Zeitschrift: Helvetica Physica Acta
Band: 41 (1968)
Heft: 1

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DOI: https://doi.org/10.5169/seals-113875

On the Derivation and Commutation of Operator Functionals

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Abstract. We give compact, formal expressions for $\partial/\partial s\, f(F(s))$ and $[H, f(A)]$, where $f(.)$ is an arbitrary function analytic in some domain, and H, A, $F(s)$ are arbitrary operators.

Introduction

Operator functions appear every day in a theoretician work, and it is often difficult to compute explicitly their derivatives or their commutator with some other operator. R. Wilcox [1] has recently written a good review of new and old formulas. Since two key formulas which we did derive some time ago do not seem to be known, we think that it might be useful to make them more widely available. These formulas have, of course, only formal significance and their applicability has to be checked by the user.

Notations

We shall denote by $\Omega_n(A, B)$ the multiple commutator of A and B, defined recursively by

$$\Omega_0(A, B) = B \qquad \Omega_{n+1}(A, B) = [A, \Omega_n(A, B)]$$

(an other frequently used notation is $\{A^n, B\} \equiv \Omega_n(A, B)$.)

Preliminary Formula

We first show that

$$A\, B^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j\, B^{n-j}\, \Omega_j(B, A) \tag{1}$$

and

$$B^n\, A = \sum_{j=0}^{n} \binom{n}{j}\, \Omega_j(B, A)\, B^{n-j} \tag{2}$$

where $\binom{n}{j}$ denotes the usual binomial coefficient. We proceed by induction, clearly, the formula holds for $n = 1$.

Suppose it to be true for $n - 1$, then

$$\begin{aligned} A\, B^{n-1}\, B &= \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j\, B^{n-j-1}\, \Omega_j(B, A)\, B \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j\, B^{n-j}\, \Omega_j(B, A) - \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j\, B^{n-j-1}\, \Omega_{j+1}(B, A) \\ &= \sum_{j=0}^{n} \binom{n}{j} (-1)^j\, B^{n-j}\, \Omega_j(B, A)\,. \end{aligned}$$

The formula (2) is derived in exactly the same way.

1st Formula. We want to show that

$$\frac{\partial}{\partial s} f\big(F(s)\big) = \sum_{j=0}^{\infty} f^{(j+1)}\big(F(s)\big) \frac{(-1)^j}{(j+1)!} \Omega_j\Big(F(s), \frac{\partial}{\partial s} F(s)\Big)$$

$$= \sum_{j=0}^{\infty} \frac{1}{(j+1)!} \Omega_j\Big(F(s), \frac{\partial}{\partial s} F(s)\Big) f^{(j+1)}\big(F(s)\big) . \tag{3}$$

Where $f^{(k)}(F(s))$ is the k^{th} functional derivative of $f(F(s))$ with respect to $F(s)$.

We suppose $f(F(s))$ to be given by its power expansion $f(F(s)) = \sum_{n=0}^{\infty} a_n F(s)^n$ from which follows

$$\frac{\partial}{\partial s} \sum_{n=0}^{\infty} a_n F(s)^n = \sum_{n=1}^{\infty} a_n \sum_{j=0}^{n-1} \binom{n}{j+1} (-1)^j F(s)^{n-j-1} \Omega_j \Big(F(s), \frac{\partial}{\partial s} F(s)\Big)$$

$$= \sum_{n=1}^{\infty} a_n \sum_{j=0}^{n-1} \binom{n}{j+1} \Omega_j\Big(F(s), \frac{\partial}{\partial s} F(s)\Big) F(s)^{n-j-1}$$

$$= \sum_{k=0}^{\infty} F(s)^k \sum_{l=0}^{\infty} a_{l+k+1} \binom{l+k+1}{l+1} (-1)^l \Omega_l \Big(F(s), \frac{\partial}{\partial s} F(s)\Big)$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{l+k+1} \binom{l+k+1}{l+1} \Omega_l\Big(F(s), \frac{\partial}{\partial s} F(s)\Big) F(s)^k \tag{4}$$

and hence (3).

2nd Formula. It is as elementarily derived as the preceeding one

$$[H, f(A)] = \sum_{j=1}^{\infty} f^{(j)}(A) \frac{(-1)^j}{j!} \Omega_j(A, H) = -\sum_{j=1}^{\infty} \frac{1}{j!} \Omega_j(A, H) f^{(j)}(A) . \tag{5}$$

We have

$$\Big[H, \sum_{n=0}^{\infty} a_n A^n\Big] = \sum_{n=1}^{\infty} a_n \sum_{j=1}^{n} \binom{n}{j} (-1)^j A^{n-j} \Omega_j(A, H) = -\sum_{n=1}^{\infty} a_n \sum_{j=1}^{n} \binom{n}{j} \Omega_j(A, H) A^{n-j}$$

$$= \sum_{k=0}^{\infty} A^k \sum_{l=1}^{\infty} \binom{l+k}{l} (-1)^l a_{l+k} \Omega_l(A, H)$$

$$= -\sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \binom{l+k}{l} a_{l+k} \Omega_l(A, H) A^k. \tag{6}$$

and hence (5).

For applications of these formulas to physical problems, we refer to coming publications of the author together with G. VELO.

Note added in proofs: I have been informed by R. WILOX and W. BRITTIN that a formula equivalent to (3) has also been derived by W. E. BRITTIN and J. DREITLEIN. I thank R. WILCOX and W. BRITTIN for correspondence.

References

[1] R. M. WILCOX, Jour. Math. Phys. *8*, 962 (1967).