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Internal Manifolds, a Description of Exact and Broken Symmetries, Incorporating Rotational Excitations as Implied by the Hypothesis of Regge-Recurrences[1])

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Abstract. An attempt is made to introduce the concept of an internal manifold in elementary particle physics, and to realize the simultaneous action of the Poincaré group and of an internal SU 3 symmetry group on this manifold. A restricted class of manifolds is investigated, Riemannian globally symmetric spaces of type II, on which the internal symmetry group acts through the adjoint representation.

The interplay of the two groups supplemented by several auxiliary assumptions determines the manifold uniquely. The abstract manifold and the two Lie transformation groups acting on it are reexamined considering a linear boundary value problem on the manifold, stripped partly of its structure as homogeneous space, to allow the symmetry substitutions of the solutions determine the actions of the two transformation groups. The boundary value problem gives rise to a spectrum of masses depending on spin and internal quantum numbers. Meson and baryon masses are calculated defining special models, in which a continuation to complex angular momenta is carried out.

The situation for space like momenta is investigated and the restrictions imposed on the potentials defining the aforementioned models, by demanding that no solutions exist for space like momenta, are studied. The differential equation on the manifold is separable and is reduced to a second order linear differential equation in one dimension. The location of the bound solutions is determined from an associated Jost function. The differential equation is studied by mapping it on an analog potential scattering equation. The analog energy and potential strength appear as algebraic functions of mass, spin and internal quantum numbers.

The breaking of symmetry is treated as a perturbation. Mass splittings within meson and baryon SU 3 multiplets are obtained in first approximation with respect to the strength of the breaking.

1. Introduction

a) General Considerations Initiating the Present Investigations

An elementary particle or a state with specified quantum numbers is treated in an analogous way to an atom, conceived as a first hypothesis to be elementary as a consequence of assumed ignorance of the fact that the atom is composed of a nucleus and a surrounding electron cloud.

All the information about a possible internal structure of the atom is therefore to be obtained from scattering experiments. To outline the analogy further the resonance scattering of light by an atom is compared with the πN-scattering near the energies

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of the observed resonances. If the resonance scattering is mainly elastic we have the well known formula for the total elastic cross section

$$\sigma_{el}^{R}(k) \cong \frac{\pi}{k^2} \frac{(2\,j_R+1)}{2\,(2\,j_G+1)} \frac{\Gamma_R^2}{(k-E_R)^2+(\Gamma_R^2/4)} \qquad [1] \qquad (I.1)$$

where j_R is the angular momentum of the excited state of the atom, j_G the angular momentum of the ground state, Γ_R the width of the resonance, E_R its energy and k the energy of the incident photon. σ_{el}^{R} is to be compared with the peaks in the $\pi^+ p$ or $\pi^- p$ elastic cross sections, $\sigma_{\pi\pm p}$, near the Δ and $\mathcal{N}^*$ resonances, allowing for increasing inelasticities for the states with high masses [2].

The interpretation of the spectroscopic patterns in terms of a spectrum of energy eigenvalues, the corresponding eigenstates displaying the detailed structure of the electron shell, will serve us as a guiding model in looking for an internal manifold, describing strongly interacting particles and resonances and their energy spectrum.

The remarkable validity of an approximate SU 3 symmetry scheme or the 'Eightfold Way', first conceived by Gell-Mann [3] and Ne'eman [4], and the splitting of masses within the SU 3 multiplets as antagonistic principles have led to two distinct descriptions of broken symmetries, the group-theoretic approach and investigations of current commutation relations.

The first method, which is best called marriage of the Poincaré group to the internal symmetry group proved to be inconsistent with general principles, except in trivial cases [5, 6]. The strength of the objections relies on the fact, as stressed by Jost [6], that a unitary representation of the comprising group implies the existence of the Garding domain, which is dense in the Hilbert space carrying the representation, and on which the enveloping algebra of the corresponding Lie algebra is generated by essentially self-adjoint operators.

The basically identical situation is apparent from S. Coleman's critic of the relativistic SU 6 group [7]. In this work special representations are studied. Coleman supposes the widths of the resonances, which are to be looked upon as stable particles with respect to strong interactions, to vanish. This enables one to extract these irreducible representations of the Poincaré group from th ebackground of multi-particle states in the continuous spectrum of the mass operator $M = \int \sqrt{p^2}\, dE(P)$ a procedure which in general proves to be impossible in axiomatic field theory.

Restricted to the one particle states, of which only a finite number is supposed to lie within a finite mass interval, the representation represents faithfully a group of the form $G = K \otimes (\Lambda \times A)$ Coleman conjectures. K is a compact group, Λ contains the homogeneous Lorentz transformations, A is an abelian normal subgroup of $\Lambda \times A$, containing the group of space and time translations. ($\otimes$ denotes the direct product, $\times$ the semi-direct product.)

Once the algebraic 'Überbau' of group theory is lost, broken symmetries seem to become ununderstandable. We are therefore led to search for a mechanism giving rise to the breaking of a given group which in our case we take to be $\tilde{P} \otimes$ SU 3 ($\tilde{P} =$ Poincaré group).

It is appropriate, we think, to recall here a much older but similar situation with respect to the hydrogen atom. As is well known and first demonstrated by Pauli [8], the Kepler problem can be described in an entirely algebraic way, both in the classical and in the quantum mechanical framework. The method consists of introducing the operators

$$L = \xi \wedge p\,; \; B = \frac{1}{2}\,(p \wedge L - L \wedge p) - \mu_e\, e^2\, \frac{\xi}{r}$$

$$H = \frac{P^2}{2\,\mu_e} - \frac{e^2}{r}\,; \; B^2 = 2\,\mu_e\, H\,(L^2 + \hbar^2\,\mathbf{1}) + \mu_e\, e^2\,\mathbf{1}$$

(ξ: position coordinate of the electron relative to the proton, p: electron momentum, B: Lenz vector, $\mu_e = m_e\, m_p/m_e + m_p$: reduced mass of the electron, e: electron charge), which obey the following commutation relations

$$[L_i, L_k] = i\,\varepsilon_{ikl}\,\hbar\, L_l\,; \; [B_i, B_k] = i\,\varepsilon_{ikl}\,(-\,2\,\mu_e\, H)\, L_e$$

$$[L_i, B_k] = i\,\varepsilon_{ikl}\,\hbar\, B_e\,; \; L\, B = B\, L = 0\,. \tag{I.2}$$

For states with negative energy the relations I.2 can be expressed by normalized operators

$$M = \frac{1}{\hbar}\, L, \quad N = \frac{1}{\hbar}\, \frac{1}{\sqrt{-\,2\,\mu_e\, H}}\, B$$

$$[M_t, M_s] = i\,\varepsilon_{tsr}\, M_r \quad [N_t, N_s] = i\,\varepsilon_{tsr}\, M_r$$

$$[M_t, N_s] = i\,\varepsilon_{tsr}\, N_r \quad M\, N = N\, M = 0 \tag{I.3}$$

$$(-\,2\,\mu_e\,\hbar^2\, H)\,(M^2 + N^2 + \mathbf{1}) = (\mu_e\, e^2)^2\,\mathbf{1}\,.$$

The operators $J_\pm = (M \pm N)\,1/2$ subjected to the condition $J_+^2 = J_-^2$ generate the group O 4, as is well known. From I.3 all facts about the discrete spectrum of the hydrogen atom considered as a Kepler problem can be obtained. Looking however at the relativistic problem or at the real hydrogen spectra, displaying the fine structure of the levels, the group O 4 is found to be broken, and this happens in a way which cannot be described by any reasonable algebraic means.

Historically the memorable foundations of wave mechanics by SCHRÖDINGER [9] have revealed a new aspect of the internal structure of the hydrogen atom, on which the relevant equation and energies appear as an eigenvalue problem, i.e. as the Schrödinger equation.

$$H\,\psi(\xi, t) = i\,\partial_t\,\psi(\xi, t) = \left(-\,\frac{\hbar^2}{2\,\mu_e}\,\Delta_\xi - \frac{e^2}{r}\right)\psi \tag{I.4}$$

with the boundary condition that $\psi(\xi, t)$ be square integrable over R_3.

The same properties as in the algebraic treatment are found back again, but this is not the end of the story in this theory, because now the boundary value problem can be considered as the starting point towards a relativistic wave equation including spin, the Dirac equation.

b) Characteristics of the Present Approach

The present work is an attempt to introduce the concept of an internal manifold in elementary particle physics. The internal manifold is the carrier space of the representations, as coordinate transformation groups, of two groups, the POINCARÉ group and an internal SU 3 symmetry group (our results can be generalized without difficulty to include any internal compact semisimple Lie group).

These two representations are not a priori related to any conservation laws. The action of the POINCARÉ group in our approach generates the corresponding exact conservation laws. However the transformations of the internal group generate exactly conserved quantities if and only if they commute as coordinate transformations with the action of the POINCARÉ group on the internal manifold. Thus a breaking mechanism is easily realized.

The interplay of the above two groups on the internal manifold proves to impose rather restrictive conditions on it. We investigate a class of manifolds, which is mathematically well understood and completely classified, Riemannian globally symmetric spaces of type II. We show that the above mentioned restrictions supplemented by the requirement of minimal dimension for the manifold M, determine M uniquely.

On the Riemannian globally symmetric spaces of type II a given group (in our case the internal symmetry group $\tilde{K}$) acts through the adjoint representation $\mathrm{ad}^{\tilde{K}}$. The kernel of the homomorphism h, $h: \tilde{K} \to \mathrm{ad}^{\tilde{K}}$ is the center of $\tilde{K}$, $Z_{\tilde{K}}$. The group represented on M is therefore $\tilde{K}/Z_{\tilde{K}}$ (in the case of SU 3: $Z_{SU_3} = Z_3$, Z_3: cyclic group of three elements). This implies that no state with fractional charge (e.g. quarks) will appear.

Once the manifold is determined, on given coordinates an arbitrary coordinate transformation can be performed. This means that if a choice of coordinates displays in a simple way the group transformations, the physical interpretation of these coordinates can be quite obscure, whereas intuition indicates, that the internal manifold is related to a space time structure of particles.

We therefore start anew retaining only the general informations on the internal manifold gained from the purely mathematical construction of M. We consider a linear boundary value problem on M. The coordinates which render this problem most simple will in general not coincide with the special coordinates used before.

We investigate particularly the dependence of meson and baryon masses on complex angular momenta (Regge trajectories) in various models. We show that the a priori arbitrary potentials defining these models are restricted by the requirement that no solutions exist with space like momenta or imaginary masses.

We do not expect that these models allow to compute the masses of physically observed particles, but we wanted to show that it is possible to obtain Regge trajectories with reasonable characteristics.

We show further, that the possible values of the mass are given by the zeros of the JOST function associated with an analog potential scattering differential equation of second order in one dimension, if the analog energy and potential strength are expressed by appropriate rational functions of the mass, spin and internal quantum numbers of the physical solution in question.

Composite particles as nuclei evidently show an internal manifold. The coordinates of this internal manifold of nuclei can be given by the relative positions of the constituent particles or as in the droplet model by the position of a volume element of the droplet. The proposed structure does not a priori postulate compositeness but rather constitutes a mathematical model which can display features related to a composite particle.

II. The Internal Manifold, Assumptions and Uniqueness

1. *Definitions*

Let $\psi(x, z)$ be a wave function which shall be the solution of a partial linear differential equation, and satisfies the boundary conditions which define the eigenvalue problem.

$x = (x^\mu)$; $\mu = 0, 1, 2, 3$, is a four vector which is to be related to the center of mass coordinates of the particles described by ψ.

$z = (z_1, \dots, z_N)$ represents a point of the internal manifold M of dimension N, as mapped locally on a Euclidean space E_N with coordinates $z_1, \dots, z_N$.

The abstract POINCARÉ group will be denoted by $\tilde{P}$ with elements $\pi = (a, \Lambda)$ its covering group by $Q\tilde{P}$ with elements $\varrho = (a, A)$, the homogeneous Lorentz group by $\tilde{\Lambda}$ with elements Λ, its covering group by $S\,4 = \mathrm{SL}(2, \mathrm{C})$ with elements A.

$T_{\alpha,\Lambda} = T_\pi$ denotes the action of the Lorentz transformation (a, Λ) on the manifold (x, z). We assume the action $T_\pi: (x, z) \to (\Lambda x + a, z')$ to be a diffeomorphism (i.e. T_π is a differentiable mapping 1 to 1 and T_π^{-1} is differentiable).

The set of transformations T_π considered as mappings of M onto M will be denoted by $\tilde{\tau} \cdot \tilde{\tau}$ becomes a representation of $\tilde{P}$ by the composition law

$$T_{(a_2, \Lambda_2)}\, T_{(a_1, \Lambda_1)} = T_{(a_2 + \Lambda_2 a_1, \Lambda_2 \Lambda_1)}\,. \tag{II.1}$$

The abstract SU 3 group will be denoted by $\tilde{K}$, its elements by $\varkappa$. The symbol $S_\varkappa$ denotes the action of $\tilde{K}$ on (x, z), $S_\varkappa$: $(x, z) \to (x, z')$ shall be a diffeomorphism. The set of transformations $S_\varkappa$ from M onto M will be called $\tilde{S}$. The $S_\varkappa$ satisfy the composition law

$$S_{\varkappa_1}\, S_{\varkappa_2} = S_{\varkappa_1 \varkappa_2} \quad \varkappa_{1,2} \in \tilde{K}\,. \tag{II.2}$$

2. *The Action of S*

Assumption A 1: M is a globally symmetric analytic Riemannian manifold of type II (not necessarily irreducible).

Hence there exists a semisimple, compact, connected Lie group $\tilde{G}$ such that $M = \tilde{G} \otimes \tilde{G}/D_G$, where D_G is the subgroup of $\tilde{G} \otimes \tilde{G}$ formed by the diagonal elements $g \otimes g$, $g \in \tilde{G}$. $\tilde{G} \otimes \tilde{G}/D_G$ can be mapped canonically on $\tilde{G}$:

$$g_1 \otimes g_2 \to g_1\, g_2^{-1}\,.$$

The action of $\tilde{G} \otimes \tilde{G}$ on M is given by

$$g_1 \otimes g_2: \quad h \to g_1\, h\, g_2^{-1}\,. \tag{II.3}$$

Restricted to D_G this gives

$$g \otimes g: \quad h \to g\, h\, g^{-1}\,. \tag{II.4}$$

Introducing locally normal coordinates on M which are again denoted by $h = (h_1, \dots, h_N)$ the action of D_G is represented by

$$g \in D_G,\, g: \quad h \to g\, h\, g^{-1} = \mathrm{ad}^{\tilde{G}}(g)\, h \tag{II.5}$$

For the notions of differential geometry and symmetric spaces reference is made of [10].

$\{\mathrm{ad}^{\tilde{G}}(g) \mid g \in \tilde{G}\}$ constitute the adjoint representation of $\tilde{G}$. We now assume $\tilde{K} \subset \tilde{G}$ as a topological Lie subgroup. This reduces $\tilde{S}$ to the following transformation group:

$$S_\varkappa \in \tilde{S} \qquad S_\varkappa: \quad h \to \varkappa\, h\, \varkappa^{-1} = \mathrm{ad}^{\tilde{G}}(\varkappa)\, h\,. \tag{II.6}$$

A 1 seems to be a rather restrictive assumption which together with II.4 and II.5 reduce the action of $\tilde{S}$ to the set of linear transformations $\{\mathrm{ad}^{\tilde{G}}(\varkappa) \mid \varkappa \in \tilde{K}\}$. They form a subgroup of $\mathrm{ad}^{\tilde{G}}$ which itself is a subgroup of $SO(N)$. In the case $\tilde{G} = \tilde{K}$ the right and left translations $D_\varkappa$, $L_\varkappa$ induce separately isometric transformations on M.

Since $\tilde{G}$ is semisimple it is the direct product of a finite number of simple connected Lie groups $\tilde{G} = \tilde{G}_1 \otimes \tilde{G}_2 \otimes \ldots \otimes \tilde{G}_n$. The Lie algebra of $\tilde{G}$ is the direct sum of the ideals $\Gamma_1, \ldots, \Gamma_n$, the Lie algebras of $\tilde{G}_1, \ldots, \tilde{G}_n$ respectively $\Gamma = \Gamma_1 \oplus \ldots \oplus \Gamma_n$. The generators of $\tilde{K}$, $q_1, \ldots, q_8$ decompose in a natural way in the ideals $\Gamma_1, \ldots, \Gamma_n$

$$q_i = q_i^{(1)} \oplus q_i^{(2)} \oplus \cdots \oplus q_i^{(n)} \; q_i^{(k)} \in \Gamma_k,\; i = 1, \ldots, 8\,.$$

The $q_i^{(k)}$, $k = 1, \ldots, n$ satisfy the commutation relations of SU 3

$$[q_i^{(k)}, q_j^{(k')}] = \delta^{kk'} f_{ijs}\, q_s^{(k)} \tag{II.7}$$

f_{ijs} structure constants of SU 3.

Γ_k is of course left invariant, as an ideal, a fortiori under the action of $\tilde{S}$. This leads to the second assumption.

A 2: In Γ there is no vector which is left invariant under the action of $\tilde{S}$.

A 2 implies $q_i^{(k)} \neq 0 \;\forall\; k$. The $q_i^{(k)}$ are linearly independant $\forall\; k$. Therefore $\tilde{K}_k \subset \tilde{G}_k$. $\tilde{K}_k$ is generated by the Lie algebra $\{q_i^{(k)}\}$, $i = 1, \ldots, 8$ and is (locally) isomorphic to $\tilde{K}$. The dimension of d_k of Γ_k has to be $\geqslant 8$. We have therefore reduced the possible manifolds Γ_k to be simple compact Lie algebras which contain the Lie algebra of $\tilde{K}_k$, the k corresponding, as a Lie subalgebra.

A 3: The irreducible spaces Γ_k are of minimal dimension compatible with A 1, 2.

Hence $\Gamma_k \cong Q$, $\forall\; k$. Q: Lie algebra of $\tilde{K}$.

A 1, 2, 3 determine M up to a coordinate transformation to be the direct sum of n Lie algebras $\Gamma_1, \ldots, \Gamma_n$; $\Gamma_k \cong Q$, $k = 1, \ldots, n$.

$$M = \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_n\,. \tag{II.8}$$

We now consider the exponential mapping of the tangent space M locally onto $\tilde{G}$ to redefine $\tilde{G}_u$ globally as the universal covering group of $\tilde{G}$

$$\tilde{G}_u = \tilde{G}_{1u} \otimes \tilde{G}_{2u} \otimes \cdots \otimes \tilde{G}_{nu} \tag{II.9}$$

$\tilde{G}_{iu}$ is isomorphic to the universal covering group $\tilde{K}_u$ of $\tilde{K}$ (in our case $\tilde{K}_u = \tilde{K}$).

The action of $\tilde{S}$ on M is given by the group of inner automorphisms of $\tilde{G}_u$ induced by the elements of $D_G \subset \tilde{G}$ of the form $\varkappa \otimes \varkappa \otimes \ldots \otimes \varkappa = \varkappa_D$, $\varkappa \in \tilde{K}$:

$$\varkappa_D(n): \quad g_1 \otimes \cdots \otimes g_n \to \varkappa\, g_1\, \varkappa^{-1} \otimes \cdots \otimes \varkappa\, g_n\, \varkappa^{-1}\,. \tag{II.10}$$

We denote by $\mathrm{Int}_D\,(\tilde{K})$ the group of inner automorphisms of $\tilde{G_u} = \tilde{G}$ associated eith D_G. The mapping ϕ: $n \to \mathrm{Int}_D(n)$ is an isomorphism of $\tilde{K}/Z_{\tilde{K}}$ onto $\mathrm{Int}_D\,(\tilde{K})$ ($Z_{\tilde{K}}$: center of $\tilde{K}$). We conclude that the action of $\tilde{S}$ on M does not represent faithfully the group $\tilde{K} = \mathrm{SU}\,3$, but

$$\tilde{S} \leftrightarrow \tilde{K}/Z_{\tilde{k}} = \mathrm{SU}\,3/Z_3 \quad \leftrightarrow \text{ means represents faithfully.} \tag{II.11}$$

Z_3 is the center of SU 3 composed of the three matrices

$$e^{i\nu(2\pi/3)} \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}. \qquad \nu = 0, 1, 2$$

Hence A 1, 2, 3 imply that $Z_{\tilde{K}}$ is not represented by $\tilde{S}$.

It is to be stressed at this point that A 1, 2, 3 can be considered as ad hoc assumptions which reduce M and S to the form given by II.8, 9, 10. On the other hand if an approach along similar lines leading to other choices of M and $\tilde{S}$ (e.g. when quarks are required to be present) is followed up, A 1, 2, 3 must be modified. This means that the beautiful theory of symmetric spaces cannot be applied to its full content to the problem.

3. *The Action of* $\tilde{\tau}$

Let F denote the abelian subgroup of SU 3 generated by Γ^3 and Γ^8 corresponding to the third component of isotopic spin and hypercharge respectively.

A 4: The elements of $\tilde{S}$, $S_\varkappa$ with $\varkappa \in F$ commute as coordinate transformations with all $T_\pi \in \tilde{\tau}$, $\pi \in \tilde{P}$.

A 4 is equivalent to the fcllowing commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{T_\pi} & M \\ f\downarrow & & \downarrow f \\ M & \xrightarrow[T_\pi]{} & M \end{array} \qquad \begin{array}{l} \forall f \in F \\ T_\pi \in \tilde{\tau}\,. \end{array} \tag{II.12}$$

Figure 1

A 4 is motivated by the assumption that a theory of hadrons is thinkable which does not take into account the weak interactions, as a good approximation to the real world. In this approximation I_3 and Y are exactly conserved. I_3 and Y are related to the unitary representation of F in Hilbert space.

$$f = \exp\,(a\,\Gamma^3 + b\,\Gamma^8) \to \exp\left(-\,i\,a\,I_3 - i\,b\frac{\sqrt{3}}{2}\,Y\right) = U(f)\,.$$

A 4 cannot be exploited immediately because the eigenspaces of the matrices Ad Γ^3, Ad Γ^8 ($\Gamma^3 = -\,i\,I_3$, $\Gamma^8 = -\,i\,\sqrt{3}/2\;Y$) have complex components relative to the normal coordinates we have chosen on M. Hence we are lead to consider the complexification M_C of the Lie algebra of $\tilde{G} = \tilde{G_1} \otimes \dots \otimes \tilde{G_n}$. The complex structure J_M on M_C is induced by the mapping

$$M_C = M \oplus M \qquad J_M\,(X \oplus Y) = (-\,Y \oplus X) \qquad X, Y \in M\,.$$

The restriction of M_C to real coordinates will be denoted by M_R. We will also consider the complexification of the POINCARÉ group $\tilde{P}_C$, or better of its covering group $Q\tilde{P}$, $Q\tilde{P}_C$, obtained by complexifying the Lie algebra Π_R of $\tilde{P}_R$ and introducing the complex structure J_P on Π_C:

$$\Pi_C = \Pi_R \otimes \Pi_R ; \quad J_P (\varrho_1 \oplus \varrho_2) = (-\varrho_2 \oplus \varrho_1) \quad \varrho_{1,2} \in \Pi_R .$$

We cannot conclude that the action of $\tilde{\tau}$ as a Lie transformation group acting on M_R can be analytically continued to the action of $\tilde{\tau}_C$ on M_C, especially since $\tilde{\tau}$ will in general not act transitively on M_R ($\tilde{\tau}_C$ denotes the complexification of the POINCARÉ group as a Lie transformation group, see e.g. (11)). The complexified groups to which our investigations are extended are:

$$\tilde{G}_R = \tilde{G}_{1R} \otimes \ldots \otimes \tilde{G}_{n_R} \to \tilde{G}_C = \tilde{G}_{1C} \otimes \ldots \otimes \tilde{G}_{n_C}$$

$$\tilde{G}_{i_C} \cong \tilde{K}_C ; \quad \tilde{K}_C = \mathrm{SL}(3, \mathrm{C})$$

$$\tilde{P}_R = \tilde{\Lambda} \times \mathrm{Tr}_R (4) \to \tilde{P}_C = \tilde{\Lambda}_C \times \mathrm{Tr}_C (4) \qquad \text{(II.13)}$$

$$Q\,\tilde{P}_R = \mathrm{SL}(2, \mathrm{C}) \times \mathrm{Tr}_R (4) \to \{\mathrm{SL}(2, \mathrm{C}) \otimes \mathrm{SL}(2, \mathrm{C})\} \times \mathrm{Tr}_C (4) = {}_C\tilde{\Lambda}_C \times \mathrm{Tr}_C (4)$$

${}_C\tilde{\Lambda}_C$ denotes the universal covering group of the group of complex Lorentz transformations $\Lambda_C \in \tilde{\Lambda}_C$ which preserve the scalar product

$$(x, y) = x^\mu g_{\mu\nu} y^\nu ; \; x, y \in C_4 : \quad (\Lambda_C x)^\mu g_{\mu\nu} (\Lambda_C y)^\nu = x^\mu g_{\mu\nu} y^\nu .$$

$$\tilde{\Lambda}_C = \mathrm{SL}(2, \mathrm{C}) \otimes \mathrm{SL}(2, \mathrm{C}) / Z_2$$

Tr_C (4) is the group of complex translations in C_4. $\times$ denotes the semi-direct product.

In order to ascertain the analytical structure of the action of $\tilde{\tau}$ on M_R we make the following assumption:

A 5: $\exists$ a Lie group $\tilde{Q}_R$ which contains $\tilde{P}_R$ as a Lie subgroup and which acts transitively on M_R. The analytic structure induced on M_R by the identification mapping $M_R \leftrightarrow \tilde{Q}_R/\tilde{H}_{x_0}$ where $\tilde{H}_{x_0}(\tilde{Q}_R)$ is the isotropy group of an arbitrary point $x_0 \in M_R$ with respect to $\tilde{Q}_R$, is compatible with the analytic structure of M_R as the Lie algebra of $\tilde{G}_R$.

In other words, if we represent the coordinate transformation induced by T_q, $q \in \tilde{Q}_R$ using locally normal coordinates $q_1, \ldots, q_\sigma$ for the elements of $\tilde{Q}_R$, by the functions

$$T_q : \quad x^i_{(k)} \to x_q{}^i_{(k)} = \varphi^i_{(k)} \{x^s_{(t)} ; \; q_1, \ldots, q_\sigma\}$$

$\varphi^i_{(k)}$ are analytic functions of all its arguments for x and q in a complex neighbourhood of the origins in the respective spaces.

A 5 could probably be replaced by a weaker assumption.

It permits to consider the action of $\tilde{\tau}_C$ on M_C, which is the restriction to $\tilde{P}_C$ of the action of $\tilde{Q}_C$ as an analytic Lie transformation group on M_C. A 4 can now be extended to the complexified abelian group F_C generated by Γ^3, $i\,\Gamma^3$, Γ^8, $i\,\Gamma^8$, and $\tilde{\tau}_C$ acting on M_C:

$$\begin{array}{ccc} M_C & \xrightarrow{T_{\pi C}} & M_C \\ f_C \downarrow & & \downarrow f_C \\ M_C & \xrightarrow[T_{\pi C}]{} & M_C \end{array} \qquad \forall\, f_C \in F_C\, T_{\pi_C} \in \tilde{\tau}_C\,. \tag{II.14}$$

Figure 2

We consider the consequences of II.14 in the root spaces of the Cartan subalgebra of $SL(3, C)_1 \otimes \dots \otimes SL(3, C)_n$ generated by $\{\Gamma^3_{(k)}, \Gamma^8_{(k)}\}$, $k = 1, \dots, n$ as a basis over the complex numbers. Let the vectors $\in \Gamma_{(k)}$ belonging to the roots $(1, 0)$, $(1/2, \sqrt{3}/2)$, $(-1/2, \sqrt{3}/2)$, $(-1, 0)$, $(-1/2, -\sqrt{3}/2)$, $(1/2, -\sqrt{3}/2)$ be called $E_1(k)$, $E_4(k)$, $E_6(k)$, $E_{-1}(k)$, $E_{-4}(k)$, $E_{-6}(k)$, $k = 1, \dots, n$ respectively. We look at the following subspaces of M_C:

$$M_C^l = \{z \in M_C \mid z = z_1\, E_e(1) \oplus z_2\, E_e(2) \oplus \cdots \oplus z_n\, E_e(n)\}$$
$$l = \pm 1, \pm 4, \pm 6; \quad z_i \in C\,.$$

II.14 implies $\tilde{\tau}_C M_C^l \subset M_C^l$. $\tilde{\tau}_C$ induces on M_C analytic coordinate transformations:

$$T_\pi \in \tilde{\tau}_C: \quad (z_1^n, \dots, z_n^e) \to (z_1^{e,\pi}, \dots, z_n^{e,\pi})$$
$$z_i^{e,\pi} = \varphi_i^e\,(z_1, \dots, z_n; p_1, \dots, p_{10}) \quad \pi = (p_1, \dots, p_{10})\,. \tag{II.15}$$

Let f_C be $\exp[+2\,i\,\lambda\,\Gamma^3]$, $\lambda = \lambda_1 + i\,\lambda_2$ and $l = 4$. f_C induces on M_C^4 the mapping $\exp(2\,i\,\Gamma^3)\, z^{(4)} = \exp[2\,i\,\mathrm{Ad}\,\Gamma^3]\, z^{(4)} = \exp\lambda\, z^{(4)}$. II.14 implies

$$\begin{array}{ccc} z_1^{(4)}, \dots, z_n^{(4)} & \xrightarrow{\exp 2\,i\,\lambda\,\Gamma^3} & \exp\lambda\,(z_1^{(4)}, \dots, z_n^{(4)}) \\ & & \downarrow T_\pi \\ T_\pi \downarrow & & \varphi_1\,[\exp\lambda\, z^4, \pi], \dots, \varphi_n \\ \varphi_1(z^{(4)}, \pi), \dots, \varphi_n & \xrightarrow{\exp 2\,i\,\lambda\,\Gamma^3} & \exp\lambda\,(\varphi_1[z^4, \pi], \dots, \varphi_n)\,. \end{array} \tag{II.16}$$

Figure 3

Let $z = z^{(4)}$ and $\mu = \exp\lambda$. II.16 is equivalent to

$$\varphi_i(\mu\, z, \pi) = \mu\, \varphi_i(z, \pi)\,. \tag{II.17}$$

We have been careless in defining the topology on M_C and $\tilde{\tau}_C$ as can be seen from II.17, which implies for $z_1 \neq 0$

$$\varphi_i(z, \pi) = z_k\, \frac{\partial \varphi_i}{\partial z_k}\,(z, \pi) = \sum_k z_k\, \chi_k^i(z, \pi) = z_1 \sum \frac{z_k}{z_1}\, \chi_k^i(z, \pi) = z_1\, \psi^i(z, \pi)$$
$$\psi^i(\mu\, z, \pi) = \psi^i(z, \pi); \quad i = 1, 2, \dots, n\,. \tag{II.18}$$

$z_1\, \psi^i$ is interpreted to induce locally an analytic transformation of $Pr\,(z_1, \dots, z_n)$ onto itself. $Pr\,(z_1, \dots, z_n) = Pr\,(n)$ denotes the complex projective space of n complex variables $z_1, \dots, z_n = \mu\, z_1, \dots, \mu\, z_n$, $\mu \in C$.

A 6: $\tilde{\tau}_C$ acts locally transitively on the spaces $Pr\,(n, M_C^l)$, $l = \pm 1, \pm 4, \pm 6$.

$Pr\,(n, M_C^l)$ denoting the projective space $Pr(n)$ associated with the n dimensional complex vector space M_C^l.

As a consequence of A 5 and A 6 the spaces $Pr(n, M_C^l)$ can be locally mapped on the spaces

$$V_C^e = \tilde{P}_C / \tilde{H}_C(z_0, l) \qquad \tilde{H}_C(z_0, l) \subset \tilde{P}_C \tag{II.19}$$

$\tilde{H}_C(z_0, l)$ denotes the isotropy group of a point $z_0 \in M_C^l$, contained in $\tilde{P}_C$. We use normal coordinates for the elements $\Pi \in P_c$:

$$(p_1, \dots, p_{10}) \in \Pi_C = \Pi_R \oplus \Pi_R$$
$$J_P:\ (p_1, \dots, p_{10}) \to (i\,p_1, \dots, i\,p_{10})\,.$$

The operation of complex conjugation * extends to an involutory automorphism of $\tilde{P}_C$ onto itself:

$$*:\ (p_1, \dots, p_{10}) \to (p_1^*, p_2^*, \dots, p_{10}^*)\,.$$

A 7: $H_C(z_0, l)$ is stable under the * operation, $l = \pm 1, \pm 4, \pm 6$.

A 7 implies that $\tilde{H}_R(z_0, l)$, the coordinates of which are real with respect to a basis in Π_R, the Lie algebra of $\tilde{P}_R$, is a Lie subgroup of $\tilde{P}_R$. Therefore A 5, 6, 7 imply the existence of

$$V_R^e = \tilde{P}_R / \tilde{H}_R(z_0, l) \qquad \dim V_R^e = n - 1\,. \tag{II.20}$$

To determine n by a minimality requirement use is made of a further assumption.

A 8: The action of $\tilde{\tau}_R(\tilde{\Lambda})$ of the homogeneous Lorentz transformations on M_R commutes as coordinate transformation group with the action of $\tilde{S}$.

Let m_3, m_8 denote the following linear subspaces of M:

$$m_t = \{z / z = z^1\,\Gamma_1^t \oplus z^2\,\Gamma_2^t \oplus \cdots \oplus z^n\,\Gamma_n^t\}; \qquad t = 3, 8$$

m_t can be considered as m_t^C, vector space over C or as m_t^R, vector space over R. Since the failure of $\tilde{\tau}_R$ and $\tilde{S}_R$ to commute comes about through the representation of translations on the spaces $m_t^{C/R}$ which can as a consequence of A 8 be identified with $M_{C/R}^l$, we conclude that the representation of $\tilde{\Lambda}$ on M_R^l and therefore also on V_R^l must be extendable to a representation of $\tilde{P}_R$ which does not represent trivially the translation group. This does not mean however, that the actual representation on a given M_R^l or V_R^l represents in a nontrivial way the translation group.

3. *Determination of the Dimension of the Spaces* m_t^R, M_R^l; $t = 3, 8$, $l = \pm 1, \pm 4, \pm 6$

(1) $n = 1, 2$ is excluded.

Proof: $n = 1$. The dimensions of M_R^l and m_t^R being all 1, $\tilde{P}_R$ is trivially represented in M_R. Therefore $\tilde{\tau}$ commutes with $\tilde{S}$, which is in disagreement with our assumptions.

$n = 2$. $\dim V_R^l = 1$. This implies that the homogeneous Lorentz transformations are represented trivially in M_R. Looking at the representation of $\tilde{P}_R$ in m_t^R the above implies the existence of a Lie subgroup of $\tilde{P}_R$ of dimension $\geqslant 8$, in which the translation group is not entirely contained. Since there is no such subgroup $n = 2$ can be excluded q.e.d.

(2) $n = 3$ can be excluded.

Proof: $\dim V_R^l = 2$. From this we conclude, that the translations are trivially represented in M_R^l.

In m_t^R the transformation group representing $\tilde{P}_R$ which does not represent all translations trivially is uniquely determined up to a coordinate transformation (see appendix): Let λ be a light like vector ($\lambda^2 = 0$), σ a scalar. Consider the real projective space $Pr\,(3)$:

$$(\lambda, \sigma) \cong (a\,\lambda, a\,\sigma) \qquad a \in R_1\,.$$

The dimension of $Pr\,(3)$ is three. The action of $\tilde{\tau}_R$ on $Pr\,(3)$ is given by

$$(a, \Lambda): \quad (\lambda, \sigma) \to (\Lambda\,\lambda, \sigma + (a, \Lambda\,\lambda))\,. \tag{II.21}$$

II.21 can be represented on the light cone by the following substitution:

$$\sigma \neq 0 \quad \lambda_1 = \frac{\lambda}{\sigma}; \quad (a, \Lambda): \quad \frac{\lambda}{\sigma} \to \frac{\Lambda\,\lambda}{\sigma + (a, \Lambda\,\lambda)} = \frac{\Lambda\,\lambda_1}{1 + (a, \Lambda\,\lambda_1)} \quad \lambda_1^2 = 0\,. \tag{II.22}$$

A 5, 6, 7 limit M_R to the following form for $n = 3$:

$$M_R = \{z/z = z_k^1\,\Gamma_1^k \oplus z_k^2\,\Gamma_2^k \oplus z_k^3\,\Gamma_3^k\}\,,\ z_k^i \in R_1\,.$$

The actions of $\tilde{\Lambda}$ and $\tilde{S}$ are given by

$$(z_k^1)^2 + (z_k^2)^2 + (z_k^3)^2 = |z_k|^2$$

$$\Lambda \in \tilde{\Lambda}: \ z_k^t \to \Lambda_s^t\,z_k^s + \Lambda_0^t\,|z|_k \quad T_\varkappa \in \tilde{S}: \ z_k^t \to \mathrm{ad}(\varkappa)_k^e\,z_e^t \quad (z_k^1, z_k^2, z_k^3) = \mathbf{z}_k \tag{II.23}$$

A 8 implies $|\mathrm{ad}(\varkappa)_k^l\,\mathbf{z}_l| = \mathrm{ad}(\varkappa)_k^l\,|z^l|$; $\forall\,\varkappa$, $\forall\,\mathbf{z}_m$. This is clearly impossible q.e.d.

(3) The case $n = 4$.

We are led to consider $\tilde{\tau}$ on m_t^R with dimension 4. The isotropy group $\tilde{H}(x_0, t)$ of a point $x_0^t \in m_t^R$ has the dimension 6. Let LH denote the Lie algebra of $\tilde{H}$. We choose the following basis in Π_R: $T_0, \ldots, T_3$ generate the translations in the $0, \ldots, 3$ direction respectively.

$$M^{\mu\nu} = \begin{pmatrix} 0 & N_1 & N_2 & N_3 \\ -N_1 & 0 & L_3 & -L_2 \\ -N_2 & -L_3 & 0 & L_1 \\ -N_3 & L_2 & -L_1 & 0 \end{pmatrix}$$

are the generators of infinitesimal Lorentz transformations.

The generators in LH have the following components in Π_R:

$$h^i \in L\,H \quad h^i = \xi^{(i)s}\,L_s + \eta^{(i)s}\,N_s + p^{(i)\mu}\,T_\mu$$

$$h^i = (\xi^{(i)}, \eta^{(i)}, p^{(i)}) \qquad i = 1, \ldots, 6\,.$$

Another basis for the Lie algebra L of $\tilde{\Lambda}$ is sometimes used $b_{1,2,3} = L_{1,2,3}$, $b_{4,5,6} = N_{1,2,3}$. (ξ, η) is abbreviated by ζ.

The homogeneous part of $\tilde{H}$ is a Lie subgroup of $\tilde{\Lambda}$. The following two cases have to be distinguished:

(α) ζ^i $i = 1, \ldots, 6$ are linearly independent.

(β) ζ^i $i = 1, \ldots, 6$ are linearly dependent.

(α): The general element of $\tilde{H}$ is of the form $g \in \tilde{H}$: $g = (a(\Lambda), \Lambda)$, Λ arbitrary. As a consequence of the group properties of $\tilde{H}$ it follows

$$a(\Lambda_2 \Lambda_1) = \Lambda_2\, a(\Lambda_1) + a(\Lambda_2) \,. \tag{II.24}$$

Further information about $\tilde{H}$ will be gained considering the representation of $\tilde{P}_R$ by 5×5 matrices

$$\mu:\ (a, \Lambda) \to \left(\begin{array}{c|c} \Lambda^\mu_\nu & a^\mu \\ \hline 0 & 1 \end{array}\right) \tag{II.25}$$

Π_R is mapped by μ on the matrices

$$\mu:\ (\zeta, p) \to \left(\begin{array}{c|c} \lambda^\mu_\nu(\zeta) & p^\mu \\ \hline 0 & 0 \end{array}\right). \tag{II.25a}$$

The commutation law in Π_R is given by

$$[(\zeta, p), (\eta, q)] = ([\zeta, \eta], \lambda(\zeta)\, q - \lambda(\eta)\, p) \,. \tag{II.26}$$

The elements of LH are of the form $h = (\zeta, p(\zeta))$, $\zeta \in L$, arbitrary. II.24 implies

$$\lambda(\zeta)\, p(\eta) - \lambda(\eta)\, p(\zeta) = p([\zeta, \eta]) \tag{II.27}$$

λ and p are linear functions of their arguments (e.g. $\lambda^\mu_\nu = \lambda^\mu_{\nu a}\, \xi^a$, $a = 1, \dots, 6$).

Proposition: Any $p(\xi)$ satisfying II.27 is of the form

$$p(\zeta) = \lambda(\zeta)\, q \,, \tag{II.28}$$

q independant of ζ.

Proof: (i) II.28 implies II.27. If $p(\zeta) = \lambda(\zeta)\, q$ then

$$\lambda(\zeta)\, p(\eta) - \lambda(\eta)\, p(\zeta) = [\lambda(\zeta), \lambda(\eta)]\, q = \lambda([\zeta, \eta])\, q = p([\zeta, \eta]) \,.$$

(ii) II.27 implies II.28. Let us denote by α the basis elements $M^{\mu\nu}$ of L: $\xi = \xi^{\alpha'}$, $\alpha', \alpha = 1, \dots, 6$, and let $\lambda_\alpha = \lambda(\alpha)$, $p_\alpha = p(\alpha)$. Then II.27 takes the form

$$\lambda_\alpha\, p_\beta - \lambda_\beta\, p_\alpha = c^\gamma_{\alpha\beta}\, p_\gamma; \quad \lambda_\alpha \lambda_\beta - \lambda_\beta \lambda_\alpha = c^\gamma_{\alpha\beta}\, \lambda_\gamma \tag{II.27a}$$

$c^\gamma_{\alpha\beta}$: structure constants of the Lorentz group:

$$(\mathrm{Ad}\,\alpha)\, \beta = [\alpha, \beta] = c^\gamma_{\alpha\beta}\, \gamma = (\mathrm{Ad}\,\alpha)^\gamma_\beta\, \gamma \qquad (\mathrm{Ad}\,\alpha)^\gamma_\beta = c^\gamma_{\alpha\beta} \,.$$

Since the connected part of Λ is semisimple the Killing form is non-singular. For semisimple groups $c_{\alpha\beta\gamma} = g_{\gamma\gamma'}\, c^{\gamma'}_{\alpha\beta}$, $g_{\alpha\beta} = -\, Sp(Ad\, \alpha, Ad\, \beta)$ is totally antisymmetric with respect to the indices α, β, γ. From II.27a it follows

$$g^{\alpha\sigma}(\lambda_\sigma \lambda_\alpha\, p_\beta - \lambda_\sigma \lambda_\beta\, p_\alpha) = g^{\alpha\sigma}\, c^\gamma_{\alpha\beta}\, \lambda_\sigma\, p_\gamma \,.$$

Since the matrices λ_α constitute an irreducible representation of L, the Casimir operator $g^{\alpha\sigma}\, \lambda_\alpha \lambda_\sigma$ is a multiple of the unit matrix. Therefore

$$d p_\beta - g^{\alpha\sigma}\, \lambda_\sigma \lambda_\beta\, p_\alpha = c^{\sigma\gamma}_\beta\, \lambda_\sigma\, p_\gamma$$

$$\begin{aligned} d p_\beta = {} & \lambda_\beta\, (g^{\alpha\sigma}\, \lambda_\sigma\, p_\alpha) + g^{\alpha\sigma}\, c^\gamma_{\sigma\beta}\, \lambda_\gamma\, p_\alpha \\ & + c^{\sigma\gamma}_\beta\, \lambda_\sigma\, p_\gamma = \lambda_\beta\, q_1; \;\; q_1 = g^{\alpha\sigma}\, \lambda_\sigma\, p_\alpha \mid d\, \mathbf{1} = g^{\alpha\beta}\, \lambda_\alpha \lambda_\beta \,. \end{aligned} \tag{II.29}$$

It remains to be shown that $d \neq 0$. This can be seen directly since

$$g^{\alpha\beta} \lambda_\alpha \lambda_\beta \propto \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - (\lambda_4^2 + \lambda_5^2 + \lambda_6^2) = 3\,\mathbf{1}$$

which implies $d \neq 0$ and finally

$$p_\beta = \lambda_\beta q; \; q = \frac{1}{d} g^{\alpha\sigma} \lambda_\sigma p_\alpha \qquad \text{q.e.d.} \tag{II.30}$$

Using the representation II.25 and exponentiating the infinitesimal matrices we have $p(\zeta) = \lambda(\zeta)\, q$

$$\exp \left(\begin{array}{c|c} \lambda(\zeta) & p(\zeta) \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} \exp(\lambda(\zeta)) & (\exp \lambda(\zeta) - \mathbf{1})\, q \\ \hline 0 & 1 \end{array}\right). \tag{II.31}$$

From II.31 we deduce the form of a (Λ):

$$a_q(\Lambda) = \Lambda\, q - q\,. \tag{II.32}$$

From II.32 one easily verifies II.24. We conclude: to the isotropy group $\tilde{H}_{x_0}$ of a point $x_0 \in m_t^R$ is associated a four vector $q(x_0)$ such that

$$\tilde{H}_{x^0,\, q(x0)} = \{a_q(\Lambda), \Lambda\}\,.$$

Since no pure translation is in $\tilde{H}_{x_0}$ and since dim $m_t^R = 4$, the translations act, at least locally, transitively on m_t^R. Let us identify a point $x \in m_t$ with a if $T(a, \mathbf{1})$ sends x_0 into x. The isotropy group of x will be

$$\tilde{H}_x = \{(a, \mathbf{1})\,(\Lambda\, q - q, \Lambda)\,(-a, \mathbf{1})\} = \{(\Lambda\,(q - a) - (q - a), \Lambda)\}\,. \tag{II.33}$$

Performing a translation $T\,(a = q, \mathbf{1})$ and calling z_0 the point $T\,(q, \mathbf{1})\, x_0$ the isotropy group of z_0 becomes $H_{z_0} = \{0, \Lambda\}$ i.e. the homogeneous Lorentz group. Identifying m_t^R with P_R/H_{z_0} we have

$$m_t \cong R_4 \;\; x \in R_4\!: \;\; (a, \Lambda)\!: x \to \Lambda\, x + \tau_t\, a \tag{II.34}$$

τ_t scale factor.

The above result can be stated equivalently: a six-dimensional Lie subgroup of $\tilde{P}_R$ which does not contain any pure translation is equivalent to the homogeneous Lorentz group.

The scale factors τ_t can not be chosen arbitrarily because the relative scale of the spaces m_t is given by the requirement that $x_t \in m_t^R$ can be incorporated into the direct sum of four Lie algebras isomorphic to the Lie algebra of SU 3, in which the structure constants fix the scale. The above reasoning holds only for the four-dimensional subspace of $m_3 \oplus m_8$ in which the translations are actually represented ($\tau \neq 0$). This will turn out to be m_8^R.

(β) The generators of the homogeneous part of LH ξ^i, $i = 1, \ldots, 6$ are linearly dependent.

Let (ξ^i, p^i) be a basis in LH. $\exists\, \lambda_i$ such that $\lambda_i\, \xi^i = 0$. Since (ξ^i, p^i) are taken linearly independent

$$\sum_i \lambda_i\, p^i = p^* \neq 0\,.$$

We conclude that in $\tilde{H}$ there is at least one pure translation $(p^*, \mathbf{1})$. We call $\tilde{H}_h$ the group of homogeneous Lorentz transformation Λ such that $(a, \Lambda) \in \tilde{H}$ for some a. The pure translations in LH span a linear vector space denoted by $E_H \subset R_4$. Let $(a, \Lambda) \in \tilde{H}$, $p \in E_H$. Then

$$(a, \Lambda)\,(p, \mathbf{1})\,(a, \Lambda)^{-1} = (\Lambda\, p, \mathbf{1}) \in \tilde{H}\,.$$

This implies that E_H is invariant under $\tilde{H}_h$. We now add another assumption to the previous ones which extends A 6 in a natural way to the spaces m_t^R.

A 9: $\tilde{\tau}_R$ acts transitively on the four dimensional subspace m of $m_3 \oplus m_8$ on which the translation group is nontrivially represented.

A 9 implies $E_H(x_0)$ which depends on $x_0 \in m$ with respect to which the isotropy group $\tilde{H}(x_0)$ is considered, is a proper subspace of R_4 for all $x_0 \in m$.

Proof: Assume that at some point $y \in m$ $E_H(y) = R_4$. This means that $Tr\,(4) \subset \tilde{H}_y$. As a consequence of A 9 we can locally identify a point x of m with $(0, \Lambda)$ modulo $\tilde{H}_y$ if $T(0, \Lambda)$ sends y into x. But $\tilde{H}_x = (0, \Lambda)\, \tilde{H}_y (0, \Lambda^{-1})$. Hence $(0, \Lambda)\; Tr(4)\; (0, \Lambda^{-1}) = Tr(4)$, $Tr(4) \subset \tilde{H}_x$ for all x in a neighbourhood of y. This contradicts our assumptions on the breaking of symmetry through the representation of at least a part of the translation group on m.

Let (a_1, Λ) and (a_2, Λ) be in $\tilde{H}_{x_0}$. Then $(a_1, \Lambda)\;(-\Lambda^{-1} a_2, \Lambda^{-1}) = (a_1 - a_2, \mathbf{1}) \in H_{x_0}$. This implies $a_1 - a_2 \in E_H$. Therefore H is a semi-direct product of a group consisting of elements $(a(\Lambda)/E_H, \Lambda)$ with $\Lambda \in H_h$ and $(E_H, \mathbf{1})$. Hence $\dim \tilde{H} = \dim \tilde{E}_H + \dim \tilde{H}_h = 6$. From the above propositions it follows that $\dim E_H \leqslant 3$, $\dim \tilde{H}_h \leqslant 4$. The possible choices of dimensions are: $\dim E_4 = 3$, $\dim \tilde{H}_h = 3$ and $\dim E_4 = 2$, $\dim \tilde{H}_h = 4$. The second combination can immediately be discarded, because the only four dimensional Lie subgroup of $\tilde{\Lambda}$, $G_4^*(1)$ (up to equivalence) does not leave invariant any two dimensional subspace of R_4. $G_4^*(1)$ is determined by a light like vector λ:

$$G_4^*(1, \lambda) = \{\Lambda \mid \Lambda\,\lambda = c_\Lambda\,\lambda,\; c_\Lambda \text{ arbitrary}\}\,.$$

We are led to consider the three dimensional Lie subgroups of $\tilde{\Lambda}$ and to determine the three dimensional subspaces which are left invariant by these groups.

The analysis is conducted through the use of the covering group of $\tilde{P}$, $Q\,\tilde{P}$, called the quantum mechanical Poincaré group in (12), with elements (a, A), $A \in \mathrm{SL}(2, \mathrm{C})$, and the composition law

$$(a_2, A_2)\;(a_1, A_1) = (a_2 + \Lambda(A_2)\; a_1, A_2\, A_1)\,.$$

$\Lambda(A)$ is the Lorentz transformation associated with A. There are three classes of three dimensional Lie subgroups of SL(2, C) and $\tilde{\Lambda}$ (see appendix):

(1) $G_3^*(4)$: $\{\Lambda \mid \Lambda\, p = p,\, p^2 = 1\}\, p$, time like vector;
(2) $G_3^*(3)$: $\{\Lambda \mid \Lambda\, \mu = \mu,\, \mu^2 = -1\}\, \mu$, space like vector;
(3) $G_3^*(2, \pm\pi)$, $G_3(2, \gamma)$, $G_3^*(2, \gamma, -\mathbf{1})$, $G_3(2, 0)$, $G_3^*(2, 0, -\mathbf{1})$, groups associated with a light vector λ.

The third class is more differentiated than the classes (1) and (2) and will be considered last.

(1) $G_3^*(4)$ is the little group of a time like vector p and is equivalent to the rotation group in three dimensional space.

E_H is determined by p: $E_H = \{\xi \mid p\,\xi = 0\}$. On m the following coordinates can be introduced m: (p, σ), $p^2 = 1$, σ: scalar

$$(a, \Lambda): \quad (p, \sigma) \rightarrow (\Lambda\, p, \sigma + (a, \Lambda\, p))\,. \tag{II.35}$$

II.35 can be compared to II.21. (p, σ) can be mapped on $V_+ + V_-$ ($V_+ + V_-$: the interior of the future and past light cones including the points at infinity).

$$\varrho_1 = \frac{p}{\sigma}; \quad (a, \Lambda): \quad \varrho_1 \rightarrow \frac{\Lambda\, \varrho_1}{1 + (a, \Lambda\, \varrho_1)} \tag{II.36}$$

(2) $G_3^*(3)$ is the little group of a space like vector μ. It is equivalent to the three dimensional Lorentz group. E_H is determined by μ: $E_H = \{\xi \mid \mu\,\xi = 0\}$. The following coordinates can be chosen on m

$$m: \quad (\mu, \sigma), \ \mu^2 = -1, \sigma \text{ scalar} \quad (a, \Lambda): \quad (\mu, \sigma) \rightarrow (\Lambda\, \mu, \sigma + (a, \Lambda\, \mu)) \tag{II.37}$$

(μ, σ) can be mapped on the complement of $V_+ + V_-$, $C\,(V_+ + V_-)$, the light cone excluded, points at infinity included

$$\varrho_2 = \frac{\mu}{\sigma}; \quad (a, \Lambda): \quad \varrho_2 \rightarrow \frac{\Lambda\, \varrho_2}{1 + (a, \Lambda\, \varrho_2)}\,. \tag{II.38}$$

If we take the union of $V_+ + V_-$, $C\,(V_+ + V_-)$, LC (LC: light cone), with the representations given by II.36, 38, 22 respectively, we obtain a space m^* on which $\tilde{\tau}_R$ does not act transitively, but which will prove to bear an important representation of $\tilde{P}_R$

$$\varrho \in m^*, \ (a, \Lambda): \quad \varrho \rightarrow \frac{\Lambda\, \varrho}{1 + (a, \Lambda\, \varrho)} \tag{II.39}$$

(3) a) $G^*(2, \pm\pi)$ is the little group of a light like vector λ. E_H is determined by λ: $E_H = \{\xi \mid \lambda\,\xi = 0\}$. The following coordinates can be chosen

$$m: \quad (\lambda, \sigma); \quad (a, \Lambda) : (\lambda, \sigma) \rightarrow (\Lambda\, \lambda, \sigma + (a, \Lambda\, \lambda))\,. \tag{II.40}$$

The mapping $(\lambda, \sigma) \rightarrow \lambda/\sigma$ is not 1 to 1 (compare with II.21, 22).

b) We will now consider a set of three dimensional Lie subgroups which if they are simultaneously Lie subgroups of $\tilde{\Lambda}$ and $\tilde{S}_4$ are denoted by a symbol *. The groups $G_3(2, \gamma)$ depend on a real parameter γ, $-2\pi \leqslant \gamma \leqslant 2\pi$ and on a light like vector λ. The group considered in (a) will turn out to be a special case of (b) when $\gamma = \pm\pi$, which has to be distinguished from the rest of the groups $G_3(2, \gamma)$. (For a detailed analysis of the Lorentz group see e.g. (12).)

The three dimensional groups associated with a light like vector λ considered in the following as subgroups of SL(2, C) are all subgroups of $G_4^*(1) \subset$ SL(2, C) associated with the same vector λ. Let (λ) denote the 2×2 matrix $\lambda^\mu\,\sigma_\mu$

$$\left\{\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_t: \ \text{Pauli matrices}, \ t = 1, 2, 3\right\}$$

$G_4^*(1, \lambda)$ is given by $\{A \mid A(\lambda)\,A^+ = c_A(\lambda)\}$. A^+ denotes the hermitian conjugate matrix to A. If a coordinate system is chosen such that λ has coordinates (1/2, 0, 0, 1/2) then

$$(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $G^*(1, \lambda)$ is given by the triangular matrices

$$B = \begin{pmatrix} c & a \\ 0 & c^{-1} \end{pmatrix} \quad \text{with} \quad B(\lambda)\, B^+ = |c|^2\, (\lambda)\,.$$

The composition law within $G_4^*(1)$ is given by

$$\begin{pmatrix} c_1 & a_1 \\ 0 & c_1^{-1} \end{pmatrix} \begin{pmatrix} c_2 & a_2 \\ 0 & c_2^{-1} \end{pmatrix} = \begin{pmatrix} c_1\, c_2 & c_1\, a_2 + a_1\, c_2^{-1} \\ 0 & (c_1\, c_2)^{-1} \end{pmatrix}.$$

The following sets of matrices depending on the parameter γ constitute the three dimensional Lie subgroups of $G_4^*(1)$:

$$\begin{matrix} b \ \textit{real} \\ a \ \textit{complex} \end{matrix} \quad G_3(2, \gamma, \lambda) = \left\{ B \mid B = \begin{pmatrix} \exp\,[b\, e^{i\gamma/2}] & a \\ 0 & \exp\,[-\, b\, e^{i\gamma/2}] \end{pmatrix} \right\}. \tag{II.41}$$

A distinction has to be made between the groups containing the matrix $-\mathbf{1} \in \mathrm{SL}(2, \mathrm{C})$, and those which do not. Since the translation a accompanying $B \in \mathrm{SL}(2, \mathrm{C})$ depends only on $\Lambda(B) = \Lambda(-\, B)$, $\tilde{H}_{x_0}$ as a subgroup of $Q\, P_R$ contains the element $(a = 0, -\, \mathbf{1})$ if and only if $(-\, \mathbf{1})$ is an element of $\tilde{H}_h$.

In the representations considered so far (II.22, 34, 36, 38, 39, 40), $\tilde{H}_{x_0}$ contains in each case the element $(a = 0, -\, \mathbf{1})$. Therefore these representations are one-valued representations of $\tilde{P}_R$.

For $\gamma \neq \pm\, \pi$, $G_3(2, \gamma, \lambda)$ does not contain the element $(-\, \mathbf{1})$, the corresponding representation $\tilde{\tau}_R$ of $\tilde{P}_R$ is therefore two-valued.

$\tilde{H}_{x_0}(\gamma)$ is characterized by a vector $\zeta \in C_2$ $(\zeta = (\zeta_1, \zeta_2))$. ζ can be mapped on the future light cone

$$\zeta \to \lambda(\zeta): \quad (\lambda^\mu(\zeta)\, \sigma_\mu)_{ik} = \zeta_i\, \overline{\zeta_k}\,. \tag{II.42}$$

The group $G_3(2, \gamma, \zeta)$ is given by

$$G_3(2, \gamma, \zeta) = \{B \mid B\, \zeta = \exp\,[b\, e^{i\gamma/2}]\, \zeta\}\,. \quad b \ \text{real arbitrary} \tag{II.43}$$

On the following coordinates a representation of $Q\, \tilde{P}_R$ is realized. σ: scalar

$$(\zeta, \sigma)\,; \quad (a, A): \quad (\zeta, \sigma) \to (A\, \zeta, \sigma + (a, \Lambda(A)\, \lambda(\zeta)))\,. \tag{II.44}$$

$\Lambda(A)$ is the Lorentz transformation associated with $A \in \mathrm{SL}(2, \mathrm{C})$. Identifying (ζ, σ) with $\{\exp b\, e^{\,i\gamma/2} \zeta, \sigma_\gamma^*\, (b, \sigma)\}$ where $\sigma_\gamma^*(b, \sigma)$ is a function to be determined, by an identification mapping $i(b, \gamma)$ we have

$$\begin{array}{ccc} (\zeta, \sigma) & \xrightarrow{\ (a,\, A)\ } & A\, \zeta, \sigma + (a, \Lambda(A)\, \lambda(\zeta)) \\ & & \big\downarrow{\scriptstyle i(b,\, \gamma)} \\ {\scriptstyle i(b,\, \gamma)}\big\downarrow & & \{\exp\,[b\, e^{i\gamma/2}]\, A\, \zeta, \sigma_\gamma^*\, (b, \sigma + (a, \Lambda(A)\, \lambda(\zeta)\} \\ \exp\,[b\, e^{i\gamma/2}]\, \zeta, \sigma_\gamma^*(b, \sigma) & \xrightarrow[(a,\, A)]{} & \exp\,[b\, e^{i\gamma/2}]\, A\, \zeta, \sigma_\gamma^*(b, \sigma) \\ & & +\, (a, \Lambda(A)\, \lambda\, \{\exp\,[b\, e^{i\gamma/2}]\, \zeta\}). \end{array} \tag{II.45}$$

Figure 4

Equating the two expressions on the lower right of the above commutative diagram one obtains

$$\sigma_\gamma^* (b, \sigma + [a, \Lambda(A)\, \lambda(\zeta)]) = \sigma_\gamma^* (b, \sigma) + \exp [2\, b \cos \gamma/2]\, (a, \Lambda(A)\, \lambda(\zeta)) . \tag{II.46}$$

If we demand $\sigma_\gamma^* (b, \sigma = 0) = 0$, II.46 reduces to

$$\sigma_\gamma^* (b, q) = \exp [2\, b \cos \gamma/2]\, q ; \quad q = (a, \Lambda(A)\, \lambda(\zeta)) . \tag{II.47}$$

Since q in II.47 is arbitrary $\sigma_\gamma^* (b, q)$ is thereby determined. The mapping $i(b, \gamma)$ identifies the following coordinates

$$m(\gamma) : \quad (\zeta, \sigma) \underset{i(b, \gamma)}{\cong} (\exp [b\, e^{i\gamma/2}]\, \zeta, \exp [2\, b \cos \gamma/2]\, \sigma) \tag{II.48}$$

b real arbitrary.

$$(a, \Lambda) : \quad (\zeta, \sigma)_\gamma \to (A\, \zeta, \sigma + (a, \Lambda(A)\, \lambda(\zeta))_\gamma .$$

The index γ reminds one, that the identification mapping $i(b, \gamma)$ depends on γ as a parameter.

The groups $G_3^* (2, \gamma, -\mathbf{1})$ are obtained from $G_3(2, \gamma)$ by adjoining the matrix $-\mathbf{1}$, they are not Lie subgroups of SL(2, C) but can be mapped by h: $(A, -A) \to \Lambda(A) = \Lambda(-A)$ on corresponding Lie subgroups of $\tilde{\Lambda}$.

Let us resume the four dimensional spaces m and the corresponding representations of P_R or $Q \tilde{P}_R$:

(α) $\quad m \cong R_4,\ x \in R_4,\ (a, \Lambda) : \ x \to \Lambda\, x + \tau\, a \qquad \tau$ scale factor

(β) $\quad m \cong (V_+ + V_-) \cup C\, (V_+ + V_-) \cup L\, C \qquad x \in m,\ (a, \Lambda) : \ x \to \dfrac{\Lambda\, x}{1 + \tau(a, \Lambda\, x)}$

(γ) $\quad m \cong V_+ + V_-,\ x \in V_+ + V_-,\ (a, \Lambda) : \ x \to \dfrac{\Lambda\, x}{1 + \tau(a, \Lambda\, x)}$

(δ) $\quad m \cong C(V_+ + V_-),\ x \in C(V_+ + V_-) \qquad (a, \Lambda) : \ x \to \dfrac{\Lambda\, x}{1 + \tau(a, \Lambda\, x)}$

(ε_γ) $\quad m \cong \{(\zeta, \sigma)_\gamma \underset{i(b, \gamma)}{\cong} (\exp [b\, e^{i\gamma/2}]\, \zeta, \exp [2\, b \cos \gamma/2]\, \sigma)\}$

$$(a, \Lambda) : \quad (\zeta, \sigma) \to (A\, \zeta, \sigma + \tau(a, \Lambda(A)\, \lambda(\zeta)) . \tag{II.49}$$

Proposition: Only the manifolds (α) and (β) satisfy A 8, i.e. the commutability of the actions of $\tilde{\tau}(\tilde{\Lambda})$ and $\tilde{S}$.

Sketch of a proof: We proove that certain collections of charts on the manifolds (γ), (δ), ($\varepsilon\, \gamma$) do not satisfy A 8.

$$(\gamma), (\delta) : \ M = \{z_i^\mu ; \ i = 1, \dots, 8, \mu = 0, \dots 3\} \qquad (\gamma) : \ (z_i)^2 > 0 \qquad (\delta) : \ (z_i^2) < 0 .$$

The above conditions cannot be maintained through the action of $\tilde{S}$.

$$(\varepsilon_\gamma) : \ M = \{(\zeta_k, \sigma_k) \underset{i(b_k, \gamma)}{\cong} \exp [b_k\, e^{i\gamma/2}]\, \zeta_k, \exp [2\, b_k \cos \gamma/2]\, \sigma_k\} .$$

b_k real arbitrary $k = 1, \dots, 8$.

The mapping $i(b_1, \gamma) \otimes \ldots \otimes i(b_8, \gamma) = \underline{i}(\underline{b}, \gamma)$ does not commute with $\tilde{S}$: $\gamma \in S$

$$\begin{array}{ccc}
(\zeta_k, \sigma_k) & \xrightarrow{\quad g \quad} & g_{ke}\,\zeta_{ke}, g_{ke}\,\sigma_e \\
\underline{i}(\underline{b}, \gamma) \downarrow & & \not\downarrow\ \underline{i}(\underline{b}', \gamma) \\
\exp[b_k\, e^{i\gamma/2}]\,\zeta_k, \exp[2\, b_k \cos\gamma/2]\,\sigma_k & \xrightarrow[g]{} & g_{ke} \exp[b_e\, e^{i\gamma/2}]\,\zeta_e, \\
 & & g_{ke} \exp[2\, b_e \cos\gamma/2]\,\sigma_e\,.
\end{array}$$

Figure 5

The above reasoning does not provide a complete proof of the above stated proposition. The possibility that other than the above charts are better suited to satisfy A 8 is not excluded. Nevertheless we will not pursue this question further but rather concentrate on the cases (α) and (β). In both cases $\tilde{\Lambda}$ is represented by the same transformations and the respective manifolds are identical:

$$\begin{aligned}
M((\alpha), (\beta)) &= \{z_{i\,\mu};\ i = 1, \ldots, 8, \mu = 0, \ldots, 3\} \quad z_i \in R_4 \\
(\alpha):\quad (a, \Lambda)&: \{z_i\} \to \{\Lambda\, z_i + \tau_i\, a\} \\
g \in \tilde{K}&: \{z_k\} \to \{g_{ke}\, z_e\} \\
(\beta):\quad (a, \Lambda)&: \{z_i\} \to \left\{\frac{\Lambda\, z_i}{1 + \tau_i(a, \Lambda\, z_i)}\right\} \\
g \in \tilde{K}&: \{z_i\} \to \{g_{ie}\, z_e\}\,. \qquad \text{(II.50)}
\end{aligned}$$

It follows from II.50 that in both cases A 8 is satisfied.

The goal to determine the internal manifold M is hereby reached under the assumptions A 1 … 8 and the auxiliary assumption A 9, which in case (β) does not hold. Contrasting with the assumptions A 1–3, A 4–8 should be regarded as the consequences of general principles which underlie these investigations.

We now change the viewpoint that the action of $\tilde{S}$ and $\tilde{\tau}$ constitute primary notions, and reconsider the internal manifold M in the light of a boundary value problem on the space (x, z), $z \in M$. The solutions of this problem will admit appropriate substitutions which generate the representations of $\tilde{K}$ and $\tilde{P}_R$.

III. Partial Differential Equation, Boundary Conditions, Breaking of Symmetry as a Perturbation

1. *Mesons*

Let $\psi(x, z)$, $z \in M$ be a scalar or pseudoscalar wave function. The discussion of a wave function of this type bears a close ressemblance to a theory of nonlocal fields as described by YUKAWA [13]. It has been shown by FIERZ [14] that the nonlocal field discussed in [13] can be decomposed into irreducible fields, transforming under the POINCARÉ group as free fields of a common mass μ and spin s ($s = 0, 1, \ldots$), which obey the well known local commutation relations of free fields. Wave functions of the above type were also discussed by WIGNER [15] displaying the unusual representations of the POINCARÉ group in the case of mass 0. In [13] and [15] the internal manifold was given by one four vector z restricted by the conditions $p\,z = 0, p^2 = 0$ (p: energy-momentum four vector, $p^2 = 0$).

The following operators are available to construct a differential equation

$$p_\mu = i\,\partial_{x\mu},\ w_\mu^{jk} = 1\,\varepsilon_{\mu\nu\sigma\tau}\,p^\nu\,z^{\sigma/j}\,i\,\partial_z^{\tau/k} \quad D^{jk} = z^{\mu/j}\,i\,\partial_{z\mu}^k,\ D'^{jk} = i\,\partial_{z\mu}^k\,z^{\mu/j} \quad V(z^{\mu/j})\ ;$$

V arbitrary function of $z^{\mu/j}$ alone.

As in [13] and [15] an equation for ψ only has a particle interpretation, if the following subsidiary conditions are satisfied

$$z^{\mu/j}\,\partial_{x\mu}\,\psi = 0; \quad j = 1, \ldots, 8\,. \tag{III.1}$$

Therefore only such operators are admitted which commute with $(z^{\mu/j}\,\partial_{x\mu})$; $j = 1, \ldots, 8$, or when commuted with $(z^{\mu/j}\,\partial_{x\mu})$ give rise to an operator which vanishes if applied to ψ as a consequence of III.1. We will only use the following combinations of the above operators

$$-\Box_x = p^2 = t \quad w_\mu = \sum_j w_\mu^{jj},\quad \hat{\varrho} = w^2 = w_\mu\,w^\mu \quad D' = \sum_j D'^{jj};$$

$$C = f_{ijk}\,D^{jk}\,f_{ij'k'}\,D^{j'k'} \quad V^* = V^*(z^2),\quad z^2 = \sum_k z_\mu^k\,z^{\mu/k} \tag{III.2}$$

f_{ijk} structure constants of SU 3.

A partial differential equation of the following type will be considered

$$\left\{\begin{matrix} f(\hat{\varrho})\,V_1^* + t\,V_2^* + V_3^* + V_4^*\,C + \gamma\,0_8 \\ + \{D'\,V_5^*\}^2 \end{matrix}\right\}\psi = 0 \tag{III.3}$$

$$z^{\mu/j}\,\partial_{x\mu}\,\psi = 0$$

O_8 is an operator which transforms like the eighth component of a member of an SU 3 octet, built from the operators in III.2

$$(\text{e.g. } d_{8jk}\,w_\mu^{jk}\,w^\mu,\ f_{8jk}\,w_\mu^{jk}\,w^\mu,\ d_{8jk}\,z_\mu^j\,z^{\mu/k}\,U(z^2),\ f_{8jk}\,D^{jk},\ \ldots)$$

d_{ijk} are the totally symmetric Clebsch-Gordan coefficients generating the mapping $(\underline{8} \otimes 8)_s \to 8$. (8: octet representation of SU 3, s denotes the symmetrical product space.)

γ reflects the strength of the breaking and should not be confused with the γ labeling the groups $G_3(2, \gamma)$, which do not enter the discussion of III.3.

Remarks: (i) the symmetry breaking operators associated with w_μ^{jk} violate the postulate that the homogeneous Lorentz transformations commute with $\tilde{S}$.

(ii) the symmetry breaking will be treated in the following as a perturbation. Hence mass formulas have to be looked at as first approximations and not as in other approaches as the exact result of algebraic identities [16].

(iii) $D^{jk} = D'^{jk}$ for $j \neq k$ is a selfadjoint differential operator. For $j = k$ special care has to be taken because of the subsidiary conditions III.1.

We perform the following Fourier transformation on ψ:

$$\psi(x, z) = N \cdot \int \exp(-i\,p\,x)\,\phi(p, z)\,d_p^4;$$

N: normalization constant. Equation II.3 becomes

$$\left\{\begin{matrix} (f(\hat{\varrho})V_1^* + t\,V_2^* + V_3^* + V_4^*\,C + \gamma\,O_8) \\ + (D'\,V_5^*)^2 \end{matrix}\right\} \phi = 0 \qquad \text{(III.4)}$$

$$(z^j\,p)\,\phi = 0\,.$$

Let $\Lambda^D(p)$ be defined by

$$e_p = \frac{p}{|p|} \qquad \Lambda^D(p) = \begin{cases} \begin{pmatrix} \frac{p_0}{\sqrt{p^2}} & -\frac{|p|}{\sqrt{p^2}}\,e_{pi} \\ -\frac{|p|}{\sqrt{p^2}}\,e_{pk} & \delta_{ik} - e_{pi}\,e_{pk} + \frac{p_0}{\sqrt{p^2}}\,e_{pi}\,e_{pk} \end{pmatrix} & p^2 > 0,\ p_0 > 0 \\ \Lambda^D(-p) & p^2 > 0,\ p_0 < 0 \\ \begin{pmatrix} \frac{|p|}{\sqrt{-p^2}} & -\frac{p_0}{\sqrt{-p^2}}\,e_{pi} \\ -\frac{p_0}{\sqrt{-p^2}}\,e_{pk} & \delta_{ik} - e_{pi}\,e_{pk} + \frac{|p|}{\sqrt{-p^2}}\,e_{pi}\,e_{pk} \end{pmatrix} & p^2 < 0 \end{cases} \qquad \text{(III.5)}$$

$\Lambda^D(p)$ is not defined on the light cone, $p^2 = 0$. Let $\Lambda(p)$ be defined by

$$\Lambda(p) = R(e_p)\,\Lambda^D(p)\,; \quad R^{-1}(e_p) = R_z(\varphi_{ep})\,R_y(\vartheta_{ep})\,R_z(-\varphi_{ep}) \qquad \text{(III.6)}$$

$R(e_p)$ is the rotation through the axis perpendicular to e_p and e_z (e_z along the positive z-axis) which turns e_p into e_z. For $p^2 > 0$, $\Lambda^{-1}(p)$ is the generalized boost operator used in the construction of helicity amplitudes [17]. For $p^2 \neq 0$ $\Lambda(p)$ has the following properties

$$p^2 > 0,\ p_0 > 0\!: \ \Lambda(p)\,p = (\sqrt{p^2}, 0, 0, 0)$$

$$p^2 > 0,\ p_0 < 0\!: \ \Lambda(p)\,p = (-\sqrt{p^2}, 0, 0, 0)$$

$$p^2 < 0 \quad \Lambda(p)\,p = (0, 0, 0, \sqrt{-p^2})\,. \qquad \text{(III.7)}$$

Consider the coordinate transformation regular everywhere except on the light cone

$$(p, z^k) \to (p, z'^k = \Lambda(p)\,z^k) \qquad \text{(III.8)}$$

The case $p^2 = 0$ will not be discussed here. Even if the unusual representations for zero rest mass or massless particles with definite spin constitute solutions of III.3 they are incorporated without difficulty, the problem in this connection is to exclude solutions for $p^2 < 0$, which if they would appear shatter our hopes to describe a mass spectrum of physical states.

Using the coordinate (p, z'^k) III.4 becomes

$$\left\{\begin{matrix} f[\hat{\varrho}(z')]\ V_1^*(z'^2) + t\,V_2^*)z'^2) + V_3^*(z'^2) \\ + V_4^*(z'^2)\ C(z') + (D'(z')\ V_5^*(z'))^2 + \gamma\,O_8(z') \end{matrix}\right\} \phi = 0 \qquad \text{(III.9)}$$

$$(z'^k\,\Lambda(p)\,p)\,\phi = 0\,.$$

1.1. *The Case* $p^2 = t > 0$

The subsidiary conditions are $z_0'^j \phi = 0$. We take the following form for ϕ: $\phi = \prod_k \delta(z_0'^k)\, \varphi(p, \mathbf{z}'^j)$. The δ functions imply a redefinition of the operators

$$w^{jk}(p, z^j, i\, \partial_z^k),\ D'^{jk}(z^j, i\, \partial_z^k), \ldots :$$

$$w^{jk}(p, z^j, i\, \partial_z^k) = \Lambda^{-1}(p)\, w_k^j\, (\Lambda(p)\, p, z'^j, i\, \partial_{z'}^k)$$

$$w^{jk}(\Lambda(p)\, p, z'^j, i_{z'}^k, \partial) = \varepsilon(p_0) \sqrt{t}\, (0, \tau^{jk})$$

$$\varepsilon(x) = \begin{Bmatrix} +1 & x > 0 \\ -1 & x < 0 \end{Bmatrix}, \quad \tau^{jk} = \mathbf{z}'^j \wedge \frac{1}{i}\, \boldsymbol{\nabla}'^k$$

$$D'^{jk}\,(z^j, i\, \partial_z^k) = D'^{jk}\,(z'^j, i\, \partial_j'^k) = i\, \boldsymbol{\nabla}'^k\, \mathbf{z}'^j\,. \tag{III.10}$$

In order to display the self adjoint character of the operators in question, especially of $D'\, V_5^*$, let us consider a scalar product for the wave functions φ_1, φ_2, p being fixed

$$(\varphi_1 \varphi_2)_p = \int d^3z_1 \ldots d^3z_8\, \overline{\varphi}_1(p, z_1 \ldots z_8)\, \varphi_2(p, z_1 \ldots z_8)\,. \tag{III.11}$$

The adjoint of D'^{kk} is

$$(D'^{kk})^+ = (i\, \boldsymbol{\nabla}^k\, \mathbf{z}^k)^+ = D^{kk} = D'^{kk} - i\, 3\, 1 \tag{III.12}$$

and

$$(D'\, V_5^*)^+ = V_5^*\, D = D'\, V_5^* = D'\, V_5^* - i\, [r\, (\partial_r\, V_5^*) + 24\, V_5^*]\,.$$

Demanding $(D'\, V_5^*)^+ = D'\, V_5^*$ one obtains

$$r\, [\partial_r\, V_5^*(r)] + 24\, V_5^* = 0;\ V_5^* = A\, r^{-24}\,.$$

We will put the normalization constant $A = 1$. III.10 becomes

$$u = f(-t\, \mathbf{L}^2)\, V_1^* + t\, V_2^* + V_3^* + V_4^*\, C + \gamma\, O_8$$

$$\left\{ r^{-46} \left[\left(\frac{\partial}{\partial r}\right)^2 - 23\, \frac{1}{r}\, \frac{\partial}{\partial r} \right] - u \right\} \varphi = 0\,. \tag{III.13}$$

Substituting $u_1 = r^{46}\, u$ we have

$$\left[\left(\left(\frac{\partial}{\partial r}\right)^2 - 23\, \frac{1}{r}\, \frac{\partial}{\partial r} \right) - u_1 \right] \varphi = 0$$

and replacing φ by $r^b\, \chi$ gives for $b = 23/2$

$$\left(\left(\frac{\partial}{\partial r}\right)^2 - u_2 \right) \chi = 0 \qquad u_2 = u_1 + \frac{c}{r^2} = r^{46}\, u + \frac{c}{r^2}$$

$$c = 6 \cdot 24 - 1/4$$

$$u_2 = V_1(r)\, f(-t\, \mathbf{L}^2) + t\, V_2 + V_3 + V_4\, C + \gamma\, \tilde{O}_8$$

$$V_{1,2,4}(r) = r^{46}\, V_{1,2,4}^*(r)\,, \qquad V_3(r) = r^{46}\, V_3^*(r) + \frac{c}{r^2}\,, \qquad \tilde{O}_8 = r^{46}\, O_8\,. \tag{III.14}$$

Before analyzing III.14 further we look into the situation for $t < 0$.

1.2. *The Case* $p^2 = t < 0$

The subsidiary conditions are $z_3'^k \Phi = 0$. We choose the following form for ϕ: $\phi = \prod_n \delta(z_3'^n) \varphi$.

The important modification compared to 1.1 concerns the operators

$$w^{jk}(p, z^j, \partial_z^k) = \Lambda^{-1}(p)\, w^{jk}(\Lambda(p)\, p, z'^j, \partial_z'^k),$$

$$w(\Lambda(p)\, p, z'^j, \partial_z'^k) = \sqrt{-t}\,(L_3^{jk}, N_2^{jk}, -N_1^{jk}, 0) \tag{III.15}$$

$$M^{\mu\nu/jk} = z^{\mu/j}\, i\, \partial_z^{\nu/k} - z^{\nu/j}\, i\, \partial_z^{\mu/k} = \begin{pmatrix} 0 & N_1^{jk} & N_2^{jk} & N_3^{jk} \\ -N_1^{jk} & 0 & L_3^{jk} & -L_2^{jk} \\ -N_2^{jk} & -L_3^{jk} & 0 & L_1^{jk} \\ -N_3^{jk} & +L_2^{jk} & -L_1^{jk} & 0 \end{pmatrix}.$$

The operators $L_3 = \sum_j L_3^{jj}$, $N_{1,2} = \sum_j N_{1,2}^{jj}$ generate a three dimensional Lorentz group $\tilde{\Lambda}_3$.

$$[L_3, N_1] = i\, N_2 \quad [L_3, N_2] = -i\, N_1 \quad [N_1, N_2] = -i\, L_3$$

$$\hat{\varrho} = w^2 = (-t)\, \hat{\nu}^2,\ \hat{\nu}^2 = L_3^2 - N_1^2 - N_2^2\,. \tag{III.16}$$

Since the argument $z'^2 = \sum_j (z'^j)^2$ of $V_i^*(z'^2)$ in 1.1 is negative we are as yet free to continue these functions for positive values of their argument in an arbitrary fashion. For $(z')^2$ positive let $V_i^*\{(z')^2\} = \infty$, thereby excluding the possibility that the wave function ϕ penetrates to these values of z'^j.

$$r = \sqrt{-\sum_j (z'^j)^2}$$

is again defined. The rest of the analysis can be taken over from 1.1 which leads to the potential

$$u_2 = f(-t\,\hat{\nu}^2)\, V_1(r) + t\, V_2(r) + V_3(r) + V_4(r)\, C + \gamma\, \tilde{O}_8 \qquad t < 0$$

$V_i(r)$ have the same form as in III.14.

The following two equations arise in the cases $t > 0$ and $t < 0$ respectively:

$$\left\{\left(\frac{\partial}{\partial r}\right)^2 - [f(-t\, L^2)\, V_1 + t\, V_2 + V_3 + V_4\, C + \gamma\, \tilde{O}_8]\right\} \varphi = 0\,, \quad t > 0$$

$$\left\{\left(\frac{\partial}{\partial r}\right)^2 - [f(-t\,\hat{\nu}^2)\, V_1 + t\, V_2 + V_3 + V_4\, C + \gamma\, \tilde{O}_8]\right\} \varphi = 0 \quad t < 0\,. \tag{III.17}$$

We now neglect $\gamma\, \tilde{O}_8$ which allows to separate variables in both cases

$$\varphi_+ = F_+(r, l, q)\, Y_+\left(\frac{z_k}{r}, l, q\right), \qquad \varphi_- = F_-(r, v, q)\, Y_-\left(\frac{z_k}{r}, v, q\right)$$

$l\,(l+1)$, $\nu\,(\nu+1)$, q^2 denote the eigenvalues of the operators L^2, $\hat{\nu}^2$, C respectively. Since C is a positive operator, its eigenvalues are non-negative. C as defined in III.2

is one of the two Casimir operators of SU 3, namely F^2 in the notation of DE SWART [18]. Denoting by G^3 the other Casimir operator, C can be generalized

$$C' = \chi\,(\alpha\, F^2 + \beta\, G^3)\,. \tag{III.18}$$

χ is an arbitrary function. In order to maintain the positivity of C' the simplest choice for χ is

$$\chi(x) \propto x^2\,.$$

If the possibility of unequal masses for conjugate representations of SU 3 (e.g. the 10 and $\overline{10}$ representations) is to be included in the equations corresponding to III.17 for baryons, β must be chosen different from 0.

$Y_\pm$ are generalized spherical functions depending on polar coordinates $\vartheta_{\pm 1}, \ldots$ $\ldots, \vartheta_{\pm h-1}$; $n = \dim(z_1, \ldots, z_8) = 24$.

ϑ_{+i} are coordinates on a compact manifold, ϑ_{-i} on a non-compact manifold. The detailed structure of the functions $Y_\pm$ is of no relevance to the further analysis and will not be considered here.

The radial functions F satisfy the equations

$$\left[\left(\frac{d}{dr}\right)^2 - \{f[-t\,l\,(l+1)]\,V_1 + t\,V_2 + V_3 + V_4\,q^2\}\right] F_+(r) = 0$$

$$\left[\left(\frac{d}{dr}\right)^2 - \{f[-t\,v\,(v+1)]\,V_1 + t\,V_2 + V_3 + V_4\,q^2\}\right] F_-(r) = 0\,. \tag{III.19}$$

The spectrum of L^2 is a discrete spectrum with non-negative eigenvalues $l\,(l+1)$, $l = 0, 1, 2, \ldots$. The spectrum of $\hat{v}^2$ contains a continuous and a discrete part. This spectrum has been studied on special manifolds in the context of general non-compact rotation groups of arbitrary dimension by RACZKA, LIMIC and NIEDERLE [19]. To determine the spectrum of $\hat{v}^2$ let us consider the following coordinates in the space $(z_1, \ldots, z_8)$ in analogy with a similar construction in the case of the rotation group.

Let (z_1, z_2) be linearly independent and let E_{12} denote the plane spanned by the two vectors. E_{12} contains

(a) space like and time like vectors and two linearly independent light like vectors,
(b) only space like vectors,
(c) space like vectors and one light like vector.

Excluding the singular case (c) is no loss of generality, since we excluded already the situation where (z_1, z_2) are linearly dependent. (a) Let us first exclude further the situation where both z_1 and z_2 are light like, e.g. $z_1^2 \neq 0$. Let us denote by z_2' a vector in E_{12}, orthogonal to z_1. If $z_1^2 > 0$, $z_2'^2 < 0$ and vice versa. We normalize z_1, z_2' to hyperbolic length 1 and -1 corresponding to $z_1^2 \gtrless 0$, $z'^2 \lessgtr 0$.

Let z_1^* denote the vector out of the pair (z_1, z_2) with $z^2 > 0$, and z_2^* the other one. We call e_3 the unit vector along z_1^*, e_1 the unit vector along z_2^* with lengths ± 1 respectively. From z_1, z_2 one can construct $z'_{3\mu} = \varepsilon_{\mu\nu\sigma}\, z_1^\nu\, z_2^\sigma$. z_3' lies along e_2 defined by $e_{2\mu} = \varepsilon_{\mu\nu\sigma}\, e_3^\nu\, e_1^\sigma$ with $e_2^2 = -1$. We have thus constructed a hyperbolic vector triplet $e_1(z_1, z_2)$, $e_2(z_1, z_2)$, $e_3(z_1, z_2)$, $\varepsilon_{\mu\nu\sigma}$: totally antisymmetric tensor. It can be reached from the standard vector triplet (e_x, e_y, e_z) along the x, y, z axes respectively, through a Lorentz transformation which is thereby uniquely determined

$$\Lambda_3 \in \tilde{\Lambda}_3,\ \Lambda_3(\varphi, \chi, \psi) = R_z(\varphi)\; S_y(\chi)\; R_z(\psi)$$

$R_z(\varphi)$ denotes a rotation around the z-axis through the angle φ, $S_y(\chi)$ a pure Lorentz transformation in the y-direction through the hyperbolic angle χ. The coordinates (φ, χ, ψ) are, when interpreted with the proper caution, analogs of the Euler angles defining the general position of a Cartesian vector triplet. (b) $z_1^2 < 0$, $z_2^2 < 0$, $z_\mu'^3 = \varepsilon_{\mu\nu\sigma} z_1^\nu z_2^\sigma$. We can again find an orthogonal vector to z_1, z_2' in E_{12}, and orient a hyperbolic vector triplet (e_1, e_2, e_3) along the vectors z_1, z_2', z_3', which is mapped on the coordinates (φ, χ, ψ). The manifold described by the coordinates (φ, χ, ψ) can be identified with the group manifold of $\tilde{\Lambda}_3$. Locally the following coordinates can be introduced on $(z_1, \ldots, z_8)$:

$$\left(\sum(z_1, z_2), z_1^2, z_1 z_2, z_2^2, z_t\left(\sum\right), t = 3, 4, \ldots, 8\right).$$

$\Sigma(z_1, z_2)$ denotes the hyperbolic triplet as defined above by (z_1, z_2). $z_t(\Sigma)$ denote the coordinates of z_t with respect to $\Sigma(z_1, z_2)$. Let us call I the set of invariant coordinates z_1^2, z_2^2, $z_1 z_2$, $z_t(\Sigma)$, $t = 3, 4, \ldots, 8$. Locally a mapping μ exists which maps $(z_1, \ldots, z_8)$ on the space (Σ, I). The action of $\tilde{\Lambda}_3$ on $(z_1, \ldots, z_8)$ or (Σ, I) can be represented by the following diagram

$$\begin{array}{ccc} (z_1, \ldots, z_8) & \xrightarrow{\Lambda_3} & (\Lambda_3 z_1, \ldots, \Lambda_3 z_8) \\ \mu \downarrow & & \downarrow \mu \\ \left(\sum, I\right) & \xrightarrow[\Lambda_3]{} & \left(\Lambda_3 \sum, I\right) \end{array} \qquad \text{(III.20)}$$

Figure 6

$\Sigma \to \Lambda_3 \Sigma$ are the left translations induced by the mapping

$$\Lambda_3\colon \Lambda(\varphi, \chi, \psi) \to \Lambda_3 \Lambda(\varphi, \chi, \psi) \qquad \text{(III.21)}$$

An infinitesimal Lorentz transformation $\Lambda(\alpha_1, \alpha_2, \alpha_3) = \mathbf{1} - i\,[\alpha_1 N_1 + \alpha_2 N_2 + \alpha_3 L_3]$ induces the following coordinate transformation on Σ:

$$\begin{aligned} \delta\varphi(\varphi, \chi, \psi) &= \alpha_3 - \coth\chi\,(\cos\varphi\,\alpha_1 + \sin\varphi\,\alpha_2) \\ \delta\chi(\varphi, \chi, \psi) &= -\sin\varphi\,\alpha_1 + \cos\varphi\,\alpha_2 \\ \delta\psi(\varphi, \chi, \psi) &= \frac{1}{\operatorname{sh}\chi}\,(\cos\varphi\,\alpha_1 + \sin\varphi\,\alpha_2)\,. \end{aligned} \qquad \text{(III.22)}$$

The generators of $\tilde{\Lambda}_3$ are mapped on the following differential operators on Σ:

$$\begin{aligned} L_3 &\to \frac{1}{i}\,\partial_\varphi \\ N_1 &\to \frac{1}{i}\left[-\coth\chi\cos\varphi\,\partial_\varphi - \sin\varphi\,\partial_x + \frac{1}{\operatorname{sh}\chi}\cos\varphi\,\partial_\varphi\right] \\ N_2 &\to \frac{1}{i}\left[-\coth\chi\sin\varphi\,\partial_\varphi + \cos\varphi\,\partial_x + \frac{1}{\operatorname{sh}\chi}\sin\varphi\,\partial_4\right] \end{aligned} \qquad \text{(III.23)}$$

$\Delta = L_3^2 - N_1^2 - N_2^2$ is the Laplace-Beltrami operator on Σ the metric tensor given by

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\mathrm{sh}^2 \chi}, & -\frac{\mathrm{ch}\,\chi}{\mathrm{sh}^2 \chi} \\ 0 & -\frac{\mathrm{ch}\,\chi}{\mathrm{sh}^2 \chi} & \frac{1}{\mathrm{sh}^2 \chi} \end{pmatrix} \begin{matrix} \chi \to i,\, k = 1 \\ \varphi \to i,\, k = 2 \\ \psi \to i,\, k = 3\,. \end{matrix}$$

The invariant volume element on $\tilde{\Lambda}_3$ or the Haar measure is therefore given by $d\mu = \sqrt{|g|}\, d\varphi\, d\chi\, d\psi = d\varphi\, |\mathrm{sh}\,\chi|\, d\chi\, d\psi$

$$\Delta = \frac{1}{\sqrt{|g|}}\, \partial_i \sqrt{|g|}\, g^{ij}\, \partial_j = \partial_\chi^2 + \coth\chi\, \partial_\chi + \frac{1}{\mathrm{sh}^2 \chi}\, [\partial_\varphi^2 + \partial_\psi^2 - 2\, \mathrm{ch}\,\chi\, \partial_\varphi\, \partial_\psi]\,. \quad \text{(III.24)}$$

The eigenvalues of $\Delta^* = L_3^2 - N_1^2 - N_2^2$ on $(z_1, \ldots, z_8)$ are therefore reduced on (Σ, I) to the eigenvalues of Δ on Σ because Δ^* has the form $\Delta^* = \Delta \otimes \mathbf{1}$ on $(\Sigma \otimes I)$. We are led to consider the following eigenvalue equation on Σ:

$$\Delta \varrho^\nu(\varphi, \chi, \psi) = \sigma(\nu)\, \varrho^\nu(\varphi, \chi, \psi), \qquad \sigma(\nu) = \nu\, (\nu + 1)\,. \quad \text{(III.25)}$$

The periodicity requirements in the compact variables φ, ψ which are global requirements will be determined from the general analysis of the unitary irreducible representations of the groups $\tilde{\Lambda}_3$ and $\tilde{S}_3$ ($\tilde{S}_3$: covering group of Λ_3). We take over the results of the complete analysis of the above groups by BARGMANN [20]. Following the ideas of this work, $\varrho^\nu(\varphi, \chi, \psi)$ can be realized as a matrix element $(u_2, U^{(\nu)}(\Lambda[\varphi, \chi, \psi])\, u_1)$ where $U^{(\nu)}(\Lambda[\varphi, \chi, \psi])$ is the unitary transformation which represents $\Lambda[\varphi, \chi, \psi]$. $u_{1,2}$ are elements of a suitably chosen Hilbert space $\mathfrak{H}^{(\nu)}$. ϱ^ν can be decomposed using the basis of eigenstates of L_3, u_m in $\mathfrak{H}^\nu$ with $L_3\, u_m = m\, u_m$

$$\varrho^\nu_{m\, m'} = (u_m\,,\, u^{(\nu)}(\varphi, \chi, \psi)\, u_{m'}) = e^{-im\varphi}\, d^\nu_{m\, m'}(\chi)\, e^{-im'\psi} \quad \text{(III.26)}$$

If we demand that the functions $\varrho^\nu_{m\, m'}$ $(\int d\, \nu\, f\, (\nu)\, \varrho^\nu_{m\, m'})$ are square integrable with respect to the left and right invariant measure

$$d\mu(\varphi\, \chi\, \psi) = |\mathrm{sh}\,\chi|\, d\chi\, d\varphi\, d\psi$$

the following is the complete list of unitary irreducible representations labelled by the parameter ν:

(1) C_ν^0: $\{m = 0, \pm 1, \pm 2, \ldots;\ \nu = -1/2 + i\, \alpha,\ \alpha$ real arbitrary$\}$
$\Delta \varrho^\nu_{m\, m'} = \sigma(\nu)\, \varrho^\nu_{m\, m'}, -\infty < \sigma(\nu) \leq -1/4$

(2) $C_\nu^{1/2}$: $\{m = \pm 1/2, \pm 3/2, \ldots, \nu = -1/2 + i\, \alpha,\ \alpha$ real arbitrary
$-\infty < \sigma(\nu) \leq -1/4$

(3) D_ν^+: $\{m = \nu + 1, \nu + 2, \ldots;\ \nu = 0, 1/2, 1, 3/2, \ldots\}$ $\sigma(\nu) \geqslant 0$

(4) D_ν^-: $\{m = -(\nu + 1), -(\nu + 2), \ldots;\ \nu = 0, 1/2, 1, 3/2 \ldots\}$ $\sigma(\nu) \geqq 0$ (III.28)

(1) C^ν gives rise to functions $\varrho^\nu_{m\, m'}(1)$, m, m' integers with $N_\pm = N_1 \pm i\, N_2$, $(N_\pm)^a\, \varrho^\nu_{mm'}(1) \neq 0$ for $a = 0, 1, 2, \ldots$.

(2) $C_\nu^{1/2}$ gives rise to functions $\varrho^\nu_{m\, m'}(2)$, m, m' half-integers, with $(N_\pm)^a\, \varrho^\nu_{m\, m'}(2) \neq 0$ for $a = 0, 1, 2, \ldots$.

(3) D_ν^+ gives rise to functions $\varrho^\nu_{m\,m'}(3)$, with $N_- \, \varrho^\nu_{\nu+1,m'} = 0$.
(4) D_ν^- gives rise to functions $\varrho^\nu_{m\,m'}(4)$, with $N_+ \, \varrho^\nu_{-(\nu+1),m'} = 0$.

Integer m, m' correspond to one-valued, half integer m, m' to two-valued representations of $\tilde{\Lambda}_3$. On our manifold $(z_1, \ldots, z_8)$ we can exclude the two-valued representations. Collecting all possible values for ν and $\sigma(\nu)$ occurring in the four cases above, we obtain

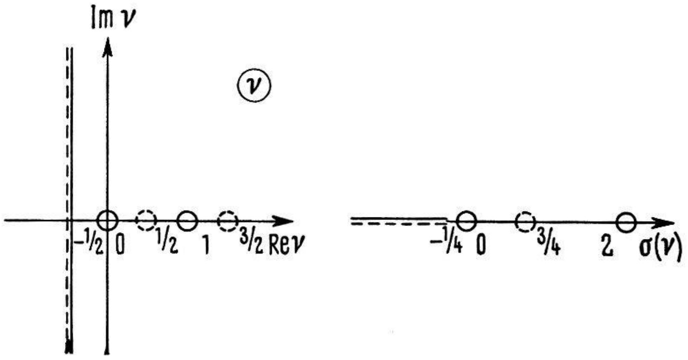

Figure 7

○ discrete values of ν, $\sigma(\nu)$ ⎫
— continuous values of ν, $\sigma(\nu)$ ⎭ from one-valued representations of $\tilde{\Lambda}_3$;

◌ discrete values of ν, $\sigma(\nu)$ ⎫
······ continuous values of ν, $\sigma(\nu)$ ⎭ from two-valued representations of $\tilde{\Lambda}_3$.

We now return to III.19. Let $\tilde{\varrho}/t$ be the eigenvalue of L^2 for $t > 0$ and of $\hat{\nu}^2$ for $t < 0$

$$\tilde{\varrho} = \begin{cases} l\,(l+1)\,t & l = 0, 1, \ldots, t > 0 \\ \nu\,(\nu+1)\,t \begin{cases} \nu = -1/2 + i\,\alpha, & -\infty < \alpha < +\infty \\ \nu = k, & k = 0, 1, 2, \ldots \end{cases} & t < 0 . \end{cases} \qquad \text{(III.28)}$$

We rewrite III.19 in a different form, redefining the potentials V_i

$$\left\{\left(\frac{d}{dr}\right)^2 - [(\tilde{\varrho} + a)^2\, \tilde{V}_1 + t\, \tilde{V}_2 + V_3 + \tilde{\varrho}\, V_4]\right\} F_\pm = 0 \,.$$

$$a > 0 \qquad \text{(III.29)}$$

The difference between $t > 0$ and $t < 0$ lies in the different ranges for $\tilde{\varrho}$ as given by III.28 and in the change of sign of the term $t\, V_2$. Let us assume $V_1 = W_1^2$, $W_1 > 0$ for $0 \leqslant r < \infty$ and let ω denote $(\tilde{\varrho} + a)^2$. Then the following substitutions simplify III.29

$$\int_0^r W_1(r')\, dr' = W_2(r), \quad r_2 = W_2(r), \quad \frac{d}{dr} = W_1 \frac{d}{dr_2} \qquad F_\pm = \psi = \frac{1}{(W_1)^{1/2}}\, \chi$$

$$\left\{\left(\frac{d}{dr_2}\right)^2 - [\omega + t\, V_2' + V_3' + \varrho\, V_4']\right\} \chi = 0$$

$$V_2' = \frac{V_2}{V_1} \qquad V_3' = \frac{V_3}{V_1} + \frac{1}{(W_1)^{1/2}} \left(\frac{d}{dr^2}\right)^2 (W_1)^{1/2} \,. \qquad \text{(III.30)}$$

In order to gain further insight let us consider a solvable example taking

$$V_2' = -\frac{\mu}{r} \quad V_3' = \frac{b}{r^2} + \frac{\nu}{r} \quad V_4' = \frac{c}{r^2}\,.$$
$$\mu, \nu, l, c > 0, \quad r_2 \rightarrow r\,.$$

With this choice of potentials III.30 describes the quantum mechanical Kepler problem (the index 2 of r_2 has been dropped)

$$\left\{\left(\frac{d}{dr^2}\right)^2 - \left[\omega - \frac{\mu\, t - \nu}{r} + \frac{b + \tilde{\varrho}\, c}{r^2}\right]\right\} \chi = 0\,. \tag{III.31}$$

For $t < \nu/\mu$ the Coulomb potential is repulsive so for $t < \nu/\mu$ the solutions correspond to the continuous spectrum with positive energy. In III.31 the energy is given by $E = -\omega$ and is always negative. Therefore there are no solutions of III.31 for $t < \nu/\mu$. The Hamilton function for the above problem in two dimensional phase space is given by

$$H(p, r) = p^2 - \frac{\mu\, t - \nu}{r} + \frac{b + \tilde{\varrho}\, c}{r^2}\,.$$

The substitutions $\omega = \varkappa^2$, $z = 2\,\varkappa\, r$, $\lambda\,(\lambda + 1) = b + \tilde{\varrho}\, c$, $\chi = e^{-z/2}\, z^{\lambda+1} f_\lambda(z)$ lead to the equation for $f_\lambda(z)$:

$$z\left(\frac{d}{dz}\right)^2 f_\lambda(z) + (2\,\lambda + 2 - z)\, f_\lambda(z) - \left(\lambda + 1 - \frac{\mu\, t - \nu}{2\,\varkappa}\right) f_\lambda(z) = 0\,. \tag{III.32}$$

The regular solution of III.32 is given by the confluent hypergeometric function

$$f_\lambda(z) = C f\left(\lambda + 1 - \frac{\mu\, t - \nu}{2\,\varkappa}, \quad 2\,\lambda + 2, z\right). \tag{III.33}$$

The condition establishing the normalizability of χ reads

$$-n_r = \lambda + 1 - \frac{\mu\, t - \nu}{2\,\varkappa} \; n_r = 0, 1, 2, \ldots \tag{III.34}$$

n_r is the number of zeros of $\chi(n_r, \lambda, z)$, $f(-n_r, z\,\lambda + 2, z)$ being a polynomial of degree n_r with only real zeros. III.34 implies

$$\left(\frac{\mu\, t - \nu}{2}\right)^2 = (\tilde{\varrho} + a)^2\, [n_r + 1/2 + (b + 1/4 + \tilde{\varrho}\, c)^{1/2}]^2\,.$$
$$\mu\, t > \nu \tag{III.35}$$

For $\tilde{\varrho} \gg 1$ and $c \neq 0$ the right hand side of III.35 behaves like $c\,\tilde{\varrho}^3$ and III.35 can be asymptotically for $\tilde{\varrho} \rightarrow \infty$ replaced by

$$\left(\frac{\mu\, t - \nu}{2}\right)^2 \cong c\,\tilde{\varrho}^3 \text{ or } \left(\frac{\mu\, t}{2}\right)^2 \cong c\, t^3\, l^3\, (l + 1)^3$$
$$\varrho, t \rightarrow \infty\,.$$

This implies

$$t \cong \left(\frac{\mu}{2}\right)^2 \frac{1}{c}\, \frac{1}{l^3}\, \frac{1}{(l+1)^3}\,.$$

Therefore if $c \neq 0$ l goes to zero for $t \to \infty$, which is not a reasonable spectrum. Hence we demand $c = 0$.

$$\left(\frac{\mu t - \nu}{2}\right)^2 = (\varrho + a)^2 (n_r + 1 + d)^2 ; \quad d + 1/2 = (b + 1/4)^{1/2}$$
$$\mu t > \nu \qquad\qquad d > 0$$

and substituting $\alpha(n_r) = n_r + 1 + d$ we obtain

$$t = \frac{a + \nu/2\,\alpha(n_r)}{\mu/2\,\alpha(n_r) - l(l+1)} \qquad l(l+1) = \frac{\mu}{2\,\alpha(n_r)} - \frac{a + (\nu/2\,\alpha(n_r))}{t}$$
$$t > \nu/\mu . \tag{III.36}$$

Remark: If we assume $l(t)$ to be an analytic function of t except for poles and cuts, then III.36 is unsatisfactory, for $l(t = -0)$ necessarily is divergent, which from other considerations is impossible. Since however III.36 defines $l(t)$ only for $t > \nu/\mu$, nothing can be said in this framework about the continuation of $l(t)$ to other values of t.

Discussion of III.36

Let $t_0(n_r)$ denote the value of t, for which $l(l+1)[n_r] = 0$, $t_0(n_r) = \nu/\mu + 2\,\alpha(n_r)\,a/\mu$; $t_0(n_r)$ is always bigger than ν/μ and increases linearly with increasing n_r. $l(l+1)[n_r]$ approaches asymptotically for $t \to \infty$ the value $\mu/2\,\alpha(n_r)$.
The pole of $l(t)$ at $t = 0$ can be avoided, looking at another example:

$$V_2' = -\mu \frac{c}{r^{2-\alpha}} \qquad V_3' = \nu \frac{c}{r^{2-\alpha}} + \frac{b}{r^2} \qquad \mu, \nu, b, c > 0; \quad 0 < \alpha < 1$$

$$\left\{\left(\frac{d}{dr}\right)^2 - \left[\omega - (\mu t - \nu)\frac{c}{r^{2-\alpha}} + \frac{b}{r^2}\right]\right\}\chi = 0 . \tag{III.37}$$

The substitution $z = f(t)\, r$ leads to

$$\left\{\left(\frac{d}{dz}\right)^2 - \left[\frac{\omega}{f^2(t)} - \frac{\mu t - \nu}{f^\alpha(t)}\frac{c}{z^{2-\alpha}} + \frac{b}{z^2}\right]\right\}\chi = 0 .$$

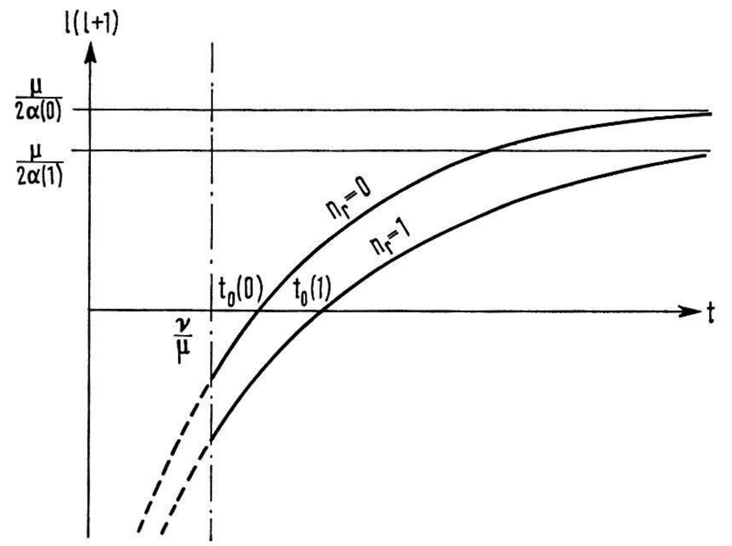

Figure 8

Choosing $f = (\mu\, t - \nu)^{1/\alpha}$ we obtain

$$\left\{\left(\frac{d}{dz}\right)^2 - \left[\frac{\omega}{(\mu\, t - \nu)^{2/\alpha}} - \frac{c}{z^{2-\alpha}} + \frac{b}{z^2}\right]\right\} \chi = 0\,. \tag{III.38}$$

The ansatz $\omega(t) = D^2\, (\mu\, t - \nu)^{2/\alpha}$ gives an equation independant of t:

$$\left\{\left(\frac{d}{dz}\right)^2 - \left[D^2 - \frac{c}{z^{2-\alpha}} + \frac{b}{z^2}\right]\right\} \chi = 0\,. \tag{III.39}$$

If $D, c, b > 0$ are chosen appropriately, III.39 has a bound solution. The spectrum is determined by

$$\tilde{\varrho} = a = D\,(\mu\, t - \nu)^{1/\alpha}; \quad t > \nu/\mu \tag{III.40}$$

If $1/\alpha = 2\,N$, N integer, III.40 leads to

$$l\,(l+1) = \frac{D}{t}\,(\mu\, t - \nu)^{2N} - \frac{a}{t}\,. \tag{III.41}$$

The pole for $l\,(l+1)$ at $t = 0$ is avoided choosing $D\,\nu^{2N} = a$. This restricts the values of $l(0)$ since

$$(l(0) + 1/2)^2 = 1/4 - \frac{2\,N\,\mu\,a}{\nu}\,; \quad l(0) = -\,1/2 \pm \sqrt{\frac{1}{4} - \frac{2\,N\,\mu\,a}{\nu}}\,.$$

Remarks: (i) Because b in III.37 is independent of l the solution $F_+(r)\; Y_+(l, q, z_i/r)$ will be irregular at $r = 0$ for $l > l^*$, $l^*\,(l^* + 1) = b$.

(ii) The fact that in the two examples considered one does not find a wide class of mass spectra is related to the interpolation of $l(t)$ for noninteger l, which is real in both cases.

The General Problem

Given a first function $R(\hat{\varrho}, t)$ modulo a second function $T\,(l = (\hat{\varrho}/t + 1/4)^{1/2} - 1/2)$, ϱ, $t > 0$ such that

$$T\,(l) = 0 \quad \text{for} \quad l = 0, 1, 2, \ldots$$

Determine the class of equivalent potentials $V(r)$ together with constants A, B, C, D such that the regular, square integrable solution of the equation

$$0 = \left\{\left(\frac{d}{dr}\right)^2 - \left[(\tilde{\varrho} + a)^2 + t\,V(r) + \frac{A\,\tilde{\varrho}^2 + B\,\tilde{\varrho} + C\,t + D}{r^2}\right]\right\} \chi \tag{III.42}$$

gives rise to a mass spectrum in the form of the relation

$$R(\tilde{\varrho}, t) + T(\tilde{\varrho}, t) = 0\,. \tag{III.43}$$

If $A = B = C = 0$, $D = \lambda\,(\lambda + 1)$, $R + T$ is the Jost function $f_\lambda(k, t)$[4]), where k is taken on the physical sheet, $k = i\,(\tilde{\varrho} + a)$. f_λ depends on t which plays the role of coupling strength and is as a consequence of a general theorem by POINCARÉ [21] an analytic function of t for fixed k for a wide class of potentials and certainly if:

$$\operatorname{Im} k \geqslant 0 \quad \text{and} \quad \int_0^\infty dr\, |V(r)|\, \frac{r}{1 + |k|\, r}$$

[4]) The general analysis of the S-matrix in terms of appropriate solutions of the Schrödinger equation is due to JOST [22].

exists. In addition it is required that $(A\,\tilde{\varrho}^2 + B\,\tilde{\varrho} + c\,t)\,[l]$ increases for $l \to \infty$ at least as fast as $l\,(l+1)$ and that for $t < 0$ the operator

$$\left\{\left(\frac{d}{dr}\right)^2 + t\,V(r) + \frac{A\,\tilde{\varrho}^2 + B\,\tilde{\varrho} + C\,t + D}{r^2}\right\}$$

in $L_2(R_1)$ has no eigenstates belonging to the discrete or continuous spectrum, for negative energy.

2. *Baryons*

Let $\tilde{\Psi}(x, z)$, $z \in M$ be a spinor wave function. The construction of a wave equation for $\tilde{\Psi}$ proceeds close to the path followed in the scalar or pseudoscalar case.

The following operators can be used

$$(p\,\gamma) = p_\mu\,\gamma^\mu\,; \quad w_\mu\,\gamma_5\,\gamma^\mu = \gamma_5(w\,\gamma) = k\,. \tag{III.44}$$

γ_5 is defined as $\gamma_5 = i^{\gamma^0\gamma^1\gamma^2\gamma^3}$ with $\gamma_5^2 = 1$, $\gamma_5^T = \gamma_5^+ = \gamma_5$. The equation for $\tilde{\Psi}$ shall have the form

$$\left\{\begin{matrix} f_1(w^2)\,V_1^*(\sum_k (z_k)^2) + f_2(w^2)\,V_2^*\,f_3(k) \\ + V_3^* + (p\,\gamma)\,V_4^* + g(F^2, G^3)\,V_5^* + \gamma\,O_8 \\ + (D'\,V_5^*)^2 \end{matrix}\right\}\tilde{\Psi} = 0$$

$$(z^j\,p)\,\tilde{\Psi} = 0 \tag{III.45}$$

As in 1 the Fourier transform of $\tilde{\Psi}$ is considered

$$\tilde{\Psi}(x, z) = N \int d^4p \exp(-\,i\,p\,x)\,\Psi(p, z)\,.$$

2.1. *The Case* $p^2 = t > 0$

Let $\Lambda(p)$ be defined as in III.5,6 and let $S(\Lambda)$ be the spinor representation of the $1-1$ component of $\tilde{\Lambda}$ such that $S^{-1}(\Lambda)\,\gamma^\mu\,S(\Lambda) = \Lambda^\mu_\nu\,\gamma^\nu$. As in 1 the following coordinate transformation is performed

$$(p, z^k) \to (p, z^{i\,k} = \Lambda(p)\,z^k)\,. \tag{III.8}$$

The transformation of III.45 to the rest frame is completed by the substitutions $\Psi_1(p, z'^k) = S^{-1}(\Lambda(p))\,\Psi(p, z^k)$

$$f_1(w^2)\,V_1^* + g\,V_6^* + V_3^* + \gamma O_8 + (D'\,V_5^*)^2 = U$$

to give

$$\{U\,\gamma_0 + \varepsilon(p_0)\,[V_4^*\sqrt{t} + f_2(w^2)\,V_2^*\,f_3(\sqrt{t}\,\boldsymbol{L}\sum)\}\,\Psi_1 = 0 \tag{III.46}$$

$(z'^j)^0\,\Psi_1 = 0$ provided that f_3 is an odd function of its argument.

Σ_k is the matrix

$$\begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}, \quad k = 1, 2, 3\,.$$

Substituting

$$\Psi_1 = \prod_k \delta(z'^{k0})\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$

where φ and χ are two component spinors, in a representation of the Γ-matrices in which γ_0 is diagonal one obtains

$$\{U + \varepsilon(p_0)\,[V_4^* \sqrt{t} + f_2(-t\,L^2)\,V_2^*\,f_3(\sqrt{t}\,\mathbf{L}\textstyle\sum)\}\,\varphi = 0$$
$$\{U - \varepsilon(p_0)\,[V_4^* \sqrt{t} + f_2(-t\,L^2)\,V_2^*\,f_3(\sqrt{t}\,\mathbf{L}\textstyle\sum)\}\,\chi = 0\,. \qquad \text{(III.47)}$$

If (φ_+, χ_+) is a solution of III.47 for $p_0 > 0$, then $(\varphi_-, \chi_-) = (\chi_+, \varphi_+)$ is a solution for $p_0 < 0$. The corresponding equations to III.47 for a free Dirac wave function are

$$(-m^* + \varepsilon(p_0)\sqrt{t})\,\varphi = 0 \qquad (-m^* - \varepsilon(p_0)\sqrt{t})\,\chi = 0\,.$$

Thus $\sqrt{t}$ is determined from the first equation for $p_0 > 0$ and the solutions are of the form $(\varphi_+, \chi_+ = 0)$. It is to be noted, that the equations for φ and χ are decoupled, all the odd operators vanishing in the rest frame.

φ and χ can be decomposed further, specifying the eigenvalues of (J^2, J_z, π), of angular momentum and parity

$$\varphi_{jm}^{q+} = \begin{Bmatrix} \sqrt{\frac{l+1/2+m}{2l+1}} & Y_{l,m-1/2}^{q_1}\left(\frac{z_k}{r}\right) \\ \sqrt{\frac{l+1/2-m}{2l+1}} & Y_{lm+1/2}^{q_1}\left(\frac{z_k}{r}\right) \end{Bmatrix} R_{j+}^{q_1}(r) \quad j = l + 1/2$$

$$\varphi_{jm}^{q-} = \begin{Bmatrix} \sqrt{\frac{l+1/2-m}{2l+1}} & Y_{lm-1/2}^{q_2}\left(\frac{z_k}{r}\right) \\ \sqrt{\frac{l+1/2+m}{2l+1}} & Y_{lm-1/2}^{q_2}\left(\frac{z_k}{r}\right) \end{Bmatrix} R_{j-}^{q_2}(r) \quad j = l - 1/2\,.$$

Similarly the functions $T_{jq}^{\pm}(r)$ can be associated with χ.

Since there are several vectors z^k, $k = 1, \ldots, 8$ available for the construction of irreducible harmonic polynomials, the parity of Y_{eme}^{q} is not necessarily $(-1)^l$ (e.g. $z^k \wedge z^{k'}$ has $l = 1$ and $\pi = +1$). Let $\varkappa_\pm(j) = \pm(J + 1/2)$. Substituting the above expressions for φ and χ we obtain

$$\{U + \varepsilon(p_0)\,[V_4^* \sqrt{t} + f_2(-t\,L_\pm^2)\,V_2^*\,f_3(\sqrt{t}\,(\varkappa_\pm - 1))]\}\,R_{jq}^{\pm} = 0$$
$$\{U - \varepsilon(p_0)\,[V_4^* \sqrt{t} + f_2(-t\,L_2^{\pm})\,V_2^*\,f_3(\sqrt{t}\,(\varkappa_\pm - 1))]\}\,T_{jq}^{\pm} = 0$$
$$L_\pm^2 = l_\pm\,(l_\pm + 1)\,. \qquad \text{(III.48)}$$

In III.48 the $(\pm)$ signs in the upper and lower equation are independent. The same substitutions as in III.13–15 lead to

$$\begin{Bmatrix} \left[f_1(-t\,L_\pm^2)\,V_1 + V_3 + g\,V_6 + \gamma\,\tilde{O}_8 - \left(\frac{d}{dr}\right)^2\right] \\ + \varepsilon(p_0)\,[V_4 \sqrt{t} + f_2(-t\,L_\pm^2)\,V_2\,f_3[\sqrt{t}\,(\varkappa_\pm - 1)]] \end{Bmatrix} \check{R}_{jq}^{\pm} = 0$$

$$\Big\{\left[f_1(-t\,L_\pm^2)\,V_1 + V_3 + g\,V_6 + \gamma\,\tilde{O}_8 - \left(\frac{d}{dr}\right)^2\right]$$
$$- \varepsilon(p_0)\,[V_4 \sqrt{t} + f_2(-t\,L_\pm^2)\,V_2\,f_3\,[\sqrt{t}\,(\varkappa_\pm - 1)]\Big\}\,T_{jq'}^{\pm} = 0 \qquad \text{(III.49)}$$

$$V_{1,2,4,6} = r^{46}\,V_{1,2,4,6}^*\,, \qquad V_3 = r^{46}\,V_3^* + c\,\frac{1}{r^2} \qquad \tilde{O}_8 = r^{46}\,O_8\,, \quad c \text{ as in (III.14)}$$

2.2. *The Case* $p^2 = t < 0$

As in III.15 $w^2 = -t\,\hat{v}^2$

$$w(\Lambda(p)\,p, z', \partial'_z) = \sqrt{-t}\,(L_3, N_2, -N_1, 0)\,.$$

The spinor representation of SL(2, C) being a non-unitary representation $\tilde{S}_3$ is represented in a non-unitary way. The following operators can be diagonalized simultaneously

$$\delta^2 = \left(L_3 + \frac{1}{2}\sum\nolimits_3\right)^2 - \left(N_1 + \frac{i}{2}\,\alpha_1\right)^2 - \left(N_2 + \frac{i}{2}\,\alpha_2\right)^2$$

$$J_3 = L_3 + \frac{1}{2}\sum\nolimits_3\,, \quad \Delta = L_3^2 - N_1^2 - N_2^2\,.$$

Special care will have to be taken when the continuous spectrum of Δ is considered, because the corresponding functions $\varrho^{\nu}_{m\,m'}(\varphi, \chi, \psi)$ are not normalizable within the scalar product given by

$$(\varrho_1, \varrho_2) = \int_0^{2\pi} d\varphi \int_0^{2\pi} d\psi \int_0^{\infty} d\,c\,h\,\chi\,(\bar{\varrho}_1\,\varrho_2)\,.$$

The generators of $\tilde{S}_3$ are

$$J_3 = L_3 + \frac{1}{2}\sum\nolimits_3\,, \quad \nu_1 = N_1 + \frac{i}{2}\,\alpha_1\,, \quad \nu_2 = N_2 + \frac{i}{2}\,\alpha_2\,.$$

The equation $J_3\,\psi = m\,\psi$ reduces ψ to the following form

$$\psi^{\nu,q}_m = \begin{pmatrix} c^{\nu}_{+\,m} \cdot \varrho^{\nu}_{m-1/2,\,s+} \otimes \varphi^q_{m\,+} \\ c^{\nu}_{-\,m} \cdot \varrho^{\nu}_{m+1/2,\,s-} \otimes \varphi^q_{m\,-} \\ d^{\nu}_{+\,m} \cdot \varrho^{\nu}_{m-1/2,\,t+} \otimes \psi^q_{m\,+} \\ d^{\nu}_{-\,m} \cdot \varrho^{\nu}_{m+1/2,\,t-} \otimes \psi^q_{m\,-} \end{pmatrix}.$$

The index q represents all other quantum numbers, not fixed by the behaviour of ψ under Lorentz transformations.

The equation $\delta^2\,\psi^{\beta\nu q}_m = \beta\,(\beta+1)\,\psi^{\beta\nu q}_m$ can be brought to the form

$$i\left(N_-\,\frac{\alpha+}{2} + N_+\,\frac{\alpha-}{2}\right)\psi^{\beta\nu q}_m = \left(m\sum\nolimits_3 - \tau\right)\psi^{\beta\nu q}_m$$

$$N_\pm = N_1 \pm i\,N_2\,, \quad \alpha_\pm = \alpha_1 \pm i\,\alpha_2\,, \quad \tau = \beta\,(\beta+1) - \nu\,(\nu+1) - 1/4\,. \quad \text{(III.50)}$$

The phases of $\varrho^{\nu}_{a\,b}$ can be defined such that

$$N_\pm\,\varrho^{\nu}_{a\,b} = C^{\nu}_{\pm\,a}\,\varrho^{\nu}_{a\pm1,\,b} \quad C^{\nu}_{\pm\,a} > 0 \quad C^{\nu}_{+,\,a} = C^{\nu}_{-,\,a+1} = [a\,(a+1) - \nu\,(\nu+1)]^{1/2}. \quad \text{(III.51)}$$

$a\,(a+1) - \nu\,(\nu+1) > 0$ for both the discrete and continuous spectrum. III.50 takes the form

$$(\tau - m)\,c^{\nu}_{+\,m}\,\varrho^{\nu}_{m-1/2,\,s+}\,\varphi^q_{m+} = -\,i\,C^{\nu}_{-\,m+1/2}\,d^{\nu}_{-\,m}\,\varrho^{\nu}_{m-1/2,\,t-}\,\psi^q_{m-}$$

$$(\tau + m)\,c^{\nu}_{-\,m}\,\varrho^{\nu}_{m+1/2,\,s-}\,\varphi^q_{m-} = -\,i\,C^{\nu}_{+,\,m-1/2}\,d^{\nu}_{+\,m}\,\varrho^{\nu}_{m-1/2,\,t+}\,\psi^q_{m+}$$

$$(\tau - m)\,d^{\nu}_{+\,m}\,\varrho^{\nu}_{m-1/2,\,t+}\,\psi^q_{m+} = -\,i\,C^{\nu}_{-,\,m+1/2}\,c^{\nu}_{-\,m}\,\varrho^{\nu}_{m-1/2,\,s-}\,\varphi^q_{m-}$$

$$(\tau + m)\,d^{\nu}_{-\,m}\,\varrho^{\nu}_{m+1/2,\,t-}\,\psi^q_{m-} = -\,i\,C^{\nu}_{+,\,m+1/2}\,c^{\nu}_{+\,m}\,\varrho^{\nu}_{m+1/2,\,s+}\,\varphi^q_{m+}\,.$$

Therefore we obtain $s^+ = t^-$, $s^- = t^+$, $\varphi^q_{m\pm} = \psi^q_{m\mp}$ and substituting

$$C^{\nu}_{+,m-1/2} = C_{-m+1/2} = [(m - 1/2)\,(m + 1/2) - \nu\,(\nu + 1)]^{1/2} = D^{\nu}_m :$$

$$\begin{aligned}(\tau - m)\, c^{\nu}_{+m} &= (-i)\, D^{\nu}_m\, d^{\nu}_{-m}\\ (\tau + m)\, c^{\nu}_{-m} &= (-i)\, D^{\nu}_m\, d^{\nu}_{+m}\\ (\tau - m)\, d^{\nu}_{+m} &= (-i)\, D^{\nu}_m\, c^{\nu}_{-m}\\ (\tau + m)\, d^{\nu}_{-m} &= (-i)\, D^{\nu}_m\, c^{\nu}_{+m}\end{aligned} \qquad \text{(III.52)}$$

III.52 determines τ

$$\tau^2 - m^2 = -\,(D^{\nu}_m)^2 = (\nu + 1/2)^2 - m^2$$

$$\tau_\pm = \pm\,(\nu + 1/2)$$

$$\beta_\pm = \nu \pm 1/2\,. \qquad \text{(III.53)}$$

The discrete spectrum of Δ: $\nu = 0, 1, \ldots$ gives rise to discrete eigenvalues of δ^2: $\nu = 1/2^+, 3/2^\pm, 5/2^\pm, \ldots$ in analogy with the situation for $t > 0$. The continuous values of ν: $\nu = -1/2 + i\,\alpha$, α real arbitrary, give rise to two bands of values for β

$$\beta_+ = i\,\alpha\,, \quad \beta_- = -1 + i\,\alpha\,.$$

Let us consider the operator $k_1 = \gamma_0\,\gamma_5\,w_\mu\,\gamma^\mu$ which commutes with (J_3, ν_1, ν_2). k_1 is a symmetric operator if the scalar product of two spinors Ψ_1, $\Psi_2(\varphi, \chi, \psi)$ is given by

$$(\Psi_2, \Psi_1) = \int d\,\mu(\varphi, \chi, \psi)\,(\Psi_2^+\,\Psi_1),\;\; d\mu(\varphi, \chi, \psi) = d\varphi\, d\psi \times |\mathrm{sh}\,\chi|\, d\chi\,.$$

Multiplying III.50 by $\Sigma_3\,\gamma_5$ we obtain

$$(-N_1 \textstyle\sum_2 + N_2 \sum_1)\,\psi^{\beta_\pm\,\nu q}_m\,\beta = (L_3 + \frac{1}{2}\sum_3 - \tau_\pm \sum_3)\,\gamma_5\,\psi^{\beta_\pm\,\nu q}_m\,.$$

Since $1/\sqrt{-t}\,k_1 = -\gamma_5\,L_3 + N_2\,\Sigma_1 - N_1\,\Sigma_2$ the above equation is equivalent to

$$\frac{1}{\sqrt{-t}}\,k_1\,\psi^{\beta_\pm\,\nu, q}_m = (1/2 - \tau_\pm) \textstyle\sum_3 \gamma_5\,\psi^{\beta_\pm\,\nu, q}_m\,. \qquad \text{(III.54)}$$

In the case of discrete values of ν, III.54 is a proper equation. In the continuous case $\tau_\pm = \pm\,i\,\alpha$ and the operator on the right hand side of III.54 is not symmetric. This apparent contradiction is a consequence of the fact that the spinors $\Psi^{\beta_\pm, \nu, q}_m$, $\nu = -1/2 + i\,\alpha$ are not normalizable within the scalar product as defined above. Thus only wave packets of the form $\int d\alpha\, c^\pm_\alpha\, \Psi^{\beta_\pm(\alpha), \nu, q}_m$ are normalizable ($\alpha = \mathrm{Im}\,\beta_\pm = \mathrm{Im}\,\nu$).

III.54 implies that $c^\pm_\alpha$ has to be chosen such that

$$\int d\alpha\,\alpha\, c^\pm_\alpha\, \Psi^{\beta_\pm(\alpha), \nu, q}_m = 0$$

k_1 can then be redefined on $\psi^{\beta_\pm(\alpha), \nu, q}_m$ without changing the corresponding operator in Hilbert space, to be

$$\frac{1}{\sqrt{-t}}\,k_1\,\psi^{\beta_\pm\,\nu, q}_m = \begin{cases}(1/2 - \tau_\pm) \sum_3 \gamma_5\, \psi^{\beta_\pm\,\nu q}_m\,, & \nu = 0, 1, 2, \ldots\\ 1/2 \sum_3 \gamma_5\, \psi^{\beta_\pm\,\nu q}\,, & \nu = -1/2 + i\,\alpha\,.\end{cases} \qquad \text{(III.55)}$$

Let $\tau^*_\pm(\nu)$ be defined by

$$\tau^*_\pm(\nu) = \begin{cases} \pm(\nu + 1/2), & \nu = 0, 1, 2, \ldots \\ 0, & \nu = -1/2 + i\alpha . \end{cases}$$

III.55 now reads

$$\frac{1}{\sqrt{-t}} k_1 \psi_m^{\alpha\pm,\nu,q} = (1/2 - \tau^*) \sum\nolimits_3 \gamma_5 \psi_m^{\beta\pm\nu q} . \tag{III.56}$$

III.45 has the form

$$\{u_1\gamma_0 - \alpha_3[\sqrt{-t}\, V_4 - f_3[(1/2 - \tau^*_\pm)\sqrt{-t}]\, f_2(-t\,\hat{\nu}^2)\, V_2]\}\, \psi_m^{\beta\beta\mp 1/2,q} = 0$$

$$\psi_m^{\beta\beta\mp 1/2,q} = \begin{pmatrix} c^{\nu,\beta}_{+m}\, \varrho^{\nu}_{m-1/2,s^+}\, \varphi^q_{+m} \\ c^{\nu,\beta}_{-m}\, \varrho^{\nu}_{m+1/2,s^-}\, \varphi^q_{-m} \\ d^{\nu,\beta}_{+}\, \varrho^{\nu}_{m-1/2,s^-}\, \varphi^q_{-m} \\ d^{\nu,\beta}_{-}\, \varrho^{\nu}_{m+1/2,s^+}\, \varphi^q_{+m} \end{pmatrix} . \tag{III.57}$$

$$\alpha_3 = \gamma_5 \sum\nolimits_3 = \gamma_0\, \gamma^3$$

Substituting

$$u_1^* = \sqrt{-t}\, V_4 - f_3[(1/2 - \tau^*_\pm)\sqrt{-t}]\, f_2(-t\,\hat{\nu}^2)\, V_2$$

we obtain

$$s^- = s^+ \qquad u_1\, \varphi^{\nu,q}_{+m} = u_1^*\, \varphi^{\nu,q}_{1-m} \qquad \varphi^{\nu q}_{1-m} = \frac{d^\nu_{+m}}{c^\nu_{+m}}\, \varphi^{\nu q}_{-m}$$

$$u_1\, \varphi^{\nu,q}_{1-m} = -u_1^*\, \varphi^{\nu,q}_{+m} \qquad \frac{d^\nu_{+m}}{c^\nu_{+m}} = \frac{d^\nu_{-m}}{c^\nu_{-m}} . \tag{III.58}$$

The same substitutions as in III.13–15 are used to reduce further III.58 and III.49

$$t > 0 \quad \begin{aligned} \tilde{u} &= f_1(-t\,L^2)\, V_1 + g\, V_6 + V_3 + \gamma\, \tilde{O}_8 - \left(\frac{d}{dr}\right)^2 \\ \tilde{u}^* &= V_4\sqrt{t} + f_2(-t\,L^2) f_3[\sqrt{t}\,(\varkappa'_\pm - 1/2)]\, V_2,\ \varkappa'_\pm = \pm(l + 1/2) \end{aligned}$$

$$t < 0 \begin{cases} \tilde{u}_1 = f_1(-t\,\hat{\nu}^2)\, V_1 + g\, V_6 + V_3 + \gamma\, \tilde{O}_8 - \left(\frac{d}{dr}\right)^2 \\ \tilde{u}_1^* = V_4\sqrt{-t} - f_2(-t\,\hat{\nu}^2)\, f_3[(1/2 - \tau^*_\pm)\sqrt{-t}]\, V_2 . \end{cases} \tag{III.59}$$

Substituting the operators $\tilde{u}$, $\tilde{u}^k$, $\tilde{u}_1$, $\tilde{u}_1^k$ as defined in III.59 we obtain

$$t > 0, p_0 > 0 \begin{cases} \tilde{u}\, \check{R}^{j,l,q}_m = -\tilde{u}^*\, \check{R}^{j,l,q}_m \\ \tilde{u}\, \check{T}^{j,l,q}_m = \tilde{u}^*\, \check{T}^{j,l,q}_m \end{cases}$$

$$t < 0 \begin{cases} \tilde{u}_1\, \varphi^{\beta,\nu,q}_{+m} = \tilde{u}_1^*\, \varphi^{\beta,\nu,q}_{1-m} \\ \tilde{u}_1\, \varphi^{\beta,\nu,q}_{1-m} = -\tilde{u}_1^*\, \varphi^{\beta,\nu,q}_{+m} \end{cases}$$

or dropping the indices $\binom{j,l,q}{m}$, $\binom{\beta,\nu,q}{m}$ and replacing $(\check{R}^{j,l,q}_m, \check{T}^{j,l,q}_m)$ by (x, y), $\varphi^{\beta,\nu,q}_m - i\,\varphi^{\beta\nu q}_{1-m}$ by x_1, $\varphi^{\beta,\nu,q}_{+m} + i\,\varphi^{\beta,\nu,q}_{1-m}$ by y_1:

$$\begin{aligned} &t > 0, p_0 > 0 \quad (\tilde{u} + \tilde{u}^*)\, x = 0;\ (\tilde{u} - \tilde{u}^*)\, y = 0 \\ &t < 0 \qquad\qquad (\tilde{u}_1 + i\,\tilde{u}_1^*)\, x_1 = 0;\ (\tilde{u}_1 - i\,\tilde{u}_1^*)\, y_1 = 0 . \end{aligned} \tag{III.60}$$

The potentials in III.59 have to be chosen such that the only equation in III.60 possessing non-trivial solutions is $(\tilde{u} + \tilde{u}^*)\, x = 0$.

Example: We take $\gamma = 0$, $f_1(x) = x^2 - 2\, a\, x$, $V_1 \equiv 1$, $V_2 \equiv 0$

$$g\, V_6 + V_3 = a^2 + \frac{\nu}{r^s} + \frac{\lambda\,(\lambda+1)}{r^2}\,, \qquad V_4 = -\frac{\mu}{r^s}$$

$$\tilde{\varrho} = t\,\{L^2 \quad t > 0 \quad \hat{\nu}^2 \quad t < 0; \quad 1 \le s < 2, \quad \mu, \nu, a, \lambda > 0 \quad \lambda\text{: integer.}$$

III.60 becomes

$$\omega = (\tilde{\varrho} + a)^2$$

$$\left\{\left(\frac{d}{dr}\right)^2 - \left[\omega - \frac{q_m - \nu}{r^s} + \frac{\lambda\,(\lambda+1)}{r^s}\right]\right\} f_m = 0 \qquad m = 1, 2, 3, 4$$

$$\begin{aligned}
&m = 1\colon\ q_1 = \mu\sqrt{t}\,,\quad t > 0,\quad p_0 > 0,\\
&f_1 = x\\
&m = 2\colon\ q_2 = -\mu\sqrt{t}\,,\quad t > 0,\quad p_0 > 0,\\
&f_2 = y\\
&m = 3\colon\ q_3 = i\,\mu\sqrt{-t}\,,\quad t < 0,\\
&f_3 = x_1\\
&m = 4\colon\ q_4 = -i\,\mu\sqrt{-t}\,,\quad t < 0,\\
&f_4 = y_1\,.
\end{aligned} \qquad \text{(III.61)}$$

It follows immediately that the equation $m = 2$ has no non-trivial solutions and that for $m = 1$ $\sqrt{t} > \nu/\mu$. To discuss III.61 further we substitute $z = \varkappa\, r$, $\varkappa = \sqrt{\omega}$:

$$\left\{\left(\frac{d}{dz}\right)^2 - \left[1 - \tilde{\gamma}\,\frac{1}{z^s} + \frac{\lambda\,(\lambda+1)}{z^s}\right]\right\} f$$

$$\tilde{\gamma} = \frac{q - \nu}{\varkappa^\alpha} \qquad s = 2 - \alpha\,. \qquad \text{(III.62)}$$

Since the boundary conditions for f in III.62 do not depend on the parameter $\tilde{\gamma}$ we know by Poincaré's theorem [21] that $f_\lambda(z, \tilde{\gamma})$ is an analytic function of $\tilde{\gamma}$. Let us consider III.62 to be a special case of the equation

$$\left\{\left(\frac{d}{dz}\right)^2 + \left[k^2 + \tilde{\gamma}\,\frac{1}{z^s} - \frac{\lambda\,(\lambda+1)}{z^2}\right]\right\} f = 0 \qquad \text{(III.63)}$$

for $k = i$.

Following the general discussion of potential scattering and the construction of the S-matrix from the Jost functions [22] let us introduce the special solutions of III.63 $f_{\lambda\pm}(z, k, \tilde{\gamma})$ with

$$\lim_{z\to\infty} e^{\mp i k z}\, f_{\lambda\pm}(z, k, \tilde{\gamma}) = 1$$

and the Jost functions

$$f_{\lambda\pm}(k, \gamma) = \lim_{z\to\infty} z^\lambda\, f_{\lambda\mp}(z, k, \tilde{\gamma}) \qquad \text{(III.64)}$$

$f_{\lambda\pm}(z, k, \tilde{\gamma}), f_{\lambda\pm}(k, \gamma)$ satisfy the identities

$$f^*_{\lambda\pm}(z, -k^*, \tilde{\gamma}^*) = f_{\lambda\pm}(z, k, \tilde{\gamma}) \qquad f^*_{\lambda\pm}(-k^*, \tilde{\gamma}^*) = f_{\lambda\pm}(k, \tilde{\gamma})\,. \tag{III.65}$$

The regular solution of III.63 defined by

$$\lim_{z\to 0} z^{-\lambda-1}\,\varphi(z, k, \tilde{\gamma}) = 1$$

has the representation

$$\varphi(z, k, \tilde{\gamma}) = \frac{1}{2\,i\,k}\,[f_{\lambda-}(k, \tilde{\gamma})\,f_{\lambda+}(z, k, \tilde{\gamma}) - f_{\lambda+}(k, \tilde{\gamma})\,f_{\lambda-}(z, k, \tilde{\gamma})]\,. \tag{III.66}$$

The location of the bound solution of III.63 is determined by

$$f_{\lambda+}(k, \tilde{\gamma}) = f^*_{\lambda+}(-k^*, \tilde{\gamma}^*) = 0 \tag{III.67}$$

for $k = i\,\alpha$. Since $(-i\,\alpha)^* = i\,\alpha$ we conclude that if the pair $(\alpha, \tilde{\gamma})$ is a root of III.67, so is $(\alpha, \tilde{\gamma}^*)$. Equations of the type III.63 with complex couplings $\tilde{\gamma}$ have been discussed by NATAF and CORNILLE [23]. From their results it follows that if the potential (in our case $V(z) = -1/z^s$) does not change sign, for a bound solution

$$\gamma = \tilde{\gamma}^* \tag{III.68}$$

necessarily holds. III.68 implies that there are no non-trivial solutions of III.61 for $m = 3, 4$.

Let us now consider III.67 for $k = i$. γ_n, $n = 1, 2, \ldots$ denote the roots of $f_{\lambda+}(i, \tilde{\gamma})$ such that $f_{\lambda+}(i, \gamma_n) = 0$. Thereby the trajectories for ω as a function of $\sqrt{t}$ are determined

$$\tilde{\gamma} = \frac{\mu\sqrt{t} - \nu}{\varkappa^\alpha} = \gamma_n \qquad \sqrt{t} > \nu/\mu \qquad 0 < \alpha \leq 1/4\,. \tag{III.69}$$

In order to avoid trajectories for which $l(t) \to 0$ for $t \to \infty$ it is necessary that $\alpha \leqslant 1/4$. The discussion of III.69 is then the same as in the case of integer spin.

IV. Mass Formulas as Approximate Relations

1. *Integer Spin*

We consider the equation for φ as defined in III.11–14

$$\varphi = \varphi(q, i_3, y; p, z_1, \ldots, z_8) \tag{IV.1}$$

q labels the representation of SU 3, (i_3, y) denote the values of the third component of isotopic spin and hypercharge respectively.

$$\{(D'\,V^*_5)^2 + f\,[-t\,l\,(l+1)]\,V^*_1 + t\,V^*_2 + \gamma\,O_8 + V^*_3 + C(q)\,V_4\}\,\varphi = 0\,. \tag{IV.2}$$

The perturbation $\gamma\,O_8$ causes a splitting of masses within a multiplet. We develop $t = t(l, \gamma, y)$ in powers of γ and determine t in first approximation $t \cong t_0(l, q) + \gamma\,t_1(l, q, y)$

keeping l fixed. The wave function $\varphi = \varphi(q, i_3, y, \gamma, p, z_k)$ is determined by the usual perturbation methods. $\varphi_0(q, i_3, y, p, z_k)$ denotes the solution of IV.2 for $\gamma = 0$, the equation considered in the preceding section. IV.2 is of the general type

$$\left(\sum f_i\,[t(\gamma)]\,A_i + \gamma\,B + C\right)\varphi(\gamma) = 0 \tag{IV.3}$$

C, A_L, B denote self adjoint operators. IV.3 is a generalized eigenvalue problem (as considered also in (9)). The usual perturbation methods imply

$$t_1(q, l, y) = \frac{(\varphi_0(q, i_3, y)/-O_8/\varphi_0(q, i_3, y))}{(\varphi_0/V_2^* - l\,(l+1)\,f'[-t_0\,l\,(l+1)]\;V_1^*/\varphi_0)}\,. \tag{IV.4}$$

The scalar product (φ_1, φ_2) is defined as in III.11. For small values of l such that

$$l\,(l+1)\,\left|(\varphi_0\,|f'\,V_1^*\,|\,\varphi_0)\right| \ll \left|(\varphi_0\,|V_2^*\,|\,\varphi_0)\right|$$

$t_1(l, q, y)$ is approximatively given by

$$t_1(q, l, y) \cong \frac{(\varphi_0(q, i_3, y)/-O_8/\varphi_0(q, i_3, y))}{(\varphi_0/V_2^*/\varphi_0)}\,. \tag{IV.5}$$

In this approximation $t_1(l, q, y)$ is independent of t_0, which distinguishes the case where t is developed in powers of γ from choosing other functions of t for a perturbation expansion.

2. *Half-integer Spin*

Let us go back to Equation III.47 for φ and χ. As follows from the discussion of III.61 $\chi \equiv 0$ for $p_0 > 0$.

$$0 = \left\{(D'\,V_5^*)^2 + f\,[-t\,l\,(l+1)]\,V_1^* + V_3^* + g\,V_6^* + \gamma\,O_8 + \sqrt{t}\,V_4^*\right\}\varphi(q, i_3, y, \gamma) \tag{IV.6}$$

In IV.6 the term $f_2(-t\,L^2)\;V_2^*\,f_3(\sqrt{t}\,\mathbf{L}\,\sum)$ has been neglected. The scalar product for spinors φ_1, φ_2 is given by

$$(\varphi_1, \varphi_2) = \int d^3z_1 \ldots d^3z_8 \quad \varphi_1^+(q, i_3, y, p, \mathbf{z})\;\varphi_2(q, i_3, y, \mathbf{z}) \tag{IV.7}$$

The term $\sqrt{t}\,V_4^*$ indicates to develop $\sqrt{t} = m(q, l, y, \gamma)$ in powers of γ, $m \cong m_0(q, l) + \gamma\,m_1(q, l, y)$. One readily obtains

$$m_1(q, l, y) = \frac{(\varphi_0(q, i_3, y)/-O_8/\varphi_0(q, i_3, y))}{(\varphi_0/V_4^* - 2\,f'm_0(q, l)\,l\,(l+1)\;V_1^*/\varphi_0)}\,. \tag{IV.8}$$

As in 1, for values of l such that

$$\left|2\,f'\,m_0(q, l)\,l\,(l+1)\;(\varphi_0\,|V_1^*\,|\,\varphi_0)\right| \ll \left|(\varphi_0\,|V_4^*\,|\,\varphi_0)\right|$$

$m_1(q, l, y)$ is given by

$$m_1(q, l, y) \cong \frac{(\varphi_0(q, i_3, y)/(-O_8)/\varphi_0(q, i_3, y))}{(\varphi_0/V_4^*/\varphi_0)}\,. \tag{IV.9}$$

A reason for the above mass splittings within multiplets to be comparable to the mass difference of distinct multiplets can be sought in the smallness of the denominators in IV.5 and IV.9

$$(\varphi_0 \mid V_2^* \mid \varphi_0),\ (\varphi_0 \mid V_4^* \mid \varphi_0)\ .$$

In ordinary perturbation calculations this term is replaced by $(\varphi_0, \varphi_0) = 1$.

Unless more evidence is gained for the discussed manifold to come close to reality the approximate formulas IV.4,5,8,9 can not be taken as indications that the Gell-Mann-Okubo mass formula applies to the square of the mass in the case of mesons and to the mass in the case of baryons.

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After this work was completed I became aware that E. MAJORANA [24] had discussed relativistic equations incorporating an internal Lorentz group. There have been several articles which take MAJORANA's investigations as a starting point [25, 26].

Appendix

A complete classification of the Lie subgroups of SL(2, C) and $\tilde{\Lambda}$ up to equivalence will be given and some properties thereof established. Proofs will be omitted in order not to lengthen unduly this outline.

A Lie subgroup of SL(2, C) will be denoted by $G_d(n)$ (d: dimension of $G_d(n)$, n: label of $G_d(n)$). If the matrix $-\mathbf{1} \in$ SL(2, C) is contained in $G_d(n)$ this property will be indicated by $G_d^*(n)$. If $-\mathbf{1} \notin G_d(n)$ the adjunction of $-\mathbf{1}$ to $G_d(n)$ leads to an enlarged group, denoted by $G_d^*(n, -\mathbf{1})$ which is no longer a Lie subgroup of SL(2, C).

$G_d^*(n)$ as well as $G_d^*(n, -\mathbf{1})$ can be identified by a two to one homomorphism h from SL(2, C) onto $\tilde{\Lambda}$, with a Lie subgroup of $\tilde{\Lambda}$.

$$h: \quad A, -A \to \Lambda(A) = \Lambda(-A) \tag{1}$$

In the Lie algebra L of SL(2, C) or $\tilde{\Lambda}$ the following basis is chosen

X_i, $i = 1, 2, 3$ generators of SU 2, or generators of the rotation group,

Y_i, $i = 1, 2, 3$ generators of pure Lorentz transformations in the i-direction.

In the self-representation of SL(2, C) to (X_k, Y_k) correspond

$$\frac{1}{2i}\,\sigma_k\,, \quad \frac{1}{2}\,\sigma_k$$

respectively. The commutation relations are

$$[X_s, X_t] = \varepsilon_{str} X_r \quad [X_s, Y_t] = \varepsilon_{str} Y_r \quad [Y_s, Y_t] = -\varepsilon_{str} X_r . \tag{2}$$

An element ζ of L is given by its components

$$\zeta = (x_k, y_k) = \sum_k (x_k X_k + y_k Y_k) .$$

There are two invariant bilinear, indefinite forms on L

$$(\zeta_1 \zeta_2)_1 = \boldsymbol{x}_1 \boldsymbol{x}_2 - \boldsymbol{y}_1 \boldsymbol{y}_2; \quad (\zeta_1, \zeta_2)_2 = (\boldsymbol{x}_1 \boldsymbol{y}_2 + \boldsymbol{x}_2 \boldsymbol{y}_1) \tag{3}$$

L can also be realized as the complexification of the algebra generated by X_k, $k = 1, 2, 3$ identifying Y_k with $i X_k$. ζ then has three complex components

$$\zeta = (z_k = x_k + i y_k) = \sum z_k X_k = \sum_k (x_k X_k + y_k Y_k) .$$

The bilinear form in L_C is given by

$$(\zeta_1, \zeta_2)_3 = \sum_k z_{1k} z_{2k} \quad (\zeta_1, \zeta_2)_1 = Re\{(\zeta_1, \zeta_2)_3\}; \quad (\zeta_1, \zeta_2)_2 = Im\{(\zeta_1, \zeta_2)_3\} . \tag{4}$$

In the following $A \in \mathrm{SL}(2, \mathrm{C})$, $\Lambda \in \tilde{\Lambda}$. The Lie subgroups of SL(2, C) and $\tilde{\Lambda}$ can be arranged as follows:

1. $G_4^*(1) = \left\{ A = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\}$ — $a, b,$ complex arbitrary

$G_4^*(1) \xrightarrow{h} \{\Lambda \mid \Lambda \lambda = c_\Lambda \lambda\}$. — λ light like vector; c_Λ real arbitrary

2.0. $G_3(2, \gamma = 0) = \left\{ A = \begin{pmatrix} e^\gamma & b \\ 0 & e^{-\gamma} \end{pmatrix} \right\}$ — r real arbitrary; b complex arbitrary

$G_3^*(2, \gamma = 0, -1) \xrightarrow{h} \{\Lambda \mid \Lambda = \exp[a Y_3 + b (Y_1 - X_2) + c (X_1 - Y_2)]\}$. — a, b, c real arbitrary

2.γ. $G_3(2, \gamma) = \left\{ A = \begin{pmatrix} e^{[r e^{i\gamma/2}]} & b \\ 0 & e^{-[r e^{i\gamma/2}]} \end{pmatrix} \right\}$ — r real arbitrary; b complex arbitrary; γ real fixed

$\gamma \neq 0$, $\gamma \neq \pm \pi$,

$G_3^*(2, \gamma, -1) \xrightarrow{h} \{\Lambda \mid \Lambda = \exp[a(\cos\gamma/2\, Y_3 - \sin\gamma/2\, X_3) + b(Y_1 - X_2) + c(X_1 - Y_2)]\}$. — a, b, c real arbitrary

2.$\gamma = \pm\pi$. $G_3^*(2, \gamma = \pm\pi) = \left\{ A = \begin{pmatrix} e^{i\varphi/2} & b \\ 0 & e^{-i\varphi/2} \end{pmatrix} \right\}$ — φ real arbitrary; b complex arbitrary

$G_3^*(2, \gamma = \pm\pi) \xrightarrow{h} \{\Lambda \mid \Lambda \lambda = \lambda, \lambda^2 = 0\}$. — λ light like vector

3. $G_3^*(3) = \{A \mid A = \bar{A}\}$, $\quad$ $\bar{A}$ complex conjugate matrix to A

$G_3^*(3) \xrightarrow{h} \tilde{\Lambda}_3$.

4. $G_3^*(4) = \{A \mid A^+ A = \mathbf{1}\}$,

$G_3^*(4) \cong \mathrm{SU}\,2 \xrightarrow{h} R$ (rotation group) .

5. $G_2^*(5) = \left\{ A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \right\}$, $\quad$ α complex arbitrary (abelian)

$G_2^*(5) \xrightarrow{h} \{\Lambda \mid \Lambda = \exp(a\, X_3 + b\, Y_3)\}$, $\quad$ a, b real arbitrary

6. $G_2(6) = \left\{ A = \begin{pmatrix} e^r & \varrho \\ 0 & e^{-r} \end{pmatrix} \right\}$, $\quad$ ϱ, r real arbitrary

$G_2^*(6, -\mathbf{1}) \xrightarrow{h} \{\Lambda \mid \Lambda = \exp(a\, Y_3 + b\, (Y_1 - X_2))\}$. $\quad$ a, b real arbitrary

7. $G_2(7) = \left\{ A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}$, $\quad$ b complex arbitrary (abelian)

$G_2^*(7, -\mathbf{1}) \xrightarrow{h} \{\Lambda \mid \Lambda = \exp(a\, (Y_1 - X_2) + b\, (X_1 - Y_2))\}$ $\quad$ a, b real arbitrary

8. $G_1(8, \gamma) = \left\{ A = \begin{pmatrix} e^r e^{i\gamma/2} & 0 \\ 0 & e^{-r} e^{i\gamma/2} \end{pmatrix} \right\}$, $\quad$ r real arbitrary, γ real fixed

$\gamma \neq 0,\ \gamma \neq \pm \pi$.

$G_1^*(8, \gamma, -\mathbf{1}) \xrightarrow{h} \{\Lambda \mid \Lambda = \exp[a\, (\cos\gamma/2\, Y_3 - \sin\gamma/2\, X_3)]\}$. $\quad$ a real arbitrary

9. $G_1(9) = \left\{ A = \begin{pmatrix} 1 & \varrho \\ 0 & 1 \end{pmatrix} \right\}$, $\quad$ ϱ real arbitrary

$G_1^*(9, -\mathbf{1}) \xrightarrow{h} \{\Lambda \mid \Lambda = \exp[a\, (Y_1 - X_2)]\}$. $\quad$ a real arbitrary

10. $G_1(10) = \left\{ A = \begin{pmatrix} e^b & 0 \\ 0 & e^{-b} \end{pmatrix} \right\}$, $\quad$ b real arbitrary

$G_1^*(10, -\mathbf{1}) \xrightarrow{h} \{\Lambda \mid \Lambda = \exp[a\, Y_3]\}$. $\quad$ a real arbitrary

11. $$G_1(11) \quad = \left\{ A = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \right\}, \qquad \varphi \text{ real arbitrary}$$

$$G_1^*(11) \longrightarrow \{\Lambda \mid \Lambda = \exp[\varphi X_3]\} . \qquad \varphi \text{ real arbitrary}$$

The block diagram in Figure 9 illustrates the interdependence of the groups $G_d(n)$, $G_d^*(n')$, $G_d^*(n'', -1)$ (if two boxes are joined by a line, the lower group is contained in the higher one).

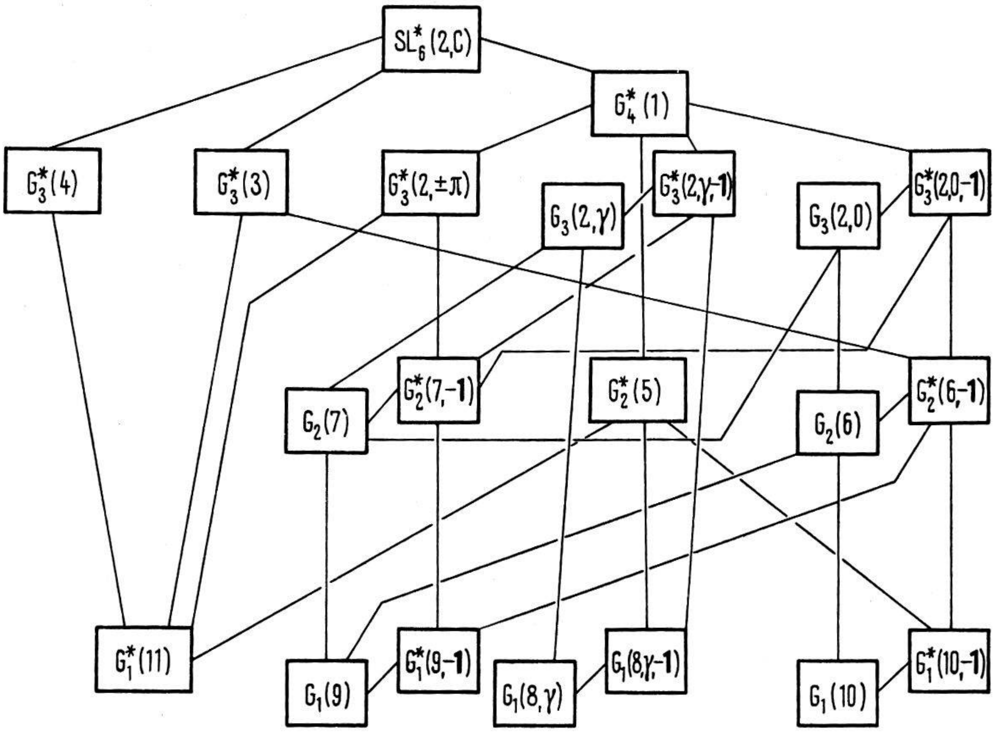

Figure 9

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