

Zeitschrift: Helvetica Physica Acta
Band: 41 (1968)
Heft: 1

Artikel: On the derivation and commutation of operator functionals
Autor: Guenin, Marcel
DOI: <https://doi.org/10.5169/seals-113875>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 02.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

On the Derivation and Commutation of Operator Functionals

by Marcel Guenin

Institut de Physique Théorique, Université de Genève

(22. IX. 67)

Abstract. We give compact, formal expressions for $\partial/\partial s f(F(s))$ and $[H, f(A)]$, where $f(\cdot)$ is an arbitrary function analytic in some domain, and $H, A, F(s)$ are arbitrary operators.

Introduction

Operator functions appear every day in a theoretician work, and it is often difficult to compute explicitly their derivatives or their commutator with some other operator. R. WILCOX [1] has recently written a good review of new and old formulas. Since two key formulas which we did derive some time ago do not seem to be known, we think that it might be useful to make them more widely available. These formulas have, of course, only formal significance and their applicability has to be checked by the user.

Notations

We shall denote by $\Omega_n(A, B)$ the multiple commutator of A and B , defined recursively by

$$\Omega_0(A, B) = B \quad \Omega_{n+1}(A, B) = [A, \Omega_n(A, B)]$$

(an other frequently used notation is $\{A^n, B\} \equiv \Omega_n(A, B)$.)

Preliminary Formula

We first show that

$$A B^n = \sum_{j=0}^n \binom{n}{j} (-1)^j B^{n-j} \Omega_j(B, A) \quad (1)$$

and

$$B^n A = \sum_{j=0}^n \binom{n}{j} \Omega_j(B, A) B^{n-j} \quad (2)$$

where $\binom{n}{j}$ denotes the usual binomial coefficient. We proceed by induction, clearly, the formula holds for $n = 1$.

Suppose it to be true for $n - 1$, then

$$\begin{aligned} A B^{n-1} B &= \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j B^{n-j-1} \Omega_j(B, A) B \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j B^{n-j} \Omega_j(B, A) - \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j B^{n-j-1} \Omega_{j+1}(B, A) \\ &= \sum_{j=0}^n \binom{n}{j} (-1)^j B^{n-j} \Omega_j(B, A). \end{aligned}$$

The formula (2) is derived in exactly the same way.

1st Formula. We want to show that

$$\begin{aligned} \frac{\partial}{\partial s} f(F(s)) &= \sum_{j=0}^{\infty} f^{(j+1)}(F(s)) \frac{(-1)^j}{(j+1)!} \Omega_j \left(F(s), \frac{\partial}{\partial s} F(s) \right) \\ &= \sum_{j=0}^{\infty} \frac{1}{(j+1)!} \Omega_j \left(F(s), \frac{\partial}{\partial s} F(s) \right) f^{(j+1)}(F(s)). \end{aligned} \quad (3)$$

Where $f^{(k)}(F(s))$ is the k^{th} functional derivative of $f(F(s))$ with respect to $F(s)$.

We suppose $f(F(s))$ to be given by its power expansion $f(F(s)) = \sum_{n=0}^{\infty} a_n F(s)^n$ from which follows

$$\begin{aligned} \frac{\partial}{\partial s} \sum_{n=0}^{\infty} a_n F(s)^n &= \sum_{n=1}^{\infty} a_n \sum_{j=0}^{n-1} \binom{n}{j+1} (-1)^j F(s)^{n-j-1} \Omega_j \left(F(s), \frac{\partial}{\partial s} F(s) \right) \\ &= \sum_{n=1}^{\infty} a_n \sum_{j=0}^{n-1} \binom{n}{j+1} \Omega_j \left(F(s), \frac{\partial}{\partial s} F(s) \right) F(s)^{n-j-1} \\ &= \sum_{k=0}^{\infty} F(s)^k \sum_{l=0}^{\infty} a_{l+k+1} \binom{l+k+1}{l+1} (-1)^l \Omega_l \left(F(s), \frac{\partial}{\partial s} F(s) \right) \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{l+k+1} \binom{l+k+1}{l+1} \Omega_l \left(F(s), \frac{\partial}{\partial s} F(s) \right) F(s)^k \end{aligned} \quad (4)$$

and hence (3).

2nd Formula. It is as elementarily derived as the preceding one

$$[H, f(A)] = \sum_{j=1}^{\infty} f^{(j)}(A) \frac{(-1)^j}{j!} \Omega_j(A, H) = - \sum_{j=1}^{\infty} \frac{1}{j!} \Omega_j(A, H) f^{(j)}(A). \quad (5)$$

We have

$$\begin{aligned} \left[H, \sum_{n=0}^{\infty} a_n A^n \right] &= \sum_{n=1}^{\infty} a_n \sum_{j=1}^n \binom{n}{j} (-1)^j A^{n-j} \Omega_j(A, H) = - \sum_{n=1}^{\infty} a_n \sum_{j=1}^n \binom{n}{j} \Omega_j(A, H) A^{n-j} \\ &= \sum_{k=0}^{\infty} A^k \sum_{l=1}^{\infty} \binom{l+k}{l} (-1)^l a_{l+k} \Omega_l(A, H) \\ &= - \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \binom{l+k}{l} a_{l+k} \Omega_l(A, H) A^k. \end{aligned} \quad (6)$$

and hence (5).

For applications of these formulas to physical problems, we refer to coming publications of the author together with G. VELO.

Note added in proofs: I have been informed by R. WILCOX and W. BRITTIN that a formula equivalent to (3) has also been derived by W. E. BRITTIN and J. DREITLEIN. I thank R. WILCOX and W. BRITTIN for correspondence.

References

- [1] R. M. WILCOX, Jour. Math. Phys. 8, 962 (1967).