Zeitschrift: Helvetica Physica Acta

Band: 42 (1969)

Heft: 1

Artikel: On the scalar field model

Autor: Guenin, Marcel / Velo, Giorgio

DOI: https://doi.org/10.5169/seals-114057

On the Scalar Field Model[1])

by **Marcel Guenin**
Institut de Physique Théorique, Université de Genève, Geneva, Switzerland
and **Giorgio Velo**[2])
Department of Physics, New York University, New York, N.Y. USA

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Abstract. The scalar field model is studied on a mathematically rigorous basis. Using algebraic techniques, we get the explicit cut-off dependent operator solution, and discuss the existence of the limit whenever the cut-off is removed. It is shown that the Wightman functions are tempered distributions in the limit of no cut-off and in a space with dimension less or equal to three.

Introduction

Even if all models actually available in quantum field theory have little physical content, the main interest in their study lies in the fact that ideas and methods can be tested on them.

Rigorously founded solutions of many models are to be found in WIGHTMAN's work [1] (see also VELO [2]). The method of WIGHTMAN is to take the set of all Wightman functions obtained by some mean, and to guess a combination of functions of free fields which reproduces them. One then checks that the field equations are in some sense verified. The only trouble with this method, is that the solution is not expressed in terms of the unperturbed (free) fields. This makes it difficult to compare the exact solution with solutions (or approximations) obtained by more conventional methods. But one does know the analyticity properties of the solution as a function of the coupling constant.

The method we shall test here has been proposed by one of us [3, 4] and already applied to the very simple case of a quadratic interaction [5]. The basic idea is to consider the time evolution as an algebraic (instead of spatial) automorphism of the algebra generated by field operators. What we want to show here is that it is possible, at least for simple enough models, to compute explicitly this automorphism, and, from this, to get all Wightman functions. We give a rigorously defined operator solution, in term of the unperturbed fields, and this should make rather easy to compare this exact solution with, say, perturbative expansions. We shall give two derivations of the basic result, not that the model is so interesting in itself, but in order to test methods which we shall apply to other cases in coming publications.

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1) Research supported in part by the National Science Foundation.
2) On leave of absence from the Istituto di Fisica dell'Università di Bologna, Italy.

The Scalar Field Model

The scalar field model is characterized by the hamiltonian

$$H = H_0 + H_I$$
$$H_0 = m_0 \int d^s \boldsymbol{p}\, \psi^*(\boldsymbol{p})\, \psi(\boldsymbol{p}) + \int d^s \boldsymbol{k}\, \omega(\boldsymbol{k})\, a^*(\boldsymbol{k})\, a(\boldsymbol{k}) = H_{0F} + H_{0B}$$
$$\omega(\boldsymbol{k}) = \sqrt{\boldsymbol{k}^2 + \mu^2}\ (= k_0)$$
$$H_I = \frac{\lambda}{(2\,\pi)^{s/2}} \int d^s \boldsymbol{p}\, d^s \boldsymbol{k}\, \frac{f(\boldsymbol{k}^2)}{\sqrt{2\,\omega(\boldsymbol{k})}}\, \psi^*(\boldsymbol{p}+\boldsymbol{k})\, \psi(\boldsymbol{p})\, \{a(\boldsymbol{k}) + a^*(-\boldsymbol{k})\} \tag{1}$$

and the commutation rules

$$[\psi(\boldsymbol{p}), \psi^*(\boldsymbol{p}')]_+ = \delta^{(s)}\,(\boldsymbol{p}-\boldsymbol{p}')$$
$$[\psi(\boldsymbol{p}), \psi(\boldsymbol{p}')]_+ = [\psi^*(\boldsymbol{p}), \psi^*(\boldsymbol{p}')]_+ = 0$$
$$[a(\boldsymbol{k}), a^*(\boldsymbol{k}')] = \delta^{(s)}\,(\boldsymbol{k}-\boldsymbol{k}')$$
$$[a(\boldsymbol{k}), a(\boldsymbol{k}')] = [a^*(\boldsymbol{k}), a^*(\boldsymbol{k}')] = 0$$
$$[\psi(\boldsymbol{p}), a(\boldsymbol{k})] = [\psi(\boldsymbol{p}), a^*(\boldsymbol{k})] = [\psi^*(\boldsymbol{p}), a(\boldsymbol{k})] = [\psi^*(\boldsymbol{p}), a^*(\boldsymbol{k})] = 0\,. \tag{2}$$

Thus this model describes spinless fermions interacting with a neutral scalar boson field. The energy of the nucleon is taken to be independant of its momentum, and this is usually refered to as 'recoiles nucleon'. At the end of this paper we shall show how it is possible for a particular choice of the cut-off function to give the exact solution of the relativistic nucleon with recoil.

$f(\boldsymbol{k})$ describes a cut-off function and s is the number of space dimensions. We shall drop from now on the suffix s in all integrals and δ-functions.

This scalar field model is known [6] to be exactly soluble in the sense that it is possible to give the exact renormalized one-particle state. The exact S-matrix elements for the scattering of a meson by a nucleon may also be computed and turns out to be trivial. Schweber [6] has also given the form for the U-matrix, and thus, in principle, the S-matrix too, but his form is ill defined for two reasons: first it contains a term proportional to the time T during which the interaction has been switched on, secondly it is not clear whether the expression given remains meaningful as the cut-off is removed.

We shall give the exact operator solution, and have it perfectly defined as an operator valued generalized function on the Fock space of the free fields, with all renormalization terms put in evidence. From this it will then be easy to give the explicit expressions for the n-points functions.

1st Method

This first method is in fact the completion of a method already partly used by Schweber, and explained in great detail by Magnus [7]. It is based on a splitting of the U-matrix into different parts.

We have:

$$H_I(t) \equiv e^{iH_0 t}\, H_I\, e^{-iH_0 t} \qquad U(t, t_0) \equiv e^{iH_0 t}\, e^{-iH(t-t_0)}\, e^{-iH_0 t_0}$$
$$i\,\partial_t\, U(t, t_0) = H_I(t)\, U(t, t_0);\ \ U(t_0, t_0) = 1\,. \tag{3}$$

Let us make the 'ansatz'

$$U(t, t_0) = e^{-i \int_{t_0}^{t} H_I(t')\, dt'} \, V(t, t_0)$$

it follows then that

$$i\, \partial_t\, V(t, t_0) = e^{i \int_{t_0}^{t} H_I(t')\, dt'} \{H_I(-t) - i\, \partial_t\}\, e^{-i \int_{t_0}^{t} H_I(t')\, dt'} \, V(t, t_0)$$

$$= \sum_{n=1}^{\infty} \frac{i n}{n!} \frac{n}{n+1}\, \Omega_n \left(\int_{t_0}^{t} H_I(t')\, dt', H_I(t) \right) V(t, t_0) \tag{4}$$

where $\Omega_n(A, B)$ denotes the multiple commutator defined recursively by $\Omega_0(A, B) = B$, $\Omega_{n+1}(A, B) = [A, \Omega_n(A, B)]$ and using the formula

$$[e^B, A] = \sum_{n=1}^{\infty} \frac{1}{n!}\, \Omega_n(B, A)\, e^B = -\, e^B \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\, \Omega_n(B, A) \tag{5}$$

and

$$\frac{\partial}{\partial s}\, e^{F(s)} = e^{F(s)} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}\, \Omega_n\Big(F(s), \frac{\partial}{\partial s}\, F(s)\Big)$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\, \Omega_n\Big(F(s), \frac{\partial}{\partial s}\, F(s)\Big)\, e^{F(s)}\, . \tag{6}$$

In our particular problem of the no-recoil scalar field, we have that

$$\Omega_2\left(\int_{t_0}^{t} H_I(t')\, dt', H_I(t) \right) = 0$$

and that

$$\int_{t_0}^{t} \Omega_1\left(\int_{t_0}^{t} H_I(t_2)\, dt_2, H_I(t_1) \right) dt_1$$

commutes with

$$\Omega_1\left(\int_{t_0}^{t} H_I(t')\, dt', H_I(t) \right).$$

We may therefore write the solution of the equation

$$i\, \partial_t\, V(t, t_0) = \frac{i}{2}\, \Omega_1\left(\int_{t_0}^{t} H_I(t')\, dt', H_I(t) \right) V(t, t_0)$$

as being

$$V(t, t_0) = \exp \left\{ \frac{1}{2} \int_{t_0}^{t} dt_1 \left[\int_{t_0}^{t} H_I(t_2)\, dt_2, H_I(t_1) \right] \right\}$$

or explicitly

$$V(t, t_0) = \exp \left\{ \frac{1}{2} \int_{t_0}^{t} d\tau \int N(\boldsymbol{x})\, N(\boldsymbol{y})\, g(\boldsymbol{x}, \boldsymbol{y}, t_0, \tau)\, d\boldsymbol{x}\, d\boldsymbol{y} \right\} \tag{7}$$

with the notations

$$N(\boldsymbol{x}) \equiv \psi^*(\boldsymbol{x})\, \psi(\boldsymbol{x}) \tag{8}$$

$$g(\boldsymbol{y}', \boldsymbol{y}, t_0, \tau) \equiv \frac{\lambda^2}{(2\pi)^s} \int_{t_0}^{\tau} dt' \int \tilde{f}(\boldsymbol{y}' - \boldsymbol{z}')\, \tilde{f}(\boldsymbol{y} - \boldsymbol{z})\, i\, \Delta(\boldsymbol{z}' - \boldsymbol{z}, t' - \tau)\, d\boldsymbol{z}\, d\boldsymbol{z}' \tag{9}$$

with

$$\tilde{f}(\boldsymbol{x}) = \frac{1}{(2\pi)^{s/2}} \int e^{i(\boldsymbol{k}\cdot\boldsymbol{x})} f(\boldsymbol{k}^2)\, d\boldsymbol{k}$$

and all other notations being those of SCHWEBER [6]. As $N(\boldsymbol{x})$ is permutable with the free hamiltonian, it is convenient to introduce the notation

$$g'(\boldsymbol{x}, \boldsymbol{y}, t_0, t) \equiv \int_{t_0}^{t} g(\boldsymbol{x}, \boldsymbol{y}, t_0, \tau)\, d\tau \tag{10}$$

from which it follows that

$$V(t, t_0) = \exp\left\{\frac{1}{2} \int N(\boldsymbol{x})\, N(\boldsymbol{y})\, g'(\boldsymbol{x}, \boldsymbol{y}, t_0, t)\, d\boldsymbol{x}\, d\boldsymbol{y}\right\}. \tag{11}$$

In contradistinction to what one usually does, we are not interested in an expression for the U-matrix, and this for reasons that have been explained at great length in [3, 4]. We shall, therefore, try to compute

$$e^{iH(t-t_0)}\, \psi^*\, e^{-iH(t-t_0)} = e^{iH_{0F}(t-t_0)}\, e^{-iH_{0B}t_0}\, U^{-1}(t, t_0)\, \psi^* U(t, t_0)\, e^{iH_{0B}t_0}\, e^{-iH_{0F}(t-t_0)}\,.$$

The first step in the computation is given by

$$e^{i\int_{t_0}^{t} H_I(t')\,dt'}\, \psi^*(\boldsymbol{x})\, e^{-i\int_{t_0}^{t} H_I(t')\,dt'} = \psi^*(\boldsymbol{x}) \exp\left\{\frac{i\lambda}{(2\pi)^{s/2}} \int_{t_0}^{t} dt' \int \tilde{f}(\boldsymbol{y})\, \phi(\boldsymbol{x} - \boldsymbol{y}, t')\, d\boldsymbol{y}\right\} \tag{12}$$

where

$$\phi(\boldsymbol{x}, t) = \frac{1}{(2\pi)^{s/2}} \int \frac{d\boldsymbol{k}}{\sqrt{2\,\omega(\boldsymbol{k})}} \left\{a(\boldsymbol{k})\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega(\boldsymbol{k})t)} + a^*(\boldsymbol{k})\, e^{-i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega(\boldsymbol{k})t)}\right\}$$

and the second one by

$$\begin{aligned} V^{-1}(t, t_0)\, \psi^*(\boldsymbol{z})\, V(t, t_0) &= \exp\left\{\frac{1}{2}\, g'(\boldsymbol{z}, \boldsymbol{z}, t_0, t) - \int N(\boldsymbol{x})\, g'(\boldsymbol{x}, \boldsymbol{z}, t_0, t)\, d\boldsymbol{x}\right\} \psi^*(\boldsymbol{z}) \\ &= \psi^*(\boldsymbol{z}) \exp\left\{-\frac{1}{2}\, g'(\boldsymbol{z}, \boldsymbol{z}, t_0, t) - \int N(\boldsymbol{x})\, g'(\boldsymbol{x}, \boldsymbol{z}, t_0, t)\, d\boldsymbol{x}\right\}. \end{aligned} \tag{13}$$

Thus the answer is

$$\begin{aligned} \psi_H^*(\boldsymbol{z}, t) = e^{iH(t-t_0)}\, \psi^*(\boldsymbol{z})\, e^{-iH(t-t_0)} = \psi^*(\boldsymbol{z}, t - t_0) \exp\Bigg\{ &-\frac{1}{2}\, g'(\boldsymbol{z}, \boldsymbol{z}, t_0, t) \\ &- \int N(\boldsymbol{x})\, g'(\boldsymbol{x}, \boldsymbol{z}, t_0, t)\, d\boldsymbol{x} + i\, \frac{\lambda}{(2\pi)^{s/2}} \int_0^{t-t_0} dt' \int \tilde{f}(\boldsymbol{y})\, \phi(\boldsymbol{z} - \boldsymbol{y}, t')\, d\boldsymbol{y}\Bigg\}. \end{aligned} \tag{14}$$

Case of the Boson

For the meson field, the answer is much simpler. Indeed, the perturbative expansion [3]

$$\phi_H(\boldsymbol{z}, t) = \phi(\boldsymbol{z}, t) + i \int_0^t d\tau \, [H_I \, (t - \tau), \phi(\boldsymbol{z}, t)]$$
$$+ i^2 \int_0^t d\tau_1 \int_0^{\tau'} d\tau_2 \, [H_I \, (t - \tau_1), [H_I \, (t - \tau_2), \phi(\boldsymbol{z}, t)]]$$
$$+ \cdots \tag{15}$$

breaks down after the first non-trivial term, so that

$$\phi_H(\boldsymbol{z}, t) = \phi(\boldsymbol{z}, t) + \frac{i \lambda}{(2 \pi)^{s/2}} \int d\boldsymbol{y} \, d\boldsymbol{x} \, N(\boldsymbol{y}) \, \tilde{f} \, (\boldsymbol{y} - \boldsymbol{x}) \int_0^t i \Delta \, (\boldsymbol{x} - \boldsymbol{z}, t' - t) \, dt' \, . \tag{16}$$

2nd Method

The trouble with the first method, is that it does not seem to allow an application to another class of models, as it is based in a very essentiel way on the fact that

$$[H_I(t_1), [H_I(t_2), H_I(t_3)]] = 0 \, .$$

Whenever this condition is fullfilled, the method will work, but it fails otherwise. Only in the case

$$[H_I(t_1), [H_I(t_2), [\ldots [H_I(t_{n-1}), H_I(t_n)] \ldots] = 0$$

for n finite, is it possible to generalize it, using the work of Magnus [7], but this does not significantly widen the physical applications.

Our second method is based on making an 'ansatz' for the solution, and then reducing the problem to the solution of elementary differential equations. One could make the ansatz directly for the solution of the Heisenberg equation:

$$\partial_t \, \psi_H(\boldsymbol{z}, t) = i \, [H, \psi_H(\boldsymbol{z}, t)] \, .$$

It turns out, however, that it is much more convenient to use the form of the interaction picture given by one of us [3, 4]. One defines

$$\psi_G(\boldsymbol{z}, t) \equiv e^{-iH_0 t} \, e^{iHt} \, \psi(\boldsymbol{z}, 0) \, e^{-iHt} \, e^{iH_0 t} \tag{17}$$

from which follows

$$\partial_t \, \psi_G(\boldsymbol{z}, t) = i \, [H_I(-t), \psi_G(\boldsymbol{z}, t)] \, .$$

Let us now make the ansatz

$$\psi_G^*(\boldsymbol{z}, t) = \psi^*(\boldsymbol{z}, 0) \exp\left\{- \alpha(\boldsymbol{z}, t) - \int N(\boldsymbol{x}) \, \beta(\boldsymbol{x}, \boldsymbol{z}, t) \, d\boldsymbol{x} + \int d\boldsymbol{y} \, \gamma(\boldsymbol{y}) \int_{-t}^0 \phi \, (\boldsymbol{z} - \boldsymbol{y}, \tau) \, d\tau\right\} . \tag{18}$$

We get, using formulas (5) and (6)

$$\partial_t \psi_G^*(\boldsymbol{z}, t) = \psi^*(\boldsymbol{z}, 0) \left(- \partial_t \alpha(\boldsymbol{z}, t) - \int N(\boldsymbol{x})\, \partial_t \beta(\boldsymbol{x}, \boldsymbol{z}, t)\, d\boldsymbol{x} - \int d\boldsymbol{y}\, \gamma(\boldsymbol{y})\, \phi\, (\boldsymbol{z} - \boldsymbol{y}, -t) \right.$$
$$\left. + \frac{1}{2} \int_{-t}^{0} d\tau \int i\, \Delta\, (\boldsymbol{y}_1 - \boldsymbol{y}_2, \tau + t)\, \gamma(\boldsymbol{y}_1)\, \gamma(\boldsymbol{y}_2)\, d\boldsymbol{y}_1\, d\boldsymbol{y}_2 \right)$$
$$\times \exp \left\{ - \alpha - \int N\, \beta + \int \gamma \int_{-t}^{0} \phi \right\}$$

$$i\, [H_I(-t), \psi_G^*(\boldsymbol{z}, t)] = \frac{i\, \lambda}{(2\, \pi)^{s/2}}\, \psi^*(\boldsymbol{z}, 0) \left(\int d\boldsymbol{y}\, \tilde{f}\, (\boldsymbol{z} - \boldsymbol{y})\, \phi(\boldsymbol{y}, -t) - \int d\boldsymbol{y}\, N(\boldsymbol{y}) \right.$$
$$\left. \times \tilde{f}\, (\boldsymbol{y} - \boldsymbol{x}) \int_{-t}^{0} i\, \Delta\, (\boldsymbol{x} - \boldsymbol{z} - \boldsymbol{y}', -t - \tau)\, \gamma(\boldsymbol{y}')\, d\boldsymbol{y}'\, d\boldsymbol{x}\, d\tau \right)$$
$$\times \exp \left\{ - \alpha - \int N\, \beta + \int \gamma \int_{-t}^{0} \phi \right\}.$$

Identifying the various operator coefficients, we get the set of equations:

$$\partial_t \alpha(\boldsymbol{z}, t) = \frac{1}{2} \int_{-t}^{0} i\, \Delta\, (\boldsymbol{y}_1 - \boldsymbol{y}_2, \tau + t)\, d\boldsymbol{y}_1\, d\boldsymbol{y}_2\, \gamma(\boldsymbol{y}_1)\, \gamma(\boldsymbol{y}_2)$$
$$- \gamma(\boldsymbol{y}) = \tilde{f}\, (\boldsymbol{z} - \boldsymbol{y})\, \frac{i\, \lambda}{(2\, \pi)^{s/2}}$$
$$\partial_t \beta(\boldsymbol{x}, \boldsymbol{z}, t) = \frac{i\, \lambda}{(2\, \pi)^{s/2}} \int_{-t}^{0} d\tau \int \tilde{f}\, (\boldsymbol{x} - \boldsymbol{y})\, i\, \Delta\, (\boldsymbol{y} + \boldsymbol{y}_1 - \boldsymbol{z}, t + \tau)\, \gamma(\boldsymbol{y}_1)\, d\boldsymbol{y}\, d\boldsymbol{y}_1$$
$$= - \frac{i^2\, \lambda^2}{(2\, \pi)^s} \int_0^t d\tau \int \tilde{f}\, (\boldsymbol{x} - \boldsymbol{y})\, \tilde{f}\, (\boldsymbol{y}_2 - \boldsymbol{z})\, i\, \Delta\, (\boldsymbol{y}_1 - \boldsymbol{y}_2, t - \tau)\, d\boldsymbol{y}_1\, d\boldsymbol{y}_2$$

and thus, using the boundary conditions $\psi_G(\boldsymbol{z}, 0) = \psi(\boldsymbol{z}, 0)$ we get

$$\alpha(\boldsymbol{z}, t) = - \frac{1}{2} \int_0^t dt_2 \int_0^{t_2} dt_1 \int i\, \Delta\, (\boldsymbol{y}_2 - \boldsymbol{y}_1, t_2 - t_1)\, \gamma(\boldsymbol{y}_1)\, \gamma(\boldsymbol{y}_2)\, d\boldsymbol{y}_1\, d\boldsymbol{y}_2$$
$$= - \frac{i^2\, \lambda^2}{(2\, \pi)^s}\, \frac{1}{2} \int_0^t dt_2 \int_0^{t_2} dt_1\, \tilde{f}\, (\boldsymbol{z} - \boldsymbol{y}_1)\, \tilde{f}\, (\boldsymbol{z} - \boldsymbol{y}_2)\, i\, \Delta\, (\boldsymbol{y}_2 - \boldsymbol{y}_1, t_2 - t_1)\, d\boldsymbol{y}_1\, d\boldsymbol{y}_2$$
$$= \frac{1}{2}\, g'(\boldsymbol{z}, \boldsymbol{z}, 0, t)\, .$$

Similarily,

$$\beta(\boldsymbol{x}, \boldsymbol{z}, t) = - \frac{i^2\, \lambda^2}{(2\, \pi)^s} \int_0^t dt_2 \int_0^{t_2} dt_1 \int d\boldsymbol{y}_1\, d\boldsymbol{y}_2\, \tilde{f}\, (\boldsymbol{x} - \boldsymbol{y}_1)\, \tilde{f}\, (\boldsymbol{z} - \boldsymbol{y}_2)\, i\, \Delta\, (\boldsymbol{y}_1 - \boldsymbol{y}_2, t - t_1)$$
$$= g'(\boldsymbol{x}, \boldsymbol{z}, 0, t)\, .$$

That is, we get exactly the same answer as with the first method.

Study of the Boson Field

We have that

$$\phi_H(\boldsymbol{z}, t) = \phi(\boldsymbol{z}, t) + \frac{i\,\lambda}{(2\,\pi)^{s/2}} \int d\boldsymbol{x}\, d\boldsymbol{y}\, N(\boldsymbol{y})\, \tilde{f}\,(\boldsymbol{y} - \boldsymbol{x}) \int_{t_0}^{t} i\,\Delta\,(\boldsymbol{x} - \boldsymbol{z}, t' - t)\, dt'$$

and it is clear that this expression is an operator valued tempered distribution.

The n-points boson function is exactly the free one, so that, as expected, there is no boson-boson scattering.

It is elementary to see that the expression above for ϕ_H is a solution of the Heisenberg equation of motion independently of the t_0 chosen. We cannot use the normalization of the 1-boson state to fix it, and the most reasonable choice is $t_0 = -\infty$. It is easy to see that this ensures that ϕ_H will satisfy an LSZ-type of asymptotic condition for $t \to \pm\infty$ between states with bounded number of (bare) fermions. This remains true if we take the physical (i.e. interacting) fermion states.

Renormalization of the Fermion Field

We have to give a definite meaning to the different factors of our formal solution (14). Note that the solution has nothing heuristical if the cut-off function f is kept finite and is sufficiently smooth. The problem arrises whether the solution still does make sense as $f \to \delta$ in $\mathcal{S}'$. We shall proceed in two steps. In the first one, we shall write the different factors of the solution in term of Wick-ordered functions. This will enable us to separate quantities which are divergent in some space dimensions and renormalize the solution. In a second step, we shall study the $\boldsymbol{z}$ and t dependence of the fermion field and determine whether it is a distribution or only some kind of generalized function (this last step only for $s \leqslant 3$).

Let us first consider the term

$$\exp\left\{\frac{i\,\lambda}{(2\,\pi)^{s/2}} \int_0^{t-t_0} dt' \int d\boldsymbol{y}\, \tilde{f}(\boldsymbol{y})\, \phi\,(\boldsymbol{z} - \boldsymbol{y}, t')\right\}.$$

Using the well known formula

$$e^{A+B} = e^A\, e^B\, e^{-\frac{1}{2}[A,B]} = e^{\frac{1}{2}[A,B]}\, e^B\, e^A$$

valid whenever

$$[A, [A, B]] = [B, [A, B]] = 0$$

we get that

$$\begin{aligned}
\exp\left\{\frac{i\,\lambda}{(2\,\pi)^{s/2}} \int_0^{t-t_0} dt' \int d\boldsymbol{y}\, \tilde{f}(\boldsymbol{y})\, \phi\,(\boldsymbol{z} - \boldsymbol{y}, t')\right\} &= \exp\left\{\frac{i\,\lambda}{(2\,\pi)^{s/2}} \int_0^{t-t_0} dt_2 \int d\boldsymbol{y}_2\, \tilde{f}(\boldsymbol{y}_2)\, \phi^{(-)}\,(\boldsymbol{z} - \boldsymbol{y}_2, t_2)\right\} \\
&\times \exp\left\{\frac{i\,\lambda}{(2\,\pi)^{s/2}} \int_0^{t-t_0} dt_1 \int d\boldsymbol{y}_1\, \tilde{f}(\boldsymbol{y}_1)\, \phi^{(+)}\,(\boldsymbol{z} - \boldsymbol{y}_1, t_1)\right\} \\
&\times \exp\left\{\frac{1}{2}\, \frac{i^2\,\lambda^2}{(2\,\pi)^s} \int_0^{t-t_0} dt_1 \int_0^{t-t_0} dt_2 \int d\boldsymbol{y}_1\, d\boldsymbol{y}_2\, \tilde{f}(\boldsymbol{y}_1)\, \tilde{f}(\boldsymbol{y}_2)\, i\,\Delta^+\,(\boldsymbol{y}_2 - \boldsymbol{y}_1, t_1 - t_2)\right\}
\end{aligned}$$

$$= :\exp\left\{\frac{i\,\lambda}{(2\,\pi)^{s/2}}\int_0^{t-t_0} dt' \int dy\,\tilde{f}(y)\,\phi\,(z-y,t')\right\}:$$
$$\times\exp\left\{\frac{1}{2}\,\frac{i^2\,\lambda^2}{(2\,\pi)^s}\int_0^{t-t_0} dt_1 \int_0^{t-t_0} dt_2 \int dy_1\,dy_2\,\tilde{f}(y_1)\,\tilde{f}(y_2)\,i\,\Delta^+\,(y_2-y_1,t_1-t_2)\right\}.$$

On the other hand,

$$\exp\left\{\frac{1}{2}\,\frac{i^2\,\lambda^2}{(2\,\pi)^s}\int_0^{t-t_0} dt_1 \int_0^{t-t_0} dt_2 \int dy_1\,dy_2\,\tilde{f}(y_1)\,\tilde{f}(y_2)\,i\,\Delta^+\,(y_2-y_1,t_1-t_2)\right\}$$
$$= \exp\left\{-\frac{1}{2}\,\frac{\lambda^2}{(2\,\pi)^s}\int dk\,\frac{|f(k^2)|^2}{k_0^3}\,(1-\cos\,(k_0\,(t-t_0)))\right\}$$
$$\int dk\,\frac{|f(k)|^2}{k_0^3}\,\cos\,(k_0\,(t-t_0))$$

is converging, and represents a continuous function in t, even in the limit $f(k) \to 1$ (for $t \neq t_0$)

$$\int dk\,\frac{|f(k)|^2}{k_0^3}$$

converges for $s \leqslant 2$, diverges logarithmically for $s = 3$, whenever $f(k) \to 1$. This quantity will appear in the field strength renormalization, but we may as well trace its origin now. Remark that if one doesn't take into account any boundary condition, $\psi_G(z, t)$ is only defined up to a multiplicative function, more exactly, up to a time-independent term which is also commuting with $H_I(-t)$. This means that the quantity

$$\exp\left\{-\frac{1}{2}\,\frac{\lambda^2}{(2\,\pi)^s}\int dk\,\frac{|f(k)|^2}{k_0^3}\right.$$

may be dropped from $\psi_G(z, t)$ and that the remaining part $\psi'_G(z, t)$ will still be a solution of

$$\partial_t\,\psi'_G(z,t) = i\,[H_I(-t),\psi'_G(z,t)]\,. \tag{17}$$

The constant factor being fixed by boundary conditions, we see that performing a field strength renormalization exactly amounts to changing the boundary which gives the right answer is the proper normalization of the 1-particle state, or alternatively, the LSZ-asymptotic condition.

To come back to the study of the first term of the formal solution, we remark that Wick ordering implies that it will be well defined on the vacuum state. In order to know its action on other vectors of the Fock space, we only need to know the commutator:

$$\left[:\exp\left\{\frac{i\,\lambda}{(2\,\pi)^{s/2}}\int_0^{t-t_0} dt' \int dy\,\tilde{f}(y)\,\phi\,(z-y,t')\right\}:\,,\ \phi(w,\tau)\right]$$
$$= :\exp\left\{\frac{i\,\lambda}{(2\,\pi)^{s/2}}\int_0^{t-t_0} dt' \int dy\,\tilde{f}(y)\,\phi\,(z-y,t')\right\}:$$
$$\times\,\frac{i\,\lambda}{(2\,\pi)^{s/2}}\int_0^{t-t_0} dt' \int dy\,\tilde{f}(y)\,i\,\Delta\,(z-y-w,t'-\tau)\,,$$

The second term of the solution (14), of the form

$$\exp\left\{-\int N(\boldsymbol{x})\, g'(\boldsymbol{x}, \boldsymbol{z}, t_0, t)\, d\boldsymbol{x}\right\}$$

seems more difficult to handle than the first one. The way to proceed is as follows: We make the 'ansatz'

$$\exp\left\{\varepsilon\int \psi^*(\boldsymbol{x})\, \psi(\boldsymbol{x})\, \varrho(\boldsymbol{x})\, d\boldsymbol{x}\right\} = :\exp\left\{\int \psi^*(\boldsymbol{x})\, \psi(\boldsymbol{x})\, \vartheta(\boldsymbol{x}, \varepsilon)\, d\boldsymbol{x}\right\}:$$

differentiating by respect to ε, leads to

$$\begin{aligned}
&\int \psi^*(\boldsymbol{x})\, \psi(\boldsymbol{x})\, \varrho(\boldsymbol{x})\, d\boldsymbol{x} \exp\left\{\varepsilon\int \psi^*(\boldsymbol{x})\, \psi(\boldsymbol{x})\, \varrho(\boldsymbol{x})\, d\boldsymbol{x}\right\}\\
&\quad = :\left\{\int \psi^*(\boldsymbol{x})\, \psi(\boldsymbol{x})\, \vartheta(\boldsymbol{x}, \varepsilon)\, d\boldsymbol{x} \exp\left\{\int \psi^*(\boldsymbol{x})\, \psi(\boldsymbol{x})\, \vartheta(\boldsymbol{x}', \varepsilon)\, d\boldsymbol{x}'\right\}\right\}:\\
&\quad = \int \psi^*(\boldsymbol{x}) :\exp\left\{\int \psi^*(\boldsymbol{x}')\, \psi(\boldsymbol{x}')\, \vartheta(\boldsymbol{x}', \varepsilon)\, d\boldsymbol{x}'\right\}: \psi(\boldsymbol{x})\, \frac{\partial}{\partial\varepsilon}\, \vartheta(\boldsymbol{x}, \varepsilon)\, d\boldsymbol{x}\\
&\quad = \int \psi^*(\boldsymbol{x}) \exp\left\{\varepsilon\int \psi^*(\boldsymbol{x}')\, \psi(\boldsymbol{x}')\, \varrho(\boldsymbol{x}')\, d\boldsymbol{x}'\right\} \psi(\boldsymbol{x})\, \frac{\partial}{\partial\varepsilon}\, \vartheta(\boldsymbol{x}, \varepsilon)\, d\boldsymbol{x}\,.
\end{aligned}$$

Multiplying from the right by

$$\exp\left\{-\,\varepsilon\int \psi^*(\boldsymbol{x}')\, \psi(\boldsymbol{x}')\, d\boldsymbol{x}'\, \varrho(\boldsymbol{x}')\right\}$$

and using the easily derived relation

$$\exp\left\{\varepsilon\int \psi^*(\boldsymbol{x})\, \psi(\boldsymbol{x})\, \varrho(\boldsymbol{x})\, d\boldsymbol{x}\right\} \psi(\boldsymbol{z}) \exp\left\{-\,\varepsilon\int \psi^*(\boldsymbol{x}')\, \psi(\boldsymbol{x}')\, \varrho(\boldsymbol{x}')\, d\boldsymbol{x}'\right\} = \psi(\boldsymbol{z}) \exp\left\{-\,\varepsilon\, \varrho(\boldsymbol{z})\right\}$$

we get the differential equation

$$\int \psi^*(\boldsymbol{x})\, \psi(\boldsymbol{x})\, \varrho(\boldsymbol{x})\, d\boldsymbol{x} = \int \psi^*(\boldsymbol{x})\, \psi(\boldsymbol{x})\, e^{-\varepsilon\varrho(\boldsymbol{x})}\, \frac{\partial}{\partial\varepsilon}\, \vartheta(\boldsymbol{x}, \varepsilon)\, d\boldsymbol{x}$$

with the boundary condition $\vartheta(\boldsymbol{x}, 0) = 0$.

Therefore

$$\vartheta(\boldsymbol{x}, \varepsilon) = e^{\varepsilon\varrho(\boldsymbol{x})} - 1$$

and thus

$$e^{\varepsilon\int \psi^*(\boldsymbol{x})\, \psi(\boldsymbol{x})\, \varrho(\boldsymbol{x})\, d\boldsymbol{x}} = :\exp\left\{\int \psi^*(\boldsymbol{x})\, \psi(\boldsymbol{x})\, (e^{\varepsilon\varrho(\boldsymbol{x})} - 1)\, d\boldsymbol{x}\right\}:.$$

Applying the result to our case, we get

$$\exp\left\{-\int N(\boldsymbol{x})\, g'(\boldsymbol{x}, \boldsymbol{z}, t_0, t)\, d\boldsymbol{x}\right\} = :\exp\left\{\int \psi^*(\boldsymbol{x})\, \psi(\boldsymbol{x})\, (e^{-g'(\boldsymbol{x}, \boldsymbol{z}, t_0, t)} - 1)\, d\boldsymbol{x}\right\}:$$

and since

$$\begin{aligned}
&:\exp\left\{\int \psi^*(\boldsymbol{x})\, \psi(\boldsymbol{x})\, (e^{-g'(\boldsymbol{x}, \boldsymbol{z}, t_0, t)} - 1)\, d\boldsymbol{x}\right\}: \psi^*(\boldsymbol{w})\\
&\qquad = \psi^*(\boldsymbol{w}) :\exp\left\{\int \psi^*(\boldsymbol{x})\, \psi(\boldsymbol{x})\, (e^{-g'(\boldsymbol{x}, \boldsymbol{z}, t_0, t)} - 1)\, d\boldsymbol{x}\right\}: \exp\left\{-\, g'(\boldsymbol{w}, \boldsymbol{z}, t_0, t)\right\}.
\end{aligned}$$

We may write:

$$\psi_G^*(\boldsymbol{z}, t) = \psi^*(\boldsymbol{z}, t_0) \exp \left\{ -\frac{1}{2} g'(\boldsymbol{z}, \boldsymbol{z}, t_0, t) - \int N(\boldsymbol{x}) \, g'(\boldsymbol{x}, \boldsymbol{z}, t_0, t) \, d\boldsymbol{x} \right.$$

$$\left. + \frac{i\lambda}{(2\pi)^{s/2}} \int_0^{t-t_0} dt' \int d\boldsymbol{y} \, \tilde{f}(\boldsymbol{y}) \, \phi(\boldsymbol{z}-\boldsymbol{y}, t'-t) \right\} = \psi^*(\boldsymbol{z}, t_0) \exp \left\{ -\frac{1}{2} g'(\boldsymbol{z}, \boldsymbol{z}, t_0, t) \right\}$$

$$\times \exp \left\{ -\frac{1}{2} \frac{\lambda^2}{(2\pi)^s} \int d\boldsymbol{k} \, \frac{|f(\boldsymbol{k})|^2}{k_0^3} (1 - \cos(k_0 (t - t_0))) \right\}$$

$$\times : \exp \left\{ \frac{i\lambda}{(2\pi)^{s/2}} \int_0^{t-t_0} dt' \int d\boldsymbol{y} \, \tilde{f}(\boldsymbol{y}) \, \phi(\boldsymbol{z}-\boldsymbol{y}, t'-t) \right\} :$$

$$\times : \exp \left\{ \int \psi^*(\boldsymbol{x}) \, \psi(\boldsymbol{x}) \, (e^{-g'(\boldsymbol{x}, \boldsymbol{z}, t_0, t)} - 1) \, d\boldsymbol{x} \right\} : .$$

At this point it is good to remember that $g'(\boldsymbol{z}, \boldsymbol{z}, t_0, t)$ in fact does not depend upon $\boldsymbol{z}$ since

$$g'(\boldsymbol{z}, \boldsymbol{z}, t_0, t) = \frac{i\lambda^2}{(2\pi)^s} \int d\boldsymbol{k} \, |f(\boldsymbol{k})|^2 \, \frac{1}{k_0^2} \left\{ (t - t_0) - \frac{1}{k_0} \sin(k_0 (t - t_0)) \right\}.$$

Before we determine the field strength renormalization, we first compute the physical one particle state, and for this we need to know the renormalized mass:

$$m = m_0 - \frac{\lambda^2}{2(2\pi)^s} \int d\boldsymbol{k} \, |f(\boldsymbol{k})|^2 \, \frac{1}{k_0^2}$$

we then get

$$\psi_H^*(\boldsymbol{z}, t) = \psi^*(\boldsymbol{z}, t) \, e^{tm(t-t_0)} \exp \left\{ \frac{\lambda^2}{2(2\pi)^s} \int d\boldsymbol{k} \, |f(\boldsymbol{k})|^2 \, k_0^{-3} \, e^{ik_0(t-t_0)} \right\}$$

$$\times \exp \left\{ -\frac{1}{2} \frac{\lambda^2}{(2\pi)^s} \int d\boldsymbol{k} \, |f(\boldsymbol{k})|^2 \, k_0^{-3} \right\} : \exp \left\{ \int \psi^*(\boldsymbol{x}) \, \psi(\boldsymbol{x}) \, (e^{-g'(\boldsymbol{x}, \boldsymbol{z}, t_0, t)} - 1) \, d\boldsymbol{x} \right\} :$$

$$\times : \exp \left\{ \frac{i\lambda}{(2\pi)^{s/2}} \int_{t_0}^{t} dt' \int d\boldsymbol{y} \, \tilde{f}(\boldsymbol{y}) \, \phi(\boldsymbol{z}-\boldsymbol{y}, t') \right\} : .$$

Notice that in the limit $f(\boldsymbol{k}) \to 1$ the mass renormalization is finite only for $s = 1$.

Consider the (mass renormalized) two-points function (in the bare vacuum)

$$W^{(2)}(\boldsymbol{z}_1 - \boldsymbol{z}_2, t_1 - t_2) \equiv \langle 0 \mid \psi_H(\boldsymbol{z}_1, t_1) \, \psi_H^*(\boldsymbol{z}_2, t_2) \mid 0 \rangle$$

$$= e^{-imt} \exp \left\{ -\frac{\lambda^2}{2(2\pi)^s} \int d\boldsymbol{k} \, |f(\boldsymbol{k})|^2 \, k_0^{-3} \, (1 - e^{-ik_0 t}) \right\} \delta(\boldsymbol{z})$$

with the notations

$$t \equiv t_1 - t_2, \; \boldsymbol{z} \equiv \boldsymbol{z}_1 - \boldsymbol{z}_2, \; A(\boldsymbol{k}) \equiv \frac{\lambda^2}{(2\pi)^s} \frac{1}{2} \, |f(\boldsymbol{k})|^2 \, k_0^{-3},$$

we get

$$W^{(2)}(\boldsymbol{z}, E) = \frac{1}{\sqrt{2\pi}} \int dt \, W^{(2)}(\boldsymbol{z}, t) \, e^{itE} = \sqrt{2\pi} \exp \left\{ -\int A(\boldsymbol{k}) \, d\boldsymbol{k} \right\}$$

$$\times \left\{ \delta(E - m) + \sum_{n=1}^{\infty} \frac{1}{n!} \int A(\boldsymbol{k}_1) \ldots A(\boldsymbol{k}_n) \, \delta\left(E - m - \sum_{j=1}^{n} k_{0j} \right) d\boldsymbol{k}_1 \ldots d\boldsymbol{k}_n \right\} \delta(\boldsymbol{z})$$

and we can write

$$\int A(\boldsymbol{k}_1) \ldots A(\boldsymbol{k}_n)\, \delta\left(E - m - \sum_{j=1}^{n} k_{0j}\right) d\boldsymbol{k}_1 \ldots d\boldsymbol{k}_n$$

$$= 4\pi \int A(\boldsymbol{k}_1) \ldots A(\boldsymbol{k}_{n-1})\, A\left(\sqrt{\left(E - m - \sum_{j=1}^{n-1} k_{0j}\right)^2 - \mu^2}\right)$$

$$\times \left(E - m - \sum_{j=1}^{n-1} k_{0j}\right) \sqrt{\left(E - m - \sum_{j=1}^{n-1} k_{0j}\right)^2 - \mu^2} \quad \theta\left(E - m - \sum_{j=1}^{n-1} k_{0j} - \mu\right)$$

$$\times\, d\boldsymbol{k}_1 \ldots d\boldsymbol{k}_{n-1}$$

which shows that the spectral weight function has the right structure, having an isolated singularity at $E = m$, and continuous parts starting at $m + \mu$, $m + 2\mu$ etc. Therefore, in order to get a one-particle state, we shall smear out $\psi_H^*(\boldsymbol{z}, t)\, |0\rangle$ with a test function $h(t)$ the support of which in the space E is contained in a neighborhood of m:

$$\operatorname{Supp} \tilde{h}(E) = \{E \mid m - \varepsilon < E < m + \varepsilon,\ \varepsilon < \mu\}$$

$$h(t) = \frac{1}{\sqrt{2\pi}} \int \tilde{h}(E)\, e^{-iEt}\, dE$$

which is possible with the particular choice of Supp $\tilde{h}$. Putting

$$B(\boldsymbol{k}, \boldsymbol{z}) = \frac{\lambda^2}{2(2\pi)^s} \frac{|f(\boldsymbol{k})|^2}{k_0^3} + \frac{\lambda}{(2\pi)^{s/2}} \frac{f(\boldsymbol{k})}{\sqrt{2 k_0^3}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{z}}\, a^*(-\boldsymbol{k})$$

we get

$$\psi_H^*(\boldsymbol{z}, t)\, |0\rangle = e^{imt} \exp\left\{-\int B(\boldsymbol{k}, \boldsymbol{z})\, d\boldsymbol{k}\right\} \exp\left\{\int B(\boldsymbol{k}, \boldsymbol{z})\, e^{ik_0 t}\, d\boldsymbol{k}\right\} \psi^*(\boldsymbol{z}, 0)\, |0\rangle$$

from which follows

$$\int h(t)\, \psi_H^*(\boldsymbol{z}, t)\, dt\, |0\rangle = \int \frac{\tilde{h}(E)}{\sqrt{2\pi}}\, e^{i(m-E)t}$$

$$\times\, dE \left\{1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int B(\boldsymbol{k}_1) \ldots B(\boldsymbol{k}_n) \exp\left\{i \sum_{j=1}^{n} k_{0j}\, t\right\} d\boldsymbol{k}_1 \ldots d\boldsymbol{k}_n\right\} dt$$

$$\times \exp\left\{-\int B(\boldsymbol{k}, \boldsymbol{z})\, d\boldsymbol{k}\right\} \psi^*(\boldsymbol{z}, 0)\, |0\rangle = \sqrt{2\pi} \int dE\, \tilde{h}(E)$$

$$\times \left\{\delta\,(E - m) + \sum_{j=1}^{\infty} \frac{1}{n!} \int B(\boldsymbol{k}_1) \ldots B(\boldsymbol{k}_n)\, \delta\left(E - m - \sum_{j=1}^{n} k_{0j}\right) d\boldsymbol{k}_1 \ldots d\boldsymbol{k}_n\right\}$$

$$\times \exp\left\{-\int B(\boldsymbol{k}, \boldsymbol{z})\, d\boldsymbol{k}\right\} \psi^*(\boldsymbol{z}, 0)\, |0\rangle$$

$$= \sqrt{2\pi} \left\{\tilde{h}(m) + \sum_{n=1}^{\infty} \frac{1}{n!} \int d\boldsymbol{k}_1 \ldots d\boldsymbol{k}_n\, \tilde{h}\left(m + \sum_{j=1}^{\infty} k_{0j}\right) B(\boldsymbol{k}_1) \ldots B(\boldsymbol{k}_n)\right\}$$

$$\times \exp\left\{-\int B(\boldsymbol{k}, \boldsymbol{z})\, d\boldsymbol{k}\right\} \psi^*(\boldsymbol{z}, 0)\, |0\rangle$$

and, using the support properties of $\tilde{h}$, we get as 1-particle state

$$\int h(t)\, \psi_H^*(\boldsymbol{z}, t)\, |0\rangle = \sqrt{2\pi}\, \tilde{h}(m) \exp\left\{-\int B(\boldsymbol{k}, \boldsymbol{z})\, d\boldsymbol{k}\right\} \psi^*(\boldsymbol{z}, 0)\, |0\rangle\,.$$

If we now look at the 2-points function,

$$W^{(2)}(\boldsymbol{z}, t) = e^{-imt} \exp\left\{-\frac{\lambda^2}{2(2\pi)^s}\int d\boldsymbol{k}\,\frac{|f(\boldsymbol{k})|^2}{k_0^3}\,(1-e^{ik_0 t})\right\}\delta(\boldsymbol{z})$$

we remark that the first and the last factors are the only ones which we would have for a free field. The second factor is equal to one for $t = 0$. This is not astonishing as we did impose to our fields to be free at time $t = 0$, or t_0; this was indeed our boundary condition. The presence of the term

$$\exp\left\{-\frac{\lambda^2}{2(2\pi)^s}\int d\boldsymbol{k}\,|f(\boldsymbol{k})|^2\,k_0^{-3}\right\}$$

makes that the one-particle state is not properly normalized. Remembering the discussion above concerning the connection between field strength renormalization and boundary condition, we see that we can renormalize the field operatos by dropping a factor

$$\exp\left\{-\frac{1}{4}\,\frac{\lambda^2}{(2\pi)^s}\int d\boldsymbol{k}\,|f(\boldsymbol{k})|^2\,k_0^{-3}\right\}.$$

The trouble with this factor is evidently that it is equal to zero in the limit $f(\boldsymbol{k}) = 1$ and in three dimensional space, and from this it follows that the two-points function is no longer defined for $t = 0$ (still if $s = 3$), that is, the interacting field operator is not defined at a sharp time, but has to be smeared out in time also.

It is easy to check that the properly smeared out renormalized two point function converges asymptotically toward the free one and that the unrenormalized does not.

The unrenormalized $2\,n$-point function is given by:

$$\begin{aligned}
&W^{(2n)}((\boldsymbol{x}_n, t_n)\ldots(\boldsymbol{x}_1, t_1), (\boldsymbol{y}_1, s_1)\ldots(\boldsymbol{y}_n, s_n)) \equiv \langle 0\,|\,\psi_H(\boldsymbol{x}_n, t_n)\ldots\psi_H(\boldsymbol{x}_1, t_1)\\
&\quad\times \psi_H^*(\boldsymbol{y}_1, s_1)\ldots\psi_H^*(\boldsymbol{y}_n, s_n)\,|0\rangle = \exp\left\{i\,m\sum_{j=1}^{n}(s_j - t_j)\right\}\\
&\quad\times \exp\left\{2\,n\left(-\frac{\lambda^2}{(2\pi)^s}\int d\boldsymbol{k}\,\frac{|f(\boldsymbol{k})|^2}{2\,k_0^3}\right)\right\}\exp\left\{\frac{\lambda^2}{(2\pi)^3}\int d\boldsymbol{k}\,\frac{|f(\boldsymbol{k})|^2}{2\,k_0^3}\sum_{j=1}^{n}\left(e^{ik_0 s_j} + e^{-ik_0 t_j}\right)\right\}\\
&\quad\times \exp\left\{\sum_{i<j=2}^{n}[g'(\boldsymbol{x}_j, \boldsymbol{x}_i, 0, t_i) - g'(\boldsymbol{y}_j, \boldsymbol{y}_i, 0, s_i) + C(\boldsymbol{y}_i, \boldsymbol{y}_j, s_i, s_j) + C(\boldsymbol{x}_j, \boldsymbol{x}_i, t_j, t_i)]\right.\\
&\quad\left.-\sum_{i,j=1}^{n} C(\boldsymbol{x}_i, \boldsymbol{y}_j, t_i, s_j)\right\}\langle 0\,|\,\psi(\boldsymbol{x}_n, 0)\ldots\psi(\boldsymbol{x}_1, 0)\,\psi^*(\boldsymbol{y}_1, 0)\ldots\psi^*(\boldsymbol{y}_n, 0)\,|0\rangle
\end{aligned}$$

with

$$\begin{aligned}
C(\boldsymbol{x}, \boldsymbol{y}, t, s) &\equiv \frac{(i\lambda)^2}{(2\pi)^s}\langle 0\,|\int_0^t d\tau_1\int d\boldsymbol{w}_1\,\tilde{f}(\boldsymbol{w}_1)\,\phi(\boldsymbol{x}-\boldsymbol{w}_1, \tau_1)\int_0^s d\tau_2\int d\boldsymbol{w}_2\,\tilde{f}(\boldsymbol{w}_2)\\
&\quad\times\phi(\boldsymbol{y}-\boldsymbol{w}_2, \tau_2)\,|0\rangle = \frac{(i\lambda)^2}{2(2\pi)^s}\int\frac{|f(\boldsymbol{k})|^2}{k_0^3}\,e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})}\left\{e^{ik_0(s-t)} - e^{ik_0 s} - e^{-ik_0 t} + 1\right\}d\boldsymbol{k}\,.
\end{aligned}$$

By computing

$$\langle 0\,|\,\psi(\boldsymbol{x}_n, 0)\ldots\psi(\boldsymbol{x}_1, 0)\,\psi^*(\boldsymbol{y}_1, 0)\ldots\psi^*(\boldsymbol{y}_n, 0)\,|0\rangle$$

one obtains

$$W^{2n}((\boldsymbol{x}_n, t_n) \ldots (\boldsymbol{x}_1, t_1)(\boldsymbol{y}_1, s_1) \ldots (\boldsymbol{y}_n, s_n))$$

$$= \exp\left\{i\, m \sum_{j=1}^{n} (s_j - t_j)\right\} \exp\left\{- n \frac{\lambda^2}{(2\pi)^s} \int d\boldsymbol{k}\, |f(\boldsymbol{k})|^2 k_0^{-3}\right\}$$

$$\times \exp\left\{\frac{\lambda^2}{(2\pi)^s} \int d\boldsymbol{k}\, |f(\boldsymbol{k})|^2 (2 k_0^3)^{-1} \sum_{j=1}^{n} \left(e^{i k_0 s_j} + e^{-i k_0 t_j}\right)\right\}$$

$$\times \sum_{P \in S_n} \exp\left\{\sum_{i<j=2}^{n} [g'(\boldsymbol{x}_j, \boldsymbol{x}_i, 0, t_i) - g'(\boldsymbol{y}_j, \boldsymbol{y}_i, 0, s_i) + C(\boldsymbol{y}_i, \boldsymbol{y}_j, s_i, s_j)\right.$$

$$\left. + C(\boldsymbol{x}_j, \boldsymbol{x}_i, t_j, t_i)] - \sum_{i,j=1}^{n} C(\boldsymbol{x}_i, \boldsymbol{y}_j, t_i, s_j)\right\} (-1)^{\sigma(P)} \prod_{l=1}^{n} \delta(\boldsymbol{y}_l - \boldsymbol{x}_{P(l)})$$

where S_n is the group of all permutations of n objects and $\sigma(P)$ is the parity of the permutation P. After inserting the expressions for g' and C, with some easy computations one gets

$$W^{(2n)}((\boldsymbol{x}_n, t_n) \ldots (\boldsymbol{y}_n, s_n)) = e^{i m \sum_{j=1}^{n} (s_j - t_j)} \exp\left\{- n \frac{\lambda^2}{(2\pi)^3} \int d\boldsymbol{k} \frac{|f(\boldsymbol{k})|^2}{2 k_0^2}\right\}$$

$$\times \sum_{P \in S_n} (-1)^{\sigma(P)} \exp\left\{\sum_{i<j=2}^{n} \frac{i \lambda^2}{(2\pi)^3} \int d\boldsymbol{k} \frac{|f(\boldsymbol{k})|^2}{k_0^3} \cos[\boldsymbol{k}(\boldsymbol{x}_j - \boldsymbol{x}_i)] (t_i - s_{P^{-1}(i)})\right\}$$

$$\times \exp\left\{\sum_{i=1}^{n} \frac{\lambda^2}{(2\pi)^3} \int d\boldsymbol{k} \frac{|f(\boldsymbol{k})|^2}{2 k_0^3} e^{i k_0 (s_{P^{-1}(i)} - t_i)} + \sum_{i<j=2}^{n} \frac{\lambda^2}{(2\pi)^3} \int d\boldsymbol{k} \frac{|f(\boldsymbol{k})|^2}{2 k_0^3} e^{i \boldsymbol{k}(\boldsymbol{x}_j - \boldsymbol{x}_i)}\right.$$

$$\left.\times \left[e^{i k_0 (t_i - t_j)} + e^{i k_0 (s_{P^{-1}(j)} - s_{P^{-1}(i)})} - e^{i k_0 (s_{P(j)} - t_i)} - e^{i k_0 (s_{P^{-1}(i)} - t_j)}\right]\right\}$$

$$\times \prod_{l=1}^{n} \delta(y_l - x_{P(l)}) .$$

The renormalized $2n$-point function is obtained by considering m as the physical mass of the fermion and by dropping the field renormalization factor

$$\exp\left\{- n \frac{\lambda^2}{(2\pi)^s} \int d\boldsymbol{k} \frac{|f(\boldsymbol{k})|^2}{2 k_0^3}\right\}.$$

Existence of the solutions as distributions for $f(k) = 1$, and $s = 1, 2, 3$

The only factors which could give some troubles in the renormalized $2n$-point function are of the form:

$$\exp\left\{\frac{\lambda^2}{2(2\pi)^s} \int d\boldsymbol{k}\, k_0^{-3} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - k_0 t)}\right\}$$

and

$$\exp\left\{\frac{i \lambda^2}{(2\pi)^s} \int d\boldsymbol{k}\, k_0^{-2} \cos(\boldsymbol{k}\cdot\boldsymbol{x})\, t\right\}.$$

If $s = 1, 2$ the integral $\int d\boldsymbol{k}\, k_0^{-3} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - k_0 t)}$ is absolutely convergent, defining a continuous bounded function of $(\boldsymbol{x}, t)$, and therefore

$$\exp\left\{\frac{\lambda^2}{2(2\pi)^s} \int \frac{d\boldsymbol{k}}{k_0^3} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - k_0 t)}\right\}$$

defines a tempered distribution in $(\boldsymbol{x}, t)$ space.

If $s = 3$ we may write:

$$\int \frac{d\boldsymbol{k}}{k_0^3} e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-k_0 t)} = 4\pi \int^{\infty} \frac{k^2\,dk}{k_0^3} \frac{\sin k\,x}{k\,x} e^{-ik_0 t}$$

$$= 2\pi \left\{ -\mu^2 \int_{-\infty}^{+\infty} \frac{dk}{k_0^3} \frac{\sin k\,x}{k\,x} e^{-ik_0 t} + \int_{-\infty}^{+\infty} \frac{dk}{k_0} \frac{\sin k\,x}{k\,x} e^{-ik_0 t} \right\}$$

where $x = |\boldsymbol{x}|$.

The first term is a continuous function of $(\boldsymbol{x}, t)$, whereas the second one, continuous for $x \neq 0$ and all t, is for $x = 0$ as singular as the two point function of the scalar field in two (1 space, 1 time) dimensions. Therefore

$$\exp\left\{\frac{\lambda^2}{2(2\pi)^3} \int \frac{d\boldsymbol{k}}{k_0^3} e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-k_0 t)}\right\}$$

is defined as a distribution not only for test functions with compact support in the energy-momentum, but also for test functions $\in S(R^4)$.

Let us study now:

$$\exp\left\{\frac{i\,\lambda^2}{(2\pi)^s} \int d\boldsymbol{k}\, k_0^{-2} \cos(\boldsymbol{k}\cdot\boldsymbol{x})\, t\right\}.$$

For $s = 1$ there is again no problem. For $s = 2$ we get:

$$\int \frac{d\boldsymbol{k}}{k_0^2} \cos \boldsymbol{k}\cdot\boldsymbol{x} \sim \int_0^{\infty} \frac{k\,dk}{k^2+\mu^2} \int_0^{2\pi} \cos(k\,x\cos\vartheta)\, d\vartheta \sim \int_0^{\infty} \frac{y\,dy}{y^2+\mu^2 x^2} J_0(y)$$

where $J_0(y)$ is the Bessel function of order 0. Since $J_0(y) \underset{y\to+\infty}{\sim} 1/\sqrt{y}$ the last integral is a continuous function of x for $x \neq 0$. At $x = 0$, it has a logarithmic singularity, as one easily checks. For $s = 3$ the integral under examination can be computed explicitely by integration in the complex plane:

$$\int d\boldsymbol{k}\, k_0^{-2} \cos \boldsymbol{k}\cdot\boldsymbol{x} \sim \int_0^{\infty} \frac{k^2}{k^2+\mu^2} \frac{\sin k\,x}{k\,x}\, dk \sim \frac{1}{x} e^{-\mu x}.$$

As a consequence in both cases $s = 2, 3$

$$\exp\left\{\frac{i\,\lambda^2}{(2\pi)^s} \int d\boldsymbol{k} \frac{1}{k_0^2} \cos(\boldsymbol{k}\cdot\boldsymbol{x})\, t\right\}$$

defines a tempered distribution in $(\boldsymbol{x}, t)$. In fact if we take a $\varphi(\boldsymbol{x}, t) \in S(R^4)$ and we put

$$\tilde{\varphi}(\boldsymbol{x}, E) = \int \varphi(\boldsymbol{x}, t)\, e^{-iEt}\, dt$$

$$\int\left(\int \exp\left\{\frac{i\,\lambda^2}{(2\pi)^s} \int \frac{d\boldsymbol{k}}{k_0^2} \cos(\boldsymbol{k}\cdot\boldsymbol{x})\, t\right\} \varphi(\boldsymbol{x}, t)\right) d\boldsymbol{x} = \int d\boldsymbol{x}\, \varphi\left(\boldsymbol{x}, \frac{\lambda^2}{(2\pi)^3} \int \frac{d\boldsymbol{k}}{k_0^2} \cos(\boldsymbol{k}\cdot\boldsymbol{x})\right)$$

converges due to the strong decreasing of $\tilde{\varphi}$ at ∞ and to the nature of the singularities of $\int d\boldsymbol{k}/k_0^2 \cos(\boldsymbol{k}\cdot\boldsymbol{x})$.

It is then proved that for $s = 1, 2, 3$, once the mass and field renormalization are performed, the Wightman functions are tempered distributions.

The Solution with Recoil

In the case the fermion field is chosen to have a momentum dependent energy, that is if

$$H_0 = \int \varepsilon(\boldsymbol{p})\, \psi^*(\boldsymbol{p})\, \psi(\boldsymbol{p})\, d\boldsymbol{p} + \int \omega(\boldsymbol{k})\, a^*(\boldsymbol{k})\, a(\boldsymbol{k})\, d\boldsymbol{k}$$

where the specific choice of $\varepsilon(\boldsymbol{p})$ doesn't matter, we can still get the explicit operator solution for $\psi_H^*(\boldsymbol{q}, t)$, but only for the special choice of the cut-off $f(\boldsymbol{k}) = \delta(\boldsymbol{k})$. This cut-off is, of course, totally unsound from a physical standpoint, since it corresponds to an absolutely non-local interaction. Nevertheless it is interesting to see that the solution is formally very similar to the solution without recoil. In fact, in our form of the interaction picture:

$$\psi_G^*(\boldsymbol{q}, t) = \psi^*(\boldsymbol{q}) \exp\left\{\alpha(t)\, a(0) + \beta(t)\, a^*(0) + \gamma(t) \int \psi^*(\boldsymbol{p})\, \psi(\boldsymbol{p})\, d\boldsymbol{p} + \eta(t)\right\}$$

with

$$\alpha(t) = \frac{\lambda}{(2\pi)^{s/2}} \frac{1}{\sqrt{2}\,\mu^2} (e^{i\mu t} - 1)$$

$$\beta(t) = \frac{-\lambda}{(2\pi)^{s/2}} \frac{1}{\sqrt{2}\,\mu^2} (e^{-i\mu t} - 1)$$

$$\gamma(t) = \frac{-\lambda^2}{(2\pi)^s} \frac{1}{2\,\mu^4}\, \delta(0)\, \{e^{i\mu t} + e^{-i\mu t} - 2\}$$

$$\eta(t) = \frac{1}{2}\, \gamma(t)\,.$$

This solution is, of course, purely formal, and we don't think it worth of trying to give it a precise mathematical meaning.

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