will be established.

In section 2 we describe two methods of calculating expectation values in the presence of an external field: the conventional one and a 'pure quantum method'. The above mentioned photon states are constructed in section 3. Section 4 shows the equivalence of the two methods. In section 5 the results are extended to non-diagonal matrix elements and to more general state vectors.

2. Expectation Values

We consider a system with Hamiltonian

$$H = H_0 + H_1 , \tag{2.1}$$

where H_0 describes the free electromagnetic field and charged particles (electrons, positrons, nucleons, mesons, etc.) without their electromagnetic interaction energy, which is given by H_1. H_1 is a functional of the electromagnetic potential A_μ (and of the other dynamical variables). In the interaction representation the expectation value of an arbitrary operator F at time t is given by

$$\langle F\rangle_t = \langle\phi_0 \mid U^{-1}(t, t_0)\ F(t)\ U(t, t_0) \mid \phi_0\rangle \tag{2.2}$$

where $\mid \phi_0\rangle$ is the state vector of the system at the initial time t_0,

$$F(t) = e^{iH_0 t} F\ e^{-iH_0 t} \qquad (\hbar = c = 1) , \tag{2.3}$$

and U is the unitary time evolution operator

$$\begin{aligned} U(t, t_0) &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t} dt_n\ T\ (H_1(t_1) \ldots H_1(t_n)) \\ U^{-1}(t, t_0) &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t} dt_n\ T^*\ \big(H_1(t_1) \ldots H_1(t_n)\big) . \end{aligned} \tag{2.4}$$

(T = time ordering operator, T^* = inverse time ordering operator). Since

$$U^{-1}(t, t_0) = [U(t, \infty)\ U(\infty, t_0)]^{-1} = U^{-1}(\infty, t_0)\ U^{-1}(t, \infty) = S^{-1}\ U(\infty, t)$$

where

$$S = U(\infty, t_0) , \tag{2.5}$$

(2.2) can be written (see Akhiezer and Berestetski [1])

$$\langle F\rangle_t = \langle\phi_0 \mid S^{-1}\ T\big(F(t) S\big) \mid \phi_0\rangle . \tag{2.6}$$

It is usual to choose $t_0 = -\infty$. The operator F will in general be a functional of A_μ and of the dynamical variables of the charged particles. Simple examples are $F = H_1$ or $A_\mu(\boldsymbol{x})$ or $j_\mu(\boldsymbol{x})$ (charged particle current) or $A_\mu(\boldsymbol{x})\ A_\nu(\boldsymbol{x}')$ (field correlation operator).

In order to calculate expectation values in the presence of a *free* external electromagnetic field the following two methods are feasible:

Method I (conventional method).

In all expressions A_μ is replaced by the sum (1.1), where A_μ^{ex} fulfils the free field equation, and the functionals $F[A_\mu]$, $S[A_\mu]$ can be written as $F[A_\mu^q + A_\mu^{ex}]$, $S[A_\mu^q + A_\mu^{ex}]$, etc. Equation (2.6) becomes

$$\langle F\rangle_t = \langle\phi_0^I \mid S^{-1}\ [A_\mu^q + A_\mu^{ex}]\ T(F\ [A_\mu^q + A_\mu^{ex}]\ S\ [A_\mu^q + A_\mu^{ex}]) \mid \phi_0^I\rangle . \tag{2.7}$$

In the initial state $|\phi_0^I\rangle$ an arbitrary number of charged particles are present, yet no photons:

$$|\phi_0^I\rangle = |\psi_0 \text{ (charged particles)}\rangle \otimes |o \text{ photons}\rangle . \tag{2.8}$$

Method II (pure quantum method).

Here the electromagnetic potential is a pure quantum field

$$A_\mu = A_\mu^q . \tag{2.9}$$

Equation (2.2) now becomes

$$\langle F\rangle_t = \langle\phi_0^{II} \mid S^{-1}[A_\mu^q]\; T\big(F[A_\mu^q]\; S[A_\mu^q]\big) \mid \phi_0^{II}\rangle . \tag{2.10}$$

As far as the charged particles are concerned, the initial state $|\phi_0^{II}\rangle$ is the same as for method I, but the photon vacuum is replaced by a certain state $|z\rangle$ in the Hilbert space of the free photons:

$$|\phi_0^{II}\rangle = |\psi_0 \text{ (charged particles)}\rangle \otimes |z\rangle . \tag{2.11}$$

$|z\rangle$ is to be determined in such a way that the two expressions (2.7) and (2.10) agree.

In the following we shall use equations (2.4) and (2.5) to expand (2.7) and (2.10) in powers of the electric charge e and then compare the two series term by term.

3. Construction of the State $|z\rangle$

Requiring the two methods to yield identical results in the cases $F = A_\mu(\boldsymbol{x})$ and $F = A_\mu(\boldsymbol{x})\, A_\nu(\boldsymbol{x}')$, we shall see that the state $|z\rangle$ is already uniquely determined by identifying in (2.7) and (2.10) the terms of lowest order in e, viz.

$$\langle 0 \mid A_\mu^q(\boldsymbol{x}, t) + A_\mu^{\text{ex}}(\boldsymbol{x}, t) \mid 0\rangle = \langle z \mid A_\mu^q(\boldsymbol{x}, t) \mid z\rangle \tag{3.1}$$

$$\langle 0 \mid \big(A_\mu^q(\boldsymbol{x}, t) + A_\mu^{\text{ex}}(\boldsymbol{x}, t)\big)\big(A_\nu^q(\boldsymbol{x}', t) + A_\nu^{\text{ex}}(\boldsymbol{x}', t)\big) \mid 0\rangle = \langle z \mid A_\mu^q(\boldsymbol{x}, t)\, A_\nu^q(\boldsymbol{x}', t) \mid z\rangle . \tag{3.2}$$

According to (2.3) $A_\mu^q(\boldsymbol{x}, t)$ has the time dependence of the free radiation field and can be expanded in the usual way:

$$A_\mu^q(\boldsymbol{x}, t) = \sum_k \big(u_\mu^{(k)}(\boldsymbol{x})\, a_k\, e^{-i\omega_k t} + u_\mu^{(k)*}(\boldsymbol{x})\, a_k^+\, e^{i\omega_k t}\big) , \tag{3.3}$$

a_k^+ and a_k being creation and annihilation operators for photons. k numbers the modes $(\boldsymbol{k}, \lambda)$, where $\boldsymbol{k}$ is the wave vector and λ labels the polarization. The functions $u_\mu^{(k)}(\boldsymbol{x})$ can be chosen to fulfil the relations

$$\sum_\mu \int d^3x\, u_\mu^{(k)*}(\boldsymbol{x})\, u_\mu^{(l)}(\boldsymbol{x}) = \frac{1}{2\,\omega_k}\, \delta_{kl} , \qquad u_\mu^{(k)*}(\boldsymbol{x}) = u_\mu^{(-k)}(\boldsymbol{x}) , \tag{3.4}$$

where we designate the mode $(-\boldsymbol{k}, \lambda)$ by $-k$.

Similarly, the free external field is expanded:

$$A_\mu^{\text{ex}}(\boldsymbol{x}, t) = \sum_k \big(u_\mu^{(k)}(\boldsymbol{x})\, \alpha_k\, e^{-i\omega_k t} + u_\mu^{(k)*}(\boldsymbol{x})\, \alpha_k^*\, e^{i\omega_k t}\big) , \tag{3.5}$$

α_k being given complex numbers.

Equation (3.1) immediately shows that

$$\langle z \mid a_k \mid z\rangle = \alpha_k \qquad \langle z \mid a_k^+ \mid z\rangle = \alpha_k^* \tag{3.6}$$

whereas equation (3.2) imposes the condition on $|z\rangle$:

$$A_\mu^{\mathrm{ex}}(\boldsymbol{x}, t)\, A_\nu^{\mathrm{ex}}(\boldsymbol{x}', t) + \langle 0 \mid A_\mu^q(\boldsymbol{x}, t)\, A_\nu^q(\boldsymbol{x},' t) \mid 0\rangle = \langle z \mid A_\mu^q(\boldsymbol{x}, t)\, A_\nu^q(\boldsymbol{x}', t) \mid z\rangle\,. \tag{3.7}$$

Writing

$$A_\mu^q\, A_\nu^q = N(A_\mu^q\, A_\nu^q) + \langle 0 \mid A_\mu^q\, A_\nu^q \mid 0\rangle$$

where $N(..)$ denotes WICK's normal product, and of course assuming $\langle z|z\rangle = 1$, we see that (3.7) is equivalent to

$$A_\mu^{\mathrm{ex}}(\boldsymbol{x}, t)\, A_\nu^{\mathrm{ex}}(\boldsymbol{x}', t) = \langle z \mid N\big(A_\mu^q(\boldsymbol{x}, t)\, A_\nu^q(\boldsymbol{x}', t)\big) \mid z\rangle\,. \tag{3.8}$$

In this equation we insert the expansions (3.3) and (3.5). Multiplying by $u_\mu^{(l)}(\boldsymbol{x}) \cdot u_\nu^{(l)*}(\boldsymbol{x}')$ (l arbitrary), integrating over $\boldsymbol{x}$ and $\boldsymbol{x}'$ and summing over μ and ν, we obtain by virtue of (3.4)

$$\big(\langle z \mid a_{-l}\, a_l \mid z\rangle - \alpha_{-l}\, \alpha_l\big)\, e^{-2i\omega_l t} + \big(\langle z \mid a_l^+\, a_{-l}^+ \mid z\rangle - \alpha_l^*\, \alpha_{-l}^*\big)\, e^{2i\omega_l t}$$
$$+ \big(\langle z \mid a_l^+\, a_l \mid z\rangle - \alpha_l^*\, \alpha_l + \langle z \mid a_{-l}^+\, a_{-l} \mid z\rangle - \alpha_{-l}^*\, \alpha_{-l}\big) = 0\,. \tag{3.9}$$

Since this has to hold for all times each of the three terms on the left hand side of (3.9) must vanish separately. Now from the inequality

$$\langle z \mid (a_l^+ - \alpha_l^*)\, (a_l - \alpha_l) \mid z\rangle \geqslant 0 \tag{3.10}$$

it follows, by virtue of (3.6) that

$$\langle z \mid a_l^+\, a_l \mid z\rangle \geqslant \alpha_l^*\, \alpha_l\,. \tag{3.11}$$

Therefore the third term in (3.9) is $\geqslant 0$ and vanishes only if the equals sign holds in (3.10) and (3.11). The necessary and sufficient condition for this is

$$(a_l - \alpha_l) \mid z\rangle = 0\,, \tag{3.12}$$

i.e. $|z\rangle$ is an eigenstate of a_l for all l. Such states are called coherent states and have been discussed by GLAUBER [2] and others. Using the tensor product of the number eigenstates $|n_l\rangle$ as a basic set in the Hilbert space of the photons we expand

$$\mid z\rangle = \sum_{n_1, n_2, \dots} C_{n_1, n_2, \dots} \mid n_1\rangle \otimes \mid n_2\rangle \otimes \dots \tag{3.13}$$

(3.12) yields for $l = 1$

$$\sum_{n_1, n_2, \dots} C_{n_1, n_2, \dots} \big(\sqrt{n_1} \mid n_1 - 1\rangle - \alpha_1 \mid n_1\rangle\big) \otimes \mid n_2\rangle \otimes \dots\,. \tag{3.14}$$

Substituting n_1 by $n_1 + 1$ in the first term and making use of the linear independence of the basis vectors, we get

$$\sqrt{n_1 + 1}\, C_{n_1+1, n_2, \dots} - \alpha_1\, C_{n_1, n_2, \dots} = 0\,. \tag{3.15}$$

The general solution is

$$C_{n_1, n_2, n_3, \ldots} = \frac{\alpha_1^{n_1}}{\sqrt{n_1!}} C'_{n_2, n_3, \ldots} \tag{3.16}$$

with arbitrary $C'_{n_2, n_3, \ldots}$. Defining the normalized GLAUBER state

$$| \alpha_l \rangle = e^{-1/2 |\alpha_l|^2} \sum_{n_l} \frac{\alpha_l^{n_l}}{\sqrt{n_l!}} | n_l \rangle \tag{3.17}$$

we can write

$$| z \rangle = | \alpha_1 \rangle \otimes \sum_{n_2, n_3, \ldots} C''_{n_2, n_3, \ldots} | n_2 \rangle \otimes | n_3 \rangle \otimes \ldots \tag{3.18}$$

By induction one finds

$$| z \rangle = \prod_l \otimes | \alpha_l \rangle . \tag{3.19}$$

Since $\langle \alpha_l | \alpha_l \rangle = 1$ we also have $\langle z | z \rangle = 1$. It is clear that, apart from an irrelevant phase factor, expression (3.19) for $| z \rangle$ follows uniquely. With the adjoint equation of (3.12):

$$\langle z | a_l^+ = \langle z | \alpha_l^* \tag{3.20}$$

one generally has for normal products

$$\langle z | a_{k_1}^+ \ldots a_{k_s}^+ a_{k_{s+1}} \ldots a_{k_n} | z \rangle = \alpha_{k_1}^* \ldots \alpha_{k_s}^* \alpha_{k_{s+1}} \ldots \alpha_{k_n} . \tag{3.21}$$

In particular all three terms in (3.9) vanish separately and (3.8) is obviously fulfilled. (3.3), (3.5) and (3.21) lead to the important result

$$\langle z | N(A_{\mu_1}^q(\boldsymbol{x}_1, t_1) \ldots A_{\mu_n}^q(\boldsymbol{x}_n, t_n)) | z \rangle = A_{\mu_1}^{\mathrm{ex}}(\boldsymbol{x}_1, t_1) \ldots A_{\mu_n}^{\mathrm{ex}}(\boldsymbol{x}_n, t_n) . \tag{3.22}$$

This unique property of the coherent states will be the key to the analysis of section 4.

4. Equivalence of the Two Methods

Using (2.4) and (2.5) one recognizes that a general term in the expansion of (2.7) or (2.10) is a multiple space-time integral over an expression of the form

$$T^*(\ldots)\, T(\ldots)$$

where each bracket contains a number of A_μ's and of particle operators. Since the latter commute with the A_μ's (interaction representation!) and $| \psi_0 \rangle$ is the same in both cases, the particle contributions are identical in corresponding terms. So only the electromagnetic parts have to be examined. Abbreviating

$$A_{\mu_i}^q(\boldsymbol{x}_i, t_i) = A_i^q , \quad A_{\mu_i}^{\mathrm{ex}}(\boldsymbol{x}_i, t_i) = A_i^{\mathrm{ex}} \tag{4.1}$$

one is left with integrands of the following form:

Method I

$$M_{r\,n}^I = \langle 0 | T^* [(A_1^q + A_1^{\mathrm{ex}}) \ldots (A_r^q + A_r^{\mathrm{ex}})]\, T [(A_{r+1}^q + A_{r+1}^{\mathrm{ex}}) \ldots (A_n^q + A_n^{\mathrm{ex}})] | 0 \rangle . \tag{4.2}$$

Method II

$$M_{rn}^{II} = \langle z \mid T^* [A_1^q \dots A_r^q] \, T [A_{r+1}^q \dots A_n^q] \mid z \rangle \,. \tag{4.3}$$

A slight generalization of WICK's theorem (compare SCHWEBER [3], p. 440ff.) states that

$$\begin{aligned} T^* (A_1 \dots A_r) \, T (A_{r+1} \dots A_n) = {} & N (A_1 \dots A_n) \\ & + N (A_1 \dots A_i \dots A_j \dots A_n) + \dots \\ & + N (A_1 \dots A_{i_1} \dots A_{j_1} \dots A_{i_2} \dots A_{j_2} \dots A_n) + \dots \\ & + \dots \end{aligned} \tag{4.4}$$

where the contractions

$$A_i A_j = \begin{cases} \langle 0 \mid T(A_i A_j) \mid 0 \rangle & \text{if } i \text{ and } j \geqslant r+1 \\ \langle 0 \mid T^*(A_i A_j) \mid 0 \rangle & \text{if } i \text{ and } j \leqslant r \\ \langle 0 \mid A_i A_j \mid 0 \rangle & \text{if } i \leq r \text{ and } j \geqslant r+1 \end{cases} \tag{4.5}$$

have to be taken over all possible pairs of operators.

With the aid of this theorem we now proceed to the last step:

Method I

In (4.2) the multiplication is carried out to give a sum of terms, each containing a certain number of factors A_i^q and A_i^{ex}. The products of the A_i^q are decomposed according to (4.4). The vacuum expectation values of normal products vanishing, only terms with all operators A_i^q contracted survive. Thus:

$$\begin{aligned} M_{rn}^{I} = {} & A_1^{\mathrm{ex}} \dots A_n^{\mathrm{ex}} + A_1^{\mathrm{ex}} \dots A_i^q \dots A_j^q \dots A_n^{\mathrm{ex}} + \dots \\ & + A_1^{\mathrm{ex}} \dots A_{i_1}^q \dots A_{j_1}^q \dots A_{i_2}^q \dots A_{j_2}^q \dots A_n^{\mathrm{ex}} + \dots \end{aligned} \tag{4.6}$$

Method II

(4.3) is decomposed with the aid of (4.4). Making use of (3.22) we immediately find

$$\begin{aligned} M_{rn}^{II} = {} & A_1^{\mathrm{ex}} \dots A_n^{\mathrm{ex}} + A_1^{\mathrm{ex}} \dots A_i^q \dots A_j^q \dots A_n^{\mathrm{ex}} + \dots \\ & + A_1^{\mathrm{ex}} \dots A_{i_1}^q \dots A_{j_1}^q \dots A_{i_2}^q \dots A_{j_2}^q \dots A_n^{\mathrm{ex}} + \dots \end{aligned} \tag{4.7}$$

i.e. the same as (4.6).

We see that the expressions (2.7) and (2.10) are term by term the same, which establishes their complete equivalence. Such an equality has also been found by KIBBLE [4] [1]) (see his equation (2.19)), thus showing that coherent states are a possible way of representing external fields. The present analysis shows that this is the only possible way if one wants to avoid c-number fields in a quantum theory.

1) We are indebted to Professor N. STRAUMANN who, when reading our manuscript, informed us about KIBBLE's article.

5. Generalization

The foregoing analysis can be generalized in two respects:

(1) The initial state for method I may contain an arbitrary number of photons.

(2) Instead of expectation values one can also consider transition amplitudes or other non-diagonal elements of operators F. Again in the state vectors of method I photons may be present.

However, in both cases the photons have to belong to modes whose amplitudes in the decomposition (3.5) of the external field vanish. In method II this means that these 'extra' photons are different from those building up the coherent state. [This is the case in actual physical situations: the external field is slowly varying whereas the photons observed are of much shorter wavelength]. Then, in the analysis of section 4 one has to pick out operators that take care of the necessary destruction and creation of the photons. This is done in the same way for both methods and their equivalence is again evident.

We have restricted our investigation to *source-free* external fields. In certain physical situations, as in the case of Bremsstrahlung, the external field fulfils an *inhomogeneous* wave equation, the source being a prescribed current distribution. Now GLAUBER has shown (see p. 2783–4 of [2]) that such a classical current leads to a pure coherent state. Therefore the preceding analysis can easily be carried over to this case. However, one then introduces a c-number current, in contradiction to the original aim of this paper.

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References

[1] A. I. AKHIEZER and V. B. BERESTETSKI, Quantum Electrodynamics (Interscience, New York 1965).
[2] ROY J. GLAUBER, Phys. Rev. *131*, 2766 (1963).
[3] S. S. SCHWEBER, Relativistic Quantum Field Theory (Harper and Row, New York 1961).
[4] T. W. B. KIBBLE, Cargèse Lectures in Physics, vol. *2*, p. 299–345 (Gordon and Breach, New York 1968).