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The Electromagnetic Vertex in SL(2, C)[1]

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Abstract. We derive a formula for the ratio of the electric to the magnetic form factors of a spin-1/2-particle in a model assuming SL(2, C) symmetry.

Introduction

In recent publications[2] the BUDINI-FRONSDAL [4] symmetry group $G = P' \otimes \mathrm{SL}(n, \mathrm{C})$ has been applied to three particle vertices. One particle states are represented in the space of the tensor product of the spin zero representation of the Poincaré group P' (with appropriate mass) with a unitary irreducible representation (UIR) of SL(n, C), the latter describing spin and intrinsic quantum numbers. For explicit calculations serveral methods have been proposed [1, 5, 6]. A method originally advocated in Ref. [5] used the canonical basis of SL(2, C) representations and leads to infinite series. Another technique, developped in Ref. [1], succeeded in exhibiting explicit covariance by means of homogeneous variables and leads to integrals which in general have to be treated numerically. Apart from applications given in Ref. [1b], this method has been applied [2] to the ratio of the magnetic to the electric form factor of a spin-1/2-particle in a model assuming $n = 2$. In the present paper we solve the same model, however making use of the 'canonical basis method' in order to get an explicit formula for this ratio. Our purpose is to exhibit the technical features of deriving the desired formula (18) rather than any discussion of fundamental questions associated with the symmetry group G, which can be found e.g. in Ref. [1] together with general comments concerning the representations to be used. – We should like to point out that BAMBERG [3] recently gave the closed solution of a related but considerably more complicated problem ($n = 6$) than the simpler case treated here ($n = 2$).

Canonical Basis of SL(2, C) Representations

J_i and N_i denote the generators of SL(2, C) ($i = 1, 2, 3$), where the J_i generate rotations and the N_i are associated with pure velocity transformations

$$
\begin{aligned}
[J_i, J_k] &= i\, \varepsilon_{ikl}\, J_l \\
[J_i, N_k] &= i\, \varepsilon_{ikl}\, N_l \\
[N_i, N_k] &= -i\, \varepsilon_{ikl}\, J_l .
\end{aligned}
\tag{1}
$$

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[2]) See Ref. [1, 2, 3], more applications can be found in the bibliography of these papers.

The UIR of SL(2, C) are characterized [7][3] by two parameters $[j_0, \varrho]$. The possible values of j_0 are 0, 1/2, 1, ... and the continuous variable ϱ defines the invariant

$$\frac{1}{4} \varrho^2 = j_0^2 - 1 + \Sigma N_i^2 - \Sigma J_i^2 . \tag{2}$$

It has been shown [8] that a parity operator mapping a UIR $[j_0, \varrho]$ onto itself exists only for the type $[j_0, 0]$; therefore, we restrict ourselves to these representations ($\varrho = 0$) henceforth. The canonical basis is obtained by diagonalizing $\boldsymbol{J}^2$ and J_3 (eigenvalues $j(j+1)$ and m); then, to each pair j, m in the range ($m = -j, -j+1, \ldots, j$; $j = j_0, j_0 + 1, \ldots$) there exists exactly one canonical basis state $|j, m\rangle_{j_0}$ of the UIR characterized by j_0. The action of the generators N_i is much simplified by the choice $\varrho = 0$ and reads [7][3]

$$N_3 \, |j, m\rangle_{j_0} = B_{j_0}(j+1, m) \, |j+1, m\rangle_{j_0} + B_{j_0}(j, m) \, |j-1, m\rangle_{j_0}$$
$$B_{j_0}(j, m) = (j^2 - j_0^2)^{1/2} \, (j^2 - m^2)^{1/2} \, (4 j^2 - 1)^{-1/2} . \tag{3}$$

The action of the J_i onto the states $|j, m\rangle_{j_0}$ is given according to the $(2j+1)$-dimensional representation of the rotation group.

One Particle States

For the description of the spin-1/2-baryon we have to choose $j_0 = 1/2$, this being the only representation that contains the spin-1/2-particle (note that this UIR also includes a particle of spin 3/2, 5/2, ...). The spin-1/2-baryon at rest with spin-z-component m is then represented by the direct product

$$|\boldsymbol{0}, m\rangle = |\boldsymbol{0}\rangle_M \otimes |1/2, m\rangle_{1/2} , \tag{4}$$

where $|\boldsymbol{0}\rangle_M$ originates from the representation of P' and describes a spin zero particle with mass M at rest. Applying a pure velocity transformation to (4), we obtain the state vector representing the particle with any momentum $\boldsymbol{p}$ (energy $p_0 = (M^2 + p^2)^{1/2}$, $p = |\boldsymbol{p}|$) and the same spin-z-component m:

$$|\boldsymbol{p}, m\rangle = |\boldsymbol{p}\rangle_M \otimes \exp(i \, \boldsymbol{a} \, \boldsymbol{N}) \, |1/2, m\rangle_{1/2} . \tag{5}$$

(The boost parameter $\boldsymbol{a}$ is parallel to $\boldsymbol{p}$, $M \cosh a = p_0$, $M \sinh a = p$, $a = |\boldsymbol{a}|$; subsequently we put $M = 1$). In any vertex function the spin zero state vectors $|\boldsymbol{p}\rangle_M$ will give rise to a spin-independent lorentzinvariant function f of the momentas; subsequently we omit the representations of P' keeping this function in mind.

As usual, the corresponding antiparticles are described by the complex conjugate representation; however, since $[j_0, 0]$ is unitary equivalent to its complex conjugate representation, we may treat antiparticles like the corresponding particles.

It has been proposed[4] to associate the (virtual) photon with the UIR $j_0 = 1$. Accordingly, the photon with vanishing three-momentum and spin-z-component m is assumed to transform like $|1, m\rangle_1$.

Evaluation of a Vertex Function

We consider the baryon-antibaryon-photon annihilation vertex in the rest frame of the photon (three-momentas $\boldsymbol{p}$, $-\boldsymbol{p}$, $\boldsymbol{0}$; spin-z-components m_1, m_2, m_3) and choose

[3] The symbols j_0, ϱ used here correspond to $1/2\,|m|$, ϱ in Ref. [2] and to j_0, 2λ in Ref. [7].

[4] See Ref. [2] for this specific assumption and Ref. [1b] for general comments.

the direction of the z-axis such that $\boldsymbol{p} = (0, 0, p)$. Assuming invariance under G, the amplitude will then be given by

$$A(m_1\, m_2\, t) = f(t) \langle I| \{e^{i a N_3} |1/2, m_1\rangle_{1/2} \otimes e^{-i a N_3} |1/2, m_2\rangle_{1/2} \otimes |1, m_3\rangle_1\}. \quad (6)$$

$|I\rangle$ denotes the invariant state in the direct product of the three representations $[1/2, 0]$, $[1/2, 0]$, $[1, 0]$ and is defined by

$$\text{(a)} \quad \boldsymbol{J} |I\rangle = 0; \qquad \text{(b)} \quad \boldsymbol{N} |I\rangle = 0. \quad (7)$$

The arbitrary function $f(t)$ originates from the spin zero representations of P' and is determined by G only as far as it has to be a spin-independent lorentzinvariant function of the momentas ($\sqrt{t} = 2 p_0 = 2 \cosh a$, in our frame). Since $|I\rangle$ is determined uniquely by (7) (see Equations (8), (9)), the ratio of the two amplitudes corresponding to parallel and antiparallel spins m_1, m_2 respectively will be fixed by (6) and yield the ratio of the magnetic form factor $G_m(t)$ to the electric form factor $G_e(t)$ of the spin-1/2-baryon considered in this model.

The first factor in the ansatz [5]

$$|I\rangle = \sum_{\substack{j_1\, j_2\, j_3 \\ m_1 m_2 m_3}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} [j_1\, j_2\, j_3]_{1/2\, 1/2\, 1} |j_1, m_1\rangle_{1/2} \otimes |j_2, m_2\rangle_{1/2} \otimes |j_3, m_3\rangle_1 \quad (8)$$

denotes the usual SU(2) Wigner symbols and guarantees (7a). The 'reduced' Clebsch Gordan coefficients $[j_1, j_2, j_3]_{1/2\, 1/2\, 1}$ turn out to be uniquely determined by the requirement (7b) (apart from an arbitrary factor c): inserting (8) into (7b) gives recursion relations[5]) for the $[j_1, j_2, j_3]_{1/2\, 1/2\, 1}$, from which these coefficients may be computed with the result

$$[j', j, 1]_{1/2\, 1/2\, 1} = c\, (2j + 1)^{1/2}\, (2j)^{-1/2}\, (2j + 2)^{-1/2}\, \delta_{j, j'} \quad (j = 1/2, 3/2, \ldots). \quad (9)$$

As a next step, the state $\exp(i\, a\, N_3)\, |1/2, m\rangle_{1/2}$ has to be written in terms of canonical basis vectors:

$$e^{i a N_3} |1/2, m\rangle_{1/2} = \sum_{n=0}^{\infty} D_n(a)\, |n + 1/2, m\rangle_{1/2}; \quad m = \pm 1/2. \quad (10)$$

As J_3 commutes with N_3, no summation over m becomes necessary. Furthermore, since the coefficients in (3) are not affected by the sign of m, the functions $D_n(a)$ will not depend on m. The lowest coefficient $D_0(a)$ may be obtained by standard techniques [6], and the higher coefficients can then be computed easily from the recursion relations

$$B_{j_0}(j, m)\, D_{j-3/2}(a) + B_{j_0}(j + 1, m)\, D_{j+1/2}(a) + i \frac{d}{da} D_{j-1/2}(a) = 0 \quad (11)$$

which follow by differentiating (10) with respect to a and applying (3), with the result

$$D_n(a) = 4\, i^n\, (n+1)^{1/2}\, (e^a - e^{-a})^n\, (e^a + 2 + e^{-a})^{-n-1} = 2\, i^n\, (n+1)^{1/2} p^n\, (1 + p_0)^{-n-1}. \quad (12)$$

Inserting (12), (10) and (9) into (6), one gets

$$A(m_1, m_2, t) =$$

$$c\, f(t) \sum_{n=0}^{\infty} 4\, (n + 1) \sqrt{\frac{(2n+2)}{(2n+1)\,(2n+3)}} \begin{pmatrix} n + 1/2, & n + 1/2, & 1 \\ m_1, & m_2, & m_3 \end{pmatrix} \frac{p^{2n}}{(1 + p_0)^{2n+2}}, \quad (13)$$

[5]) See Ref. [5]. Note that the phase conventions for the states $|j, m\rangle_{j_0}$ adopted here (see our Equation (3)) are those of Ref. [7] and do not correspond to those in Ref. [5].

and from this expression, using the explicit formulas for the Wigner symbols involved and carrying out the summations, we obtain

$$A\,(\pm 1/2, \pm 1/2, t) = -c(2)^{1/2} f(t)\,(1+p_0)^{-2} \times \{(1+r^2)^{-1} + 1/2\, r^{-3}\,(1+r^2)\,\mathrm{arctg}\, r - 1/2\, r^{-2}\}$$

$$A\,(\pm 1/2, \mp 1/2, t) = c\, f(t)\,(1+p_0)^{-2}\, r^{-3}\,\{r - (1-r^2)\,\mathrm{arctg}\, r\}\,, \tag{14}$$

where

$$r = p\,(1+p_0)^{-1}\,, \qquad r^2 = \left(\sqrt{t}-2\right)\left(\sqrt{t}+2\right)^{-1}.$$

On the other hand, from the general expression for the amplitude in terms of the Sachs form factors $G_m(t)$ and $G_e(t)$

$$A(m_1, m_2, t) = \bar{v}(-p, m_2)\left[F_1 \gamma^\mu + \frac{1}{4} F_2\,(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)\,(p_2 + p_1)_\nu\right] u(p, m_1)\, e_\mu(m_3) \tag{15}$$

$$G_m(t) = F_1(t) + F_2(t)\,, \qquad G_e(t) = F_1(t) + p_0^2\, F_2(t) \tag{16}$$

we have

$$A\,(\pm 1/2, \pm 1/2, t) = (2)^{1/2}\, 2\, p_0\,(1+p_0)\, G_m(t)$$

$$A\,(\pm 1/2, \mp 1/2, t) = -2\,(1+p_0)\, G_e(t)\,. \tag{17}$$

Note that (15) vanishes as well as (13) unless $m_3 = -m_1 - m_2$. Comparison of (17) with (14) finally gives the desired ratio

$$\frac{G_m(t)}{G_e(t)} = \frac{r^3}{p_0}\,\frac{(1+r^2)^{-1} + 1/2\, r^{-3}\,(1+r^2)\,\mathrm{arctg}\, r - 1/2\, r^{-2}}{r - (1-r^2)\,\mathrm{arctg}\, r}\,, \tag{18}$$

which may be compared directly with the results of a more general numerical technique as used in Ref. [2], to which paper we refer for a graphical representation[6]) of the ratio (18).

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