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Autor: Guenin, M. / Velo, G.

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Mass Renormalization as an Automorphism of the Algebra of Field Operators[1])

by **M. Guenin**

Institut de Physique Théorique de l'Université de Genève

and **G. Velo**[2])

Department of Physics, New York University New York, New York

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Abstract. In a model with only mass renormalization it is shown that the use of time translation operations not unitarily implementable enables one to overcome the difficulties connected with the Haag theorem.

I. Introduction

The use of canonical formalism seems characteristic to most field theories. One of the essential difficulties hidden in it is the fact that, as one works in the Fock space corresponding to a given mass, there exists no unitary operator representing time translations, providing one makes the usual assumptions of a relativistic theory (Haag's theorem) [1, 2, 3]. In order to bypass this difficulty, one of us [4] has proposed to consider the time translation in Fock space as an automorphism of the algebra of fields (or observables) which is not unitarily implementable. The automorphism should be reached by a limiting procedure of unitarily implementable automorphisms. These are constructed cutting off the Hamiltonian and then violating one of the hypotheses underlying the proof of Haag's theorem, namely translation invariance.

The purpose of this note is to show how this procedure can be applied successfully to the mass renormalization interaction of a spin zero neutral particle.

II. The Model

Let us consider the Fock space corresponding to a scalar neutral relativistic particle of mass m, moving in an s-dimensional space. Let H_0 be the free Hamiltonian of the system:

$$H_0 = \frac{1}{2}\int \{m^2 : \phi^2 : (\boldsymbol{x}) + : \pi^2 : (\boldsymbol{x}) + : \nabla\, \phi^2 : (\boldsymbol{x})\}\, d^s x = \int \omega(\boldsymbol{k})\, a^*(\boldsymbol{k})\, a(\boldsymbol{k})\, d^s k$$

where

$$\phi(\boldsymbol{x}) = \frac{1}{(2\,\pi)^{s/2}} \int \frac{d^s k}{(2\,\omega(k))^{1/2}}\, \{a(\boldsymbol{k}) + a^*(-\,\boldsymbol{k})\}\, e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\pi(\boldsymbol{x}) = \frac{i}{(2\,\pi)^{s/2}} \int \frac{d^s k}{(2\,\omega(k))^{1/2}}\, \omega(k)\, \{a(-\,\boldsymbol{k})^* - a(\boldsymbol{k})\}\, e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$[a(\boldsymbol{k}), a(\boldsymbol{k}')] = 0 \qquad [a(\boldsymbol{k}), a(\boldsymbol{k}')^*] = \delta\,(\boldsymbol{k} - \boldsymbol{k}') \qquad \omega(k) = \sqrt{\boldsymbol{k}^2 + m^2}$$

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and the fields are represented in a Fock space of mass m. If we now add to H_0 a term like

$$V = \frac{\delta m^2}{2} \int :\phi^2: (\boldsymbol{x}) \, d^s x$$

the new Hamiltonian $H_0 + V$ seems to correspond to the Hamiltonian of a free field of mass $(\delta m^2 + m^2)^{1/2}$. But this is not the case because we are still representing the fields in the Fock space of mass m, and in this space V, and therefore $H_0 + V$, are not well defined operators in the sense that they cannot be applied to states with a finite number of particles. In order to show that what we intuitively believe is true, that is, that in a certain sense $H_0 + V$ describes a free field with mass $(\delta m^2 + m^2)^{1/2}$, we must first put ourselves in a well defined mathematical framework. We allow as many cut offs as are necessary on the interaction Hamiltonian so as to make it a well defined operator. We define then

$$V = \int :\phi(\boldsymbol{x} + \boldsymbol{y}) \, \phi(\boldsymbol{x} - \boldsymbol{y}): f(\boldsymbol{x}) \, \varphi(\boldsymbol{y}) \, d^s x \, d^s y$$

$$H = H_0 + V \qquad \phi(\boldsymbol{x}, t) = e^{iHt} \, \phi(\boldsymbol{x}) \, e^{-iHt} .$$

We want to show that as

$$\varphi \to \delta \qquad f \to \frac{\delta m^2}{2} .$$

$\phi(\boldsymbol{x}, t)$ converges to an operator of the algebra which is linked to $\phi(\boldsymbol{x})$ by an automorphism parameterized by t.

A simpler cut off interaction Hamiltonian

$$\int :\phi^2: (\boldsymbol{x}) \, f(\boldsymbol{x}) \, d^s x$$

which would seem more natural has the disadvantage that for $s \geqslant 2$ it is not well defined. In fact if we denote by ψ_0 the no particle state and by $\tilde{f}$ the Fourier transform of f

$$\tilde{f}(\boldsymbol{k}) = \frac{1}{(2\pi)^{s/2}} \int e^{i \boldsymbol{k} \cdot \boldsymbol{x}} f(\boldsymbol{x}) \, d^s x$$

we have

$$\left\| \int :\phi^2: (\boldsymbol{x}) \, f(\boldsymbol{x}) \, d^s x \, \psi_0 \right\|^2 \sim \int \frac{d^s k_1 \, d^s k_2}{\omega(k_1) \, \omega(k_2)} \, |\tilde{f}(\boldsymbol{k}_1 + \boldsymbol{k}_2)|^2$$

which diverges for $s \geqslant 2$. The same thing is true if instead of ψ_0 we take any state with a finite number of particles. Since we are essentially interested in the limit for which the cut offs disappear, we take for f and φ the following explicit functions:

$$f(\boldsymbol{x}) = \frac{\delta m^2}{2} \prod_{i=1}^{s} e^{-x_i^2 \alpha_i/4} \quad \xrightarrow[\text{as } \{\alpha_i\} \to 0]{(\text{in } S')} \quad \frac{\delta m^2}{2}$$

$$\varphi(\boldsymbol{x}) = \left(\frac{1}{\pi}\right)^{s/2} \prod_{i-1}^{s} \frac{1}{\sqrt{\beta_i}} \, e^{-x_i^2/\beta_i} \quad \xrightarrow[\text{as } \{\beta_i\} \to 0]{(\text{in } S')} \quad \prod_{i=1}^{s} \delta(x_i) .$$

In order to compute

$$a(\boldsymbol{p}, t) = e^{iHt} \, a(\boldsymbol{p}) \, e^{-iHt}$$

we make the following ansatz:

$$a(\boldsymbol{p}, t) = \int A_1(t, \boldsymbol{p}, \boldsymbol{k})\, a(\boldsymbol{k})\, d^s k + \int A_2(t, \boldsymbol{p}, \boldsymbol{k})\, a(-\boldsymbol{k})^*\, d^s k\,. \tag{1}$$

By applying the Heisenberg equations, one gets that A_1 and A_2 must satisfy the following differential equations:

$$i \frac{\partial}{\partial t} A_1(t, \boldsymbol{p}, \boldsymbol{k}) = \omega(k)\, A_1(t, \boldsymbol{p}, \boldsymbol{k}) + \int d^s k' \frac{\tilde{f}(\boldsymbol{k}+\boldsymbol{k}')\, \tilde{\varphi}(\boldsymbol{k}-\boldsymbol{k}')}{\sqrt{\omega(k)}\, \sqrt{\omega(k')}} \times \big(A_1(t, \boldsymbol{p}, \boldsymbol{k}') - A_2(t, \boldsymbol{p}, \boldsymbol{k}')\big) \tag{2}$$

$$i \frac{\partial}{\partial t} A_2(t, \boldsymbol{p}, \boldsymbol{k}) = -\omega(k)\, A_2(t, \boldsymbol{p}, \boldsymbol{k}) + \int d^s \boldsymbol{k}' \frac{\tilde{f}(\boldsymbol{k}+\boldsymbol{k}')\, \tilde{\varphi}(\boldsymbol{k}-\boldsymbol{k}')}{\sqrt{\omega(k)}\, \sqrt{\omega(k')}} \big(A_1(t, \boldsymbol{p}, \boldsymbol{k}') - A_2(t, \boldsymbol{p}, \boldsymbol{k}')\big)$$

with the initial conditions:

$$A_1(0, \boldsymbol{p}, \boldsymbol{k}) = \delta(\boldsymbol{p} - \boldsymbol{k})\,, \quad A_2(0, \boldsymbol{p}, \boldsymbol{k}) = 0\,. \tag{3}$$

In the Appendix it will be shown in a rather sketchy way that (2) and (3) have a solution which, smeared out on the variable $\boldsymbol{p}$ with a test function belonging to $S(\boldsymbol{R}^s)$, belongs to $S(\boldsymbol{R}^s)$ in the variable $\boldsymbol{k}$. Moreover it will be shown that A_1 and A_2, smeared out in the $\boldsymbol{p}$ variable, converge in the topology of $S(\boldsymbol{R}^s)$ as $\{\alpha_i\}, \{\beta_i\} \to 0$, and that this limit represents the solution of the differential equations and boundary conditions corresponding to

$$f(\boldsymbol{x}) = \frac{\delta m^2}{2} \qquad \varphi(\boldsymbol{x}) = \prod_{i=1}^{s} \delta(x_i)\,.$$

That is, A_1 and A_2 in the limiting case are solutions of:

$$i \frac{\partial}{\partial t} A_1(t, \boldsymbol{p}, \boldsymbol{k}) = \omega(k)\, A_1(t, \boldsymbol{p}, \boldsymbol{k}) + \frac{\delta m^2}{2\,\omega(k)} \big(A_1(t, \boldsymbol{p}, \boldsymbol{k}) - A_2(t, \boldsymbol{p}, \boldsymbol{k})\big)$$

$$i \frac{\partial}{\partial t} A_2(t, \boldsymbol{p}, \boldsymbol{k}) = -\omega(k)\, A_2(t, \boldsymbol{p}, \boldsymbol{k}) + \frac{\delta m^2}{2\,\omega(k)} \big(A_1(t, \boldsymbol{p}, \boldsymbol{k}) - A_2(t, \boldsymbol{p}, \boldsymbol{k})\big)$$

$$A_1(0, \boldsymbol{p}, \boldsymbol{k}) = \delta(\boldsymbol{p} - \boldsymbol{k}) \qquad A_2(0, \boldsymbol{p}, \boldsymbol{k}) = 0\,. \tag{4}$$

And it is easy to check that

$$A_1(t, \boldsymbol{p}, \boldsymbol{k}) = \left[\left(\frac{1}{2} - \frac{\omega(k)^2 + \Omega(k)^2}{4\,\omega(k)\,\Omega(k)}\right) e^{i\Omega(k)t} + \left(\frac{1}{2} + \frac{\omega(k)^2 + \Omega(k)^2}{4\,\omega(k)\,\Omega(k)}\right) e^{-i\Omega(k)t}\right] \delta(\boldsymbol{p} - \boldsymbol{k})$$

$$A_2(t, \boldsymbol{p}, \boldsymbol{k}) = \left[\frac{\omega(k)^2 - \Omega(k)^2}{4\,\omega(k)\,\Omega(k)}\, e^{i\Omega(k)t} + \frac{\Omega(k)^2 - \omega(k)^2}{4\,\omega(k)\,\Omega(k)}\, e^{-i\Omega(k)t}\right] \delta(\boldsymbol{p} - \boldsymbol{k})$$

are the solution of (4), with $\Omega(k) = (\boldsymbol{k}^2 + m^2 + \delta m^2)^{1/2}$. So if we denote by $H(\alpha_i, \beta_j)$ the cut off Hamiltonian, explicitly exhibiting the $\{\alpha_i\}\{\beta_j\}$ dependence, and by $A_1^{(\alpha_i, \beta_j)}$, $A_2^{(\alpha_i, \beta_j)}$ the corresponding A_i, we know that by definition

$$e^{iH(\alpha_i,\beta_j)t} \int a(\boldsymbol{p})\, \psi(\boldsymbol{p})\, d^s p\, e^{-iH(\alpha_i,\beta_j)t} = \int d^s k \left(\int d^s p\, \psi(\boldsymbol{p})\, A_1^{(\alpha_i,\beta_j)}(t, \boldsymbol{p}, \boldsymbol{k})\right) a(\boldsymbol{k}) + \int d^s k \left(\int (d^s p\, \psi(\boldsymbol{p})\, A_2^{(\alpha_i,\beta_j)}(t, \boldsymbol{p}, \boldsymbol{k})\right) a(-\boldsymbol{k})^* \tag{6}$$

where $\psi(\boldsymbol{p}) \in S(R^s)$. Now, since

$$\int a(\boldsymbol{k})\,\psi(\boldsymbol{k})\,d^s k \quad \text{and} \quad \int a(\boldsymbol{k})^*\,\psi(\boldsymbol{k})\,d^s k$$

applied to a state with bounded number of particles give rise to vector-valued distributions in $\psi(\boldsymbol{k})$ [5], and

$$\lim_{\substack{\alpha_i \to 0 \\ \beta_j \to 0}} \int A_l^{(\alpha_i, \beta_j)}(t, \boldsymbol{p}, \boldsymbol{k})\,\psi(\boldsymbol{p})\,d^s p = \int A_l(t, \boldsymbol{p}, \boldsymbol{k})\,\psi(\boldsymbol{p})\,d^s p \qquad (l = 1, 2)$$

where the A_l are given by (5), we have the convergence of the l.h.s. of (6), at least on the states of a bounded number of particles.

On going to the limit $\{\alpha_i\}, \{\beta_i\} \to 0$, and writing the field $\phi(\boldsymbol{x}, t)$ as:

$$\phi(\boldsymbol{x}, t) = \frac{1}{(2\pi)^{s/2}} \int \frac{d^s k}{\sqrt{2\,\omega(k)}}\,\{a(\boldsymbol{k}, t) + a(-\boldsymbol{k}, t)^*\}\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

we obtain by some simple manipulations

$$\phi(\boldsymbol{x}, t) = \frac{1}{(2\pi)^{s/2}} \int \frac{d^s k}{\sqrt{2\,\Omega(k)}}\,\{b(\boldsymbol{k})\,e^{-i\Omega(k)t} + b(-\boldsymbol{k})^*\,e^{i\Omega(k)t}\}\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{7}$$

with

$$b(\boldsymbol{k}) = \frac{1}{2}\sqrt{\frac{\Omega(k)}{\omega(k)}}\left\{\left(1 + \frac{\omega(k)}{\Omega(k)}\right) a(\boldsymbol{k}) + \left(1 - \frac{\omega(k)}{\Omega(k)}\right) a(-\boldsymbol{k})^*\right\}. \tag{8}$$

One immediately sees that $b(\boldsymbol{k})$ and $b(\boldsymbol{k})^*$ satisfy the canonical commutation relations. One easily checks that there is no element ψ of the Fock space corresponding to mass m, having finite norm and such that $b(\boldsymbol{k})\,\psi = 0$.

III. Conclusions

Equation (7) gives the time translation on the algebra of the fields. By Haag's theorem or by a direct check one sees that the evolution in time is not unitarily implementable. If we want to represent it by unitary operators a theorem due to SEGAL [6] tells us that the only representation is that of Fock corresponding to mass $(m^2 + \delta m^2)^{1/2}$. The hypotheses are that the Hamiltonian is a positive operator with unique ground state cyclic for the fields and a certain regularity condition on the matrix elements of the field. This example shows the power of algebraic thinking, at least if one restricts oneself to the evolution of the operators. Nothing can be said in general about the vacuum state, although in this case the problem was easily solvable with the help of the Segal theorem.

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Appendix

In order to prove the statements made about Equations (6) and (7), it will be shown that the solution of (6) and (7) can be obtained by the perturbation method. If we put:

$$U(t, \boldsymbol{k}) = \int \begin{pmatrix} e^{i w(k) t} A_1(t, \boldsymbol{p}, \boldsymbol{k}) \\ e^{-i w(k) t} A_2(t, \boldsymbol{p}, \boldsymbol{k}) \end{pmatrix} \psi(\boldsymbol{p})\, d^s p$$

where $\psi(\boldsymbol{p}) \in S(\boldsymbol{R}^s)$ and

$$M(t, \boldsymbol{k}, \boldsymbol{k}') = \frac{1}{i} \frac{\tilde{\varphi}\,(\boldsymbol{k}+\boldsymbol{k}')}{\sqrt{\omega(\boldsymbol{k})}\,\sqrt{\omega(\boldsymbol{k}')}} \begin{pmatrix} e^{i(w(k)-w(k'))t} & -e^{i(w(k)+w(k'))t} \\ e^{-i(w(k)+w(k'))t} & -e^{-i(w(k)-w(k'))t} \end{pmatrix}$$

the system we are examining can be rewritten as:

$$\frac{\partial}{\partial t} U(t, \boldsymbol{k}) = \int d^s k'\, \tilde{f}\,(\boldsymbol{k}-\boldsymbol{k}')\, M(t, \boldsymbol{k}, \boldsymbol{k}')\, U(t, \boldsymbol{k}') \qquad U(0, \boldsymbol{k}) = \begin{pmatrix} \psi(\boldsymbol{k}) \\ 0 \end{pmatrix}$$

or

$$U(t, \boldsymbol{k}) = U(0, \boldsymbol{k}) + \int_0^t d\tau \int d^s k'\, \tilde{f}\,(\boldsymbol{k}-\boldsymbol{k}')\, M(t, \boldsymbol{k}, \boldsymbol{k}')\, U(\tau, \boldsymbol{k}')\,.$$

We try a solution by a successive approximation procedure:

$$U_0(t, \boldsymbol{k}) = U(0, \boldsymbol{k})$$

$$U_n(t, \boldsymbol{k}) = U(0, \boldsymbol{k}) + \sum_1^n{}_j \int d^s k_1 \ldots d^s k_j\, \tilde{f}\,(\boldsymbol{k}-\boldsymbol{k}_1) \ldots \tilde{f}\,(\boldsymbol{k}_{j-1}-\boldsymbol{k}_j)$$

$$\times \int_0^t d\tau_1\, M(\tau_1, \boldsymbol{k}, \boldsymbol{k}_1) \ldots \int_0^{\tau_{j-1}} d\tau_j\, M(\tau_j, \boldsymbol{k}_{j-1}, \boldsymbol{k}_j)\, U(0, \boldsymbol{k}_j)\,.$$

As

$$\| M(\tau, \boldsymbol{k}, \boldsymbol{k}') \| \leqslant A$$

where A is independent of τ, $\boldsymbol{k}$, $\boldsymbol{k}'$, $\{\beta_j\}$, we obtain

$$| U_n(t, \boldsymbol{k}) | \leqslant \sum_1^n{}_j \frac{B^j t^j}{j!} \| U(0, \boldsymbol{k}) \|_{L\infty}$$

where B is a constant independent of $\{\alpha_i\}$. Then we have convergence of $U_n(t, \boldsymbol{k})$ uniformly in $\boldsymbol{k}$, $\{\alpha_i\}$, $\{\beta_j\}$. In the same way one can estimate

$$\prod_{i=1}^s (1 + (k^i)^2)^p\, | U_n(t, \boldsymbol{k}) |\,.$$

This shows that $U_n(t, \boldsymbol{k})$ converges more quickly than any polynomial, uniformly in $\boldsymbol{k}$, and $\{\alpha_i\}$ $\{\beta_j\}$ as long as $\{\alpha_i\}$, $\{\beta_j\}$ vary in a compact neighborhood of zero. The same kind of estimate can also be made of the derivatives with respect to $\boldsymbol{k}$ of $U_n(t, \boldsymbol{k})$, giving the stated convergence in $S(\boldsymbol{R}^s)$.

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