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The Convergence of the Perturbation Series in a Model of Quantum Field Theory

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Abstract. We study the behaviour of the vacuum expectation value of a regularized scalar field operator as a function of the coupling constant μ^2 in the Fierz model. The vacuum expectation value exhibits an essential singularity at $\mu^2 = 0$. Therefore, the perturbation series diverges, representing only an asymptotic expansion of the vacuum expectation value under consideration.

1. Introduction

This paper is devoted to an exactly soluble model field theory of two interacting real scalar fields which was proposed by Fierz [1]. In particular, we investigate Fierz's conjecture that the perturbation expansion associated with this model is valid only in the sense of an asymptotic series[1]).

We first make use of perturbation theory in the coupling constant μ^2 to evaluate the coefficients of the power series expansion of a regularized vacuum expectation value, and show that this series diverges even for finite values of the cutoff-parameter. We then evaluate the same quantity for finite values of the coupling constant without invoking perturbation theory. For finite values of the cutoff-parameter the result is a well defined function of the coupling constant which exhibits an essential singularity at $\mu^2 = 0$. We finally verify that the result of the perturbation calculation is identical with the asymptotic expansion of this exact result.

2. The Model

The model has been proposed by Fierz [1], and describes the interaction between two massless and spinless real boson fields ϱ and ψ. The field equations[2])

$$\Box\, \varrho = \mu^2 \varrho\, \psi,{}_\nu \psi,{}^\nu$$

$$\Box\, \psi = -\,2\,\frac{1}{\varrho}\, \varrho,{}_\nu \psi,{}^\nu \tag{1}$$

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[1]) Dyson [2] has presented arguments to the effect that the perturbation expansion in quantum electrodynamics is of this nature.

[2]) *Notation:* $\eta_{\mu\nu} = (1, -1, -1, -1)$; $f,{}_\mu = \partial f/\partial x^\mu$; $*$ complex conjugate; $^+$ hermitian adjoint; operators are characterized by $\hat{\ }$.

where μ^2 is the coupling constant, can be solved by the substitution

$$1 + \mu \gamma = \varrho \, e^{i \mu \psi} . \tag{2}$$

The complex field γ then obeys the free field equation

$$\Box \gamma = 0 \tag{3}$$

and the two fields ϱ and ψ are given by the relations

$$\begin{aligned} \varrho &= \sqrt{(1 + \mu \gamma)(1 + \mu \gamma^*)} \\ \psi &= \frac{1}{2 i \mu} \ln \frac{1 + \mu \gamma}{1 + \mu \gamma^*} . \end{aligned} \tag{4}$$

In order to quantize the field theory we introduce the usual creation and annihilation operators for boson fields, and write

$$\hat{\gamma}(x) = (2 \pi)^{-3/2} \int \frac{d^3 p}{\sqrt{2 \omega_p}} \, C(\boldsymbol{p}) \, \{ e^{-i p x} \, \hat{a}(\boldsymbol{p}) + e^{i p x} \, \hat{b}^+(\boldsymbol{p}) \} \tag{5}$$

$C(\boldsymbol{p})$ is a cutoff function which has been introduced to regularize the field $\hat{\gamma}$ in the infrared and ultraviolet region and which should vanish inside a sphere with radius P_1 and outside a sphere with radius $P_2 > P_1$, being finite in between the two spheres and smooth everywhere.

The quantity we are going to evaluate in the next two sections is the vacuum expectation value of the field $\hat{\varrho}$. We consider (4) as the definition of $\hat{\varrho}$, which, since in the regularized theory $\hat{\gamma}$ and $\hat{\gamma}^+$ commute with one another at a fixed point of space-time, may now be written as

$$\hat{\varrho} = \sqrt{1 + \mu \hat{\gamma}} \, \sqrt{1 + \mu \hat{\gamma}^+} . \tag{6}$$

3. Perturbation Theory

If we expand the square roots in (6) with respect to μ, the vacuum expectation value of the field $\hat{\varrho}$ is given by

$$\langle 0 | \hat{\varrho} | 0 \rangle = \sum_{\lambda, \nu = 0}^{\infty} \binom{1/2}{\lambda} \binom{1/2}{\nu} \mu^{\lambda + \nu} \langle 0 | (\hat{\gamma})^\lambda (\hat{\gamma}^+)^\nu | 0 \rangle . \tag{7}$$

With the definition

$$\Delta = \langle 0 | \hat{\gamma} \hat{\gamma}^+ | 0 \rangle = (2 \pi)^{-3} \int \frac{d^3 p}{2 \omega_p} \, C^2(\boldsymbol{p}) \tag{8}$$

we therefore get the result

$$\langle 0 | \hat{\varrho} | 0 \rangle = \sum_{\lambda = 0}^{\infty} \binom{1/2}{\lambda}^2 \lambda! \, \mu^{2 \lambda} \Delta^\lambda \tag{9}$$

which is a power series with respect to the coupling constant μ^2. Note that, in the regularized theory, the constant Δ is finite.

It is easy to establish the divergence of the power series (9); we therefore conclude that $\langle 0 | \hat{\varrho} | 0 \rangle$, as a function of μ^2, is not analytic at the origin $\mu^2 = 0$.

4. Evaluation of the Vacuum Expectation Value without Perturbation Theory

We shall now show by an explicit calculation that the vacuum expectation value $\langle 0 \mid \hat{\varrho} \mid 0 \rangle$ is a well defined quantity within the framework of the regularized theory. If we define the operators

$$\begin{aligned}\hat{A} &= \Delta^{-1/2} (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2\omega_p}} C(\boldsymbol{p})\, e^{-ipx}\, \hat{a}(\boldsymbol{p}) \\ \hat{B} &= \Delta^{-1/2} (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2\omega_p}} C(\boldsymbol{p})\, e^{-ipx}\, \hat{b}(\boldsymbol{p}) \\ &[\hat{A}, \hat{A}^+] = [\hat{B}, \hat{B}^+] = 1\end{aligned} \tag{10}$$

we immediately see that our theory, for a fixed point of space-time, is reduced to the well known theory of the 2-dimensional harmonic oscillator. The calculation is now based on the existence of a complete set of simultaneous eigenstates of the operators $\hat{\gamma}(x)$ and $\hat{\gamma}^+(x)$ associated with a fixed space-time point. If we introduce the operators

$$\hat{\xi} = \frac{1}{2} (\hat{A} + \hat{A}^+ + \hat{B} + \hat{B}^+) \qquad \hat{\eta} = \frac{1}{2i} (\hat{A} - \hat{A}^+ - \hat{B} + \hat{B}^+) \qquad [\hat{\xi}, \hat{\eta}] = 0 \tag{11}$$

the eigenstates are then given by the simultaneous eigenstates $\mid \xi, \eta; \alpha \rangle$ of the operators $\hat{\xi}$ and $\hat{\eta}$, where α represents all the other quantum numbers belonging to the complete system. In this representation the field $\hat{\gamma}$ is given by

$$\hat{\gamma} = \Delta^{1/2} (\xi + i\,\eta)\,, \qquad \hat{\gamma}^+ = \Delta^{1/2} (\xi - i\,\eta)\,. \tag{12}$$

The spectra of $\hat{\xi}$ and $\hat{\eta}$ are the whole real axis, as can be seen by comparison with the harmonic oscillator.

The vacuum expectation value of the field $\hat{\varrho}$ can now be written as

$$\langle 0 \mid \hat{\varrho} \mid 0 \rangle = \iint_{-\infty}^{\infty} d\xi\, d\eta \sum_{\alpha} \mid \langle 0 \mid \xi, \eta; \alpha \rangle \mid^2 \sqrt{1 + 2\mu\, \Delta^{1/2}\, \xi + \mu^2 \Delta\, (\xi^2 + \eta^2)}\,. \tag{13}$$

The quantity $\langle 0 \mid \xi, \eta; \alpha \rangle$ essentially represents the wave function of the harmonic oscillator ground state; hence,

$$\sum_{\alpha} \mid \langle 0 \mid \xi, \eta; \alpha \rangle \mid^2 = \frac{1}{\pi}\, e^{-(\xi^2 + \eta^2)}\,. \tag{14}$$

The vacuum expectation value is now given by

$$\langle 0 \mid \hat{\varrho} \mid 0 \rangle = \frac{1}{\pi} \iint_{-\infty}^{\infty} d\xi\, d\eta\, e^{-(\xi^2 + \eta^2)} \sqrt{1 + 2\mu\, \Delta^{1/2}\, \xi + \mu^2 \Delta\, (\xi^2 + \eta^2)}\,. \tag{15}$$

This integral may be evaluated with the result

$$\langle 0 \mid \hat{\varrho} \mid 0 \rangle = \frac{1}{2} \sqrt{\pi\, \mu^2 \Delta}\; e^{-(1/\mu^2 \Delta)}\; \Phi\left(\frac{3}{2},\, 1;\, \frac{1}{\mu^2 \Delta}\right) \tag{16}$$

where $\Phi(\alpha, \beta; x)$ is the confluent hypergeometric series [3].

5. Perturbation Theory as Asymptotic Expansion

The function $\Phi(3/2, 1; x)$ possesses an essential singularity at infinity. Therefore, the quantity $\langle 0 \mid \hat{\varrho} \mid 0 \rangle$ considered as a function of the coupling constant μ^2 possesses an essential singularity at $\mu^2 = 0$, as conjectured from the divergence of the power series expansion (9) in section 3. In fact, this power series is just the asymptotic expansion of the vacuum expectation value $\langle 0 \mid \hat{\varrho} \mid 0 \rangle$ for small values of μ^2, as can be shown by means of the asymptotic expansion of the function $\Phi(3/2, 1; x)$. Thus, for the special case of the vacuum expectation value $\langle 0 \mid \hat{\varrho} \mid 0 \rangle$, FIERZ's conjecture is proved. A very interesting problem for further study is the question of whether or not the model may be expressed in terms of renormalized local fields which are well defined in the limit $\Delta \to \infty$.

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