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Localizability for Particles of Mass Zero

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Abstract. We investigate the consequences of the concept of weak localizability which was recently introduced by JAUCH and PIRON. It is found to be the appropriate mathematical tool for describing the localization of particles of given helicity in relativistic quantum mechanics. It treats particles of mass zero and those of positive mass on an equal footing. Particles of one fixed value of the helicity and spin $J \neq 0$ can never be described by states which are localized in a finite region of space. The neutrinos fall into this category. Particles which may exist in superpositions of states of different helicities (such as photons) can be localized in arbitrarily small volumes. We show that the localization of any particle is closely related to its energy density, but that this relation is always non-local. At large distances d from the region of localization of a particle of mass zero, its energy density does not vanish but falls off as d^{-7}. We give explicit expressions for the operators representing the number of particles localized in an arbitrary volume of space in relativistic quantum field theory. They will be compared with a similar expression given by MANDEL for the photon field.

I. Introduction

One of the basic concepts concerning elementary particles is that of *localizability*. It refers to the fact that particles can be located in physical space. In classical mechanics this property is built into the description of particles at a level so elementary that it has rarely been given a detailed analysis. In quantum mechanics the situation is quite different. If the position of an individual particle is an observable, then, according to the generally adopted rules of quantum mechanics, it should be the physical correlate of a linear operator in Hilbert space, the *position operator*.

The construction of the position operator has been one of the main problems in non-relativistic quantum mechanics. There it emerges as canonically conjugate to the momentum operator. The extension of this concept to the relativistic situation is not at all obvious, as is witnessed by an ever growing literature about this subject over the last twenty years. The study of these papers reveals that the origin of the difficulties can be traced to two different problems. One is that the relativistic invariance of particle positions has not yet found a generally accepted formulation. The second one is associated with particles of mass zero and spin $s \neq 0$: In these cases all the attempts to formulate the concept of localizability have either failed or are deficient in some respects. The following comments should clarify the reasons for this situation.

On confronting the definitions of position operators that have been proposed for particles of positive mass, one remarks that their disparity lies in different requirements of relativistic invariance (References [1–5] represent a small selection from among the numerous publications). NEWTON and WIGNER [2] prescribe the conditions that must be met by states localized at a point $\boldsymbol{x}$ at a certain time t in a given distinguished Lorentz frame. If the underlying Hilbert space is that of a unitary representation of the group of relativistic transformations, such a structure complies with Einstein's principle of special relativity which declares that all Lorentz frames are physically equivalent.

Other formulations insist in addition on manifest invariance [1, 3, 4]. This is expressed in several manners. For instance:

(a) The expectation values of $\boldsymbol{x}$ should transform like the first three components of a four-vector [1].

(b) The operator $\boldsymbol{x}$ should fulfil certain commutation rules not only with the infinitesimal generators of space-translations and space-rotations, but also with those of pure Lorentz transformations [4].

(c) States that are localized at a point $\boldsymbol{x}$ at time t in one frame should also appear localized at a *point* when viewed from any other Lorentz frame [3]. (We may mention that the localized states of Newton and Wigner do not obey this last rule.)

In all these formulations of manifest invariance, the resulting position operators on the positive energy states are either non-Hermitian or have non-commuting components. They are therefore hardly interpretable as position *observables* in the usual sense. Only the solution of Newton and Wigner embodies both Hermiticity and commutativity.

The second difficulty, that of finding the position operator for particles of mass zero, has received much less attention. NEWTON and WIGNER [2] mention that in all cases of mass $m = 0$ and spin $s \neq 0$ there exist no localized states meeting all of their prescribed conditions. Most of the other authors restrict themselves to situations with positive mass. Nevertheless, this difficulty has been tackled in three different manners at least. We shall briefly expose these three proposals.

The first one was indicated by PRYCE [1] and later elaborated by ACHARYA and SUDARSHAN [5] and by FLEMING [6]. These authors submit an operator which can be defined for particles of both positive and zero mass. Its components do not commute, and it fulfils no requirement of manifest invariance. Its longitudinal part is the same as that of the Newton-Wigner operator. This entails the existence of states which are localized on a plane of infinite extension (a 'front') orthogonal to the direction of propagation. Acharya-Sudarshan and Fleming suggest that such an operator seems more 'natural' than any other one, since it comprehends particles of zero mass on the same footing as those of positive mass.

The second proposal is due to FRONSDAL [7]. He constructs an operator with commuting components which is analogous to the Newton-Wigner operator but defined for particles of mass zero. However, this operator does not seem to be suitable since it does not transform as a vector under space-rotations.

Finally, we must cite a recent paper of JAUCH and PIRON [8]. It discusses a generalization on physical grounds of the ideas of WIGHTMAN [9], who reformulated the postulates of NEWTON and WIGNER in a more convenient mathematical language. The fundamental concept in Wightman's work is that of a *transitive system of imprimitivities* (MACKEY [10]). Localization of a particle in a region Δ of Euclidean three-space R^3 is described by a projection operator E_Δ acting in the Hilbert space of the states of the particle. The operators E_Δ obey certain transformation laws under translations and rotations of R^3. Moreover, all position measurements are compatible. Accordingly all of the E_Δ commute with one another. Wightman proved that particles of positive mass are localizable in this sense, but that no such system of imprimitivities exists for the cases of mass $m = 0$ and spin $s \neq 0$ (this result coincides strictly with that of NEWTON and WIGNER). In order to be able to define localizability also for $m = 0$, Jauch and Piron argue that there exists good physical justification for assuming commutativity for the projection operators E_{Δ_1} and E_{Δ_2} as long as the regions Δ_1 and Δ_2 are either disjoint, or one of them is a subdomain of the other. No such physical reason exists if Δ_1 and Δ_2 overlap. On that account they recommend to weaken Wigthman's conditions for localizability by admitting that E_{Δ_1} and E_{Δ_2} may not commute if Δ_1 and Δ_2 are two overlapping regions. They refer in this context to *weak localizability* and verify that the photon is indeed weakly localizable.

After this sifting of the relevant literature we must take up the question of the appropriate position observable in relativistic quantum mechanics. Firstly, there seems to be no physical reason for imposing manifest invariance. It is a requirement of mathematical convenience which goes beyond that of the principle of special relativity. On the other hand there are ample motives for demanding a position observable for $m = 0$. Indeed, photons (even individual ones [11]) are well known to be localizable experimentally. (The localization of neutrinos in the laboratory is connected with much greater difficulties.) Of the three solutions proposed for photons, only the one by JAUCH and PIRON [8] yields a satisfactory description of localization in a finite volume of space. Furthermore it is essentially the same as the one for positive mass found by NEWTON and WIGNER [2] and WIGHTMAN [9], which represents the only possible position *observable* for $m > 0$ (within the framework of localizability in a region Δ at a *point* of time t).

We conclude that the Newton-Wigner operator and its generalization by Jauch-Piron define the appropriate mathematical objects for describing localizability of relativistic elementary particles. The characteristics of weak localizability have been investigated to a modest extent only. In the following sections we shall therefore explain some notable consequences of these weakened postulates applied to simple relativistic systems.

Relativistic invariance will always involve a unitary representation of the Poincaré group $\mathcal{P}$ (WIGNER [12, 13]). One distinguishes four types of irreducible representations [14]. Those of imaginary mass and those of energy-momentum $p \equiv 0$ are easily discarded because of these unphysical attributes. Those of positive mass were treated in great detail by WIGHTMAN [9]. Nevertheless, weak localizability introduces a new feature for them: a (non-invariant) position observable for particles of given helicity. This novel aspect is intimately connected with the definition of localizability for the fourth type of irreducible representations of $\mathcal{P}$, those of mass zero. These fall into two classes: representations of discrete spin and of continuous spin respectively.

Particles of continuous (or infinite) spin are not known in experimental physics. Their peculiarity resides in an infinite degree of internal freedom. One might presume that these representations could be excluded from physical reality through the circumstance that they would not be localizable. Within the framework of weak localizability, it turns out that this expectation is not fully justified. Indeed, these representations admit an infinite number of inequivalent position observables in the weak sense. This fact, however, runs counter to the idea that a concept like localizability of a physical system should be described mathematically in a unique manner (up to unitary equivalence).

Particles of mass zero and discrete spin are known to occur in nature: the photon ($s = \pm 1$), the neutrinos ($s = -1/2$), the antineutrinos ($s = +1/2$), and maybe the graviton ($s = \pm 2$). We shall prove that such particles are weakly localizable if they can exist in states which are superpositions of states of both helicities $\pm s$ ($s \neq 0$). The projection operators which describe their position observables will be determined explicitly. We have likewise investigated the properties of particles of mass zero and helicity $s \neq 0$ which do not superpose with states of the opposite helicity $-s$. Such particles are *not* weakly localizable (in the sense that there exist no states which are localized in a *finite* region of space).

Our last statement of the non-localizability of particles of mass $m = 0$ and helicity $s \neq 0$ may appear surprising at first sight. Actually, it is nothing else than an expression of the fact that weak localizability combines the representations of mass zero and those of positive mass into a fully unified theory. Even for particles of $m > 0$ there exist no states belonging to one value $s \neq 0$ of the helicity which are in addition localized (in the sense of Wightman) in a finite volume (i.e. the projection operator onto the subspace of states of helicity $s \neq 0$ and the operator E_Δ for a finite volume Δ have no non-trivial common eigenstate). More generally, we shall demonstrate that weak localizability applies to particles of mass zero and spin s in precisely the same way as to particles of positive mass, spin $|s|$ and helicity s.

As a consequence of the *quantum-mechanical* description of non-interacting particles, one remarks that the relation between their localization (the 'particle density') and the corresponding energy density (at equal time) must be non-local. For particles of mass $m > 0$, the energy density falls off as $1/r^2 (\lambda\!\!\!^{-} r)^{-5/2} \exp(-2\, r/\lambda\!\!\!^{-})$ at large distances r from the volume of localization. ($\lambda\!\!\!^{-} = (\hbar/m\,c)$ is the Compton wavelength.) For $m = 0$, this decrease will turn out to behave as r^{-7}.

Our discussion of localizability will be confined to considerations about *free* elementary systems. On the other hand, the experimental determination of the position of such a physical system involves of necessity its interactions with the measuring apparatus. Such interactions are always given in terms of the field operators and their derivatives. The absorptive (positive frequency) part of the field operator transforms a one-particle state $|\varphi\rangle$ into the vacuum state multiplied by the usual coordinate-space wave-function corresponding to $|\varphi\rangle$. Newton and Wigner [2] proved that the coordinate-space wave-function belonging to a *localized* state of mass $m > 0$ extends over all space (at equal time). We shall derive a similar result for systems of mass zero and see that it is identical with that of Newton and Wigner taken in the limit $m \to 0$.

The preceding remarks explain that the description of the absorption or creation of an elementary particle relies on the coordinate-space wave-function, which is related to the $\boldsymbol{x}$-representation of the corresponding state vector in a non-local manner. We also mentioned that there exist no neutrino states which are localized in a bounded region of space. We see now that this latter fact is by no means unfortunate, since the *coordinate-space wave-function* of a neutrino state may well have the property of vanishing everywhere outside some finite three-dimensional volume at a certain instant of time. (One could also argue that this fact speaks in favour of a four-component theory of the neutrinos in place of the presently accepted two-component formulation. In the four-component theory localized states exist also for finite volumes.) For photons, one obtains localized states for arbitrary volumes. We shall find that the counting rates of a *macroscopic* photon-counter are approximately proportional to the number of photons localized in the volume that this apparatus occupies. The non-locality between the particle position and the corresponding coordinate-space wave-function also leads to a more intuitive understanding of the fact that (localized) particles can interact with physical systems which are situated anywhere in space and hence may transmit forces between such systems (e.g. the London-Van der Waals force between polarizable molecules [30]).

To conclude this introduction, we briefly mention the contents of the following sections. In Section II, we shall define the notion of localizability in quantum mechanics by means of six postulates of a mathematical nature, indicating as well the physical significance and some simple consequences of them. Section III presents a collection of results concerning the realizations of these postulates for elementary particles. The subsequent sections contain the mathematical elaboration of these results. Section IV introduces the unitary representations of the Poincaré group. In Section V we shall expound its irreducible representations corresponding to particles of discrete spin and construct the position observables for such particles of mass zero. Their properties will be enunciated in two theorems. These will then be proved in Section VI. The representations of continuous spin constitute the topic of Section VII. Part VIII is devoted to some remarks bearing upon relativistic invariance within our scheme. In Section IX we shall derive the mathematical form of the relation between the localization of particles of mass zero on the one hand and their energy density as well as their coordinate-space wave-function on the other hand, discuss the magnitude of the resulting discrepancy and explain its physical meaning. The final section offers some explicit expressions for the particle number operators for any three-dimensional volume in relativistic quantum field theory. For the case of the photon field, we shall compare these operators with a similar expression given by Mandel [39] and briefly mention the scope of their applicability for describing photon-counting experiments.

In Sections III–VII we shall make extensive use of the results obtained by Wightman [9]. Apart from Section IX we shall use units such that $\hbar = 1$ and $c = 1$; we set $g^{00} = +1, g^{ii} = -1$ $(i = 1,2,3)$. Four-vectors will be denoted by $p = p^\mu = (p^0, p^1, p^2, p^3) = (p^0, \boldsymbol{p})$.

II. Postulates for Localizability and Their Physical Significance

In this section we define the notion of localizability in quantum mechanics. It shall be applied in the later sections to the case of relativistic quantum mechanics only. We shall also impart some significant consequences of it.

The starting point is a distinguished reference frame (Galilean or Lorentzian), the rest-frame of the 'observer'. A quantum-mechanical system is described from this frame by assigning to it, for every value t of the time-coordinate in the chosen frame, a vector of a Hilbert space $\mathcal{H}_t$, the space of all possible states that the system under consideration may assume at time t. Furthermore, let $(\boldsymbol{a}, R)$ denote the elements of the group of motions in Euclidean three-space R^3. It is assumed that there exists in $\mathcal{H}_t$ a unitary representation $U_t(\boldsymbol{a}, R)$ of this group, such that $U_t(\boldsymbol{a}, R)\,\varphi_t$ describes the state obtained by way of rotating $\varphi_t \in \mathcal{H}_t$ according to R and then translating the resulting state by $\boldsymbol{a}$. The postulates for localizing this system are then the following [8, 9]:

(A) For every given time t there is associated with each Borel set $\Delta \subset R^3$ a projection operator $F_{\Delta,t}$ in $\mathcal{H}_t$ corresponding to the proposition 'The system is located within Δ at time t' (i.e. the expectation values of $F_{\Delta,t}$ give the probability of finding the system localized in Δ at time t).

(B)
$$F_{R^3,t} = I \tag{1}$$

(C) If $\Delta_1 \subset R^3$ and $\Delta_2 \subset R^3$ are disjoint (denoted by $\Delta_1 \perp \Delta_2$), then

$$F_{\Delta_1,t} \perp F_{\Delta_2,t} \tag{2}$$

i.e. $F_{\Delta_1,t}$ and $F_{\Delta_2,t}$ project onto two orthogonal subspaces of $\mathcal{H}_t$.

Let P_1 and P_2 denote two projection operators. Their intersection $P_1 \cap P_2$ is defined to be the projection operator onto the largest subspace which is contained in the ranges of both P_1 and P_2. This leads us to the next postulate:

(D)
$$F_{\Delta_1 \cap \Delta_2,t} = F_{\Delta_1,t} \cap F_{\Delta_2,t}\,. \tag{3}$$

(E) Let $R\Delta + \boldsymbol{a}$ denote the set obtained from Δ by carrying out the rotation R followed by the translation $\boldsymbol{a}$. Then

$$F_{R\Delta+\boldsymbol{a},t} = U_t(\boldsymbol{a}, R)\,F_{\Delta,t}\,U_t(\boldsymbol{a}, R)^{-1}\,. \tag{4}$$

(F) *Time reversal invariance:* Let $U_t(T)$ be the (unitary or anti-unitary) operator representing time reversal in $\mathcal{H}_t$. Then, for all $\Delta \subset R^3$

$$F_{\Delta,t}\,U_t(T) = U_t(T)\,F_{\Delta,t}\,.$$

The physical ideas behind these postulates consist in the following. (A) is a consequence of the basic assumption of quantum mechanics that elementary propositions are represented by projection operators in Hilbert space (describing yes-no experiments). (B) states that the system has probability one of being somewhere. (C) means that the projection operators corresponding to disjoint Borel sets commute. (D) asserts that the set of all states which are localized at time t in the intersection of any two given Borel sets is identical with the set of all states which are localized in both of these Borel sets at time t. Finally (E) expresses homogeneity and isotropy of the physical space R^3.

In the remainder of this section we are not interested in the time dependence of the notions that were introduced above. We shall therefore restrict our attention to a fixed value of t and omit this index.

A set of operators $\{U(\boldsymbol{a}, R), F_\Delta\}$ satisfying postulates (A)–(E) is called a *generalized system of imprimitivities* based on R^3. In particular instances it may occur that all of the F_Δ commute with each other. If this happens, one gets an ordinary system of imprimitivities. Such systems were defined by MACKEY [10] and applied to localizability by WIGHTMAN [9]. To fix the use of language, we state that by *localizability* we mean the existence of a generalized (or ordinary) system of imprimitivities which is also invariant under time reversal. If we wish to insist that a given system $\{U(\boldsymbol{a}, R), F_\Delta\}$ is ordinary or not, we shall talk about *ordinary* or *weak* localizability resp. For the projection operators of ordinary systems of imprimitivities, we shall sometimes write E_Δ instead of F_Δ.

An essential difference between ordinary and weak localizability was elucidated by JAUCH and PIRON [8]. An ordinarily localizable physical system can be described by a probability density in $\boldsymbol{x}$-space. For weakly localizable physical systems this is not true. The mathematical expression of these statements is as follows. Let Δ_1 and Δ_2 be two disjoint Borel sets. For ordinarily localizable systems one can infer that $F_{\Delta_1} + F_{\Delta_2} = F_{\Delta_1 \cup \Delta_2}$ for any such pair. In the case of weak localizability there exist pairs $\Delta_1 \perp \Delta_2$ for which $F_{\Delta_1} + F_{\Delta_2} < F_{\Delta_1 \cup \Delta_2}$. For photons the inexistence of such a probability density was announced a considerable length of time ago [15].

We shall often refer to the ensueing definition. A (generalized or ordinary) system of imprimitivities is said to be *irreducible* if there exists no non-trivial projection operator P in the Hilbert space under consideration satisfying

$$[P, U(\boldsymbol{a}, R)] = 0 \quad \text{for all } (\boldsymbol{a}, R)$$

and

$$[P, F_\Delta] = 0 \quad \text{for all } \Delta \subset R^3 \tag{5}$$

A reducible system of imprimitivities can always be split up into a direct sum of such irreducible ones.

To complete this section, we add two quite general comments about possible solutions F_Δ for localizability.

Let there be given a generalized (or ordinary) system of imprimitivities $\{U(\boldsymbol{a}, R), F_\Delta\}$ in a Hilbert space $\mathcal{H}$. Assume in addition that the representation $U(\boldsymbol{a}, R)$ is not irreducible, and let P be a projection operator that reduces $U(\boldsymbol{a}, R)$. If P commutes with all F_Δ as well, the system $\{U(\boldsymbol{a}, R), F_\Delta\}$ is reducible by P. On the other hand, if P does not commute with at least some of the F_Δ, one may define a different solution $\{U(\boldsymbol{a}, R), F'_\Delta\}$ of (A)–(E) (for the same representation U) by putting

$$F'_\Delta = P \cap F_\Delta \oplus (I - P) \cap F_\Delta . \tag{6}$$

Of course this new solution is reduced by P. We call $\{U(\boldsymbol{a}, R)\ P, P \cap F_\Delta\}$ the *reduction* of the original solution to the subspace $P\,\mathcal{H}$.

It is easy to understand the physical content of such a reduction. Let the F_Δ describe localizability in the space of the states of some physical system. We say that the vectors $\varphi^{(P)}$ lying in the subspace $P\,\mathcal{H}$ are distinguished by the attribute that they 'have the property P'. Of course in particular instances P will describe a well-determined physical characteristic (e.g. certain values of the helicity). A state $F_\Delta\,\varphi^{(P)}$ lies in general outside $P\,\mathcal{H}$, i.e. the property P is destroyed when one localizes $\varphi^{(P)}$.

However, all the vectors $(P \cap F_\Delta)\,\varphi^{(P)}$ remain in $P\,\mathcal{H}$. The eigenstates $\varphi_\Delta^{(P)} = (P \cap F_\Delta)\,\varphi_\Delta^{(P)}$ of $P \cap F_\Delta$ are such that they have the characteristic P and are likewise localized in Δ (i.e. they admit the simultaneous measurement of the two observables corresponding to P and to F_Δ). Therefore $P \cap F_\Delta$ is the operator describing localizability of the subsystem of $\mathcal{H}$ having the property P (as long as $[P, U(T)] = 0$).

From these observations we conclude that, if one wishes to localize a certain physical system without specifying it any further, one should look for an irreducible system of imprimitivities in the Hilbert space of all its possible states.

We now turn to the second remark. The postulates (A)–(F) can always be satisfied in a trivial way by defining

$$F_{\Delta,\,t} = 0 \quad \text{for} \quad \Delta \neq R^3\,, \qquad F_{R^3,\,t} = I\,. \tag{7}$$

It may occur that this trivial system of imprimitivities is the only solution of (A)–(F) for certain quantum-mechanical systems. It seems reasonable to call such an object unlocalizable. We shall use the specification 'not localizable' even in less trivial cases. Since position measurements are usually confined to finite regions of space, we shall call a quantum-mechanical system *unlocalizable* if there exist no states of it which are localized in a finite region, i.e. if $F_\Delta = 0$ for all bounded Borel sets Δ. In this definition we do not particularize whether F_Δ is zero or not for domains Δ that extend to infinity in some directions of space. Examples for unlocalizable systems are elementary particles with a given value ($\neq 0$) of the helicity, as we shall show later.

A physical system which is unlocalizable in the above sense may nevertheless be localizable as part of a more complex physical system. Let us assume that superpositions of its states with vectors of the other parts of the compound system are physically admissible (for example, photons of a given helicity are known to exist in superpositions with photons of the opposite helicity). Then it is conceivable that the complex system admits of localizability with $F_\Delta \neq 0$ for bounded Δ. The corresponding system of imprimitivities of the complex system would have the property that its reduction to the subspace of the part-system becomes trivial. The process of localization will not preserve one or several of the characteristics of the part-system. As an example for this, we shall find that superpositions of photons of both helicities are localizable. The measurement of the 'position' of photons of helicity $+1$ will convert them partly into photons of helicity -1.

III. Relativistic Elementary Systems (Results)

This part gives a survey of the realizations of postulates (A)–(F) for simple relativistic systems. All the definitions and mathematical details will be assembled in the subsequent sections.

We shall work from now on in the Heisenberg picture, so that the Hilbert spaces $\mathcal{H}_t$ may all be identified with some space $\mathcal{H}$. Relativistic quantum-mechanical systems necessitate the introduction of the representations of the group of relativistic transformations (the Poincaré group $\mathcal{P}$). Since scalar products should not depend on the Lorentz frame, its only representations of interest for physics are the unitary ones. This gives a restriction on the choice of $\mathcal{H}$. In the sequel $\mathcal{H}$ will always be the space corresponding to a unitary representation of $\mathcal{P}$. The irreducible such representations are said to describe *relativistic elementary systems* ('elementary' because every vector of $\mathcal{H}$ can be generated from an arbitrary fixed one by acting on it with the unitary operator belonging to some group element).

We denote the elements of the Poincaré group by (b, Λ) (b is a four-vector, Λ a Lorentz transformation). A unitary representation $U(b, \Lambda)$ of it in $\mathcal{H}$ includes the representation $U(\boldsymbol{a}, R)$ of its subgroup of Euclidean motions in R^3. $U(\boldsymbol{a}, R)$ satisfies the conditions of postulate (E) if one interpretes $U(b, \Lambda)$ in the active way. It is for this subrepresentation $U(\boldsymbol{a}, R)$ and for time $t = 0$ that we impose postulates (A)–(F). The time-evolution is described by the time-translations $U(t, I)$. Hence the solution $F_{\Delta, t}$ for $t \neq 0$ is connected with $F_{\Delta, 0}$ by

$$F_{\Delta, t} = U(t, I)\, F_{\Delta, 0}\, U(t, I)^{-1}\,. \tag{8}$$

(The relations between different Lorentz frames will be considered in Section VIII.)

The irreducible unitary representations of the Poincaré group fall into four types [14]: those of positive mass, of zero mass, of imaginary mass and those with energy-momentum $p \equiv 0$. We shall disregard the two last ones. Those of positive mass were treated extensively by Wightman [9]. His Theorem 6 asserts that such representations are always localizable in the ordinary sense. Beginning with his solution, one may construct generalized systems of imprimitivities by using Equation (6). Reducing subspaces can be obtained with the help of the helicity operator h, since $[h, U(\boldsymbol{a}, R)] = 0$ and $[h, U(T)] = 0$.

The irreducible representations of $\mathcal{P}$ for mass zero comprise two kinds: those of discrete spin denoted by $[0, s]$ ($s = 0, \pm 1/2, \pm 1, \ldots$) and those of continuous spin. Theorem 6 of Reference [9] affirms that an ordinary system of imprimitivities exists only for $[0, 0]$.

Our main issue is that the generalized system of imprimitivities for particles of mass zero and (discrete) spin s is isomorphic to the reduction onto the subspace of helicity s of the ordinary system of imprimitivities belonging to a representation of mass $m > 0$ and spin $|s|$. In this sense localizability involves no distinction between particles of positive mass and those of mass zero.

We shall also prove that particles of one value (different from zero) of the helicity are unlocalizable (i.e. a strict position measurement of such a particle in a finite region of space will necessarily involve a partial change of its helicity). The two-component neutrinos are therefore not localizable. Photons are observed in superposition states of both helicities. It will be established that all particles having this same property are localizable (in the weak sense for $s \neq 1/2$). We collect these statements in two theorems:

Theorem 1: The irreducible representations $[0, s]$, $s \neq 0$, of the Poincaré group are not localizable.

Theorem 2: The direct sum $[0, s] \oplus [0, -s]$ of two irreducible representations of opposite helicity is localizable for all values of $s = 1/2, 1, 3/2, \ldots$ For $s \neq 1/2$, the system of imprimitivities is not ordinary.

The representations of continuous spin have acquired no physical interest. They are weakly localizable, and this in an infinite number of ways. All such generalized systems of imprimitivities are reducible.

IV. The Unitary Representations of the Poincaré Group

This part is a purely mathematical one. We discuss here the structure that will be essential for the subsequent sections, namely the continuous unitary irreducible representations of the Poincaré group. These were found by WIGNER [14]. We shall outline very briefly the relevant points relating to their construction and classification. The reader who is already familiar with these ideas may skip this section.

The elements of the Poincaré group are formed of a four-vector b and a homogeneous Lorentz transformation Λ and have the multiplication law [16]

$$(b_1, \Lambda_1)\,(b_2, \Lambda_2) = (b_1 + \Lambda_1\, b_2, \Lambda_1\,\Lambda_2)\;.$$

We shall write P^0, $\boldsymbol{P}$ for the infinitesimal generators of the translations, $\boldsymbol{J}$ for those of the space-rotations and $\boldsymbol{K}$ for those of the pure Lorentz transformations.

The set of all complex 2×2 matrices A of determinant 1 is denoted by SL(2, C). Furthermore, the restricted Lorentz group $L_+^\uparrow$ is defined as the set of all Lorentz transformations which can be continuously connected to the unit transformation I (i.e. which contain no reflections). Using the Pauli matrices σ_μ ($\mu = 0, 1, 2, 3$), the formula

$$[\Lambda(A)]^\varrho_{\ \nu} = \frac{1}{2}\, Tr(\sigma_\varrho\, A\, \sigma_\nu\, A^\dagger) \qquad [^\dagger = \text{adjoint}] \tag{9}$$

defines a 2–1 homomorphism of SL(2, C) onto $L_+^\uparrow$. It is often easier to calculate with SL(2, C), so that one usually considers the two-sheeted covering group $\mathcal{R}$ of the restricted Poincaré group $\mathcal{P}_+^\uparrow$ rather than the latter one. The elements of $\mathcal{R}$ are pairs consisting of a four-vector b and a matrix $A \in$ SL(2, C). It has been proved that every continuous unitary representation up to a factor of $\mathcal{P}_+^\uparrow$ can be obtained from a continuous unitary representation of $\mathcal{R}$. We shall therefore restrict our attention to these latter ones. They are constructed in the following manner.

One first considers the abelian subgroup of all translations (b, I), where I is the 2×2 identity matrix. Every continuous unitary representation of it is unitarily equivalent to one of the form

$$[U(b, I)\,\phi]\,(p) = e^{i p\cdot b}\,\phi(p) \qquad (p\cdot b = p^\nu\, b_\nu) \tag{10}$$

in a Hilbert space which is a direct integral of spaces $\mathcal{H}_p$ (for all possible values of the energy-momentum four-vector p):

$$\mathcal{H} = \int_\oplus d\mu(p)\;\mathcal{H}_p\;.$$

If one takes into account that this must be the restriction of a representation of $\mathcal{R}$, one arrives at the conclusion that the spaces $\mathcal{H}_p$ for different p are all isomorphic, and that $d\mu(p) = d\mu(\Lambda\, p)$ for all $\Lambda \in L_+^\uparrow$. The structure of such a (quasi-invariant) measure μ is the following

$$\mu = c\,\delta(p) \oplus \int_0^\infty d\varrho_+(m)\; d\Omega_m(p) \oplus \int_0^\infty d\varrho_-(m)\; d\Omega_m(p) \oplus \int_0^\infty d\varrho(i\,m)\; d\Omega_{i\,m}(p) \qquad c \geqslant 0 \tag{11}$$

$d\Omega_m(p) = (\boldsymbol{p}^2 + m^2)^{-1/2}\, d^3p$ is the invariant measure on the hyperboloid $p^2 = m^2$ (with $p_0 > 0$ in the second and $p_0 < 0$ in the third term), and $d\Omega_{i\,m}(p)$ the invariant measure on the hyperboloid $p^2 = -m^2$. $d\varrho_+(m)$ and $d\varrho_-(m)$ are positive measures on the positive real semi-axis, $d\varrho(i\,m)$ is such a measure on the positive imaginary semi-axis.

One next represents $U(0, A)$ in such a Hilbert space (11). This is possible in the form

$$[U(0, A)\,\phi]\,(p) = Q(p, A)\,\phi(\Lambda(A)^{-1}\,p) \tag{12}$$

where $Q(p, A)$ is a unitary operator acting in the space $\mathcal{H}_p$ which may depend on p and on A. The multiplication law $U(0, A)\,U(0, B) = U(0, A\,B)$ implies for these operators

$$Q(p, A)\,Q(\Lambda(A)^{-1}\,p, B) = Q(p, A\,B)\,. \tag{13}$$

This is the only property that the $Q(p, A)$ must satisfy. One looks, however, for the most convenient form of them. For this purpose we must introduce a few more concepts.

If A is such that $\Lambda(A)^{-1}\,p = p$, Equation (13) reads

$$Q(p, A)\,Q(p, B) = Q(p, A\,B)\,. \tag{14}$$

Hence $Q(p, A)$ must be (for every fixed p) a representation of the group of all matrices $A \in \mathrm{SL}(2, \mathrm{C})$ for which $\Lambda(A)\,p = p$. The set of all such matrices is called the *little group* G_p of p. We write for its representations

$$\boldsymbol{Q}(A) \equiv Q(p, A)\,. \tag{15}$$

For any Lorentz-transformation $\Lambda \in L_+^\uparrow$, the little group of a vector p and that of $\Lambda\,p$ are isomorphic. The set of all four-vectors $\{p' \mid p' = \Lambda\,p, \Lambda \in L_+^\uparrow\}$ is called an *orbit*. It is characterized by the length of the vectors p': $p'^2 = p^2 = \text{const}$. There are four types of orbits:

$$\begin{aligned} &O_0^0 = \{p \mid p^0 = 0,\ p^2 = 0\} = \{0\}\,, && O_{m^2}^+ = \{p \mid p^0 > 0, p^2 = m^2,\ m \geqslant 0\} \\ &O_{m^2}^- = \{p \mid p^0 < 0,\ p^2 = m^2,\ m \geqslant 0\}\,, && O_{-m^2} = \{p \mid p^2 = -m^2,\ m > 0\}\,. \end{aligned} \tag{16}$$

They are related to the four contributions in Equation (11): the support of the measure μ is a (set-theoretic) union of such orbits. A representation of $\mathcal{R}$ can be irreducible only if the measure μ is concentrated on one of these orbits.

We return now to the question of choosing a suitable form for $Q(p, A)$. As we are interested in irreducible representations of $\mathcal{R}$, we assume that all vectors p appearing in (12) lie on some given orbit. On such an orbit, one then fixes an arbitrary vector k. Every other vector p on it can be obtained by applying to k any Lorentz-transformation $\Lambda_{p\leftarrow k}$ satisfying $\Lambda_{p\leftarrow k}\,k = p$. Rather, one selects for every vector p on this orbit a matrix $A_{p\leftarrow k} \in \mathrm{SL}(2, \mathrm{C})$ such that $\Lambda(A_{p\leftarrow k})\,k = p$. Obviously $A_{p\leftarrow k}^{-1}\,B\,A_{B^{-1}p\leftarrow k} \in G_k$ for all $B \in \mathrm{SL}(2, \mathrm{C})$. Furthermore, it is possible to make a unitary transformation in $\mathcal{H}_p$ with the result that $Q(p, B)$ is, for any p, given solely by a representation $\boldsymbol{Q}$ of the little group G_k of k. This leads then to the following expression for $Q(p, B)$

$$Q(p, B) = \boldsymbol{Q}(A_{p\leftarrow k}^{-1}\,B\,A_{B^{-1}\,p\leftarrow k})\,. \tag{17}$$

With this we come to the conclusion that every continuous unitary irreducible representation of $\mathcal{R}$ is unitarily equivalent to one of the form

$$[U(b, B)\,\phi]\,(p) = e^{i p\cdot b}\,\boldsymbol{Q}(A_{p\leftarrow k}^{-1}\,B\,A_{B^{-1}\,p\leftarrow k})\,\phi(\Lambda(B)^{-1}\,p) \tag{18}$$

where Q is a continuous unitary irreducible representation of the little group G_k of k, and the functions $\phi(p)$ are defined on the orbit containing k. The norm in the Hilbert space is deduced from the invariant measure on this orbit.

Different choices of the stabilized vector k on the orbit under consideration or of the matrices $A_{p \leftarrow k}$ lead to unitarily equivalent representations of $\mathcal{R}$ [18]. The functions $\phi(p)$ may be multi-component functions, depending on the representation Q of G_k. In such cases the scalar product includes a summation over the index labelling the different components.

As we mentioned before, the representations connected with the orbits O_0^0 and O_{-m^2} have no physical interest. Those based on $O_{m^2}^-$ correspond to particles of mass m and negative energy. They differ only by the sign of the energy from those defined on $O_{m^2}^+$. The sections that follow shall deal only with the representations based on the orbits $O_{m^2}^+$.

A unitary representation of $\mathcal{R}$ based on an orbit $O_{m^2}^+$ can be extended to a representation of the entire group which includes the reflections. Here we are interested only in adding the operator $U(T)$ corresponding to the time reversal T (leaving aside the parity operator and the total inversion). From the fact that the energy of an elementary system is always positive, it follows that $U(T)$ must be anti-unitary. The addition of $U(T)$ to a unitary representation of $\mathcal{R}$ either leaves the spaces $\mathcal{H}_p$ unchanged, or else the dimension of each $\mathcal{H}_p$ must be doubled. We shall treat only representations of the former type here, since those which require the doubling of $\mathcal{H}_p$ do not seem to be realized in nature. For the first type of representations one finds [17]

$$[U(T)\,\phi]\,(p^0, \boldsymbol{p}) = Q(A_{p \leftarrow k}^{-1}\, \Gamma\, A_{\Gamma^{-1} p \leftarrow k})\; \phi^*(p^0, -\boldsymbol{p}) \tag{19}$$

where

$$\Gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and belongs to SU(2, C).

V. Localizability for Particles of Discrete Spin

The physically interesting irreducible unitary representations of the covering group of the Poincaré group are those of positive or zero mass. We shall therefore discuss them in greater detail here. In order to define localizability, we must consider their subrepresentations of ε_3, the two-sheeted covering group of the group of Euclidean motions in R^3. In this way, we shall arrive at the construction of the operators F_Δ for positive mass, a construction which is due to Wightman. Next, we shall study the representations of ε_3 arising for mass zero. They will result to be unitarily equivalent to those of positive mass restricted to the subspace of states of a fixed helicity. Hence particles of mass zero and spin s are localized in the same way as particles of positive mass, spin $|s|$ and helicity s: The operators F_Δ for the former are unitarily equivalent to the reduction of Wightman's operators for the latter to the subspace corresponding to helicity s.

We treat first the case $m > 0$. It is simplest to stabilize the point $k = (m, 0, 0, 0)$ on the orbit $O_{m^2}^+$. This k remains unchanged under all space rotations. The little group G_k is therefore the covering group of the group of rotations in R^3. It corresponds to all unitary matrices of SL(2, C) and is named SU(2, C). Its continuous unitary irreducible representations are denoted D^J, labelled by an index $J = 0, 1/2, 1, 3/2, \dots$ They are all finite-dimensional. It is usual to write them as $(2J+1) \times (2J+1)$ matrices $D^J_{\alpha\beta}(B)$, $\alpha, \beta = -J, -J+1, -J+2, \dots, +J$ and $B \in SU(2, C)$. In this form they are defined recursively by means of Clebsch-Gordan coefficients [19]:

$$D^0(B) = 1\,, \quad D^{1/2}_{\alpha\beta}(B) = B_{\alpha\beta}\,. \tag{20}$$

$$D^J_{\alpha\beta}(B) = \sum_{\gamma\delta\varepsilon\varkappa} C\left(J - \frac{1}{2}, \frac{1}{2}, J \mid \gamma, \delta, \alpha\right) C\left(J - \frac{1}{2}, \frac{1}{2}, J \mid \varepsilon, \varkappa, \beta\right) D^{J-1/2}_{\gamma\varepsilon}(B)\, B_{\delta\varkappa}$$

where the notation of the Clebsch-Gordan coefficients is that of Rose [19].

The irreducible unitary representation $[m, J]$ is therefore obtained by setting $\boldsymbol{Q} = D^J$ in Equation (18). The measure in the Hilbert space $\mathcal{H}^{[m,J]}$ is

$$d\Omega_m(\boldsymbol{p}) = \frac{d^3p}{\sqrt{\boldsymbol{p}^2 + m^2}}\,. \tag{21}$$

The functions $\phi(p)$ have $2J+1$ components $\phi_\beta(p)$, $\beta = -J, \dots, +J$. The four variables p^0, p^1, p^2, p^3 are not independent. They are related by

$$p^0 = \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2 + m^2}\,. \tag{22}$$

The integral in (21) extends only over the three independent ones p^1, p^2 and p^3, with the specification that p^0 be replaced everywhere by (22). In order to remind ourselves of this rule, we shall henceforth write $\phi(\boldsymbol{p})$ for these functions.

It remains to select the matrices $A_{p\leftarrow k}$. Two of the possible choices will prove vital for defining localizability. We deal with them separately.

(a) $A_{p\leftarrow k}$ represents the pure Lorentz transformation mapping k into $p = (p^0, \boldsymbol{p})$ [16, 18]:

$$A^c_{p\leftarrow k} = \frac{1}{\sqrt{2m\,(p^0+m)}} \begin{pmatrix} m + p^0 + p^3 & p^1 - i\,p^2 \\ p^1 + i\,p^2 & m + p^0 - p^3 \end{pmatrix} \tag{23}$$

The superscript c indicates that this choice leads to the *canonical formalism:* The subscript β on the state vectors $\phi^c_\beta(\boldsymbol{p})$ labels the values of the 3-component of angular momentum:

$$J^3\, \phi^c_\beta(\boldsymbol{p}) = \beta\, \phi^c_\beta(\boldsymbol{p})\,.$$

For pure space rotations, $B \in SU(2, C)$, one may verify that [16]

$$A^{c-1}_{p\leftarrow k}\, B\, A^c_{B^{-1}p\leftarrow k} = B$$

and hence

$$Q(p, B) = D^J(B)\,.$$

The representation of ε_3 induced by $[m, J]$ is therefore the following

$$[U(\boldsymbol{a}, B)\,\phi]^c_\beta(\boldsymbol{p}) = e^{i\boldsymbol{p}\cdot\boldsymbol{a}} \sum_{\gamma=-J}^{J} D^J_{\beta\gamma}(B)\, \phi^c_\gamma(R(B)^{-1}\,\boldsymbol{p})\,. \tag{24}$$

For the time reversal one finds from (19)

$$[U(T)\,\phi]^c_\beta(\boldsymbol{p}) = \sum_{\gamma=-J}^{+J} D^J_{\beta\gamma}(\Gamma)\, \phi^{c*}_\gamma(-\boldsymbol{p})\,. \tag{25}$$

We shall now construct the ordinary system of imprimitivities for the representation (24) of ε_3. The functions

$$\varphi_\beta(\boldsymbol{p}) \equiv \frac{1}{\sqrt{P^0}}\, \phi^c_\beta(\boldsymbol{p}) = (\boldsymbol{p}^2 + m^2)^{-1/4}\, \phi^c_\beta(\boldsymbol{p}) \tag{26}$$

are square-integrable:

$$\sum_{\beta=-J}^{+J} \int d^3p \mid \varphi_\beta(\boldsymbol{p}) \mid^2 = \sum_{\beta=-J}^{+J} \int \frac{d^3p}{\sqrt{p^2+m^2}} \mid \phi^c_\beta(\boldsymbol{p}) \mid^2 < \infty .$$

Consequently, their three-dimensional Fourier transforms

$$\tilde{\varphi}_\beta(\boldsymbol{x}) \equiv (\mathcal{F}\, \varphi)_\beta\, (\boldsymbol{x}) = (2\,\pi)^{-3/2} \int d^3p\; e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\, \varphi_\beta(\boldsymbol{p}) \tag{27}$$

lie in the space $L^2_J(\boldsymbol{x})$ of square-integrable $(2\,J + 1)$-component functions over $\boldsymbol{R}^3$. (24) induces, together with the isomorphisms (26) and (27), a unitary representation of ε_3 in $L^2_J(\boldsymbol{x})$ of a well-known form:

$$[U(\boldsymbol{a}, B)\, \tilde{\varphi}]_\beta\, (\boldsymbol{x}) = \sum_{\gamma=-J}^{+J} D^J_{\beta\gamma}(B)\, \tilde{\varphi}_\gamma\, [R(B)^{-1}\, (\boldsymbol{x} - \boldsymbol{a})] . \tag{28}$$

For any Borel set Δ of $\boldsymbol{R}^3$, let χ_Δ denote its characteristic function:

$$\chi_\Delta(\boldsymbol{x}) = \begin{cases} 1 \text{ for } \boldsymbol{x} \in \Delta \\ 0 \text{ for } \boldsymbol{x} \notin \Delta \end{cases}.$$

The imprimitive projection operators E_Δ for (28) in the space $L^2_J(\boldsymbol{x})$ are given by

$$[E_\Delta\, \tilde{\varphi}]_\beta\, (\boldsymbol{x}) = \chi_\Delta(\boldsymbol{x})\, \tilde{\varphi}_\beta(\boldsymbol{x}) . \tag{29}$$

Wightman [9] proved that every irreducible ordinary system of imprimitivities for ε_3 satisfying also postulate (F) is unitarily equivalent to one of the form (28), (29). The argument $\boldsymbol{x}$ in these equations is the position of Newton and Wigner [2].

We take the inverse of the isomorphisms (26) and (27) in order to write explicitly the corresponding operators E_Δ in $\mathcal{H}^{[m, J]}$:

$$[E^c_\Delta\, \phi]^c_\beta\, (\boldsymbol{p}) = \sqrt{P^0}\, \mathcal{F}^{-1}\, E_\Delta\, \mathcal{F} (P^0)^{-1/2}\, \phi^c_\beta(\boldsymbol{p}) \tag{30}$$

(b) Our second choice for $A_{p\leftarrow k}$ consists of the product of a pure Lorentz transformation $A^c_{p_z\leftarrow k}$ mapping k into $p_z = (p^0, 0, 0, |\boldsymbol{p}|)$ and a space-rotation $X_{p\leftarrow p_z}$ in the plane $\{\boldsymbol{p}/\boldsymbol{0}\ \boldsymbol{z}\}$ turning p_z into p [18]:

$$A^h_{p\leftarrow k} = X_{p\leftarrow p_z}\, A^c_{p_z\leftarrow k} . \tag{31}$$

The rotation $X_{p\leftarrow p_z}$ is described by the unitary matrix

$$X_{p\leftarrow p_z} = \frac{1}{\sqrt{2\,|\boldsymbol{p}|\,(|\boldsymbol{p}|+p^3)}} \begin{pmatrix} |\boldsymbol{p}| + p^3 & -\,p^1 + i\,p^2 \\ p^1 + i\,p^2 & |\boldsymbol{p}| + p^3 \end{pmatrix}. \tag{32}$$

Here we are in the *helicity formalism*. The subscript β on the state-vectors corresponds to the helicity of the component ϕ^h_β of ϕ:

$$\left[\frac{\boldsymbol{J}\cdot\boldsymbol{P}}{|\boldsymbol{P}|}\, \phi\right]^h_\beta(\boldsymbol{p}) = \beta\, \phi^h_\beta(\boldsymbol{p}) .$$

The helicity operator $h = \boldsymbol{J}\cdot\boldsymbol{P}/|\boldsymbol{P}|$ commutes with space rotations. As a consequence we observe that the matrix $D^J_{\alpha\beta}(A^{h-1}_{p\leftarrow k}\, B\, A^h_{B^{-1}p\leftarrow k})$ is diagonal for $B \in \mathrm{SU}(2, \mathrm{C})$. One may parametrize SU(2, C) by two complex numbers:

$$B = \begin{pmatrix} v & w \\ -w^* & v^* \end{pmatrix} \qquad |v|^2 + |w|^2 = 1 \,. \tag{33}$$

Using (20) for D^J and (31) for $A^h_{p\leftarrow k}$, the diagonal elements just mentioned are ascertained to be

$$D^J_{\beta\beta}(A^{h-1}_{p\leftarrow k}\, B\, A^h_{B^{-1}p\leftarrow k}) = \left[\frac{(|\boldsymbol{p}|+p^3)\, v - (p^1 - i\, p^2)\, w^*}{|(|\boldsymbol{p}|+p^3)\, v - (p^1 - i\, p^2)\, w^*|}\right]^{2\beta} . \tag{34}$$

From (19) and (31) one deduces that $U(T)$ becomes also diagonal in the helicity formalism:

$$[U(T)\, \phi]^h_\beta(\boldsymbol{p}) = \left(\frac{p^1 - i\, p^2}{p^1 + i\, p^2}\right)^\beta \phi^{h*}_\beta(-\boldsymbol{p}) \,. \tag{35}$$

The prominent feature of Equations (34) and (35) is that their right-hand sides are independent of J.

The canonical and the helicity formalisms are unitarily equivalent. Since the components of ϕ^c diagonalize the angular momentum in 3-direction and those of ϕ^h the angular momentum along $\boldsymbol{p}$, it is almost evident that the unitary transformation between the two formalisms is defined through $X_{p\leftarrow p_z}$. Indeed

$$\phi^c_\alpha(\boldsymbol{p}) = \sum_{\beta=-J}^{+J} D^J_{\alpha\beta}(X_{p\leftarrow p_z})\, \phi^h_\beta(\boldsymbol{p}) \,. \tag{36}$$

Combining (36) and its inverse with (30), one can formulate the result of E_Δ acting on helicity states:

$$[E^h_\Delta \phi]^h_\alpha(\boldsymbol{p}) = \sum_{\beta,\gamma=-J}^{+J} D^J_{\alpha\beta}(X^{-1}_{p\leftarrow p_z})\, \sqrt{P^0}\, \mathfrak{F}^{-1} E_\Delta\, \mathfrak{F}\, \frac{1}{\sqrt{P^0}}\, D^J_{\beta\gamma}(X_{p\leftarrow p_z})\, \phi^h_\gamma(\boldsymbol{p}) \tag{37}$$

where E_Δ is given by (29), and $P^0 = \sqrt{\boldsymbol{P}^2 + m^2}$.

We now turn our attention to the representations of mass $m = 0$. The orbit O^+_0 is the forward light-cone $p^\nu p_\nu = 0, p^0 > 0$. Let us stabilize the point $k = (1/2, 0, 0, 1/2)$. Rotations around the 3-axis and pure Lorentz transformations along the 1- and 2-axis leave k fixed. These determine the little group G_k. It is isomorphic to the two-sheeted covering group ε_2 of the Euclidean group of the plane R^2 (WIGNER [14]). The infinitesimal generators of ε_2 are called S for the rotations and T_1, T_2 for the translations. The structure of ε_2 is similar to that of $\mathcal{R}$. Its irreducible representations are based on orbits $T_1^2 + T_2^2 = r^2 =$ const. (in the same way as $P_0^2 - \boldsymbol{P}^2 = m^2$ for $\mathcal{R}$). Again they differ for $r > 0$ and $r = 0$. One arrives at

(a) Infinite-dimensional representations, labelled by two indices (ε, r) with $\varepsilon = \pm 1$ and $r > 0$. They give rise to the representations $[0, \varepsilon, r]$ of $\mathcal{R}$, which are known as representations of *infinite* (*or continuous*) *spin*. They are single-valued for $\varepsilon = +1$ and double-valued for $\varepsilon = -1$.

(b) One-dimensional representations for $r = 0$, labelled by an index s whose possible values are $s = 0, +1/2, -1/2, +1, -1, \ldots$ These lead to the representations $[0, s]$ of $\mathcal{R}$, called representations of *discrete spin s*.

We shall treat the representations $[0, \varepsilon, r]$ of continuous spin in Section VII and proceed to discuss $[0, s]$. Any one-dimensional unitary representation is just multiplication by a complex number of modulus 1. One can show that $Q(p, B)$ of Equation (17) springing from the one-dimensional unitary representations of G_k takes the form [20]

$$Q(p, B) = \boldsymbol{Q}(A^{-1}_{p\leftarrow k}\, B\, A_{B^{-1}p\leftarrow k}) = [\varkappa(p, B)]^{2s} \tag{38}$$

where $\varkappa(p, B)$ is a unimodular complex number depending on p and on $B \in SL(2, C)$.

Let $\mathcal{H}^{[0, s]}$ denote the representation space of $[0, s]$. The state vectors $\phi(\boldsymbol{p}) \in \mathcal{H}^{[0, s]}$ are one-component functions defined on the forward lightcone ($p^2 = 0$, $p^0 > 0$). The invariant measure is

$$d\Omega_0(\boldsymbol{p}) = \frac{d^3p}{|\boldsymbol{p}|}\,. \tag{39}$$

Again there are several possibilities open for selecting $A_{p\leftarrow k}$ in (38) (they lead, however, to the same $\varkappa(p, B)$). We adopt the choice of Bargmann [20]:

$$A_{p\leftarrow k} = \frac{1}{\sqrt{|\boldsymbol{p}| + p^3}} \begin{pmatrix} |\boldsymbol{p}| + p^3 & 0 \\ p^1 + i\,p^2 & 1 \end{pmatrix}. \tag{40}$$

If one parametrizes SL(2, C) by four complex numbers as

$$B = \begin{pmatrix} v & w \\ t & u \end{pmatrix} \qquad v\,u - t\,w = 1 \tag{41}$$

one obtains [20]

$$\varkappa(p, B) = \frac{(|\boldsymbol{p}| + p^3)\,u^* - (p^1 - i\,p^2)\,w^*}{|(|\boldsymbol{p}| + p^3)\,u^* - (p^1 - i\,p^2)\,w^*|}\,. \tag{42}$$

In order to define localizability in $\mathcal{H}^{[0, s]}$ one has to consider the subrepresentation of (42) for $B \in SU(2, C)$. The unitarity of B implies in (41)

$$u = v^*, \qquad t = -\,w^* \tag{43}$$

and one is led back to the parametrization (33) of SU(2, C). Inserting (43) and (42) into (38) one finds for $B \in SU(2, C)$

$$Q(p, B) = [\varkappa(p, B)]^{2s} = \left[\frac{(|\boldsymbol{p}| + p^3)\,v - (p^1 - i\,p^2)\,w^*}{|(|\boldsymbol{p}| + p^3)\,v - (p^1 - i p^2)\,w^*|}\right]^{2s}. \tag{44}$$

For the time reversal

$$[U(T)\,\phi]\,(\boldsymbol{p}) = \left(\frac{p^1 - i\,p^2}{p^1 + i\,p^2}\right)^s \phi^*(-\,\boldsymbol{p})\,. \tag{45}$$

It is very important to notice that (44) is identical with $D^J_{s\,s}(A^{h\,-1}_{p\leftarrow k}\, B\, A^h_{B^{-1}p\leftarrow k})$ of Equation (34) for arbitrary values of $J = |s|, |s| + 1, |s| + 2, \ldots$:

$$[\varkappa(p, B)]^{2s} = D^J_{s\,s}(A^{h\,-1}_{p\leftarrow k}\, B\, A^h_{B^{-1}p\leftarrow k})\,. \tag{46}$$

This means that the representation of ε_3 in $\mathcal{H}^{[0, s]}$ is the same as the one in the subspace $\mathcal{H}^{[m, J]}_{(s)}$ of states of helicity s of $[m, J]$ for any m and all $J = |s|, |s| + 1, |s| + 2, \ldots$ (At the same time (45) coincides with (35) if we set $\beta = s$ in (35).) Actually, the measures $d\Omega_0(\boldsymbol{p})$ and $d\Omega_m(\boldsymbol{p})$ in these two Hilbert spaces respectively differ from one another. Indeed

$$d\Omega_0(\boldsymbol{p}) = \frac{d^3p}{\sqrt{\boldsymbol{p}^2}} \quad \text{and} \quad d\Omega_m(\boldsymbol{p}) = \frac{d^3p}{\sqrt{\boldsymbol{p}^2 + m^2}}\,.$$

However, both spaces may be mapped isomorphically onto $L^2(\boldsymbol{p})$:

$$i\colon\ \phi(\boldsymbol{p}) \in \mathcal{H}^{[0,s]} \to (\boldsymbol{p}^2)^{-1/4}\,\phi(\boldsymbol{p}) \in L^2(\boldsymbol{p}) \tag{47a}$$

$$j\colon\ \phi^h_s(\boldsymbol{p}) \in \mathcal{H}^{[m,J]}_{(s)} \to (\boldsymbol{p}^2 + m^2)^{-1/4}\,\phi^h_s(\boldsymbol{p}) \in L^2(\boldsymbol{p})\,. \tag{47b}$$

The isomorphisms i and j may also be interpreted as acting on the respective state vectors by $(P^0)^{-1/2}$. Since P^0 commutes with all space rotations (and with all translations), these isomorphisms do not change the form of $U(\boldsymbol{a}, B)$ for $B \in \mathrm{SU}(2, \mathrm{C})$ in either case. In $L^2(\boldsymbol{p})$, $\varkappa(p, B)$ is still given by (42), and $D^J_{s\,s}(A^{h-1}_{p\leftarrow k}\, B\, A^h_{B^{-1}p\leftarrow k})$ maintains its form (34). Hence the equality (46) holds true in $L^2(\boldsymbol{p})$, and the two representations of ε_3 induced in this space by i and j are identical (irrespective of m and J). The same conclusion may be inferred for $U(T)$.

One may now specify an infinite number of generalized systems of imprimitivities $\{U(\boldsymbol{a}, B), F^{[J]}_\Delta\}$ in $\mathcal{H}^{[0,s]}$ for the representation of ε_3 obtained from (44) (and $U(T)$ given by Equation (45)). One chooses arbitrarily $m > 0$ and a $J = |s|, |s| + 1, \ldots$ Let $F^{[m,J]}_{\Delta\,(s)}$ denote the reduction of the operators E^h_Δ of this representation $[m, J]$ onto the subspace $\mathcal{H}^{[m,J]}_{(s)}$ of the states of helicity s (in the helicity formalism!) The product of the two isomorphisms

$$\mathcal{H}^{[m,J]}_{(s)} \xrightarrow[j]{} L^2(\boldsymbol{p}) \xrightarrow[i^{-1}]{} \mathcal{H}^{[0,s]}$$

maps $F^{[m,J]}_{\Delta\,(s)}$ into $F^{[J]}_\Delta$:

$$F^{[J]}_\Delta = i^{-1}\, j\, F^{[m,J]}_{\Delta\,(s)}\, j^{-1}\, i = (\boldsymbol{P}^2)^{1/4}\,(\boldsymbol{P}^2 + m^2)^{-1/4}\, F^{[m,J]}_{\Delta\,(s)}\,(\boldsymbol{P}^2 + m^2)^{1/4}\,(\boldsymbol{P}^2)^{-1/4}\,. \tag{48}$$

These generalized systems of imprimitivities are not identical, i.e. in general $F^{[J]}_\Delta \neq F^{[J']}_\Delta$ for $J \neq J'$. However, a representation $[0, s]$ corresponds to particles of spin $|s|$. Its position observables F_Δ are therefore obtained by taking $J = |s|$, i.e. $F_\Delta \equiv F^{[|s|]}_\Delta$. We must therefore set $J = |s|$ in (48). The operators $F^{[m,|s|]}_{\Delta\,(s)}$ occurring in that equation may be calculated from (37). If $\phi^h_\gamma(\boldsymbol{p})$ describes a state in $\mathcal{H}^{[m,|s|]}$ of helicity s, then $\phi^h_\gamma(\boldsymbol{p}) = 0$ for $\gamma \neq s$. This permits to omit the summation over γ in (37) by merely putting $\gamma = s$. An eigenstate $\phi^h_s(\boldsymbol{p})$ of $F^{[m,|s|]}_{\Delta\,(s)}$ (with eigenvalue 1) then satisfies the equation

$$\phi^h_s(\boldsymbol{p}) = \sum_{\beta=-s}^{+s} D^{|s|}_{s\,\beta}\,(X^{-1}_{p\leftarrow p_z})\,(\boldsymbol{P}^2 + m^2)^{1/4}\,\mathfrak{F}^{-1}\,E_\Delta\,\mathfrak{F}\cdot(\boldsymbol{P}^2 + m^2)^{-1/4}\,D^{|s|}_{\beta\,s}(X_{p\leftarrow p_z})\,\phi^h_s(\boldsymbol{p})\,. \tag{49}$$

Combining this with (48), we see that $\phi \in \mathcal{H}^{[0,s]}$ is localized in Δ, $F_\Delta\,\phi = \phi$, if and only if

$$\phi(\boldsymbol{p}) = \sum_{\beta=-s}^{+s} D^{|s|}_{s\,\beta}(X^{-1}_{p\leftarrow p_z})\,(\boldsymbol{P}^2)^{1/4}\,\mathfrak{F}^{-1}\,E_\Delta\,\mathfrak{F}(\boldsymbol{P}^2)^{-1/4}\cdot D^{|s|}_{\beta\,s}(X_{p\leftarrow p_z})\,\phi(\boldsymbol{p}) \tag{50}$$

where E_Δ is given in (29).

Let us introduce the functions

$$\psi_\beta(\boldsymbol{p}) = \frac{1}{\sqrt{|\boldsymbol{p}|}}\, D^{|s|}_{\beta\,s}(X_{p\leftarrow p_z})\,\phi(\boldsymbol{p})\,. \tag{51}$$

One remembers that $F^{[m,|s|]}_{\Delta\,(s)}$ was the intersection of two projection operators. These manifest themselves also in Equation (50). $\mathfrak{F}^{-1}\,E_\Delta\,\mathfrak{F}$ acting on the ψ_β corresponds to the projection onto the states localized in Δ. The index s in $D^{|s|}_{s\beta}(X^{-1}_{p\leftarrow p_z})$ expresses the one onto the subspace of helicity s. However, the representation $[0, s]$ involves states of helicity s only. As a consequence of this, the second projection becomes trivial in (50). To be explicit: if $\psi_\beta(\boldsymbol{p})$ is an eigenstate of $\mathfrak{F}^{-1} E_\Delta \mathfrak{F}$, i.e. if $\mathfrak{F}^{-1} E_\Delta \mathfrak{F} \psi_\beta(\boldsymbol{p}) = \psi_\beta(\boldsymbol{p})$, then $\phi(\boldsymbol{p})$ satisfies (50), since

$$\sum_{\beta=-s}^{+s} D_{s\beta}^{|s|}(X_{p\leftarrow p_z}^{-1})\ (\boldsymbol{P}^2)^{1/4}\ (\boldsymbol{P}^2)^{-1/4}\ D_{\beta s}^{|s|}(X_{p\leftarrow p_z}) = 1\ .$$

From these remarks and Equation (29) we conclude:

$\phi(\boldsymbol{p})$ is localized in Δ if and only if the three-dimensional Fourier transforms

$$\tilde{\psi}_\beta(\boldsymbol{x}) = (\mathfrak{F}\,\psi_\beta)\ (\boldsymbol{x})$$

of the functions (51) *have support in Δ for all values of $\beta = -s, -s+1, \dots, +s$.*

Theorem 1 states that for $s \neq 0$ and any bounded region Δ of R^3, there exists no vector $\phi \neq 0$ satisfying these requirements.

The operators F_Δ for the reducible representations $[0, s] \oplus [0, -s]$ in $\mathcal{H}^{[0,s]} \oplus \mathcal{H}^{[0,-s]}$ can be derived in a manner completely analogous to the one that we just elucidated for $[0, s]$. The only difference is that $\phi = (\phi_+, \phi_-)$ has then two components (which correspond to positive and negative helicity resp.), and that the projection in $\mathcal{H}^{[m,|s|]}$ onto the states of helicity s is replaced by that onto the vectors of helicity $\pm s$. (51) is replaced by

$$\Sigma_\beta(\boldsymbol{p}) = \frac{1}{\sqrt{|\boldsymbol{p}|}}\, D_{\beta|s|}^{|s|}(X_{p\leftarrow p_z})\ \phi_+(\boldsymbol{p}) + \frac{1}{\sqrt{|\boldsymbol{p}|}}\, D_{\beta-|s|}^{|s|}(X_{p\leftarrow p_z})\ \phi_-(\boldsymbol{p}) \tag{52}$$

since $\phi_+(\boldsymbol{p})$ and $\phi_-(\boldsymbol{p})$ are superposed. (50) now splits up into two equations:

$$\phi_+(\boldsymbol{p}) = \sum_{\beta=-s}^{+s} D_{|s|\beta}^{|s|}(X_{p\leftarrow p_z}^{-1})\ (\boldsymbol{P}^2)^{1/4}\ \mathfrak{F}^{-1} E_\Delta\, \mathfrak{F}\, \Sigma_\beta(\boldsymbol{p}) \tag{53a}$$

$$\phi_-(\boldsymbol{p}) = \sum_{\beta=-s}^{+s} D_{-|s|\beta}^{|s|}(X_{p\leftarrow p_z}^{-1})\ (\boldsymbol{P}^2)^{1/4}\ \mathfrak{F}^{-1} E_\Delta\, \mathfrak{F}\, \Sigma_\beta(\boldsymbol{p})\ . \tag{53b}$$

The state $\phi = (\phi_+, \phi_-)$ is localized in Δ if and only if the three-dimensional Fourier transforms of $\Sigma_\beta(\boldsymbol{p})$ have their support contained inside Δ for all values of $\beta = -s, \dots, +s$.

We shall prove that such states exist for arbitrary volumes Δ with non-void interior.

This completes the construction of the operators F_Δ for systems of discrete spin. Our next task is to prove that these operators satisfy Theorems 1 and 2.

VI. Proofs of Theorems 1 and 2

In the preceding section we were able to construct the operators F_Δ for particles of mass zero and discrete spin. Two theorems about these operators were set forth already in Section III. They exhibit the properties of F_Δ for bounded domains Δ of R^3, namely that $F_\Delta = 0$ for the irreducible representations $[0, s]$ and $F_\Delta \neq 0$ for the reducible representations $[0, s] \oplus [0, -s]$ $(s \neq 0)$. We shall now prove these statements.

The proofs are based on the application of an important theorem from the theory of analytic functions of several complex variables. It states that the Laplace transform of a square-integrable function of compact support in R^n is an entire analytic function of exponential growth. Furthermore, the converse is also true. The precise wording is the following [21]:

Theorem (Plancherel-Pólya):

Let the function $F(p^1, \ldots, p^n)$ be square-integrable over the entire space of the real variables $p^1, \ldots, p^n$. In order that its Fourier transform

$$\tilde{F}(x^1, \ldots, x^n) = (2\pi)^{-n/2} \int dp^1 \ldots dp^n \, e^{i \sum_{j=1}^{n} p^j x^j} F(p^1, \ldots, p^n)$$

vanish almost everywhere (in the L^2-norm) outside some bounded region Δ of R^n, it is necessary and sufficient that $F(p^1, \ldots, p^n)$ be equivalent to a function $f(p^1, \ldots, p^n)$ which can be extended to an entire function $f(\pi^1, \ldots, \pi^n)$ of the complex variables $\pi^j = p^j + i\, q^j$ such that

$$|f(\pi^1, \ldots, \pi^n)| < A\, e^{d(|\pi^1| + \cdots + |\pi^n|)} \tag{54}$$

for some positive constants A and d.

The availability of the necessary as well as the sufficient conditions for a function to have compact support will turn out to be indispensable in our proofs. Before going over to these, we write down the explicit form of $D^{|s|}(X_{p \leftarrow p_z})$ which will be needed at a later stage. This is obtained by simple algebraic manipulations, using the expression (32) for $X_{p \leftarrow p_z}$ and the definition (20) of $D^{|s|}$. The result is

$$D^{|s|}_{\beta\,|s|}(X_{p \leftarrow p_z}) = (2\,|\boldsymbol{p}|)^{-|s|}\,(|\boldsymbol{p}| + p^3)^{\beta}\,(p^1 + i\,p^2)^{|s| - \beta}\, c_\beta \tag{55a}$$

$$D^{|s|}_{\beta\,-|s|}(X_{p \leftarrow p_z}) = (2\,|\boldsymbol{p}|)^{-|s|}\,(|\boldsymbol{p}| + p^3)^{-\beta}\,(-\,p^1 + i\,p^2)^{|s| + \beta}\, c_\beta \tag{55b}$$

with the constants $c_\beta = \left[\binom{2|s|}{|s|+\beta}\right]^{1/2}$.

Proof of Theorem 1:

Since $[0, -s]$ is simply the complex-conjugate representation of $[0, +s]$, it suffices to do the proof for $s > 0$.

Let Δ be any bounded set of R^3, and let $\phi(\boldsymbol{p})$ be localized in Δ, i.e. $(F_\Delta \phi)(\boldsymbol{p}) = \phi(\boldsymbol{p})$. We shall conclude from these assumptions that $\phi(\boldsymbol{p})$ is the zero-vector in the Hilbert space $\mathcal{H}^{[0,s]}$ of the representation $[0, s]$.

The considerations following Equation (51) show that $\phi(\boldsymbol{p})$ is localized in Δ if and only if the three-dimensional Fourier transforms $\tilde{\psi}_\beta(\boldsymbol{x})$ of the functions

$$\psi_\beta(\boldsymbol{p}) = \frac{1}{\sqrt{|\boldsymbol{p}|}}\, D^{|s|}_{\beta s}(X_{p \leftarrow p_z})\, \phi(\boldsymbol{p}) \tag{51}$$

have their support contained in Δ for all values of β ($\beta = -s, \ldots, +s$). Furthermore, since

$$\sum_{\beta=-s}^{+s} \int d^3x\, |\tilde{\psi}_\beta(\boldsymbol{x})|^2 = \sum_{\beta=-s}^{+s} \int d^3p\, |\psi_\beta(\boldsymbol{p})|^2 = \int \frac{d^3p}{|\boldsymbol{p}|}\, |\phi(\boldsymbol{p})|^2 < \infty$$

all of the $\tilde{\psi}_\beta(\boldsymbol{x})$ are square-integrable. The theorem of Plancherel-Pólya therefore implies that every $\psi_\beta(\boldsymbol{p})$ is equivalent (in the L^2-norm) to a function $\varphi_\beta(\boldsymbol{p})$ which has an entire extension $\varphi_\beta(\boldsymbol{\pi})$ $(\pi^j = p^j + i\, q^j)$.

In Equation (51) all of the $\psi_\beta(\boldsymbol{p})$ are obtained from one single function $\phi(\boldsymbol{p})$. For this reason the $\psi_\beta(\boldsymbol{p})$ are linearly dependent and can all be expressed in terms of one of them. This relation between the $\psi_\beta(\boldsymbol{p})$ is easily calculated from (55a) and (51). One finds

$$\psi_\beta(\boldsymbol{p}) = c_\beta \left(\frac{p^1 + i\, p^2}{|\boldsymbol{p}| + p^3}\right)^{s-\beta} \psi_s(\boldsymbol{p}) \tag{56}$$

(56) also holds almost everywhere between the $\varphi_\beta(\boldsymbol{p})$. Setting $\beta = s - 1$ in this equation and multiplying both sides by $|\boldsymbol{p}| + p^3$, one gets

$$|\boldsymbol{p}|\, \varphi_{s-1}(\boldsymbol{p}) = c_{s-1}\, (p^1 + i\, p^2)\, \varphi_s(\boldsymbol{p}) - p^3\, \varphi_{s-1}(\boldsymbol{p})\,. \tag{57}$$

The right-hand side of (57) has the entire extension

$$c_{s-1}\, (\pi^1 + i\, \pi^2)\, \varphi_s(\boldsymbol{\pi}) - \pi^3\, \varphi_{s-1}(\boldsymbol{\pi})\,.$$

Therefore $|\boldsymbol{p}|\, \varphi_{s-1}(\boldsymbol{p}) \equiv \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2}\, \varphi_{s-1}(\boldsymbol{p})$ must be extendable to an entire function. An analytic extension of it is given by

$$\sqrt{(\pi^1)^2 + (\pi^2)^2 + (\pi^3)^2}\, \varphi_{s-1}(\boldsymbol{\pi})\,. \tag{58}$$

One knows that a function which is defined on an open domain $D \subset R^n$ has at most one analytic continuation into C^n [22]. Hence (58) is the only analytic extension of $|\boldsymbol{p}|\, \varphi_{s-1}(\boldsymbol{p})$. Because of the square-root, it is not entire unless $\varphi_{s-1}(\boldsymbol{p}) \equiv 0$. This then implies that $\phi(\boldsymbol{p})$ is the zero-vector of $\mathcal{H}^{[0,s]}$: $\phi \equiv 0$ is the only solution of $F_\Delta\, \phi = \phi$ for bounded Δ. QED

Theorem 1 has been proved independently by A. GALINDO [23].

Proof of Theorem 2:

In the case of a reducible representation $[0, s] \oplus [0, -s]$, $s > 0$, the functions $\phi(\boldsymbol{p}) = (\phi_+(\boldsymbol{p}), \phi_-(\boldsymbol{p}))$ have two components instead of only one for $[0, s]$. This additional liberty will permit us to dispose of the square-root which was responsible for the negative result of Theorem 1.

The remarks after Equation (53) state that such a function $\phi(\boldsymbol{p})$ is localized within Δ if the Fourier transforms $\tilde{\Sigma}_\beta(\boldsymbol{x})$ of

$$\Sigma_\beta(\boldsymbol{p}) = \frac{1}{\sqrt{|\boldsymbol{p}|}}\, D^s_{\beta s}(X_{p \leftarrow p_z})\, \phi_+(\boldsymbol{p}) + \frac{1}{\sqrt{|\boldsymbol{p}|}}\, D^s_{\beta\, -s}(X_{p \leftarrow p_z})\, \phi_-(\boldsymbol{p}) \tag{52}$$

have support in Δ for all values of β. The theorem of Plancherel-Pólya indicates that Δ is bounded if and only of every $\Sigma_\beta(\boldsymbol{p})$ is equivalent (in the L^2-norm) to a function $\sigma_\beta(\boldsymbol{p})$ which is extendable to an entire function $\sigma_\beta(\boldsymbol{\pi})$ of exponential growth ($\beta = -s, \ldots, +s$).

The $\Sigma_\beta(\boldsymbol{p})$ are derived from the two independent functions $\phi_+(\boldsymbol{p})$ and $\phi_-(\boldsymbol{p})$. They are therefore again linearly dependent (for $s \geqslant 1$) and may all be expressed in terms of $\Sigma_s(\boldsymbol{p})$ and $\Sigma_{-s}(\boldsymbol{p})$. This is achieved by using the explicit form (55a, b) of $D^s(X_{p\leftarrow p_z})$ and (52). Simple algebra yields first $\phi_\pm(\boldsymbol{p})$ in terms of $\Sigma_{\pm s}(\boldsymbol{p})$:

$$\phi_\pm(\boldsymbol{p}) = [2\,|\boldsymbol{p}|\,(|\boldsymbol{p}| + p^3)]^s \frac{\sqrt{|\boldsymbol{p}|}}{D(\boldsymbol{p})} \{(|\boldsymbol{p}| + p^3)^{2s} \Sigma_{\pm s}(\boldsymbol{p}) - (\mp p^1 + i\,p^2)^{2s}\, \Sigma_{\mp s}(\boldsymbol{p})\} \quad (59)$$

with

$$D(\boldsymbol{p}) = (|\boldsymbol{p}| + p^3)^{4s} - (-1)^{2s}\,[(p^1)^2 + (p^2)^2]^{2s}\,. \quad (60)$$

Inserting (59) into (52) and using

$$(p^1)^2 + (p^2)^2 = |\boldsymbol{p}|^2 - (p^3)^2 = (|\boldsymbol{p}| + p^3)\,(|\boldsymbol{p}| - p^3) \quad (61)$$

we are led to

$$\Sigma_\beta(\boldsymbol{p}) = c_\beta\,\frac{1}{D(\boldsymbol{p})}\,\{a_\beta^+(\boldsymbol{p})\,\Sigma_s(\boldsymbol{p}) + a_\beta^-(\boldsymbol{p})\,\Sigma_{-s}(\boldsymbol{p})\} \quad (62)$$

where

$$a_\beta^\pm(\boldsymbol{p}) = (|\boldsymbol{p}| + p^3)^{2s}\,(\pm p^1 + i\,p^2)^{s\mp\beta}\,\{(|\boldsymbol{p}| + p^3)^{s\pm\beta} - (-1)^{s\pm\beta}\,(|\boldsymbol{p}| - p^3)^{s\pm\beta}\}\,.$$

Our next step is to show that the coefficients $[D(\boldsymbol{p})]^{-1}\,a_\beta^\pm(\boldsymbol{p})$ of $\Sigma_{\pm s}(\boldsymbol{p})$ in (62) do not contain the inconvenient square-root. They are rational functions of p^1, p^2, p^3. To see this, we define

$$R_\beta(\boldsymbol{p}) = (|\boldsymbol{p}| + p^3)^{s+\beta} - (-1)^{s+\beta}\,(|\boldsymbol{p}| - p^3)^{s+\beta} \quad (63)$$

and apply it to rewrite $D(\boldsymbol{p})$ and $a_\beta^\pm(\boldsymbol{p})$. First, using also (61), one finds

$$D(\boldsymbol{p}) = (|\boldsymbol{p}| + p^3)^{2s}\,\{(|\boldsymbol{p}| + p^3)^{2s} - (-1)^{2s}\,(|\boldsymbol{p}| - p^3)^{2s}\} = (|\boldsymbol{p}| + p^3)^{2s}\,R_s(\boldsymbol{p})\,. \quad (64)$$

Also

$$a_\beta^\pm(\boldsymbol{p}) = (|\boldsymbol{p}| + p^3)^{2s}\,(\pm p^1 + i\,p^2)^{s\mp\beta}\,R_{\pm\beta}(\boldsymbol{p})\,. \quad (65)$$

The combination of (64) and (65) leads to

$$\frac{1}{D(\boldsymbol{p})}\,a_\beta^\pm(\boldsymbol{p}) = (\pm p^1 + i\,p^2)^{s\mp\beta}\,\frac{R_{\pm\beta}(\boldsymbol{p})}{R_s(\boldsymbol{p})}\,. \quad (66)$$

Since $s + \beta$ is always a non-negative integer, we may expand $(|\boldsymbol{p}| \pm p^3)^{s+\beta}$ in (63) and write $R_\beta(\boldsymbol{p})$ as a sum of terms of the form $|\boldsymbol{p}|^m (p^3)^{s+\beta-m}$. The same terms will occur in the expansion of $(|\boldsymbol{p}| + p^3)^{s+\beta}$ and in that of $(|\boldsymbol{p}| - p^3)^{s+\beta}$, but in the latter with alternating signs. The additional sign-factor $(-1)^{s+\beta+1}$ appearing in (63) is such that the two terms $(p^3)^{s+\beta}$ cancel (for all values of s and β!) Hence all terms containing even powers of $|\boldsymbol{p}|$ disappear (because of the alternating signs). Of the remaining ones, a factor $|\boldsymbol{p}|$ may be put in evidence, such that (for $\beta \neq s$)

$$R_\beta(\boldsymbol{p}) = |\boldsymbol{p}|\,P'_\beta(|\boldsymbol{p}|^2, p^3)$$

where P'_β is a polynomial of degree $s + \beta - 1$ in the variables $|\boldsymbol{p}|^2$ and p^3. Replacing in it $|\boldsymbol{p}|^2$ by $(p^1)^2 + (p^2)^2 + (p^3)^2$, it is transformed into a polynomial $P_\beta[(p^1)^2, (p^2)^2, p^3]$ of the variables $(p^1)^2$, $(p^2)^2$ and p^3:

$$R_\beta(\boldsymbol{p}) = |\boldsymbol{p}|\,P_\beta[(p^1)^2, (p^2)^2, p^3]\,. \quad (67)$$

In (67) the square-root is present only in the factor $|\boldsymbol{p}|$. When one inserts (67) into (66), these factors $|\boldsymbol{p}|$ cancel in the quotients on the right-hand side of (66), so that $[D(\boldsymbol{p})]^{-1}\, a_\beta^{\pm}(\boldsymbol{p})$ are *rational* functions of p^1, p^2 and p^3.

Combining (66) and (62) we get ($\beta \neq s$)

$$\Sigma_\beta(\boldsymbol{p}) = \frac{c_\beta}{P_s[(p^1)^2, (p^2)^2, p^3]} \{(p^1 + i\, p^2)^{s-\beta}\, P_\beta[(p^1)^2, (p^2)^2, p^3]\, \Sigma_s(\boldsymbol{p}) + (-\,p^1 + i\, p^2)^{s+\beta}\, P_{-\beta}[(p^1)^2, (p^2)^2, p^3]\, \Sigma_{-s}(\boldsymbol{p})\} \,. \tag{68}$$

Equation (68) serves to construct states which are localized in a bounded region Δ. We remind ourselves that this is the case if and only if every $\Sigma_\beta(\boldsymbol{p})$ is equivalent to a function $\sigma_\beta(\boldsymbol{p})$ which has an entire extension of exponential growth (54). One sees from (68) that this condition is fulfilled if both $\Sigma_s(\boldsymbol{p})$ and $\Sigma_{-s}(\boldsymbol{p})$ are *entire, of exponential type* (54), *square-integrable* over R^3, and *vanish at the zeros of* $P_s[(p^1)^2, (p^2)^2, p^3]$. Such functions exist in abundance. An example is

$$P_s[(p^1)^2, (p^2)^2, p^3]\, \frac{\sin^m(p^1)}{(p^1)^n}\, \frac{\sin^m(p^2)}{(p^2)^n}\, \frac{\sin^m(p^3)}{(p^3)^n}$$

with $n > 2\,s$ and $m \geqslant n$. Introducing such functions for $\Sigma_{\pm s}(\boldsymbol{p})$ into (59) gives states $\phi(\boldsymbol{p})$ which are localized in a finite region of space. QED

The foregoing proof supplies no information about the size of the bounded domain Δ in which the constructed states are localized. We shall also mention the answer to this question.

The extension of the bounded support Δ of the function $\tilde{F}(x^1, \ldots, x^n)$ in the theorem of Plancherel-Pólya is connected with the property of growth of $f(\pi^1, \ldots, \pi^n)$: The constant d in the exponent of (54) corresponds to the maximal distance between the origin 0 of R^n and the points of Δ [24]. Hence, if one chooses d arbitrarily small, one can obtain states which are localized in an arbitrarily small region around the origin. The transformation property (4) under translations and rotations then implies that $F_\Delta \neq 0$ *for any region* Δ *of* R^3 *with non-void interior.*

The following statement represents a generalization of Theorem 1 for particles of positive mass:

Theorem 3: The reduction according to Equation (6) of the ordinary system of imprimitivities (28), (29) of a representation $[m, J]$, $m > 0$ and $J > 0$, to the subspace $\mathcal{H}_{(s)}^{[m,J]}$ of the states of helicity s is such that $F_{\Delta\,(s)}^{[m,J]} = 0$ for all bounded volumes Δ (i.e. eigenstates of the helicity operator of particles of spin $J \neq 0$ are never localized in a finite volume of space).

The proof of this statement is essentially the same as that of Theorem 1 and is therefore omitted.

VII. The Representations of Continuous Spin

We remind ourselves that the irreducible unitary representations $[0, \varepsilon, r]$, $\varepsilon = \pm 1$, $r > 0$ of $\mathcal{R}$ arise from infinite-dimensional representations of the little group ε_2. They were discussed in some detail by WIGNER [12]. The state vectors $\phi(\boldsymbol{p})$ obtain an infinite number of components. A convenient way of specifying them is to label $\phi(\boldsymbol{p})$ by an index n which takes all integer values ($n = 0, \pm 1, \pm 2, \ldots$) for single-valued representations ($\varepsilon = +1$) and all half-integer values ($n = \pm 1/2, \pm 3/2, \ldots$) for the double-valued ones ($\varepsilon = -1$) [25]. As in the case of discrete spin and $m = 0$, we again stabilize $k = (1/2, 0, 0, 1/2)$ and use the choice (40) for $A_{p\leftarrow k}$. The invariant measure on the lightcone was introduced in (39).

The expression for $Q(p, B)$ in this case is not found in the literature. However, it suffices to know $Q(p, B)$ for space rotations, i.e. for $B \in \mathrm{SU}(2, \mathrm{C})$. For such B, $Q(p, B)$ splits up into an infinite direct sum in such a way that the components ϕ_n for different values of n are not mixed. In each subspace $\mathcal{H}^{[n]}$ corresponding to a fixed value of n, the infinitesimal generators $\boldsymbol{J}$ of the space rotations are represented in the same way as in the case of a representation $[0, n]$ [26]. Therefore, for $B \in \mathrm{SU}(2, \mathrm{C})$

$$Q(p, B) = \bigoplus_n \, [\varkappa(p, B)]^{2n} \tag{69}$$

with $\varkappa(p, B)$ given by Equation (44). (A choice for $A_{p\leftarrow k}$ different from (40) would not entail the property (69). With the selection (40) the subspaces $\mathcal{H}^{[n]}$ are mapped into each other by the pure Lorentz transformations.)

One concludes from (69) that $U(\boldsymbol{a}, B)$ for $B \in \mathrm{SU}(2, \mathrm{C})$ coincides in every subspace $\mathcal{H}^{[n]}$ with the representation of ε_3 that we encountered for $[0, n]$. This fact permits the definition of arbitrarily many generalized systems of imprimitivities for $[0, \varepsilon, r]$: One first adds several of the $\mathcal{H}^{[n]}$ to form a more extensive subspace. This corresponds to superpositions of particles of mass zero and various helicities. The construction of the operators F_Δ in such cases was discussed in Section V. After having found the F_Δ in this first subspace, one combines another set of $\mathcal{H}^{[n]}$ into a subspace (which is, of course, orthogonal to the first one), and again determines the F_Δ. Continuing in this way, one ends up with a generalized system of imprimitivities for $[0, \varepsilon, r]$. Yet all such systems of imprimitivities are *reducible*. That there exist *non-trivial* ones may be deduced from Theorem 2: One simply chooses $\mathcal{H}^{[n]} \oplus \mathcal{H}^{[-n]}$ for an arbitrary value of $n \neq 0$ as one of the larger subspaces. (All of these systems of imprimitivities are also time reversal invariant.)

It remains to examine the question whether it is possible to find an irreducible generalized system of imprimitivities for $[0, \varepsilon, r]$. One may answer in the negative if one admits the validity of a conjecture of Jauch and Piron [8]. This conjecture states that every generalized system of imprimitivities $\{U(\boldsymbol{a}, B), F_\Delta\}$ defined in some Hilbert space $\mathcal{H}$ can be obtained in the following way: There exists a (minimal) extension $\mathcal{H}^+$ of $\mathcal{H}$ in such a way that an ordinary system of imprimitivities $\{U^+(\boldsymbol{a}, B), E_\Delta\}$ is given in $\mathcal{H}^+$, and the original one in $\mathcal{H} \equiv P\,\mathcal{H}^+$ is the reduction of it according to (6), i.e.

$$\begin{aligned} [U^+(\boldsymbol{a}, B), P] &= 0 \\ U(\boldsymbol{a}, B) &= U^+(\boldsymbol{a}, B)\, P \\ F_\Delta &= P \cap E_\Delta\,. \end{aligned} \tag{70}$$

(So far neither a proof nor a counter-example for this conjecture have been offered.)

Let now $\mathcal{H}$ be the representation space of $[0, \varepsilon, r]$ and $\mathcal{H}^+$ an extension satisfying the conditions of this conjecture. An ordinary system of imprimitivities $\{U^+(\boldsymbol{a}, B), E_\Delta\}$ for ε_3 is therefore defined in $\mathcal{H}^+$. All irreducible such systems are unitarily equivalent to one of the form (28), (29) [9]. The representations D^J of SU(2, C) appearing therein are all *finite-dimensional*. Since the helicity index n of the vectors $\phi_n \in \mathcal{H}$ may assume arbitrarily large values, the system $\{U^+(\boldsymbol{a}, B), E_\Delta\}$ must be infinitely reducible.

Next, let us assume that the representation $U(\boldsymbol{a}, B)$ of ε_3 in $\mathcal{H}$ has been completely reduced, and denote the irreducible subspaces by $\mathcal{H}_{(i)}$. Every one of these $\mathcal{H}_{(i)}$ must lie in one of the irreducible parts of the system $\{U^+(\boldsymbol{a}, B), E_\Delta\}$. Because of the finite-dimensionality of D^J, it is not possible to assemble all of the $\mathcal{H}_{(i)}$ into *one* irreducible part (remember that n may be infinitely large). The reduction of $\{U^+(\boldsymbol{a}, B), E_\Delta\}$ to the subspace $\mathcal{H}$ of $\mathcal{H}^+$ is therefore of necessity reducible. (One notices that these remarks are independent of postulate (F) about time-reversal invariance.)

The peculiarity of a 'physical' system corresponding to a representation of continuous spin is its infinite degree of internal freedom (*infinite spin*!) We showed that this fact entails the existence of an infinite number of inequivalent 'position observables'. None of them is distinguished by some property that could induce us to designate it as *the* position observable for systems of continuous spin. The conjecture (70) implies that there exists no position observable which would not distinguish certain parts of such a 'physical' system.

VIII. Relativistic Invariance

Relativistic invariance of particle positions in classical and quantum mechanics was discussed in great detail by Currie, Jordan and Sudarshan [27]. They explain that two different assumptions must be distinguished: Relativistic symmetry and manifest invariance. We shall first stress the principal points of their arguments without entering into mathematical particulars, and then indicate how the concept of relativistic symmetry applies to our scheme of localizability.

Relativistic symmetry refers to the principle of special relativity which states that the laws of physics must be invariant under relativistic changes of reference frames. In quantum mechanics this requirement can always be satisfied if the Hilbert space of the states of a physical system is that of a unitary representation of the group of relativistic transformations. Let us assume that one knows the description of such a system from a certain reference frame. This description can then be *transformed* to any other Lorentz frame by means of the unitary operator $U(b, \Lambda)$ that corresponds to the inhomogeneous Lorentz transformation (b, Λ) relating the two frames. The *unitarity* of this transformation guarantees that any expectation value taken in the second frame is identical with the one taken between the corresponding quantities in the original frame. The *group property* (i.e. the multiplication law) insures that this rule for transforming a description from one frame to another one is itself invariant under changes of reference frames: If one transforms a description from one frame to another one and next from this second frame to a third one, the result coincides with that obtained by passing directly from the original description to that in the third frame. Similarly one can deduce from the group property that formal relations between observables (e.g. the law of motion) are independent of the reference frame. All these ideas will be elaborated in a moment for localizability.

The second aspect of invariance is *manifest covariance*. It consists in the requirement that certain quantities transform under changes of reference frames in a particular manner, e.g. as tensors or spinors. There is no general agreement about its formulation in *quantum mechanics*. In the introduction we collected three proposals for manifest invariance in connection with the concept of position. They are apparently not equivalent because they lead to different position operators. The notion of localizability as introduced in Section II does not fulfil any such mathematical requirement but complies with the principle of special relativity. We shall now turn our attention to this question.

First, let us give a specification of the different Lorentz frames. We define $[a, \Lambda]$ to denote the frame in which $x' = \Lambda\, x + a$ describes the same point in space-time as does x in the arbitrarily chosen fixed frame $[0, I]$.

One remembers that in Sections III, V and VII the postulates (A)–(F) for localizability were applied in the following manner: One distinguishes some Lorentz frame (which we shall assume to be $[0, I]$ in the sequel). The states of a physical system as seen by an 'observer' in this frame $[0, I]$ form a Hilbert space $\mathcal{H}$, which one requires in addition to be that of a continuous unitary representation of $\mathcal{P}_+^\uparrow$ plus time reversal (for the sake of simplicity of notation we shall work with $\mathcal{P}_+^\uparrow$ in this section rather than with $\mathcal{R}$; all considerations would be the same for $\mathcal{R}$). When one considers a Lorentz frame different from $[0, I]$, two problems arise:

(a) How would an observer in the second frame localize the physical system under consideration?

(b) How does the observer in the second frame describe the measurements performed by the one in the first frame $[0, I]$?

Let us discuss these two questions separately.

(a) We decided earlier to interprete $U(b, \Lambda)$ in the active way: $U(b, \Lambda)\, \phi$ denotes the state obtained from ϕ by transforming it according to Λ and translating the resulting state by b. Here both ϕ and $U(b, \Lambda)\, \phi$ are states described from the distinguished frame $[0, I]$; b is a four-vector and Λ a Lorentz transformation in this frame $[0, I]$. The subrepresentation $U(\boldsymbol{a}, R)$ is used to define localizability in $[0, I]$.

Let $[a, \Lambda]$ designate the second frame under consideration. Then $\phi' = U(a, \Lambda)\, \phi$ will be the description by an observer in $[a, \Lambda]$ of the same physical state that is called ϕ by an observer in $[0, I]$. The states ϕ' form the Hilbert space $\mathcal{H}_{[a,\Lambda]}$ of the observer in $[a, \Lambda]$ for the physical system in question. Of course $\mathcal{H}_{[a,\Lambda]}$ and $\mathcal{H}$ are isomorphic. Therefore a unitary representation $\hat{U}_{[a,\Lambda]}$ of $\mathcal{P}_+^\uparrow$ is given in $\mathcal{H}_{[a,\Lambda]}$:

$$\hat{U}_{[a,\Lambda]}(b, M) \mathrel{\widehat{=}} U(a, \Lambda)\; U(b, M)\; U(a, \Lambda)^{-1}\,. \tag{71}$$

In (71), (b, M) is still measured from the frame $[0, I]$, and the sign $\widehat{=}$ stands for the fact that the operators to the left and to the right of it act in different Hilbert spaces but have the same mathematical form in their respective space.

The observer in $[a, \Lambda]$ would of course describe the argument (b, M) of $\hat{U}_{[a,\Lambda]}(b, M)$ from his own frame. The descriptions (b, M) and (b', M') of an objective inhomogeneous Lorentz transformation from the two frames $[0, I]$ and $[a, \Lambda]$ respectively are related by the transformation law of four-vectors. Let x, y be two points in $[0, I]$ related by

$$y = M\, x + b\,.$$

Let x' and y' be the description of these two points from $[a, \Lambda]$:

$$x' = \Lambda\, x + a\ ,$$

$$y' = \Lambda\, y + a\ .$$

The transformation (b', M') in $[a, \Lambda]$ is then defined by

$$y' = M'\, x' + b'\ .$$

These four equations determine completely the relation between (b, M) and (b', M'):

$$(b', M') = (a + \Lambda\, b - \Lambda\, M\, \Lambda^{-1}\, a, \Lambda\, M\, \Lambda^{-1})\ , \tag{72a}$$

$$(b, M) = (\Lambda^{-1}\, M'\, a + \Lambda^{-1}\, b' - \Lambda^{-1}\, a, \Lambda^{-1}\, M'\, \Lambda)\ . \tag{72b}$$

Let us define a representation $U_{[a,\Lambda]}$ in $\mathcal{H}_{[a,\Lambda]}$ as follows: $U_{[a,\Lambda]}$ is identical with $\hat{U}_{[a,\Lambda]}$, but the arguments in it are described from the frame $[a, \Lambda]$:

$$U_{[a,\Lambda]}\,(b', M') \equiv \hat{U}_{[a,\Lambda]}\,(b, M) \tag{73}$$

where (b', M') and (b, M) are related by (72a,b). Equation (71) then reads:

$$U_{[a,\Lambda]}\,(b', M') \triangleq U(a, \Lambda)\; U(b, M)\; U(a, \Lambda)^{-1}\ . \tag{74}$$

Let us decide here that, for any $[a, \Lambda]$, the arguments of $U_{[a,\Lambda]}$ must always denote transformations as viewed from the frame $[a, \Lambda]$. (Equation (74) complies with this requirement.)

One may insert the expression (72b) for (b, M) into (74). Using the multiplication law of $\mathcal{P}_+^\uparrow$, one obtains

$$U_{[a,\Lambda]}\,(b', M') \triangleq U(b', M')\ . \tag{75}$$

We remember the interpretation of these two operators: $U_{[a,\Lambda]}(b', M')$ in $\mathcal{H}_{[a,\Lambda]}$ corresponds to the transformation (b', M') in the frame $[a, \Lambda]$, $U(b', M')$ in $\mathcal{H}$ to the transformation (b', M') in $[0, I]$. These two Lorentz transformations are therefore not identical from the physical point of view. Equation (75) states, however, that the two unitary operators belonging to them have the same mathematical form in their respective Hilbert space. This fact is very important. It signifies that *the infinitesimal generators of the Poincaré group are the same in all Lorentz frames,* an absolute necessity if these frames are to be physically equivalent.

We now proceed to define localizability in $[a, \Lambda]$. The unitary representation $U_{[a,\Lambda]}(b', M')$ in $\mathcal{H}_{[a,\Lambda]}$ induces a representation $U_{[a,\Lambda]}(\boldsymbol{c}, R)$ of the Euclidean group of $\boldsymbol{R}^3$ (one notes that $\boldsymbol{c}$ and R denote translations and space-rotations respectively in the frame $[a, \Lambda]$!) One imposes postulates (A)–(F) for this $U_{[a,\Lambda]}(\boldsymbol{c}, R)$ in $\mathcal{H}_{[a,\Lambda]}$, where the arguments Δ and t of $F_{\Delta,t}$ signify a Borel set Δ and a time coordinate t in the frame $[a, \Lambda]$. We shall henceforth write $F_{\Delta,t[a,\Lambda]}$ for these operators: $F_{\Delta,t[a,\Lambda]}$ corresponds to the proposition 'The system is localized in Δ at time t in the frame $[a, \Lambda]$.' Since $U_{[a,\Lambda]}$ in $\mathcal{H}_{[a,\Lambda]}$ has the same mathematical form as U in $\mathcal{H}$, the operators $F_{\Delta,t[a,\Lambda]}$ also have this property with respect to $F_{\Delta,t[0,I]} \equiv F_{\Delta,t}$.

One concludes then that *localizability is given by operators of the same form in all Lorentz frames.* It does not distinguish any particular frame.

(b) It remains to reply to the second question: 'How does an observer in the frame $[c, N]$ describe the position measurements performed in a different frame $[a, \Lambda]$?' The answer will express how the description in the frame $[a, \Lambda]$ of the physical system in question is transformed to the frame $[c, N]$. It derives from the unitary operator which corresponds to the relativistic transformation relating these two frames. In the frame $[0, I]$, this transformation is just

$$(b_0, M_0) = (c, N)\,(a, \Lambda)^{-1}\,. \tag{76}$$

Its description (b_0', M_0') from $[a, \Lambda]$ is obtained by inserting (76) into (72a). The measurement $F_{\Delta, t\,[a, \Lambda]}$, when described from $[c, N]$, corresponds to

$$U_{[a, \Lambda]}\,(b_0', M_0')\,F_{\Delta, t\,[a, \Lambda]}\,U_{[a, \Lambda]}\,(b_0', M_0')^{-1}\,.$$

We shall introduce for it the projection operator $F^{[c, N]}_{\Delta, t\,[a, \Lambda]}$ in $\mathcal{H}_{[c, N]}$. This operator is the correlate of the proposition 'The system is localized in Δ at time t in the frame $[a, \Lambda]$, but one describes this measurement from $[c, N]$'. Therefore

$$F^{[c, N]}_{\Delta, t\,[a, \Lambda]} \mathrel{\widehat{=}} U_{[a, \Lambda]}\,(b_0', M_0')\,F_{\Delta, t\,[a, \Lambda]}\,U_{[a, \Lambda]}\,(b_0', M_0')^{-1}\,. \tag{77}$$

We stress that in general $F^{[c, N]}_{\Delta, t\,[a, \Lambda]}$ bears no relation to the operators $F_{\Delta, t\,[c, N]}$ which define localizability in the frame $[c, N]$ (such a relation would correspond to manifest invariance). In certain cases there exist of course such connections. To indicate some pertinent examples, we simplify by selecting for $[a, \Lambda]$ in (77) the frame $[0, I]$. The imprimitivity relation (4) immediately implies together with (77)

$$F^{[\boldsymbol{a}, R]}_{\Delta, t\,[0, I]} \mathrel{\widehat{=}} U(\boldsymbol{a}, R)\,F^{[0, I]}_{\Delta, t\,[0, I]}\,U(\boldsymbol{a}, R)^{-1} = F_{R\,\Delta + \boldsymbol{a}, t\,[0, I]}\,. \tag{78}$$

Another example is

$$F^{[0, \Lambda]}_{\Delta, 0\,[0, I]} \mathrel{\widehat{=}} F^{[0, I]}_{\Delta, 0\,[0, \Lambda^{-1}]} \tag{79}$$

(79) is an expression of the fact that the transformation of descriptions of a physical system between Lorentz frames depends only on the relative position of the two frames and not on the particular choice of one of them. The two projection operators in (79) have the same mathematical form in their respective Hilbert space, but physically they correspond to two entirely different measurements!

One may interpret the operators $F^{[c, N]}_{\Delta, t\,[a, \Lambda]}$ of Equation (77) also in connection with moving frames: $F^{[c, N]}_{\Delta, t\,[a, \Lambda]}$ is the correlate of a position observation from the frame $[c, N]$ in a moving volume Δ at a point of time t, with the specification that Δ and t are determined in the rest-frame $[a, \Lambda]$ of this volume.

Finally, we direct our attention to the formal relations between the position observables. By definition, these relations are expressed by Equations (1)–(4), (F) and (8). Let us assume that these equations hold true between the operators $F_{\Delta, t\,[0, I]}$, $U(\boldsymbol{c}, R)$, $U(t, I)$ and $U(T)$ describing localizability in the frame $[0, I]$. The transformation of $F_{\Delta, t\,[0, I]}$ to a frame $[a, \Lambda]$ was defined in (77). The operator corresponding in $[a, \Lambda]$ to $U(\boldsymbol{c}, R)$ was constructed in (74):

$$U_{[a, \Lambda]}\,(c', R') \mathrel{\widehat{=}} U(a, \Lambda)\,U(\boldsymbol{c}, R)\,U(a, \Lambda)^{-1}$$

where (c', R') and $(\boldsymbol{c}, R)$ are connected by (72a, b).

Using the multiplication law of $U(b, \Lambda)$, one verifies that these formal relations are maintained by the transformation of the description of localizability from $[0, I]$ to $[a, \Lambda]$: Equations (1)–(4), (F) and (8) hold also between

$$F^{[a,\Lambda]}_{\Delta, t\,[0,I]},\ U_{[a,\Lambda]}(c', R')\ ,\ U_{[a,\Lambda]}(t', I)\ \text{ and }\ \hat{U}_{[a,\Lambda]}(T)\ .$$

Here (c', R') is calculated from $(\boldsymbol{c}, R)$ by means of (72a): It is the description from $[a, \Lambda]$ of the objective Euclidean transformation $(\boldsymbol{c}, R)$ in $[0, I]$ (i.e. in general c' is a four-vector and R' a Lorentz transformation).

A last remark in this section must be devoted to the time translations: A state which is localized in a finite volume at time $t = t_0$ in the frame $[a, \Lambda]$ spreads out over all space at any later instant $t > t_0$ in $[a, \Lambda]$. This behaviour was found already in investigations of localizability for particles of positive mass (WEIDLICH-MITRA [2]). Our discussion following Equation (46) shows that the same must be true for particles of mass zero. Nevertheless, the expectation values of the velocity operator associated with the position operator of Newton and Wigner never exceed the velocity of light (BERG [2]).

IX. Localizability and Energy Density

The appearance of particles of mass zero in physics dates back to 1905, when Einstein proposed that light rays could be viewed as 'consisting of a finite number of energy quanta which are localized at points in space, which move without dividing, and which can only be produced and absorbed as complete units' [28]. Thus, Einstein's heuristic formulation combined already the two notions of localization and energy density. In the subsequent development of quantum mechanics, his idea was moulded into the concept of *photons* as particles. The difference between this modern interpretation of photons as corpuscles and the original view of energy quanta becomes apparent as soon as one considers their local properties. This difference is by no means a peculiarity of photons but a basic characteristic of the quantum-mechanical description of any free particle. In fact, the localization of a particle in a region Δ at time t is characterized by a projection operator $F_{\Delta, t}$, its total energy by the Hamiltonian H. For non-interacting particles, H describes in addition their time evolution. Therefore H and $F_{\Delta, t}$ cannot commute in such cases (otherwise a free particle which is localized in a volume Δ at time $t = t_0$ would never spread beyond the boundaries of Δ at any later instant $t > t_0$). The energy density corresponding to an eigenstate $\varphi = F_{\Delta, t}\,\varphi$ of $F_{\Delta, t}$ will therefore not be zero outside the volume Δ at time t: *The energy density of an elementary particle is related to the position of this particle in a non-local manner.*

It should be pointed out here that similar features are encountered already in classical physics, namely in connection with the concept of a *field*. We mention two pertinent examples. In Newtonian mechanics, a particle of mass M which is localized at a point $\boldsymbol{x}$ of space is surrounded by its gravitational field which extends over all space. It is through this field that M interacts with other massive bodies, and it is this field which contains the gravitational energy of M. In the same way, a point charge $\boldsymbol{Q}$ represents the source of an electrostatic field which again pervades all space. This field bears the electrostatic energy of $\boldsymbol{Q}$, and it is responsible for the interactions between $\boldsymbol{Q}$ and other charges. The essential properties of these classical fields are therefore the following:

(a) The field of a localized 'source' extends over all space.

(b) An energy is associated with the field. The energy density can be written as a quadratic expression in the field or its derivatives. The field energy of a localized source spreads therefore through all space.

(c) Different sources interact with each other through their respective fields.

Quantum field theory is a combination of the theory of classical fields with quantum mechanics. One expects therefore quite naturally that it should also exhibit these features (a)–(c). This is indeed the case. In a relativistic quantum field theory, the *sources* of the field are the individual *particles:* Every particle is accompanied by a field which we shall call its *particle field*. This field contains the energy of the particle and causes its interactions with other particles. We shall also show that the particle field (and hence also the energy density) of a localized particle reach out through all space. As an example, we mention here that one may view the quantized electromagnetic field as the (classically measurable) particle field of its photons (this point will be discussed in more detail in Section X).

What we have denoted by 'particle field' is the same quantity which NEWTON and WIGNER [2] call the 'coordinate-space wave-function'. As we shall show later for the special case of the photons (cf. Section X), the particle field is related to the expectation values of the field operator in the quantum field theory of such particles: The absorptive (positive frequency) part of the field operator $\Phi(\boldsymbol{x}, t)$ transforms a one-particle state $|\varphi\rangle$ into the vacuum state $|0\rangle$ multiplied by the particle field corresponding to $|\varphi\rangle$. This particle field is therefore given by $\langle 0 \,|\Phi(\boldsymbol{x}, t)|\, \varphi\rangle$.

Local considerations play an important role also in axiomatic field theory. The notion of *locality* which is used there always refers to a property of the particle *field* and not of the particles themselves. The motive for this is that particles are observed through their particle field (since it is through this field that they interact with the measuring apparatus). Consequently, local observables should refer to the position of the particle field.

The particle field of a particle with positive mass is qualitatively different from that of a particle with restmass zero: A particle of positive mass m can be characterized by a fundamental length, its Compton wave-length $\lambda\!\!\!^{-} = \hbar/(m\,c)$. One expects therefore an exponential decrease of the particle field and the energy density proportional to $\exp(-\,r/\lambda\!\!\!^{-})$ at large distances r from the region of localization of the corresponding wave-function. Such a behaviour was indeed found by Newton and Wigner for the particle field, which decreases as $1/r\,(\lambda\!\!\!^{-}\,r)^{-3/4}\exp(-\,r/\lambda\!\!\!^{-})$ for $r \to \infty$ [29]. For particles of mass zero, on the other hand, one cannot derive such a fundamental length. In this case, the decrease of the particle field must be characterized by a dimensionless number. It will be seen that this law assumes the form $r^{-5/2}$ for $r \to \infty$. The energy density of a particle of mass zero is a quadratic expression in the first derivatives of its particle field. Hence it falls off as r^{-7} at large distances r from the region of localization of such a particle.

The fact that their particle fields extend over all space may be helpful in visualizing how localized particles may interact simultaneously with two widely separated physical systems S_1 and S_2 and hence may transmit forces between these two systems. If such a force is carried by particles of mass zero, one may infer from the considerations of the preceding paragraph that it will decrease as some negative power r^{-n} of the distance r between S_1 and S_2. (The exponent $-n$ will depend on the way in which these particles interact with S_1 and S_2.) Such long-range forces have indeed been calculated in relativistic quantum field theory as arising from the exchange of virtual photons or neutrinos between S_1 and S_2 [30, 31]. We mention especially the attractive London-Van der Waals force between two neutral but polarizable molecules which results from the exchange of two virtual photons and decreases asymptotically as r^{-8} [30], and which was measured by SPAARNAY [32] on a macroscopic scale (i.e. as the attractive force between flat plates).

We shall now derive the relation between localization and energy density at a fixed instant of time t for particles of mass zero and helicity s. The result will be used to discuss the energy density of states which are localized in a finite region of space. We shall specifically stress two points: the behaviour of the energy density in the region of localization, and the manner of its decrease at large distances from this region. Since physical dimensions play an important role in these considerations, we shall introduce the constants $\hbar$ and c wherever they are required. For the sake of simplifying the notation we set $t = 0$.

As we explained in connection with Equation (39), the Hilbert space $\mathcal{H}^{[0,s]}$ consists of all functions $\phi(\boldsymbol{k})$ defined on the forward lightcone ($k^2 = 0$, $k_0 > 0$) such that

$$(\phi, \phi) \equiv \int \frac{d^3k}{|\boldsymbol{k}|} \, |\phi(\boldsymbol{k})|^2 < \infty . \tag{80}$$

The dimension of k^i ($i = 1, 2, 3$) is that of a reciprocal length, so that $p^i = \hbar k^i$ describes the corresponding component of the momentum. The $(2s+1)$-component *position-space wave-function* $\tilde{\psi}_\beta(\boldsymbol{x})$ corresponding to the state $\phi(\boldsymbol{k}) \in \mathcal{H}^{[0,s]}$ is

$$\tilde{\psi}_\beta(\boldsymbol{x}) = (2\pi)^{-3/2} \int d^3k \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \, \psi_\beta(\boldsymbol{k}) \tag{81}$$

where $\psi_\beta(\boldsymbol{k})$ is given by (51)

$$\psi_\beta(\boldsymbol{k}) = \frac{1}{\sqrt{k}} D^{|s|}_{\beta s}(X_{k \leftarrow k_z}) \, \phi(\boldsymbol{k}) \tag{51}$$

and the norm (80) is equal to

$$(\phi, \phi) = \sum_{\beta=-s}^{+s} \int d^3k \, |\psi_\beta(\boldsymbol{k})|^2 = \sum_{\beta=-s}^{+s} \int d^3x \, |\tilde{\psi}_\beta(\boldsymbol{x})|^2 . \tag{82}$$

Let us define

$$P_\phi(\boldsymbol{x}) = (\phi, \phi)^{-1} \sum_{\beta=-s}^{+s} |\tilde{\psi}_\beta(\boldsymbol{x})|^2 . \tag{83}$$

We explained in Sections II and III that $P_\phi(\boldsymbol{x})$ may be interpreted as the particle density if $s = 0$. In all other cases ($s \neq 0$) there exists no particle density, i.e. the integral

$$\int_\Delta d^3x \, P_\phi(\boldsymbol{x})$$

is not in general identical with the probability of finding the state ϕ localized in the volume Δ at time $t = 0$. One may understand this immediately by remembering that the operator F_Δ is not simply multiplication by the characteristic function χ_Δ of the

Borel set Δ in $\boldsymbol{x}$-space (because $\chi_\Delta(\boldsymbol{x})\,\tilde{\psi}_\beta(\boldsymbol{x})$ is not in general a state of helicity s). Actually, the definition of F_Δ in Section V involved a second projection operator, that onto the subspace of the states of helicity s. As a consequence of this, the fact that $F_{\Delta_1} + F_{\Delta_2} < F_{\Delta_1 \cup \Delta_2}$ for certain couples of Borel sets $\Delta_1 \perp \Delta_2$ (cf. Section II) assumes now the following expression in $\boldsymbol{x}$-space:

$$\int_\Delta d^3x\, P_\phi(\boldsymbol{x}) \geqslant (\phi, \phi)^{-1}\, (\phi, F_\Delta\, \phi)\ . \tag{84}$$

If $F_\Delta\, \phi = \phi$, then the equality sign holds in (84). Thus the denomination 'position-space wave-function' for $\tilde{\psi}_\beta(\boldsymbol{x})$ is justified, since $\tilde{\psi}_\beta(\boldsymbol{x}) = 0$ for all vectors $\boldsymbol{x}$ in some volume Δ implies that this state has probability zero of being localized in Δ.

The total energy E_ϕ of a state $\phi \in \mathcal{H}^{[0,s]}$ with $\|\phi\| = 1$ is defined as

$$E_\phi = (\phi, H\,\phi) = \int \frac{d^3k}{|\boldsymbol{k}|}\, \phi^*(\boldsymbol{k})\, \hbar\, c\, |\boldsymbol{k}|\, \phi(\boldsymbol{k}) = \hbar\, c \int \frac{d^3k}{|\boldsymbol{k}|} \left| \sqrt{|\boldsymbol{k}|}\, \phi(\boldsymbol{k}) \right|^2 . \tag{85}$$

Using (82) and the fact that H commutes with $D^{|s|}(X_{k \leftarrow k_z})$, this may be written as

$$E_\phi = \hbar\, c \sum_{\beta=-s}^{+s} \int d^3k \left| \sqrt{|\boldsymbol{k}|}\, \psi_\beta(\boldsymbol{k}) \right|^2 . \tag{86}$$

If $E_\phi < \infty$, then the Fourier transform $\hat{\psi}_\beta(\boldsymbol{x})$ of $\sqrt{|\boldsymbol{k}|}\, \psi_\beta(\boldsymbol{k})$ can be used to express the energy density in $\boldsymbol{x}$-space:

$$E_\phi = \hbar\, c \sum_{\beta=-s}^{+s} \int d^3x\, |\hat{\psi}_\beta(\boldsymbol{x})|^2 = \int d^3x\, \mathcal{H}_\phi(\boldsymbol{x}) \tag{87}$$

with

$$\hat{\psi}_\beta(\boldsymbol{x}) = (2\,\pi)^{-3/2} \int d^3k\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \sqrt{|\boldsymbol{k}|}\, \psi_\beta(\boldsymbol{k}) \tag{88}$$

(87) gives the following expression for the energy density

$$\mathcal{H}_\phi(\boldsymbol{x}) = \hbar\, c\, (\phi, \phi)^{-1} \sum_{\beta=-s}^{+s} |\hat{\psi}_\beta(\boldsymbol{x})|^2\ . \tag{89}$$

(For $s = \pm 1$, $\mathcal{H}_\phi(\boldsymbol{x})$ corresponds to the conventional expression for the energy density of the electromagnetic field. We shall introduce this conventional formalism in Section X.)

One would now like to express $\hat{\psi}_\beta(\boldsymbol{x})$ in the form of a convolution integral containing the position-space wave-function $\tilde{\psi}_\beta(\boldsymbol{x})$ as one factor. This can be achieved by introducing in the integrand (88) a factor $\exp(-\,\varepsilon\,|\boldsymbol{k}|)$, $\varepsilon > 0$, and interchanging the integral over d^3k with the limit $\varepsilon \to +0$. This is legitimate for a dense set $\mathcal{D}$ of functions $\psi_\beta(\boldsymbol{k}) \in L^2_{|s|}(\boldsymbol{k})$ for which

$$\int d^3k\, \sqrt{|\boldsymbol{k}|}\, |\psi_\beta(\boldsymbol{k})| < \infty \quad \text{for all} \quad \beta = -\,s, \ldots, +\,s \tag{90}$$

(90) insures at the same time that all functions of $\mathcal{D}$ give rise to a finite total energy.

Therefore, using also (81), we obtain for $\psi_\beta(\boldsymbol{k}) \in \mathfrak{D}$:

$$\hat{\psi}_\beta(\boldsymbol{x}) = (2\pi)^{-3/2} \int d^3k \lim_{\varepsilon \to +0} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \sqrt{|\boldsymbol{k}|}\, e^{-\varepsilon|\boldsymbol{k}|}\, \psi_\beta(\boldsymbol{k})$$

$$= (2\pi)^{-3/2} \lim_{\varepsilon \to +0} \int d^3k\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \sqrt{|\boldsymbol{k}|}\, e^{-\varepsilon|\boldsymbol{k}|}\, \psi_\beta(\boldsymbol{k})$$

$$= \lim_{\varepsilon \to +0} \int d^3y\, G_\varepsilon(\boldsymbol{x} - \boldsymbol{y})\, \tilde{\psi}_\beta(\boldsymbol{y}) \tag{91}$$

with [33]

$$G_\varepsilon(\boldsymbol{r}) = (2\pi)^{-3} \int d^3k\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \sqrt{|\boldsymbol{k}|}\, e^{-\varepsilon|\boldsymbol{k}|}$$

$$= \frac{3}{8} \frac{1}{\sqrt{\pi^3}} \frac{1}{r} \frac{1}{(r^2+\varepsilon^2)^{5/4}} \sin\left(\frac{5}{2} \operatorname{arctg} \frac{r}{\varepsilon}\right) \quad r = |\boldsymbol{r}|. \tag{92}$$

These functions $G_\varepsilon(r)$ relate the amplitude $\hat{\psi}_\beta(\boldsymbol{x})$ of the energy density to the position-space wave-function $\tilde{\psi}_\beta(\boldsymbol{x})$ according to Equation (91). If $\tilde{\psi}_\beta(\boldsymbol{x})$ corresponds to the superposition $\phi = (\phi_+, \phi_-)$ of two states of opposite helicity, the relations (87)–(92) are still valid (with $\psi_\beta(\boldsymbol{k})$ replaced by the functions $\Sigma_\beta(\boldsymbol{k})$ of Equation (52)). This is so because H commutes with the helicity operator, so that the energy densities corresponding to ϕ_+ and to ϕ_- may simply be added to give $\mathcal{H}_\phi$:

$$\mathcal{H}_\phi(\boldsymbol{x}) = \mathcal{H}_{\phi_+}(\boldsymbol{x}) + \mathcal{H}_{\phi_-}(\boldsymbol{x}) \,. \tag{93}$$

$\mathcal{H}_\phi(\boldsymbol{x})$ may be interpreted as a real density even in those cases where the same is not true for $P_\phi(\boldsymbol{x})$. The reason for this is that the projection operator onto the subspace of the states of some given value of the helicity of a particle of spin $J \neq 0$ commutes with the Hamiltonian H but not with the projection operators $E_{\Delta, t}$ describing the localization of such a particle. Furthermore, $\mathcal{H}_\phi(\boldsymbol{x})$ transforms under Euclidean motions of R^3 in the same way as the wave-function ϕ, i.e.

$$\mathcal{H}_{U(\boldsymbol{a}, R)\phi}(\boldsymbol{x}) = \mathcal{H}_\phi\, [R^{-1}(\boldsymbol{x} - \boldsymbol{a})] \,.$$

This follows immediately from the fact that the kernels $G_\varepsilon(\boldsymbol{r})$ of Equation (92) depend only on the length r of $\boldsymbol{r}$.

The first thing to notice about these functions $G_\varepsilon(r)$ is their asymptotic behaviour. For $r \gg \varepsilon$, they can be approximated by

$$G_\varepsilon(r) \approx G(r) \equiv -\frac{3}{8} \frac{1}{\sqrt{2\pi^3}} r^{-7/2} \,. \tag{94}$$

Since ε is arbitrarily small, (94) may be assumed to be correct for all $r \neq 0$. For $r = 0$, the function (94) has a pole of order $r^{-7/2}$. The exact kernels (92) are simply the regularization (in the sense of the theory of distributions [34]) of (94). A plot of $G_\varepsilon(r)$ can be found in Figure 1. (The kernel $G(r)$ of Equation (94) is the limit $m \to 0$ of the corresponding expression for particles of positive mass, which, apart from a numerical factor, takes the form $(m c/(\hbar r))^{7/4} K_{7/4}((m c/\hbar) r)$. Its asymptotic form becomes $1/r(m c/(\hbar r))^{5/4} \exp(-(m c/\hbar) r)$. $K_{7/4}(z)$ is a Kelvin function.)

If the support of $\tilde{\psi}_\beta(\boldsymbol{x})$ lies in a finite region Δ of space for all values of β, the corresponding energy density $\mathcal{H}(\boldsymbol{x})$ *will not be zero outside Δ but decrease as r^{-7}* at large distances r from Δ (i.e. for $r \gg d(\Delta)$, where $d(\Delta)$ stands for the linear extension of Δ). This is immediately verified by inserting (94) into (91) and using the expression (89) for the energy density.

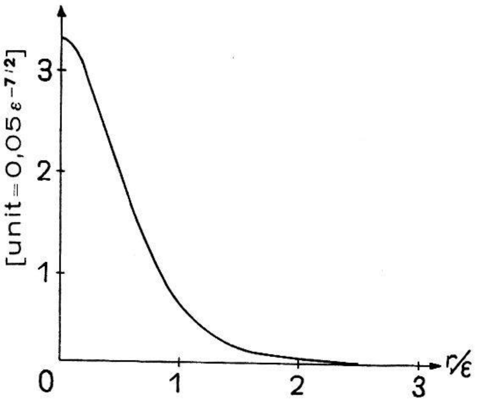

Figure 1a
The function $G_\varepsilon(r)$ of Equation (92).

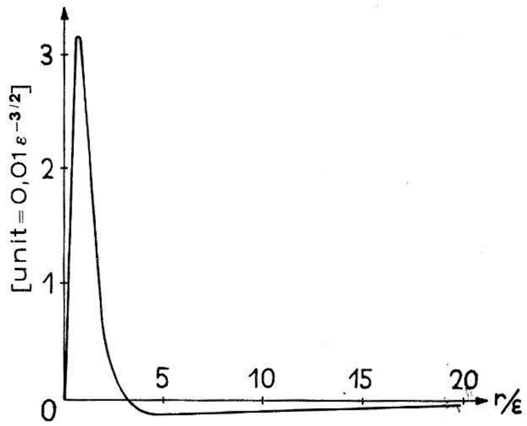

Figure 1b
The function $r^2\, G_\varepsilon(r)$.

In the region of localization one must use the exact kernels (92) for the passage from $\tilde{\psi}_\beta(\boldsymbol{x})$ to $\hat{\psi}_\beta(\boldsymbol{x})$. They fall off as $r^{-7/2}$ for $r \neq 0$, so that the main contribution to $\hat{\psi}_\beta(\boldsymbol{x})$ arises from integrating over some neighbourhood of $\boldsymbol{x}$. The function $\mathcal{H}_\phi(\boldsymbol{x})$ will therefore resemble $P_\phi(\boldsymbol{x})$: *Energy density and 'particle density' are essentially proportional to one another*. Deviations from this proportionality appear in regions where the position-space wave-function $\tilde{\psi}_\beta(\boldsymbol{x})$ increases locally so fast as to compensate the decrease of the kernel (92). Since the convolution integral (91) is three-dimensional, this happens if locally $|\tilde{\psi}_\beta(\boldsymbol{r})| \gtrsim |\boldsymbol{r}|^{3/2}$.

In order to obtain an idea about the magnitude of these deviations, we have evaluated explicitly the convolution integral (91) with the exact kernels $G_\varepsilon(r)$, $\varepsilon > 0$, for the following one-component position-space wave-function

$$\tilde{\psi}(\boldsymbol{x}) = \begin{cases} \lambda \left[1 - \dfrac{\boldsymbol{x}^2}{R^2}\right]^2 & \text{for } |\boldsymbol{x}| \leqslant R \\ 0 & \text{for } |\boldsymbol{x}| \geqslant R \end{cases} \tag{95}$$

λ is a normalization constant. The Fourier transform of $\tilde{\psi}(\boldsymbol{x})$ belongs to the dense set $\mathcal{D}$ defined by Equation (90). $\tilde{\psi}(\boldsymbol{x})$ is spherically symmetric and has as its support the sphere of radius R centered at the origin. Near this point it behaves locally as r^2, i.e. $|\tilde{\psi}(\boldsymbol{r}) - \tilde{\psi}(\boldsymbol{0})| \approx r^2$ for $|\boldsymbol{r}| \ll R$. (Wave-functions of similar local behaviour may be produced in diffraction experiments.) In this case, the convolution integral

$$\hat{\psi}^{(\varepsilon)}(\boldsymbol{r}) = \int d^3\varrho\, G_\varepsilon(|\boldsymbol{\varrho}|)\, \tilde{\psi}\, (|\boldsymbol{r} - \boldsymbol{\varrho}|)$$

is reducible to elementary integrals by introducing spherical coordinates centered at $\boldsymbol{r}$ and integrating first over the angles. The particle density $P(r)$ and the energy density $\mathcal{H}(r)$ corresponding to this state (95) are shown together in Figure 2. The scales are linear, and the units were chosen arbitrary but such that the two curves coincide on the plot for $r = 2/5\, R$. The two curves are seen to be similar in shape but not exactly proportional to each other. The deviations from proportionality amount to a few per cent.

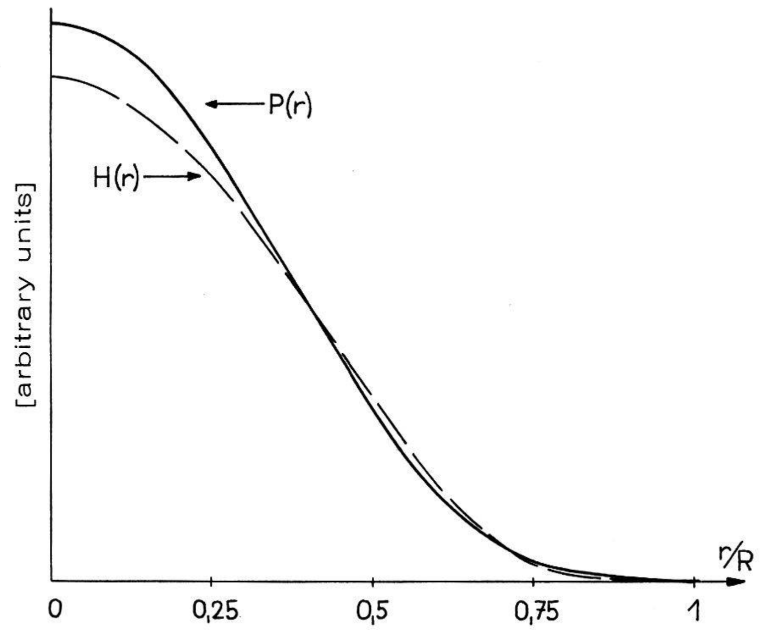

Figure 2

Energy density (– – – –) and particle density (————) of the state (95).

Finally, we give a brief analysis of the *particle field* (the configuration-space wave-function) belonging to a particle of mass zero and helicity s. It may be defined in the one-component or in the $(2\,s + 1)$-component formalism. We restrict our attention to the latter case. The definition of the particle field $\phi_\beta(\boldsymbol{x}, t)$ belonging to a state $\phi(\boldsymbol{k}) \in \mathcal{H}^{[0,\,s]}$ is then

$$\phi_\beta(\boldsymbol{x}, t) = \sqrt{\frac{\hbar\, c}{2(2\,\pi)^3}} \int \frac{d^3k}{|\boldsymbol{k}|}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x} - ic|\boldsymbol{k}|t}\, D^{|s|}_{\beta s}(X_{k\leftarrow k_z})\, \phi(\boldsymbol{k})\,. \tag{96}$$

In the same way as we obtained the amplitude $\hat{\psi}_\beta(\boldsymbol{x})$ of the energy density as a convolution integral between the kernels $G_\varepsilon(\boldsymbol{r})$ and the position-space wave-function $\tilde{\psi}_\beta(\boldsymbol{x})$, one may derive a similar representation for $\phi_\beta(\boldsymbol{x}, 0)$:

$$\phi_\beta(\boldsymbol{x}, 0) = \lim_{\varepsilon\to+0} \int d^3y\, L_\varepsilon\,(|\boldsymbol{x} - \boldsymbol{y}|)\, \tilde{\psi}_\beta(\boldsymbol{y}) \tag{97}$$

with [33]

$$L_\varepsilon(r) = \sqrt{\frac{\hbar\, c}{2(2\,\pi)^6}} \int d^3k\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \frac{1}{\sqrt{|\boldsymbol{k}|}}\, e^{-\varepsilon|\boldsymbol{k}|} = \frac{1}{4}\sqrt{\frac{\hbar\, c}{2\,\pi^3}}\, \frac{1}{r}\, \frac{1}{(r^2+\varepsilon^2)^{3/4}} \sin\left(\frac{3}{2}\,\mathrm{arctg}\,\frac{r}{\varepsilon}\right) \tag{98}$$

The asymptotic behaviour $(r \gg \varepsilon)$ of $L_\varepsilon(r)$ is the following

$$L_\varepsilon(r) \approx L(r) = \frac{1}{8}\sqrt{\frac{\hbar\, c}{\pi^3}}\, r^{-5/2}\,. \tag{99}$$

One notices that $\phi_\beta(\boldsymbol{x}, 0)$ decreases differently from the amplitude $\hat{\psi}_\beta(\boldsymbol{x})$ of the energy density. This is so because the expression for the energy density involves derivatives of the particle field. We also wish to stress once more that the argument $\boldsymbol{x}$ of $\phi_\beta(\boldsymbol{x}, t)$ stands for the position of the particle field and not for that of the particles.

X. Applications in Quantum Field Theory

The results of the preceding sections have some bearing on certain topics of relativistic quantum field theory. In this part we first submit an explicit expression for the operator $N_{\Delta,t}$ corresponding to the number of particles localized in a volume Δ at time t. This operator will then be compared with similar expressions that have appeared in the literature. Finally, we shall comment briefly upon its use for the description of photon-counting experiments.

In a second-quantized theory of a non-interacting relativistic field, the underlying Hilbert space may be represented as

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)} \tag{100}$$

where $\mathcal{H}^{(n)}$ denotes the space of all n-particle states. $\mathcal{H}^{(1)}$ corresponds to the Hilbert space of a relativistic elementary system, and $\mathcal{H}^{(n)}$ is the symmetrized or anti-symmetrized n-fold tensorial product of $\mathcal{H}^{(1)}$:

$$\mathcal{H}^{(n)} = \mathcal{S}\, \mathcal{H}^{(1)\otimes n} \quad \text{for a boson field}$$

$$\mathcal{H}^{(n)} = \mathcal{A}\, \mathcal{H}^{(1)\otimes n} \quad \text{for a fermion field}$$

where $\mathcal{S}$ is the symmetrizing and $\mathcal{A}$ the anti-symmetrizing operator.

The operator $N_{\Delta,t}$ may be specified by indicating how it acts on a basis of vectors of $\mathcal{H}$. Let $\{\varphi_i\}$, $i = 1, 2, \ldots$, be a basis of $\mathcal{H}^{(1)}$. A basis $\{\varphi_{i_1 \ldots i_n}\}$ of $\mathcal{H}^{(n)}$ for $n > 1$ is then given by

$$\varphi_{i_1 \ldots i_n} = \binom{\mathcal{S}}{\mathcal{A}} \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_n} \quad (i_k = 1, 2, \ldots)\,. \tag{101}$$

We suppose that the operators $F_{\Delta,t}$ in $\mathcal{H}^{(1)}$ are already known. For the physically interesting cases they were constructed explicitly in Section V. The operator $N_{\Delta,t}$ then acts on a vector $\varphi_{i_1 \ldots i_n}$ as follows [35]:

$$\begin{aligned} N_{\Delta,t}\, \varphi_{i_1 \ldots i_n} = \binom{\mathcal{S}}{\mathcal{A}} \{ &(F_{\Delta,t}\, \varphi_{i_1}) \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_n} \\ &+ \varphi_{i_1} \otimes (F_{\Delta,t}\, \varphi_{i_2}) \otimes \varphi_{i_3} \otimes \cdots \otimes \varphi_{i_n} \\ &+ \cdots \\ &+ \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes (F_{\Delta,t}\, \varphi_{i_n}) \}\,. \end{aligned} \tag{102}$$

Thus $N_{\Delta,t}$ transforms each subspace $\mathcal{H}^{(n)}$ into itself. In $\mathcal{H}^{(0)}$, $N_{\Delta,t}$ is defined to be the zero operator.

One concludes from (102) that

$$(\varphi_{i_1 \ldots i_n}, N_{\Delta,t}\, \varphi_{i_1 \ldots i_n}) = \sum_{k=1}^{n} (\varphi_{i_k}, F_{\Delta,t}\, \varphi_{i_k})\,. \tag{103}$$

This equation makes plain that the expectation values of the operator $N_{\Delta,t}$ (for normalized states) coincide with the number of particles which are present in the volume Δ at time t. In particular, if m ($m \leqslant n$) of the states $\varphi_{i_1}, \ldots, \varphi_{i_n}$ occurring in (102) are localized in Δ at time t (i.e. they satisfy $F_{\Delta,t}\, \varphi = \varphi$), and the remaining

$n - m$ vectors are orthogonal to the subspace of $\mathcal{H}^{(1)}$ determined by $F_{\Delta,t}$, then it follows from (102) or (103) that

$$N_{\Delta,t}\,\varphi_{i_1\cdots i_n} = m \cdot \varphi_{i_1\cdots i_n}\,.$$

Furthermore, if $\Delta = R^3$, then $F_{R^3,t} = I$, and hence $N_{R^3,t}$ acting on any n-particle state is just multiplication by n, i.e. $N_{R^3,t}$ is the usual operator corresponding to the total number of particles.

The restriction of the operator $N_{\Delta,t}$ to the subspace $\mathcal{H}^{(n)}$ can also be written formally as a sum of n operators:

$$N^{(n)}_{\Delta,t} = F_{\Delta,t} \otimes I \otimes \cdots \otimes I + I \otimes F_{\Delta,t} \otimes I \otimes \cdots \otimes I$$
$$+ \cdots + I \otimes I \otimes \cdots \otimes I \otimes F_{\Delta,t}$$
$$N^{(0)}_{\Delta,t} = 0 \tag{104}$$

Consequently

$$N_{\Delta,t} = \bigoplus_{n=0}^{\infty} N^{(n)}_{\Delta,t}\,. \tag{105}$$

For the sake of completeness, we indicate the relation between the space $\mathcal{H}^{(1)}$ of one-particle states and the Hilbert space $\mathcal{H}^{[0,s]}$ of the corresponding relativistic elementary system of mass $m = 0$ and helicity s. (Similar formulae hold for $m > 0$.) Let $a^\dagger(\boldsymbol{k}, s)$ stand for the creation operator, $a(\boldsymbol{k}, s)$ for the annihilation operator of a particle of mass $m = 0$, helicity s and momentum $\boldsymbol{k}$, normalized in such a way that

$$[a(\boldsymbol{k}, s), a^\dagger(\boldsymbol{k}', s)]_{\mp} = |\boldsymbol{k}|\, \delta^3(\boldsymbol{k} - \boldsymbol{k}') \tag{106}$$

(where the two signs distinguish between bosons and fermions). The space $\mathcal{H}^{(1)}$ is spanned by the vectors

$$|\varphi\rangle = \int \frac{d^3k}{|\boldsymbol{k}|}\, \varphi(\boldsymbol{k})\, a^\dagger(\boldsymbol{k}, s)\, |0\rangle \tag{107}$$

where $\varphi(\boldsymbol{k}) \in \mathcal{H}^{[0,s]}$. The scalar product in $\mathcal{H}^{(1)}$ corresponds to that in $\mathcal{H}^{[0,s]}$. This follows immediately from (107) and (106):

$$\langle\psi \mid \varphi\rangle = \int \frac{d^3k}{|\boldsymbol{k}|} \int \frac{d^3q}{|\boldsymbol{q}|}\, \psi^*(\boldsymbol{q})\, \varphi(\boldsymbol{k})\, \langle 0 \mid a(\boldsymbol{q}, s)\, a^\dagger(\boldsymbol{k}, s) \mid 0\rangle$$
$$= \int \frac{d^3k}{|\boldsymbol{k}|} \int \frac{d^3q}{|\boldsymbol{q}|}\, \psi^*(\boldsymbol{q})\, \varphi(\boldsymbol{k})\, |\boldsymbol{q}|\, \delta^3(\boldsymbol{q} - \boldsymbol{k})$$
$$= \int \frac{d^3k}{|\boldsymbol{k}|}\, \psi^*(\boldsymbol{k})\, \varphi(\boldsymbol{k}) \equiv (\psi, \varphi)\,. \tag{108}$$

We next express $N_{\Delta,t}$ in terms of the creation and annihilation operators. For this purpose, we denote by $\{\psi^i_{\Delta,t}\}$, $i = 1, 2, \ldots,$ a basis of vectors in the range of the projection operator $F_{\Delta,t}$ in $\mathcal{H}^{[0,s]}$. This leads to the following formula for the action of $F_{\Delta,t}$ on a state $\varphi \in \mathcal{H}^{[0,s]}$:

$$[F_{\Delta,t}\,\varphi]\,(\boldsymbol{k}) = \sum_i{}' (\psi^i_{\Delta,t}, \varphi)\, \psi^i_{\Delta,t}(\boldsymbol{k})\,. \tag{109}$$

In order to arrive at a similar expression for $N_{\Delta,t}$, we define for every $\psi^i_{\Delta,t}$ two operators in the Fock space $\mathcal{H}$ by

$$a(\psi^i_{\Delta,t}) \equiv \int \frac{d^3k}{|\boldsymbol{k}|}\, \psi^{i*}_{\Delta,t}(\boldsymbol{k})\, a(\boldsymbol{k}, s) \tag{110a}$$

$$a^\dagger(\psi^i_{\Delta,t}) = \int \frac{d^3k}{|\boldsymbol{k}|}\, \psi^{i}_{\Delta,t}(\boldsymbol{k})\, a^\dagger(\boldsymbol{k}, s)\,. \tag{110b}$$

Using (109), it is a matter of simple algebra to verify that

$$N_{\Delta,t} = \sum_i a^\dagger(\psi^i_{\Delta,t})\, a(\psi^i_{\Delta,t})\,. \tag{111}$$

Equation (111) is our final expression for $N_{\Delta,t}$. We wish now to relate it to similar proposals for a 'number of particles' operator and also to point out some confusion in the literature on relativistic quantum field theory as regards the particle density.

We begin with a scalar field, $s = 0$. The field operator $\Phi(\boldsymbol{x}, t)$ is defined as

$$\Phi(\boldsymbol{x}, t) = \Phi^{(+)}(\boldsymbol{x}, t) + \Phi^{(-)}(\boldsymbol{x}, t) \tag{112}$$

with

$$\Phi^{(+)}(\boldsymbol{x}, t) = \frac{1}{\sqrt{2(2\pi)^3}} \int \frac{d^3k}{|\boldsymbol{k}|}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i|\boldsymbol{k}|t}\, a(\boldsymbol{k}, 0) \tag{113a}$$

$$\Phi^{(-)}(\boldsymbol{x}, t) = \frac{1}{\sqrt{2(2\pi)^3}} \int \frac{d^3k}{|\boldsymbol{k}|}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x} + i|\boldsymbol{k}|t}\, a^\dagger(\boldsymbol{k}, 0)\,. \tag{113b}$$

HENLEY and THIRRING [36] use the following expression for the operator $N(\boldsymbol{x}, t)$ corresponding to the particle density of this field:

$$N(\boldsymbol{x}, t) = i\,\{\Phi^{(-)}(\boldsymbol{x}, t)\, \dot{\Phi}^{(+)}(\boldsymbol{x}, t) - \dot{\Phi}^{(-)}(\boldsymbol{x}, t)\, \Phi^{(+)}(\boldsymbol{x}, t)\}\,. \tag{114}$$

The expectation values of this operator are not positive definite [37]. It is therefore not legitimate to interpret

$$N(\Delta, t) = \int_\Delta d^3x\, N(\boldsymbol{x}, t)$$

as the operator describing the number of particles which are present in the volume Δ at time t. This is now well understandable, since we explained in Section IX that the argument $\boldsymbol{x}$ in the field operator corresponds to the position of the field and not to that of the individual particles. The correct particle density for $s = 0$ is given by SCHWEBER [38] as follows

$$N(\Delta, t) = \int_\Delta d^3q\, \tilde{\Phi}^\dagger(\boldsymbol{q}, t)\, \tilde{\Phi}(\boldsymbol{q}, t) \tag{115}$$

with

$$\tilde{\Phi}(\boldsymbol{q}, t) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{|\boldsymbol{k}|}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{q} - i|\boldsymbol{k}|t}\, a(\boldsymbol{k}, 0) \tag{116}$$

(116) differs from (113a) by a factor $\sqrt{|\boldsymbol{k}|}$ in the integrand. (115) is identical with our expression (111) for $N_{\Delta,t}$: One takes for $\psi^i_{\Delta,t}(\boldsymbol{k})$ the states $(2\pi)^{-3/2}\sqrt{|\boldsymbol{k}|}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{q} + i|\boldsymbol{k}|t}$ (i.e. the localized states at the point $(\boldsymbol{q}, t)$ of NEWTON and WIGNER [2]) and replaces $\sum_i$ by the integral over the volume Δ.

For the electromagnetic field, $s = \pm 1$, MANDEL [39, 40] applied a definition similar to (115), (116). He introduced a 'detection operator' [41]

$$\tilde{\boldsymbol{A}}(\boldsymbol{x}, t) = (2\pi)^{-3/2} \sum_{\sigma} \int \frac{d^3k}{\sqrt{|\boldsymbol{k}|}} e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i|\boldsymbol{k}|t} \boldsymbol{\varepsilon}_{\boldsymbol{k},\sigma} a_{\boldsymbol{k},\sigma} \tag{117}$$

and defined

$$N(\Delta, t) = \int_{\Delta} d^3x \, \tilde{\boldsymbol{A}}^{\dagger}(\boldsymbol{x}, t) \cdot \tilde{\boldsymbol{A}}(\boldsymbol{x}, t) . \tag{118}$$

Here $\boldsymbol{\varepsilon}_{\boldsymbol{k},\sigma}$ ($\sigma = 1, 2$) are the polarization unit vectors [42], and $a_{\boldsymbol{k},\sigma}$ the annihilation operators for photons of momentum $\boldsymbol{k}$ and polarization σ. (The projection operators $F_{\Delta,t}$ in this conventional formalism for photons were given by JAUCH and PIRON [8]. Their procedure is of course equivalent with the group theoretical approach which constitutes the basis of the present investigation. One should keep in mind, though, that the polarization σ of this conventional formalism differs from the helicity.)

The one-particle states assume the form

$$|\varphi\rangle = \sum_{\sigma} \int \frac{d^3k}{|\boldsymbol{k}|} \varphi(\boldsymbol{k}, \sigma) a^{\dagger}_{\boldsymbol{k},\sigma} |0\rangle . \tag{119}$$

The action of the detection operator $\tilde{\boldsymbol{A}}(\boldsymbol{x}, t)$ on such a state becomes

$$\tilde{\boldsymbol{A}}(\boldsymbol{x}, t) \, |\varphi\rangle = (2\pi)^{-3/2} \sum_{\sigma} \int \frac{d^3k}{\sqrt{|\boldsymbol{k}|}} \varphi(\boldsymbol{k}, \sigma) \, \boldsymbol{\varepsilon}_{\boldsymbol{k},\sigma} \, e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i|\boldsymbol{k}|t} \, |0\rangle . \tag{120}$$

Here the function

$$\boldsymbol{\varphi}(\boldsymbol{x}, t) = (2\pi)^{-3/2} \sum_{\sigma} \int \frac{d^3k}{\sqrt{|\boldsymbol{k}|}} \varphi(\boldsymbol{k}, \sigma) \, \boldsymbol{\varepsilon}_{\boldsymbol{k},\sigma} \, e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i|\boldsymbol{k}|t}$$

represents the correct position-space wave-function belonging to $\varphi(\boldsymbol{k}, \sigma)$ as defined also by JAUCH and PIRON [8] (it corresponds to our Equation (81)). However, the quantity

$$\int_{\Delta} d^3x \, |\tilde{\boldsymbol{\varphi}}(\boldsymbol{x}, t)|^2 \equiv \langle \varphi \, | N(\Delta, t) | \, \varphi \rangle$$

differs in general from the probability of finding the photon $|\varphi\rangle$ localized in the volume Δ at time t. The reason for this lies in the fact that $F_{\Delta,t}$ is *not* simply multiplication by the characteristic function $\chi_{\Delta}(\boldsymbol{x})$ of the Borel set Δ.

We infer from these remarks that the operator $N(\Delta, t)$ of Equation (118) should not be regarded as the true correlate of the number of photons in the volume Δ at time t. Indeed, it is impossible to write the correct operator $N_{\Delta,t}$ for photons as a simple integral over the volume Δ, since that would imply that $F_{\Delta_1,t}$ and $F_{\Delta_2,t}$ commute for all pairs Δ_1, Δ_2 of Borel sets of R^3, which contradicts Theorem 2. We made mention of the implications of the non-commutativity of $F_{\Delta_1,t}$ and $F_{\Delta_2,t}$ for certain couples Δ_1, Δ_2 already in Section IX. There it was responsible for the inequality (84). (A representation of $N_{\Delta,t}$ as a double integral over Δ would allow for this inequality. Such an expression will be given below.) This relation (84) entails immediately that Mandel's $N(\Delta, t)$ of Equation (118) represents an upper bound for the exact particle number operator $N_{\Delta,t}$: For all states $|\phi\rangle \in \mathcal{H}$ one has

$$\langle \phi \, | N(\Delta, t) | \, \phi \rangle \geqslant \langle \phi \, | N_{\Delta,t} | \, \phi \rangle . \tag{121}$$

It is possible to write $N_{\Delta, t}$ in terms of the detection operator $\tilde{\boldsymbol{A}}(\boldsymbol{x}, t)$. To this end we rewrite the expression (111) for $N_{\Delta, t}$ in the conventional formalism for photons. The basis vectors $\psi^i_{\Delta,t}$ in the range of $F_{\Delta, t}$ have then two components $\psi^{i,\sigma}_{\Delta,t}$ ($\sigma = 1, 2$) corresponding to the two directions of the polarization. Equations (110a, b) are replaced by

$$\boldsymbol{a}(\psi^i_{\Delta, t}) \equiv \sum_\sigma \int \frac{d^3k}{|\boldsymbol{k}|} \psi^{i,\sigma *}_{\Delta,t}(\boldsymbol{k}) \, a_{\boldsymbol{k},\sigma} \tag{122}$$

and its adjoint. The position-space representation $\tilde{\boldsymbol{\psi}}^i_\Delta(\boldsymbol{x})$ of these functions $\psi^i_{\Delta,t}$ is defined such that (cf. Equation (120))

$$\psi^{i,\sigma}_{\Delta,t}(\boldsymbol{k}) = (2\pi)^{-3/2} \sqrt{|\boldsymbol{k}|} \int d^3x \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x} + i|\boldsymbol{k}|t} \, \boldsymbol{\varepsilon}_{\boldsymbol{k},\sigma} \cdot \tilde{\boldsymbol{\psi}}^i_\Delta(\boldsymbol{x}) \tag{123}$$

Let r, s label the three space-components of the detection operator and the states $\tilde{\boldsymbol{\psi}}^i_\Delta(\boldsymbol{x})$. Inserting (122) and its adjoint into (111) and using (123) and (117) then leads to

$$N_{\Delta, t} = \sum_{r,s=1}^{3} \int d^3x \int d^3y \, K^{rs}_\Delta(\boldsymbol{x}, \boldsymbol{y}) \, \tilde{A}^\dagger_r(\boldsymbol{x}, t) \, \tilde{A}_s(\boldsymbol{y}, t) \tag{124}$$

with

$$K^{rs}_\Delta(\boldsymbol{x}, \boldsymbol{y}) = \sum_i \tilde{\psi}^{i,r}_\Delta(\boldsymbol{x}) \, \tilde{\psi}^{i,s*}_\Delta(\boldsymbol{y}) \, . \tag{125}$$

This kernel is such that

$$K^{rs}_\Delta(\boldsymbol{x}, \boldsymbol{y}) = 0 \ \text{ if } \ \boldsymbol{x} \notin \Delta \ \text{ or } \ \boldsymbol{y} \notin \Delta \, . \tag{126}$$

Furthermore, it is non-local (i.e. in general $K^{rs}_\Delta(\boldsymbol{x}, \boldsymbol{y}) \neq 0$ for all pairs $\boldsymbol{x}, \boldsymbol{y} \in \Delta$) and difficult to handle mathematically, since it requires the knowledge of a complete basis of eigenvectors of $F_{\Delta, t}$. Putting $K^{rs}_\Delta(\boldsymbol{x}, \boldsymbol{y}) = \delta_{rs} \, \delta^3 \, (\boldsymbol{x} - \boldsymbol{y}) \, \chi_\Delta(\boldsymbol{x})$ in (124) leads to the operator $N(\Delta, t)$ of Equation (118) which is distinguished by its much simpler mathematical form. We have verified that for monochromatic radiation the difference $\langle \phi \,|N(\Delta, t)|\, \phi \rangle - \langle \phi \,|N_{\Delta, t}|\, \phi \rangle$ becomes negligeable if the linear dimensions of the volume Δ are much larger than the wave-length. For many practical purposes one may therefore employ the approximate but simple operator $N(\Delta, t)$ of Equation (118) to characterize the number of photons localized in the volume Δ at time t.

To conclude this part, we wish to point out the extent to which these photon number operators are the appropriate means for the description of photon-counting experiments. The physical basis of the usual photon-counters is the photoelectric effect. Such apparatus measure therefore the electric field strength $\boldsymbol{\varepsilon}(\boldsymbol{x}, t)$ (averaged over the volume that the counter occupies). In quantum field theory, the electric field is represented by an operator $\boldsymbol{E}(\boldsymbol{x}, t)$ acting in $\mathcal{H}$. In the radiation gauge, $\boldsymbol{E}(\boldsymbol{x}, t)$ is simply the time-derivative of the vector-potential operator $\boldsymbol{A}(\boldsymbol{x}, t)$:

$$\boldsymbol{A}(\boldsymbol{x}, t) = \boldsymbol{A}^{(+)}(\boldsymbol{x}, t) + \boldsymbol{A}^{(-)}(\boldsymbol{x}, t) \tag{127}$$

with

$$\boldsymbol{A}^{(+)}(\boldsymbol{x}, t) = \frac{1}{\sqrt{2(2\pi)^3}} \sum_\sigma \int \frac{d^3k}{|\boldsymbol{k}|} \, e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i|\boldsymbol{k}|t} \, \boldsymbol{\varepsilon}_{\boldsymbol{k},\sigma} \, a_{\boldsymbol{k},\sigma} \tag{128a}$$

$$\boldsymbol{A}^{(-)}(\boldsymbol{x}, t) = \frac{1}{\sqrt{2(2\pi)^3}} \sum_\sigma \int \frac{d^3k}{|\boldsymbol{k}|} \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x} + i|\boldsymbol{k}|t} \, \boldsymbol{\varepsilon}_{\boldsymbol{k},\sigma} \, a^\dagger_{\boldsymbol{k},\sigma} \tag{128b}$$

and

$$\boldsymbol{E}(\boldsymbol{x}, t) = - \frac{\partial \boldsymbol{A}(\boldsymbol{x}, t)}{\partial t} = \boldsymbol{E}^{(+)}(\boldsymbol{x}, t) + \boldsymbol{E}^{(-)}(\boldsymbol{x}, t) . \tag{129}$$

Photon-counting experiments are characterized by the expectation values of products of operators $E_i^{(\pm)}(\boldsymbol{x}, t)$ (i labels the three components) of the following form [43]

$$\langle \phi \,| E_{i_1}^{(-)}(\boldsymbol{x}_1, t_1) \ldots E_{i_n}^{(-)}(\boldsymbol{x}_n, t_n)\, E_{j_1}^{(+)}(\boldsymbol{y}_1, \tau_1) \ldots E_{j_m}^{(+)}(\boldsymbol{y}_m, \tau_m) \,|\, \phi \rangle .$$

They are usually evaluated for states $|\phi\rangle$ of the *free* electromagnetic field. One may therefore relate these expressions to the localization of the photons of the quantum state $|\phi\rangle$. For this purpose, we calculate the action of $\boldsymbol{E}^{(+)}(\boldsymbol{x}, t)$ on a one-photon state $|\varphi\rangle$ of the form (119):

$$\boldsymbol{E}^{(+)}(\boldsymbol{x}, t)\, |\varphi\rangle = \frac{1}{\sqrt{2 (2\pi)^3}} \sum_{\sigma} \int d^3k\, \varphi(\boldsymbol{k}, \sigma)\, \boldsymbol{\varepsilon}_{\boldsymbol{k}, \sigma}\, e^{i \boldsymbol{k} \cdot \boldsymbol{x} - i |\boldsymbol{k}| t} |0\rangle . \tag{130}$$

The position-space state belonging to $|\varphi\rangle$ was found in (120). The two expressions are again connected by the kernels (92):

$$\boldsymbol{E}^{(+)}(\boldsymbol{x}, t)\, |\varphi\rangle = \frac{1}{\sqrt{2}} \lim_{\varepsilon \to +0} \int d^3y\, G_\varepsilon(\boldsymbol{x} - \boldsymbol{y})\, \tilde{\boldsymbol{A}}(\boldsymbol{y}, t)\, |\varphi\rangle . \tag{131}$$

(One may see this upon comparing (130), (120) with the corresponding expressions (88), (81) and (51) of the preceding section.) Therefore *the electric field of a localized photon decreases as $r^{-7/2}$ at large distances from the volume of localization.*

Photons interact with a counter through their electric field. The number of photons which can be absorbed by a counter occupying the volume Δ is therefore proportional to the expectation values of the operator $\int_\Delta d^3x\, \boldsymbol{E}^{(-)}(\boldsymbol{x}, t) \cdot \boldsymbol{E}^{(+)}(\boldsymbol{x}, t)$ [44]. Since the electric field of a photon is related to its position-space wave-function by means of a non-local expression, these absorption rates are not directly proportional to the expectation values of the operator $N_{\Delta, t}$ which indicate the number of photons that are present in the volume Δ at time t. However, for most states (in particular for approximations of plane waves) and for sufficiently large linear dimensions of Δ, the quotients of these two expectation values for different volumes Δ differ at most by a few per cent. Consequently, under the above-mentioned restriction on the size of Δ, one may use $N_{\Delta, t}$ or $N(\Delta, t)$ as operators representing approximately the number of photons which can be absorbed by a counter occupying the volume Δ. (A counter consisting of a single atom does not satisfy this condition on Δ, since its diameter is much smaller than the wave-lengths of visible light.) (Mandel [40] claims that the expectation values of the operator (118) coincide with Glauber's first-order correlation functions [43]:

$$\langle \phi | \int_\Delta d^3x\, \tilde{\boldsymbol{A}}^\dagger(\boldsymbol{x}, t) \cdot \tilde{\boldsymbol{A}}(\boldsymbol{x}, t) \,|\, \phi \rangle = 2 \int_\Delta d^3x \langle \phi \,| \boldsymbol{A}^{(-)}(\boldsymbol{x}, t) \cdot \boldsymbol{A}^{(+)}(\boldsymbol{x}, t) \,|\, \phi \rangle .$$

This equation is not correct. In Mandel's derivation [40], the detection operator $\tilde{\boldsymbol{A}}(\boldsymbol{x}, t)$ appearing on the left-hand side was used erroneously for the field operator $\boldsymbol{A}^{(+)}(\boldsymbol{x}, t)$ of the right-hand side.)

Finally, we should mention that the difference between localization and energy density is present also for the quantum states that belong to a classical electromagnetic field. Such a state $|\phi_{cl}\rangle$ has the properties that

$$\langle\phi_{cl}\,|\,\boldsymbol{A}(\boldsymbol{x},t)\,|\,\phi_{cl}\rangle = \mathcal{A}_{cl}(\boldsymbol{x},t)$$

$$\langle\phi_{cl}\,|\,\mathcal{H}(\boldsymbol{x},t)\,|\,\phi_{cl}\rangle = E_{cl}(\boldsymbol{x},t)$$

where $\mathcal{A}_{cl}(\boldsymbol{x},t)$ is a solution of the classical Maxwell equations, $E_{cl}(\boldsymbol{x},t)$ the corresponding classical energy density, and $\mathcal{H}(\boldsymbol{x},t)$ the operator representing the energy density of the quantized electromagnetic field. Such a classical (or coherent) state assumes the following form [43]:

$$|\,\phi_{cl}\rangle = \lambda \exp\left\{\sum_{\sigma}\int \frac{d^3k}{|\boldsymbol{k}|}\,\varphi(\boldsymbol{k},\sigma)\,a^{\dagger}_{\boldsymbol{k},\sigma}\right\}\,|\,0\rangle \tag{132}$$

where λ is a normalization constant.

$|\phi_{cl}\rangle$ is determined by a single one-photon wave-function $\varphi(\boldsymbol{k},\sigma)$. The n-photon component of $|\phi_{cl}\rangle$ consists of n photons all of which are in the state $\varphi(\boldsymbol{k},\sigma)$, and has the weight $1/\sqrt{n!}$. The energy density of the state $|\phi_{cl}\rangle$ is simply proportional to the energy density corresponding to the one-photon state $\varphi(\boldsymbol{k},\sigma)$ and shows therefore the same non-local behaviour as we discussed in Section IX.

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