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Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **40 (1967)**

Heft 7

PDF erstellt am: **29.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-113799

Phenomenological Thermodynamics V: The 2nd Law Applied to Extensive Functionals with the Use of Lagrange Multipliers

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(30. V. 67)

Abstract. In most of our preceding articles (I–IV) (see References [1]–[4]), we split the 2nd law into two parts: a) the law of evolution of an adiabatically isolated system Σ ($= \Sigma_0$), and b) the law of equilibrium of an isolated Σ ($= \Sigma_{00}$). This article is mainly devoted to the 2nd law part b, where we demonstrate that the maximum of the entropy functional $S[\ldots]$ may be found by the use of Lagrange multipliers expressing the conservation of the energy functional $H[\ldots]$ and, in general, of several other conserved extensive quantities.

Introduction and Conclusions

In practically all textbooks, the sign of physical concepts in systems Σ, like absolute temperature T, heat conductivity $\varkappa$ and capacity c are assumed positive definite. Also the mechanical quantities (mass M, viscosities ξ and η, and the elastic moduli) are, as to the question of their sign, in most all treatises [5], taken mostly from empirical fact in *phenomenological theories* (= PT). But we have shown that these signs follow from the 2nd law if it is stated in a more precise manner. So we have solved the question, left open in Pauli's [6] standard article on relativity in the 'Encyclopädie der mathematischen Wissenschaften' (where he says: '...when the static compressibility ($\equiv a$) approaches Herglotz's and Lamla's limit the phenomenological equations will probably become incorrect...'), showing that the sound velocity in a fluid $c_{\|}(\boldsymbol{x}\, t) \leqslant c_{\text{light}}$.

However, the proof we gave of our 'Maximum theorem of the entropy $S(t)$' (Appendix to III) was too short and left some ambiguity.

Thus we devote the present paper to a more rigorous proof which should satisfy and physicist. Another paper, based on well-known mathematical methods is in preparation [7].

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1) Supported by the Swiss National Fund.

2) The present article is a thesis for the obtention of the 'Dr ès Sc.' degree by the University of Geneva.

In order to understand why the theorem, mentioned in the title, is a fundamental concept of phenomenological thermodynamics ($\equiv$ PT), we recall our earlier papers, I, II, III and IV (see bibliography), in which we determined the sign of all local state functions occurring in the equations of motion of a fluid [non-relativistic (Galilei covariant) ($\equiv$ n.r.), relativistic (restricted = Lorentz covariant, including time reversal) ($\equiv$ r.r.) and in general relativity ($\equiv$ g.r.)] in terms of the sign of absolute temperature T and of the signature of a positive or negative definite space metric $\{g_{ik}\} \lesseqgtr 0$[3]).

We split the 2nd law into two parts:

2nd law a: Principle of evolution.

If a system Σ is adiabatically insulated ($\Sigma = \Sigma_0$), there exists an extensive functional $S(t)$, the entropy of Σ, which increases monotonously with time t:

$$\dot{S}(t) = \frac{dS(t)}{dt} \geqslant 0 \text{ if } \Sigma = \Sigma_0 . \tag{0.1}$$

This law (2nd(a)) determines the sign of all 'frictional' state functions (such as heat conductivity $\varkappa$, the two viscosities ξ and η, coefficients of diffusion and the coefficients of velocity for chemical reactions) in terms of the sign of absolute temperature T. 2nd(a) is, however, not sufficient to determine the arrow of time ($-\infty \to t \to +\infty$) because time reversal $\underline{T}$ in PT is

$$\underline{T} = \begin{cases} 't &= -t \\ 'S('t) &= -S(t) \end{cases} \qquad (0.2\,\underline{T})\,^{4})^{5})$$

and leaves invariant (0.1)

$$'\dot{S}('t) = \frac{d'S('t)}{d't} \geqslant 0 \text{ if } \Sigma = \Sigma_0 \qquad ('0.1)$$

and gives to the arrow of time $'t$ the inverse direction ($+\infty \leftarrow {'t} \leftarrow -\infty$): past and future may thus be interchanged if only the 2nd law a is used. We shall call the two frames t ($'t$) ortho- (pseudo-) chronous frames. We thus have to add a

2nd law b: Principle of equilibrium.

If a system Σ is (totally) insulated ($\Sigma = \Sigma_{00}$), the entropy $S(t)$ increases monotonically with time t to a finite maximum value (which is generally only asymptotically reached) in the orthochronous frame:

$$\lim S\,(t \to +\infty) = S_{max} < +\infty \text{ if } \Sigma = \Sigma_{00} . \tag{0.2}$$

This 2nd law b determines the sign of inertial mass density m (= enthalpy in r.r. and g.r.) and of the elastic modulus a, again in terms of the sign of absolute temperature T and of the signature of the metric.

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3) If $a_{ik} = a_{(ik)}$ is symmetric, $\{a_{ik}\} \lesseqgtr 0$ stands for

$$a_{ik}\, x^i\, x^k \lesseqgtr 0 \quad (\boldsymbol{x} = \{x^i\} \neq 0) \tag{0.1*}$$

4) Notation: Primes to the left: t, $'t$, $''t, \ldots$ indicate a change of frame. Primes to the right: t, t', $t'', \ldots$ are different points on the t axis, while $'t$, $'t'$, $'t''$ are the corresponding values of the same points on the $'t$ axis.

5) In r.r. and g.r. (see III), $\overset{\cup}{S}[\tau()] = \int_{\tau(y)=0} (d\overset{\cup}{\sigma}_\alpha\, j^\alpha_S)\,(y)$, ($\alpha\,\beta = 1234$, $i\,k \ldots = 123$) is a pseudochronous scalar ($g_{i\,i} = -g_{44} = 1$; $g_{\alpha \neq \beta} = 0$) defined on a timelike hypersurface $\tau(y) = 0$ in x space-time $\{x^\alpha\}$, where $(d\overset{\cup}{\sigma}_\alpha\, d\overset{\cup}{\sigma}^\alpha)\,(y) > 0$ and $d\overset{\cup}{\sigma}_4\,(y) > 0$ in every frame.

1st law:

In order to define the two conditions 'adiabatically insulated' ($\Sigma = \Sigma_0$) and '(totally) insulated' ($\Sigma = \Sigma_{00}$), we have to apply the 1st law. It introduces the state functional $H(t)$, the (total, including kinetic) energy of Σ. The 1st law affirms, for a non-insulated system Σ

$$\dot{H}(t) \equiv P(t) = P_Q(t) + P_A(t) \tag{0.3}$$

where $P(t)$ is the power furnished to Σ, dividing itself into two terms:

1) 'non-geometrical' power

$$P_Q(t) = \frac{\delta Q(t)}{\delta t} = \oint_{V(t)} d^{d-1}\, P_Q(\boldsymbol{y}\, t) = -\oint_{V(t)} (d\sigma_i\, q^i)\, (\boldsymbol{y}\, t) \tag{0.4}$$

where $\delta Q(t)$ is the 'non-geometrical' infinitesimal energy, commonly called *heat*, furnished during the infinitesimal time interval δt, at an epoch t.

2) 'geometrical' power

$$\begin{aligned} P_A(t) = \frac{\delta A(t)}{\delta t} &= \oint_{V(t)} d^{d-1}\, P_A(\boldsymbol{y}\, t) + \int_{V(t)} d^d\, P_A(\boldsymbol{x}\, t) \\ &= \oint_{V(t)} (d\sigma_k\, \tau^k{}_i\, v^i)\, (\boldsymbol{y}\, t) + \int_{V(t)} (dV\, k_i\, v^i)\, (\boldsymbol{x}\, t) \end{aligned} \tag{0.5}$$

where $\delta A(t)$ is the 'geometrical' infinitesimal energy, commonly called *work*, furnished during the infinitesimal time interval δt, at an epoch t.

$V(t)$ is a region in d dimensional 'affine space' $\boldsymbol{x} = \{x^i\}: i\, k \ldots = 12 \ldots d$ (see Section 1), enclosed by a moving surface $C(\boldsymbol{y}, t) = 0$[6]. ($\boldsymbol{q} = \{q^i\}$) $(\boldsymbol{x}, t)$ is the density of heat current, ($d\boldsymbol{\sigma} = \{d\sigma_i\}$) $(\boldsymbol{y}, t)$ is a covariant vector, the covariant surface element of the boundary, $dV(\boldsymbol{x})$ is the (scalar) volume element, $\tau^i{}_k(\boldsymbol{x}, t)$ is the mixed tensor of tension, and ($\boldsymbol{k} = \{k_i\}$) $(\boldsymbol{x}, t)$ the density of an external force (at distance, e.g., gravitation): ($\boldsymbol{v} = \{v^i\}$) $(\boldsymbol{x}, t)$ [or $(\boldsymbol{y}, t)$] is the velocity of the fluid in $V(t)$ [or the velocity of the boundary $C(\boldsymbol{y}, t) = 0$]. Now we see why

$$\begin{aligned} \delta A(t) = \delta t\, P_A(t) &= \oint_{V(t)} d^{d-1}\, \delta A(\boldsymbol{y}\, t) + \int_{V(t)} d^d\, \delta A(\boldsymbol{x}\, t) \\ &= \oint_{V(t)} (d\sigma_i\, \tau^i{}_k\, \delta r^k)\, (\boldsymbol{y}\, t) + \int_{V(t)} (dV\, k_i\, \delta r^i)\, (\boldsymbol{x}\, t) \end{aligned} \tag{0.6}$$

is 'geometrical work' (furnished to Σ) because it is proportional to the d infinitesimal geometric displacements ($\delta t > 0$) δr^i

$$\delta \boldsymbol{r}(\boldsymbol{x}\, t)\ [\text{or } (\boldsymbol{y}\, t)] = \delta t\, \boldsymbol{v}(\boldsymbol{x}\, t)\ [\text{or } (\boldsymbol{y}\, t)] \tag{0.7}$$

while

$$d^{d-1}\, \delta Q(\boldsymbol{y}\, t) = -\, (d\boldsymbol{\sigma}, \boldsymbol{q})\, (\boldsymbol{y}\, t)\, \delta t \tag{0.8}$$

can never be reduced to a form '(generalized) force $\times$ (generalized) displacement'.

[6]) We use, generally, $\boldsymbol{x}$ for an interior point ($\boldsymbol{x} \in V(t)$), and $\boldsymbol{y}$ for a boundary point on $C(\boldsymbol{y}, t) = 0$.

[Note that the second Equations (0.4) and (0.5) apply to the case of a determined phase of a fluid of only one chemical component A. If more than one component $A\ B \ldots = 12 \ldots C$ is present, $\delta A(t)$ contains terms of the form $d^{d-1}A(\boldsymbol{y}, t) = \Sigma_A(d\boldsymbol{\sigma}\,\mu_A, \delta\boldsymbol{r}_A\, n_A)\,(\boldsymbol{y}, t)$, where $\mu_A(\boldsymbol{x}, t)$ is the chemical potential, $n_A(\boldsymbol{y}, t)$ the density of substance A, and $\delta\boldsymbol{r}_A(\boldsymbol{y}, t) = \delta t\,(\boldsymbol{v}_A - \boldsymbol{v})\,(\boldsymbol{y}, t)$ ($\boldsymbol{v}_A$ = velocity of A, $\boldsymbol{v}$ = velocity of the centre-of-mass). Now $\delta\boldsymbol{r}_A$ is a geometrical quantity, and n_A can also be determined by geometrical means (part of A in a well-defined crystalline state). Analogous considerations apply to electrodynamical work furnished to Σ.]

Now we are prepared to define *adiabatically insulated systems*

$$\Sigma = \Sigma_0 \text{ implies: } d^{d-1}\,P_Q(\boldsymbol{y}\,t) = -\,(d\boldsymbol{\sigma}, \boldsymbol{q})\,(\boldsymbol{y}\,t) = 0 \qquad \forall\,(\boldsymbol{y}, t)\,. \tag{0.9}$$

A (*totally*) *insulated system* is defined, by adding to (0.9)

$$\begin{matrix}\Sigma = \Sigma_{00} \text{ implies} \\ \text{(0.9) and}\end{matrix} \begin{cases} d^{d-1}\,P_A(\boldsymbol{y}\,t) = (d\sigma_k\,\tau^k{}_i\,v^i)\,(\boldsymbol{y}\,t) = 0 & \forall\ \boldsymbol{y} \in C(\boldsymbol{y}\,t) = 0 \\ d^d\,P_A(\boldsymbol{x}\,t) = (dV\,k_i\,v^i)\,(\boldsymbol{x}\,t) = 0 & \forall\ \boldsymbol{x} \in V(t)\,. \end{cases} \tag{0.10}$$

Thus Σ_{00} is the stronger condition

$$„\Sigma = \Sigma_{00}\text{''} \subset „\Sigma = \Sigma_0\text{''}\,. \tag{0.11}$$

From (0.10) follows

$$\dot{H}(t) = 0 \text{ for } \Sigma = \Sigma_{00} \tag{0.12}$$

which states the *conservation of energy*

$$H(t) = H' \text{ for } \Sigma = \Sigma_{00} \tag{0.13}$$ [7]

i.e., the (total) energy is a constant of motion. This property is, in analytical dynamics, quantum theory and g.r. (in r.r. approximation), closely connected to homogeneity of time t.

We pass now to d further constants of motion, closely connected to homogeneity of affine space: if the system $\Sigma = \Sigma_{00}$ is free to move in space (no 'container'), the quantity of movement $\boldsymbol{\Pi} = \{\Pi_i\}$ (also d extensive quantities) is conserved:

$$\dot{\Pi}_i(t) = 0 \qquad \Pi_i(t) = \Pi_i' \text{ for } \Sigma = \Sigma_{00}\,. \tag{0.14}$$ [7]

Furthermore, if a constant metric

$$g_{ik} = g_{(ik)};\ g_{ik}\,g^{kl} = \delta_i^l = g_i^l \tag{0.15}$$

of arbitrary signature[8]) is introduced, we have a space metric which is [for $(\boldsymbol{x}, t)$ independent metric] isotropic. Isotropy of metric space is again closely connected to the conservation of the $^1/_2\,d\,(d-1)$ independent components $M_{ik} = M_{[ik]}$ of angular momentum (also an extensive quantity)

$$\dot{M}_{ik}(t) = 0;\quad M_{ik}(t) = M_{ik}' \text{ for } \Sigma = \Sigma_{00}\,. \tag{0.16}$$

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[7]) $H', \Pi_i', \ldots$ (on the right-hand side) are constants of motion.

[8]) The definite signature follows (see III and IV) [$\{g_{ik}\} \gtrless 0$ or signature $(g..) = \pm\,(+1, +1, \ldots, +1 = 1)$] from 2nd(a) and from 2nd(b) and leads to Euclidean space in the n.r. and r.r. case.

Further, from Galilei covariance (see IV) and Newton axiom [see (0.18)] it follows that the total inert mass $M(t)$ is also a constant of movement for any Σ [$\boldsymbol{v}(\boldsymbol{x}, t)$ is the velocity of the centre-of-mass in $dV(\boldsymbol{x})$ if several chemical components $A, B, \ldots$ are present]. This is the Lavoisier Law for chemical reactions. It is also an extensive quantity:

$$\dot{M}(t) = 0 \qquad M(t) = M' \text{ for any } \Sigma . \tag{0.17}$$

Thus, if in a one-component fluid we take entropy density $s(\boldsymbol{x}, t)$, inert mass density $m(\boldsymbol{x}, t)$ and the d velocity components of $\boldsymbol{v}(\boldsymbol{x}, t)$ as $2 + d$ independent local state functions, the 2nd law a, plus the $1 + d + {}^1/_2\, d\,(d - 1) + 1$ conservation laws (generalized 1st law), respectively their Galilei covariant continuity laws (see IV), the $2 + d$ laws of evolution are given if the Newton axiom for the density of quantity of movement $\Pi_i(\boldsymbol{x}, t)$ is introduced using inert mass density $m(\boldsymbol{x}, t)$

$$\Pi_i(\boldsymbol{x}\, t) = (m\, v_i)\, (\boldsymbol{x}\, t) . \tag{0.18}$$

The extensive quantities of the generalized 1st law are simple integrals over densities

$$\begin{aligned}
H(t) &= \int\limits_{V(t)} (dV\, h)\, (\boldsymbol{x}\, t) \rightarrow \left(h\, [s\, m\, v.] = \frac{1}{2}\, m\, v_i\, v^i + u[s\, m]\right) (\boldsymbol{x}\, t)\,^{9)} \\
\Pi_i(t) &= \int\limits_{V(t)} (dV\, \pi_i)\, (\boldsymbol{x}\, t) \\
M_{ik}(t) &= \int\limits_{V(t)} (dV\, (x_i\, \pi_k - x_k\, \pi_i)\, (\boldsymbol{x}\, t) \rightarrow (\tau_{ik} = \tau_{(ik)})\, (\boldsymbol{x}\, t) \\
M(t) &= \int\limits_{V(t)} (dV\, m)\, (\boldsymbol{x}\, t) .
\end{aligned} \tag{0.19}$$

In the 2nd law the entropy has an analogous form:

$$S(t) = \int\limits_{V(t)} (dV\, s)\, (\boldsymbol{x}\, t) . \tag{0.20}$$

We repeat the equations of motion for completeness:

2nd law a:

$$\begin{aligned}
(\partial_t s + \operatorname{div}\, (s\, \boldsymbol{v} + \boldsymbol{j}_S))\, (\boldsymbol{x}\, t) &= (\dot{s} + s \operatorname{div} \boldsymbol{v} + \operatorname{div} \boldsymbol{j}_S)\, (\boldsymbol{x}\, t) \equiv i(\boldsymbol{x}\, t) \geqslant 0 \\
&= \left[T^{-1} \left(\tau^{ik(fr)(0)}\, v_{ik}^{(0)} + (-\, j_S^i\, \partial_i T) + \frac{1}{d}\, \tau_l^{l(fr)}\, v_k^k\right)\right] (\boldsymbol{x}\, t)\,^{10)}
\end{aligned}$$

$$(T[s\, m] \equiv \partial u[s\, m]/\partial s \equiv u,_s[s\, m])\, (\boldsymbol{x}\, t) \qquad (\mu[s\, m] \equiv \partial u[s\, m]/\partial m \equiv u,_m[s\, m])\, (\boldsymbol{x}\, t)$$

$$(2\, v_{ik} \equiv \partial_i\, v_k + \partial_k\, v_i = 2\, v_{(ik)})\, (\boldsymbol{x}\, t)$$

$$v_l^{l(0)}\, (\boldsymbol{x}\, t) = 0; \quad \tau_l^{l(fr)(0)} = 0^{11)} \quad (\boldsymbol{j}_S = T^{-1}\, \boldsymbol{q})\, (\boldsymbol{x}\, t) . \tag{0.22}$$

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9) $u[s\, m]$ = (Galilei invariant) density of internal energy.

10) $\dot{s}$, $\dot{m}$ and $\dot{v}_i$, see (0.24).

11) $(a_{ik}^{(0)} = a_{ik} - (1/d)\, g_{ik}\, a_l^l)(\boldsymbol{x}, t)$ is the traceless (irreducible) part of $(a_{ik} = a_{(ik)})(\boldsymbol{x}, t)$, $\tau^{ik(fr)}(\boldsymbol{x}, t)$ is the frictional, $-\, g^{ik}\, p(\boldsymbol{x}, t)$ is the elastic part of the tensor of tensions $(\tau^{ik} = \tau^{(ik)})\, (\boldsymbol{x}, t)$.

$T(\boldsymbol{x}, t)$ and $\mu(\boldsymbol{x}, t)$ are the local absolute temperature and the local chemical potential (per unit of mass).

The generalized 1st law leads further to:

$$\begin{aligned}
&(\partial_t m + \operatorname{div}(\boldsymbol{v}\, m) = \dot{m} + m \operatorname{div} \boldsymbol{v})\,(\boldsymbol{x}\, t) = 0^{12)} \\
&(m\,(\partial_t\, v_i + v^k\, \partial_k\, v_i) = m\, \dot{v}_i = \partial_k\, \tau^k{}_i)\,(\boldsymbol{x}\, t) = 0 \\
&(\tau^{i\,k} = -\, g^{i\,k}\, p + \tau^{i\,k(fr)})\,(\boldsymbol{x}\, t)^{13)} \\
&(p[s\, m] = s\, u,_s[s\, m] + m\, u,_m[s\, m] - u[s\, m] = s\, T + m\, \mu - u)\,(\boldsymbol{x}\, t) \\
&\qquad (\equiv \text{pressure})\,. \qquad (0.23)
\end{aligned}$$

All equations containing substantial fluxions $\dot{f}$ $[\partial_t f(\boldsymbol{x}, t) \equiv$ local fluxion$]$

$$(\dot{f} = \partial_t\, f + v^k\, \partial_k\, f)\,(\boldsymbol{x}\, t) \qquad (0.24)$$

contain but Galilei covariant terms [if $f(\boldsymbol{x}, t)$ is Galilei covariant]. The substantial acceleration $\dot{v}_i(\boldsymbol{x}, t)$ in (0.23) is equally Galilei covariant.

We repeat that with the exception of the Newton axiom (0.18) all the $2 + d$ Galilei covariant equations of motion follow from 2nd law a and the generalized 1st law [conservation laws for H, Π_i and $M_{i\,k} = M_{[i\,k]}$ and $M(t)$]. $i(\boldsymbol{x}, t) \geqslant 0$, in (0.22), is the (Galilei covariant) density of irreversibility or source density of entropy. The three terms are (Galilei covariant) bilinear terms (thermodynamical currents $\times$ thermodynamical forces) of an irreducible tensor, a vector and a scalar. The fluid being isotropic, the positive definiteness of $i(\boldsymbol{x}, t)$ requires that each term is positive definite[14]. Therefore

$$(\tau^{i\,k(fr)(0)} = +\, 2\, \eta[s\, m]\, v^{i\,k(0)})\,(\boldsymbol{x}\, t);\; (T^{-1}\, \eta)\, [s\, m]\, (\boldsymbol{x}\, t) \geqslant 0 \qquad (0.25)$$

$$(j^i_S = T^{-1}\, q^i)\,(\boldsymbol{x}\, t); \quad (q^i = \mp\, \varkappa[s\, m]\, \partial^i T)\,(\boldsymbol{x}\, t);\; \varkappa[s\, m]\, (\boldsymbol{x}\, t) \geqslant 0 \qquad (0.26)$$

$$(\tau^{l(fr)}_l = \pm\, \xi[s\, m]\, v^l_l)\,(\boldsymbol{x}\, t);\; (T^{-1}\, \xi)\, [s\, m]\, (\boldsymbol{x}\, t) \geqslant 0 \qquad (0.27)$$

according to the choice

$$\{g_{i\,k}\} \gtrless 0\,. \qquad (0.27a)$$

Therefore $\eta(\boldsymbol{x}, t)$ ('transverse' viscosity) and $\xi(\boldsymbol{x}, t)$ ('longitudinal' viscosity) have the sign of $T(\boldsymbol{x}, t)$; $\varkappa(\boldsymbol{x}, t)$ (heat conductivity) is always positive definite if we choose $\{g_{i\,k}\} \gtrless 0$[15].

Now, in the equations of movement, written in linear approximation, we decompose

$$(\boldsymbol{v} = \boldsymbol{v}_\perp + \boldsymbol{v}_\parallel = \mathbf{rot}\, \boldsymbol{a} - \mathbf{grad}\, \varphi)\,(\boldsymbol{x}\, t) \quad \operatorname{div} \boldsymbol{v}_\perp = 0 \quad \mathbf{rot}\, \boldsymbol{v}_\parallel = 0 \qquad (0.28^{16)})$$

1) transverse 'waves'

$$(m_0\, \partial_t\, \boldsymbol{v}_\perp - \eta_0\, \Delta\, \boldsymbol{v}_\perp)\,(\boldsymbol{x}\, t) = 0 \qquad \Delta \equiv \partial_i\, \partial^i \qquad \partial^i = g^{i\,k}\, \partial_k \qquad (0.29)$$

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[12]) See p. 891, footnote [10]).

[13]) See p. 891, footnote [11]).

[14]) This requires, first of all, that $\{g_{i\,k}\} \gtrless 0$, i.e., a definite metric. Thus space is Euclidean, and we may choose, without loss of generality $\{g_{i\,k}\} > 0$ and pose: $g^{i\,k} = g^k_i = +\delta^k_i$, $\overleftarrow{\boldsymbol{a}} = \vec{\boldsymbol{a}}$, etc.

[15]) See p. 892, footnote [14]).

[16]) This is written for $d = 3$. An analogous decomposition is possible for any $d > 1$.

2) heat 'waves' [if $\boldsymbol{v}_{\|}(\boldsymbol{x}, t) = 0$]

$$(c_0\, \partial_t s - \varkappa_0 \Delta\, s)\; (\boldsymbol{x}\; t) = 0 \tag{0.30}$$

with

$$(c = T/T_{,s})\; [s\; m]\; (\boldsymbol{x}\; t) \tag{0.31}$$

(heat capacity per unit volume)

3) longitudinal waves [if $\varkappa(\boldsymbol{x}, t) = 0$]

$$\left(m_0\, \partial_t^2\, \boldsymbol{v}_{\|} - a_0 \Delta\, \boldsymbol{v}_{\|} - \left(\xi + \frac{d-2}{d}\, \eta\right)_0 \Delta\, \partial_t\, \boldsymbol{v}_{\|}\right)\; (\boldsymbol{x}\; t) = 0 \tag{0.32}$$

with

$$a[s\; m] = (s^2\, u_{,ss} + 2\, s\, m\, u_{,sm} + m^2\, u_{,mm})\; [s\; m]\; (\boldsymbol{x}\; t)\; . \tag{0.33}$$

(isentropic elastic modulus of compression)

The index $_0$ in m_0, η_0, ξ_0, $\varkappa_0$, c_0 and a_0 is the constant value of the fluid at rest [$\boldsymbol{v}(\boldsymbol{x}, t) = 0$] in 'equilibrium' [$s(\boldsymbol{x}, t) = s_0$, $m(\boldsymbol{x}, t) = m_0$, etc.].

Now, these equations cannot be discussed unless the yet undetermined signs of m, $c[s\; m]$ and $a[s\; m]$ are given in terms of the sign of T.

It is our 2nd law b which gives us this lacking information.

In all cases considered so far[17]), the equilibrium is stationary (more exactly static)

$$\partial_t\big(m(\boldsymbol{x}\; t),\; s(\boldsymbol{x}\; t),\; v^{\cdot}(\boldsymbol{x}\; t)\big) = 0 \qquad \partial_t\; V(t) = 0 \rightarrow V(t) = V\; . \tag{0.34}$$ [18])

Therefore S, H, Π_i, M_{ik} are functionals of the $d + 2$ local state functions $s(\boldsymbol{x})$, $m(\boldsymbol{x})$ and $\{v_i(\boldsymbol{x})\} \equiv v.(\boldsymbol{x})$[19]). V is a time independent region[19]). Thus we write

$$S[s(\,)] = \int_V (dV\; s)\; (\boldsymbol{x})$$

$$H[s(\,),\, m(\,),\, v.(\,)] = \int_V (dV\; h)\; (\boldsymbol{x})$$

$$\Pi_i[m(\,),\, v.(\,)] = \int_V (dV\; \pi_i)\; (\boldsymbol{x})$$

$$M_{ik}[m(\,),\, v.(\,)] = \int_V (dV\; \mu_{ik})\; (\boldsymbol{x})$$

$$M[m(\,)] = \int_V (dV\; m)\; (\boldsymbol{x})$$

$$(\mu_{ik} = x_i\, \pi_k - x_k\, \pi_i)\; (\boldsymbol{x})\; . \tag{0.35}$$

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[17]) It is an open question to us whether non-stationary equilibria $\partial_t(m(\boldsymbol{x}, t)\ldots) \neq 0$ exist. They certainly do exist for solid (or rigid) bodies, where the constant angular momentum M'_{ik} is not proportional to ω_{ik}.

[18]) $V(t)$ is an extensive quantity if $V(t)$ symbolizes also the volume of the region $V(t) = \int_{V(t)} dV(\boldsymbol{x})$, with density $v(\boldsymbol{x}) = 1$. Therefore we should write $V(t) = V'$. Or, for certain reasons we omit here the prime.

[19]) See p. 893, footnote [17]).

Now, a necessary condition for the maximum of a functional is that it has a stationary value (extremum)

$$S[s(\,)] = \text{extr.} \tag{0.36}$$

under the $1 + d + {}^1/_2\, d\,(d-1) + 1$ functional constraints

$$\begin{aligned} &H[s(\,),\, m(\,),\, v.(\,)] = H' \\ &\Pi_i[m(\,),\, v.(\,)] = \Pi_i' \\ &M_{ik}[m(\,),\, v.(\,)] = M_{ik}' \\ &M[m(\,)] = M'\,. \end{aligned} \tag{0.37}$$

This extremum can be found by the use of $(1 + d + {}^1/_2\, d\,(d-1) + 1)$ LM's, using the symbols ϑ, $-\zeta^i$, $-\omega^{ik} = -\omega^{[ik]}$ and $-\beta$. We form the extremum, by postulating stationarity for

$$\begin{aligned} \Psi[s(\,),\, m(\,),\, v.(\,)] &= \left(S + \vartheta H - \zeta^i \Pi_i - \frac{1}{2}\,\omega^{ik} M_{ik} - \beta\, M\right)[s(\,),\, m(\,),\, v.(\,)] \\ &= \int_V (dV\,\psi)\,(\boldsymbol{x}) \overset{\text{must}}{=} \text{extr.}\,^{20)} \end{aligned} \tag{0.38}$$

with the density

$$\begin{aligned} \psi(\boldsymbol{x}) = s(\boldsymbol{x}) + \vartheta \left(\frac{1}{2}\, m(\boldsymbol{x})\, g^{ik}(v_i\, v_k)\,(\boldsymbol{x}) + u[s(\boldsymbol{x}), m(\boldsymbol{x})]\right) \\ - \zeta'^{\,i}(\boldsymbol{x})\, m(\boldsymbol{x})\, v_i(\boldsymbol{x}) - \beta\, m(\boldsymbol{x})\,. \end{aligned} \tag{0.39}$$

The antisymmetry of ω^{ik} allows to write the ζ^i and ω^{ik} terms as a single term

$$\zeta'^{\,i}(\boldsymbol{x}) = \zeta^i + x_k\, \omega^{ki}\ [\boldsymbol{\zeta}'(\boldsymbol{x}) = \boldsymbol{\zeta} + [\boldsymbol{\omega} \wedge \boldsymbol{x}] \text{ for } d = 3]\,. \tag{0.40}$$

Stationarity requires that the first variation [linear in $\delta s(\boldsymbol{x})$, $\delta m(\boldsymbol{x})$ and the $\delta v_i(\boldsymbol{x})$] vanishes for arbitrary and small variations $\delta s(\boldsymbol{x})$, $\delta m(\boldsymbol{x})$ and $\delta v_i(\boldsymbol{x})$

$$\delta^{(1)}\, \Psi[\ldots] = \int_V dV\,(\psi,_s\, \delta s + \psi,_m\, \delta m + \psi,^i\, \delta v_i)\,(\boldsymbol{x}) = 0\,. \tag{0.41}$$

This implies

1) $$\psi,_s(\boldsymbol{x}) = (1 + \vartheta u,_s)\,(\boldsymbol{x}) \overset{\text{must}}{=} 0\,. \tag{0.42}$$

Thus

$$u,_s(\boldsymbol{x}) = T(\boldsymbol{x}) = -\,\vartheta^{-1} \equiv T\,. \tag{0.43}$$

The temperature of Σ is constant, or more exactly, 'Σ has a temperature T'.

2) $$\psi,^i(\boldsymbol{x}) = \vartheta(m\, v^i)\,(\boldsymbol{x}) - \zeta'^{\,i}(\boldsymbol{x})\, m(\boldsymbol{x}) \overset{\text{must}}{=} 0\,. \tag{0.44}$$

Thus, if $m(\boldsymbol{x}) \neq 0$

$$v^i(\boldsymbol{x}) = \vartheta^{-1}\, \zeta'^{\,i}(\boldsymbol{x}) = \vartheta^{-1}\,(\zeta^i + x_k\, \omega^{ki}) \qquad [\boldsymbol{v}(\boldsymbol{x}) = \vartheta^{-1}\,(\boldsymbol{\zeta} + [\omega \wedge \boldsymbol{x}]) \text{ for } d = 3]\,. \tag{0.45}$$

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20) Normally one writes $\Psi[\cdots] = (S + \vartheta\,(H - H') - \cdots)\,[\cdots]$. But the LM's being kept constant, this form would differ from (0.38) but for a constant $+\vartheta H' - \zeta^i \Pi_i' \ldots$, which can be omitted.

The velocity of the fluid is a constant vector upon which a rotational motion with constant angular velocity is superposed.

3) $$\psi,_m(\boldsymbol{x}) = \vartheta\left(\frac{1}{2}\, v_i\, v^i + u,_m\right)(\boldsymbol{x}) - \zeta'^{\,i}(\boldsymbol{x})\, v_i(\boldsymbol{x}) - \beta \overset{\text{must}}{=} 0\,.$$

Substituting $\vartheta^{-1}\,\zeta'^{\,i}(\boldsymbol{x}) = v^i(\boldsymbol{x})$, and $u,_m(\boldsymbol{x}) = \mu(\boldsymbol{x})$, one obtains the result for the chemical potential

$$\mu(\boldsymbol{x}) = \mu_0 + \frac{1}{2}\,(v_i\, v^i)\,(\boldsymbol{x})\,; \quad \mu_0 = \vartheta^{-1}\,\beta \tag{0.46}$$

relation, which will be used later.

It is by no means proved that the solutions following from the stationarity condition satisfy the equations of motion. Therefore, let us write the equations of motion for $v^i(\boldsymbol{x}, t)$. It is, in its exact form, (0.23)

$$\left(m\,(\partial_t\, v_i + v^k\, \partial_k\, v_i) + \partial_i\, p - \partial_k\, \tau_i^{k(fr)}\right)(\boldsymbol{x}\, t) = 0 \tag{0.47}$$

or, from the definition of p in (0.23) follows

$$(\partial_i\, p = s\, \partial_i\, T + m\, \partial_i\, \mu)\,(\boldsymbol{x}\, t)\,. \tag{0.48}$$

In our case $\tau^{ik(fr)}(\boldsymbol{x})$ disappears because of (0.25) and $v_{ik}(\boldsymbol{x}) = 0$ from (0.45). Thus, as $\partial_t v_i(\boldsymbol{x}) = 0$, we are, on account of (0.23), left with

$$\begin{aligned}\Big(m\, v^k\, \partial_k\, v_i + \partial_i\, p &= m\, v^k\, \partial_k\, v_i + m\, \partial_i\, \mu = m\left(v^k\, \partial_k\, v_i + \partial_i\left(\frac{1}{2}\, v_k\, v^k\right)\right) \\ &= m\, v^k\, v_{ik}\Big)\,(\boldsymbol{x}) = 0\,.\end{aligned} \tag{0.49}$$

The equations of motion $s(\boldsymbol{x}, t)$ are given in (0.22). Because of $v_{ik}(\boldsymbol{x}) = 0$ $(\operatorname{div} \boldsymbol{v} = v_l^l)\,(\boldsymbol{x}) = 0$, $(j_{S\,i} = -\,T^{-1}\,\varkappa(\partial_i T))\,(\boldsymbol{x}) = 0$, the equation of motion is: $\dot{s}(\boldsymbol{x}) = 0$. This implies that the source density of entropy vanishes: $i(\boldsymbol{x}) = 0$ [because of $v_{ik}(\boldsymbol{x}) = 0$, $\partial_i T(\boldsymbol{x}) = 0$]. The same Equation (0.23) holds for $m(\boldsymbol{x}, t)$: $\dot{m}(\boldsymbol{x}) = 0$. Now, let us change from densities (of extensive state functions) to the intensive state functions $T(\boldsymbol{x}) = T$ and $\mu(\boldsymbol{x}) = \mu_0 + {}^1/_2(v^i\, v_i)\,(\boldsymbol{x})$ and write

$$s(\boldsymbol{x}) = s[T(\boldsymbol{x}), \mu(\boldsymbol{x})]\,; \quad m(\boldsymbol{x}) = m[T(\boldsymbol{x}), \mu(\boldsymbol{x})]\,. \tag{0.50}$$

As $T(\boldsymbol{x}) = T$ is a constant, we have, for $s(\boldsymbol{x})$:

$$\begin{aligned}\dot{s}(\boldsymbol{x}) &= s,_\mu(\boldsymbol{x})\, \dot{\mu}(\boldsymbol{x}) = s,_\mu(\boldsymbol{x})\left(v^k\, \frac{1}{2}\, \partial_k\,(v_i\, v^i)\right)(\boldsymbol{x}) \\ &= s,_\mu(\boldsymbol{x})\,(v^k\, v^i\, \partial_k\, v_i)\,(\boldsymbol{x}) = (s,_\mu\, v^i\, v^k\, v_{ik})\,(\boldsymbol{x}) = 0\end{aligned} \tag{0.51.s)[21]}$$

and an analogous equation for $m(\boldsymbol{x})$

$$\dot{m}(\boldsymbol{x}) = (m,_\mu\, v^i\, v^k\, v_{ik})\,(\boldsymbol{x}) = 0\,. \tag{0.51.m}$$

Thus, the stationary extremum is compatible with the equations of motion.

We come now to the essential part of our theorem:

if the first variation of $\Psi[\ldots]$ disappears (extremum) the second variation of $S[\ldots]$ under constraints is equal to the second variation of $\Psi[\ldots]$ (with the LM's being kept constant!).

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[21]) $(v^k\, v^i\, \partial_k\, v_i = v^{(k}\, v^{i)}\, \partial_{(k}\, v_{i)} = v^i\, v^k\, v_{ik})\,(\boldsymbol{x}) = 0.$ (0.51.*)

In formulae if

$$\delta^{(1)} \Psi[\ldots] = 0 \,, \quad \delta^{(2)}_{(c)} S[\ldots] = \delta^{(2)} \Psi[\ldots] = \leqslant 0 \tag{0.52}$$

$\delta^{(2)}_{(c)}$ indicates that the variation is restricted by the $1 + d + {}^1/_2\, d\,(d-1) + 1$ constraints (0.37). A necessary condition for the maximum is therefore

$$\delta^{(2)}_{(c)} S[\ldots] = \delta^{(2)} \Psi[s(),m(),v.()] = \int\limits_{\dot V} (dV\,(\psi,_{ss}(\delta s)^2 + 2\,\psi,_{sm}\delta s\,\delta m + \psi,_{mm}(\delta m)^2$$

$$+ \psi,^k_s\, \delta v_k\, \delta s + \psi,^k_m\, \delta v_k\, \delta m + \psi,^{ik}\, \delta v_i\, \delta v_k))\;(\boldsymbol{x}) \overset{\text{must}}{\leqslant} 0\,. \tag{0.53}$$

This implies that the $2 + d$ dimensional symmetric form is negative definite

$$\left\{\begin{array}{c|cc} \psi,^{ik} & \psi,^i_s & \psi,^i_m \\ \hline \psi,^k_s & \psi,_{ss} & \psi,_{sm} \\ \psi,^k_m & \psi,_{ms} & \psi,_{mm} \end{array}\right\}(\boldsymbol{x}) \leqslant 0 \quad \forall\, \boldsymbol{x} \in V\,. \tag{0.54 [22]}$$

Or, given $\psi(\boldsymbol{x})$ in (0,39), the only quadratic forms of interest are:

1) $$\{\psi,^{ik}\}\,(\boldsymbol{x}) = \vartheta m(\boldsymbol{x})\;\{g^{ik}\} \leqslant 0\,.$$

As

$$\{g^{ik}\} > 0$$

(Euclidean choice), we have $[T^{-1}(\boldsymbol{x}) = -\,\vartheta$, (0.43)]

$$-\,\vartheta m(\boldsymbol{x}) = \frac{m(\boldsymbol{x})}{T(\boldsymbol{x})} \geqslant 0 \tag{0.55}$$

thus mass density m has the sign of absolute temperature, and:

2) $$\begin{Bmatrix} \psi,_{ss} & \psi,_{sm} \\ \psi,_{ms} & \psi,_{mm} \end{Bmatrix}(\boldsymbol{x}) = \vartheta \begin{Bmatrix} u,_{ss} & u,_{sm} \\ u,_{ms} & u,_{mm} \end{Bmatrix}(\boldsymbol{x}) \leqslant 0\,. \tag{0.56}$$

Let us first consider the diagonal element of (0.56)

$$\psi,_{ss}(\boldsymbol{x}) = \vartheta u,_{ss}(\boldsymbol{x}) = -\,\frac{T,_s(\boldsymbol{x})}{T(\boldsymbol{x})} \leqslant 0\,. \tag{0.57}$$

Now, quasistatic furniture of heat to the unit volume, at constant 'geometrical variables' ($\delta \boldsymbol{v}(\boldsymbol{x}) = 0, \delta m(\boldsymbol{x}) = 0$), defines the heat capacity per unit volume (= density of heat capacity) $c(\boldsymbol{x})$:

$$\delta h(\boldsymbol{x}) = \delta u[s,\, m]\;(\boldsymbol{x}) = T(\boldsymbol{x})\;\delta s(\boldsymbol{x})$$

$$= T(\boldsymbol{x})\;T,^{-1}_s(\boldsymbol{x})\;\delta T(\boldsymbol{x}) \equiv c[s,\, m]\;(\boldsymbol{x})\;\delta T(\boldsymbol{x})\,. \tag{0.58 [23]}$$

Thus, (0.57) states

$$c[s,\, m]\;(\boldsymbol{x}) = \left(\frac{u,_s}{u,_{ss}}\right)[s,\, m]\;(\boldsymbol{x}) \geqslant 0\,. \tag{0.59}$$

Heat capacity is always positive definite.

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[22]) A (positive or) negative definite quadratic form requires that all diagonal minors of the corresponding determinant are ($\geqslant 0$) or $\leqslant 0$.

[23]) $\delta T[s\; m]\;(\boldsymbol{x}) = T,_s[s\; m]\;(\boldsymbol{x})\;\delta s(\boldsymbol{x}) \to \delta s(\boldsymbol{x}) = T,^{-1}_s[s\; m]\;(\boldsymbol{x})\;\delta T(\boldsymbol{x})$.

Next we consider the quadratic form corresponding to (0.56)

$$\vartheta\,(u_{,ss}\,s^2 + 2\,u_{,sm}\,s\,m + u_{,mm}\,m^2)\,(\boldsymbol{x}) = -\left(\frac{a}{T}\right)(\boldsymbol{x}) \leqslant 0 \tag{0.60}$$

in which, by the definition (0.33), $a[s, m]$ is the elastic (isentropic) modulus[24]). Thus we find that the elastic modulus has the sign of the absolute temperature $T[s, m]\,(\boldsymbol{x})$

$$c_{\|}^2(\boldsymbol{x}) \equiv \left(\frac{a}{T}\right)[s, m]\,(\boldsymbol{x}) \geqslant 0 \tag{0.61}$$

$c_{\|}^2(\boldsymbol{x}) = c_{\|}^2[s, m]\,(\boldsymbol{x})$ is the velocity square of longitudinal waves in the absence of friction.

Thus, the 2nd law, (a and b) combined, gives the sign of all constants in the linear approximations [(0.29), (0.30) and (0.32)] in terms of the absolute temperature $T[s, m]\,(\boldsymbol{x})$.

We remark further that the thermal quantities: heat conductivity $\varkappa[s, m]\,(\boldsymbol{x})$ and density of heat capacity $c[s, m]\,(\boldsymbol{x})$ are, independent of choice of sign of $T[s, m]\,(\boldsymbol{x})$, positive definite.

On the other hand, the geometrical quantities: mass $m(\boldsymbol{x})$, elastic modulus $a[s, m]\,(\boldsymbol{x})$, viscosities $\xi[s, m]\,(\boldsymbol{x})$ and $\eta[s, m]\,(\boldsymbol{x})$ have all the sign of $T[s, m]\,(\boldsymbol{x})$.

Therefore, in the linear approximations (0.29), (0.30) and (0.32) all coefficients $m = m_0$; $a[s_0, m_0] = a_0$ have, in the equations of motion, the same sign[25]) (see III and IV). We have given, in IV, Section 4, the linearized equations of motion for $\partial_t s(\boldsymbol{x}, t)$, $\partial_t \boldsymbol{v}_{\|}(\boldsymbol{x}, t)$[25]) and $\partial_t^2 \boldsymbol{v}_{\perp}(\boldsymbol{x}, t)$[25]) [which are, if $\varkappa(s_0, m_0) = \varkappa_0 > 0$ (not $= 0$) coupled, even in the r.r. case], the general solutions for the equations in terms of kernels and the initial values (for $t = 0$). The general solutions exist only for the future, leading exponentially to the static equilibrium $[\boldsymbol{v}(\boldsymbol{x}, t) \to 0;\ s(\boldsymbol{x}, t) \to s_0;\ m(\boldsymbol{x}, t) \to m_0$ for $\lim t \to +\infty]$, because the different kernels exist only for $t \geqslant 0$. Therefore, for the physicist, only the future exists (future evolution = 'one-sided' Laplace determinism).

In an experiment on Σ, he chooses arbitrary initial conditions $\Sigma(t')$ at an epoch, say, $t = t'$ ($t''' \equiv 0 =$ now!), and observes the evolution $\Sigma(t)$ ($t' \leqslant t \leqslant t'' \leqslant t'''$) during a period $\{t, t''\}$ in the past (of now). This gives to man the feeling of 'free will', because

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24) In the static equilibrium $\boldsymbol{v}(\boldsymbol{x}) = 0$, we have $s(\boldsymbol{x}) = s_0 = S/V$, $m(\boldsymbol{x}) = m_0 = M/V$ and $H = U = \int_V dV\, u[s\, m] = V\, u[S/V, M/V] = U[S\, M\, V]$ from which follows

$$p[S\,M\,V] = -\,\partial U[S\,M\,V]/\partial V = (u_{,s}\,S/V + u_{,m}\,M/V - u)\,[S/V, M/V]$$

and

$$-\,\partial p[S\,M\,V]/\partial V = \partial^2 U[S\,M\,V]/\partial V^2 = V^{-1}\,a[S/V, M/V]\,.$$

Thus $a[s, m]$ is the modulus of compression.

25) In the n.r. case this statement is correct. In the r.r. case, we have for $\varkappa[s, n]\,(\boldsymbol{x}, t) > 0$ the 'premonition' analogous to Dirac's classical theory of the point electron [8] [9] [10]:

$$M\,\dot{\boldsymbol{v}}(t) \to M\,(-\alpha^{-1}\,\ddot{\boldsymbol{v}}(t) + \dot{\boldsymbol{v}}(t)) = \boldsymbol{K}(z(t), t);\ \boldsymbol{v}(t) = \dot{\boldsymbol{z}}(t)$$

where $\alpha > 0$ is an enormously small positive constant of dimension $[\alpha] = [t]^{-1}$ ($\alpha^{-1} = 2/3\,(e^2/(M\,c_{\text{light}}^3)) > 0$). In our case, the equations of motion are, if $\varkappa \neq 0$, changed in the same way $(m_0\,\partial_t\,\boldsymbol{v} \to m_0\,(-\beta_0^{-1}\,\partial_t^2\,\boldsymbol{v} + \partial_t\,\boldsymbol{v}))\,(\boldsymbol{x}, t)$ with $\beta_0^{-1} = (\varkappa\,T\,m^{-1}\,c_{\text{light}}^{-4})_0 \gtrless 0$ $(T \gtrless 0)$. GRUBER [11] has shown that such 'premonitions' exist for the coupled system of linear equations of a n.r. and a r.r. fluid.

he is under the (right or wrong) impression that he was 'creating' (to a certain extent) at the past epoch t' such initial conditions [on a limited part Σ of the finite or infinite universe ($\equiv \Sigma_\infty$)] which assured to him the most favourable future of t' up to now $t''' = 0$.

In this context arises the question: what is the method of history to trace the past evolution ($-\infty \leqslant t \leqslant 0 =$ now) (cosmogony, geology, evolution of life and history of man)? The data of history $\Sigma'(0)$ are 'documents', available now ($t = 0$) ('present-day'[26]) composition of the cosmos, stratographical composition of the earth, fossiles, archeological and historical documents). All 'documents' are more or less exactly dated as to epochs $t', < t'', < t''', < 0$ ($=$ now). Now we choose for our Σ ($t' < 0$) initial conditions at $t = t'$ in such a manner that the present data (now $= 0 > t'$), calculated as the future $\Sigma(0)$ of Σ ($t' < 0$), coincide with the real state of affairs $\Sigma'(0)$. The past events are, in general, only approximately dated, comparison between the observed states $\Sigma'(t'')$, $\Sigma'(t''')$,... with the predicted values $\Sigma(t'')$, $\Sigma(t''')$,... gives an ever increasing consistency of history. This means that the numbers $t' < t'' < \ldots < t = 0$ become more and more exact.

Let us remark here that 'miracles' in the past of 'now' are by no means surprising: some particular initial state $\Sigma(0)$ (not the general state, of course) may admit solutions for a finite past interval $\{t', 0\} \to t' \leqslant t \leqslant 0$. However, at t' we find $\Sigma(t')$ in a most singular state, [$\delta(\boldsymbol{x})$ functions, their derivatives and their integrals ($=$ step functions)], for which no documents for their ($t < t'$) can exist: this seems to us to be the definition of the miracles.

Let us finally remark that there exists, independent of the sign of $T(\boldsymbol{x})$, an upper limit for elastic waves ($v_\parallel^2 = g_{ik}\, v^i\, v^k$) ($\boldsymbol{x}$) in r.r. (see III, Section 3):

$$v_\parallel^2(\boldsymbol{x}) = \frac{a}{m}\,[s, n]\,(\boldsymbol{x}) \leqslant c_{\text{light}}^2 \tag{0.62}$$

inert mass density $= (m[s, n] = w[s, n] = (u + p)\,[s, n])\,(\boldsymbol{x}) =$
$= (T\,s + \mu\,n)\,[s, n]\,(\boldsymbol{x}) =$ enthalpy density (in the orthochronous rest frame) (0.63)

and, equally independent of the sign of $T(\boldsymbol{x})$, a lower limit for entropy density:

$$s(\boldsymbol{x}) \geqslant 0 \tag{0.64}$$

which is a rudimentary form of the 3rd law (Nernst's principle).

For simplicity, let us now call the $d + 2 \equiv \omega$ local state variables $s(\boldsymbol{x})$, $m(\boldsymbol{x})$ and $v_i(\boldsymbol{x})$ [or any $d + 2$ ($=$ or $\neq \omega$) of ω other independent state local variables]

$$\xi^{\cdot}(\boldsymbol{x}) = \{\xi^\alpha(\boldsymbol{x})\}\,, \quad \alpha\,\beta\ldots = 12\ldots\omega \quad \forall\,\boldsymbol{x} \in V \tag{0.65}$$

and consider quite generally extensive functionals or 'density type' functionals

$$F[\xi^{\cdot}(\,)] = \int_V (dV\,f)\,(\boldsymbol{x}) \equiv \int_V d^d\,F(\boldsymbol{x}) \tag{0.66}$$

where

$$f(\boldsymbol{x}) = f[\xi^{\cdot}(\boldsymbol{x}), \boldsymbol{x}] = \frac{d^d F(\boldsymbol{x})}{d\,V(\boldsymbol{x})} \tag{0.67}$$

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[26]) i.e., as observed 'now' inside the hypercone of the past!

is the density of F at a point $\boldsymbol{x}$. Let us write the partial derivatives of densities with respect to local state functions, as

$$f_{,\alpha_1 \alpha_2 \ldots \alpha_k}(\boldsymbol{x}) \equiv f_{,\alpha_1 \alpha_2 \ldots \alpha_k}[\xi^{\cdot}(\boldsymbol{x}), \boldsymbol{x}] = \left(\frac{\partial^k f[\xi^{\cdot}(\boldsymbol{x}), \boldsymbol{x}]}{\partial \xi^{\alpha_1} \partial \xi^{\alpha_2} \ldots \partial \xi^{\alpha_k}}\right)_{\xi^\alpha = \xi^\alpha(\boldsymbol{x})} \tag{0.68}$$

$S[\xi^{\cdot}(\,)]$ and the $1 + d + {}^1/_2\, d\,(d-1) + 1$ $(= m)$[27]) constraints are of this type. Let us now, for generality, write F for S, and let

$$\xi^\alpha(\boldsymbol{x}) - \xi_0^\alpha(\boldsymbol{x}) = \delta\xi^\alpha(\boldsymbol{x}) \rightarrow \text{infinitesimal } \forall\, \alpha \tag{0.68'}$$

be the variation of $\xi^\alpha(\boldsymbol{x})$. We shall assume analyticity of $f[\xi^{\cdot}, \boldsymbol{x}]$ in the small domain corresponding to (0.68'). Let $\delta\xi^\alpha_{(c)}(\boldsymbol{x})$ be any variation compatible with the m constraints.

Then, the total compatible variation of $F[\xi^{\cdot}(\,)]$ must satisfy

$$\begin{aligned} \Delta_{(c)}\, F[\xi^{\cdot}(\,)] &= F[\xi^{\cdot}(\,)] - F[\xi_0^{\cdot}(\,)] \leqslant 0 \\ &= \sum_{k=1}^{\infty} \frac{1}{k\,!} \int_V (dV\, f_{,\alpha_1 \alpha_2 \ldots \alpha_k}\, \delta\xi^{\alpha_1}_{(c)}\, \delta\xi^{\alpha_2}_{(c)} \ldots \delta\xi^{\alpha_k}_{(c)})(\boldsymbol{x}) \equiv \sum_{k=1}^{\infty} \delta^{(k)}_{(c)}\, F[\xi^{\cdot}(\,)] \end{aligned} \tag{0.69}$$

in order that $\xi_0^{\cdot}(\boldsymbol{x})$ corresponds to a maximum. In particular, we speak of an extremum of order k and of a maximum of order k, if

$$(\delta^{(1)}_{(c)}\, F = \delta^{(2)}_{(c)}\, F = \cdots = \delta^{(2k-1)}_{(c)}\, F)\, [\xi^{\cdot}(\,)] = 0; \quad \delta^{(2k)}_{(c)}\, F[\xi^{\cdot}(\,)] < 0\,. \tag{0.70}$$

We write the m extensive constraints (of the 'density type') as functionals of the $\omega\ \xi^\alpha(\boldsymbol{x})$'s

$$G^a[\xi^{\cdot}(\,)] = \int_V (dV\, g^a)\,(\boldsymbol{x}) = G'^{\,a} \qquad a\, b \ldots = 12 \ldots m \tag{0.71}$$

$$g^a(\boldsymbol{x}) = g^a[\xi^{\cdot}(\boldsymbol{x}), \boldsymbol{x}]; \qquad g^{\,a}_{,\alpha_1 \alpha_2 \ldots \alpha_k}(\boldsymbol{x}) \quad \text{See (0.68)}\,. \tag{0.72}$$

These m constraints must be linearly independent.

Then we can prove that the conditions of extremum and maximum of order k can be written as

$$\begin{aligned} \delta^{(1)}_{(c)}\, F[\xi^{\cdot}(\,)] &= \delta^{(1)}\, \Psi[\xi^{\cdot}(\,)] = 0 \\ \delta^{(2)}_{(c)}\, F[\xi^{\cdot}(\,)] &= \delta^{(2)}\, \Psi[\xi^{\cdot}(\,)] = 0 \\ \delta^{(2k-1)}_{(c)}\, F[\xi^{\cdot}(\,)] &= \delta^{(2k-1)}\, \Psi[\xi^{\cdot}(\,)] = 0 \end{aligned} \tag{0.73}$$

and

$$\delta^{(2k)}_{(c)}\, F[\xi^{\cdot}(\,)] = \delta^{(2k)}\, \Psi[\xi^{\cdot}(\,)] < 0 \tag{0.74}$$

with

$$\Psi[\xi^{\cdot}(\,)] = F[\xi^{\cdot}(\,)] + \vartheta_a\, G^a[\xi^{\cdot}(\,)]\,. \tag{0.75}$$

Therefore we may state the following theorem:

The maximum of an extensive functional (= 'density type' functional), submitted to any finite number of constraints of the same type, is equal to the result found by Lagrange multipliers.

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[27]) m (not to be confused with $m(\boldsymbol{x})$, mass density) indicates the (finite) number of constraints.

Equations (0.73) and (0.74) imply

$$\psi_{,\alpha_1}(\boldsymbol{x}) = \psi_{,\alpha_1\alpha_2}(\boldsymbol{x}) = \cdots = \psi_{,\alpha_1\alpha_2\ldots\alpha_{2k-1}}(\boldsymbol{x}) = 0$$

$$\{\psi_{,\alpha_1\alpha_2\ldots\alpha_{2k}}\}(\boldsymbol{x}) < 0 \quad \forall\, \boldsymbol{x} \in V \tag{0.76}$$ [28]

k being an arbitrary high integer, we have proved our theorem, once (0.73), (0.74) and (0.76) are shown to hold.

1. The Variation $\delta^{(k)}_{(c)}$ under m Constraints $G^a = G'^a$

The Method of Elimination

We separate, out of the region V, m (= number of constraints) arbitrarily small and non-intersecting regions $V(a)$ in physical space ($\boldsymbol{x} = \{x^i\} \in V$)

$$V = V' \cup \left(\bigcup_1^m V(a)\right) \equiv V' + \sum_{1\,a}^{m} V(a)$$

$$V' \cap V(a) = \phi \quad V(a) \cap V(b) = \phi, \quad a \neq b; \quad \forall\, a, b\,. \tag{1.1}$$

We let the $V(a)$'s tend to such small regions

$$\lim V(a) \to \text{arbitrarity small} \neq dV(\boldsymbol{a}) \to 0 \tag{1.2}$$

such that the mean value theorem [we admit but continuous densities[29] $f(\boldsymbol{x})$, $g^a(\boldsymbol{x})$]

$$\int_{V(a)} (dV\, f)\,(\boldsymbol{x}) = V(a)\, f(\boldsymbol{x}_a^{(f)}); \;\; \boldsymbol{x}_a^{(f)} \in V(a)$$

$$\int_{V(a)} (dV\, g^b)\,(\boldsymbol{x}) = V(a)\, g^b(\boldsymbol{x}_a^{(g^b)}); \;\; \boldsymbol{x}_a^{(g^b)} \in V(a) \tag{1.3}$$

$$\lim V(a) \Rightarrow \boldsymbol{x}_a^{(f)}, \boldsymbol{x}_a^{(g^b)} \to \boldsymbol{x}_a \in V(a) \quad \forall\, a, b \tag{1.3a}$$

holds for all $a\, b \ldots = 12 \ldots m$; and, in the limit, the mean co-ordinates $\boldsymbol{x}_a^{(f)}$ and $\boldsymbol{x}_a^{(g^b)}$ tend all to the same limit point $\boldsymbol{x}_a (\forall\, a, b)$ ($\equiv \boldsymbol{a}$ or a).

Now, we choose, instead of the most general variation[29] $\delta\xi^\alpha(\boldsymbol{x})$, at first, the following particular variation:

$$\delta\xi^1(\boldsymbol{x}) = \begin{Bmatrix} \delta\xi^a = \text{const}^a \\ \text{arbitrary} \end{Bmatrix} \text{ for } \boldsymbol{x} \in \begin{Bmatrix} V(a) \\ V' \end{Bmatrix}, \;\forall\, a$$

$$\delta\xi^\alpha(\boldsymbol{x}) = \begin{Bmatrix} 0 \\ \text{arbitrary} \end{Bmatrix} \text{ for } \boldsymbol{x} \in \begin{Bmatrix} V(a) \\ V' \end{Bmatrix} \;\forall\, a,\; \alpha \neq 1\,. \tag{1.4a}$$ [30]

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[28]) $\{\psi_{,\alpha_1\alpha_2\ldots\alpha_{2k}}\}(\boldsymbol{x}) < 0$ is a negative definite form of order $2\,k$, i.e.,

$$\psi_{,\alpha_1\alpha_2\ldots\alpha_{2k}}(\boldsymbol{x})\, \eta^{\alpha_1}\eta^{\alpha_2}\ldots\eta^{\alpha_{2k}} < 0 \tag{0.76}$$

if the 'vector' $\eta^\cdot = \{\eta^\alpha\} \neq 0$.

[29]) As $f(\boldsymbol{x}) = f[\xi^\cdot(\boldsymbol{x}), \boldsymbol{x}]$ is a continuous function, $\delta\xi^\alpha(\boldsymbol{x}) = \xi^\alpha(\boldsymbol{x}) - \xi^\alpha_0(\boldsymbol{x})$ should also be continuous. So (1.4a) should be considered as being the limit of continuous functions.

[30]) See footnote [30]) page 901.

Now, the m constraints on $\delta F[\ldots]$, (0.71), imply (as V is kept constant):

$$\delta G^a[\xi^{\cdot}(\,)] \equiv \delta^{(1)}\, G^a[\xi^{\cdot}(\,)] = \int\limits_V (dV\, g_{,\alpha}^{\;a}\, \delta\xi^\alpha)\,(\boldsymbol{x})$$

$$= \sum_{b=1}^{m} V(b)\, g_{,1}^{\;a}\,(\boldsymbol{x}_b)\, \delta\xi^b + \int\limits_{V'} (dV\, g_{,\alpha}^{\;a}\, \delta\xi^\alpha)\,(\boldsymbol{x}) \overset{\text{must}}{=} 0\,. \tag{1.5a}$$

We thus have m equations to determine the m $\delta\xi^{b}$' s[30]). In order to do this, we consider the *matrix*, defined by:

$$\{g_{,b}^{\;a}\} \equiv \{g_{,1}^{\;a}(\boldsymbol{x}_b)\} \tag{1.6}$$

where the element depends but on the point $\boldsymbol{x}_b$ in physical space. As the m constraints (0.71) are linearly independent (see Section 0)[31]), the determinant of this matrix is not identically (for all $\boldsymbol{x} \in V$) equal to zero. Thus, an inverse matrix $g^{-1\,b}_{\;\;a}$ exists (whose elements depend on all m $\boldsymbol{x}^{a}$' s):

$$g_{,c}^{\;a}\, g^{-1\,c}_{\;\;b} = \delta^a_b\,. \tag{1.7a}$$

Thus, dividing by $\delta\xi^\alpha(\boldsymbol{x})$ ($\boldsymbol{x} \in V'$), we may define a functional derivation of a $\xi^a[\ldots]$

$$\frac{\delta\xi^a[\xi^{\cdot}(\,)]}{\delta\xi^\alpha(\boldsymbol{x})} \equiv \xi_{,\alpha\boldsymbol{x}}^{\;a}\,[\xi^{\cdot}(\,)] \quad \boldsymbol{x} \in V'$$

$$= -\,V(a)^{-1}\, g^{-1\,a}_{\;\;b}\,(\boldsymbol{x}_1 \ldots \boldsymbol{x}_m)\, g_{,\alpha}^{\;b}\,(\boldsymbol{x})\,. \tag{1.8}$$ [32])

Let us remark that this system of m functional derivative equations cannot be solved, i.e., $\xi^a[\xi^{\cdot}(\,)]$, $\boldsymbol{x} \in V'$ does not exist, unless we introduce the further restriction

$$\xi^1(\boldsymbol{x}) = \begin{Bmatrix} \xi^a = \text{const}^a \\ \text{arbitrary} \end{Bmatrix} \text{ for } \boldsymbol{x} \in \begin{Bmatrix} V(a) \\ V' \end{Bmatrix} \;\forall\, a \tag{1.9}$$

which is by no means necessary. Equation (1.8) would then be of the form

$$\xi_{,\alpha\boldsymbol{x}}^{\;a} = \xi_{,\alpha\boldsymbol{x}}^{\;a}\,[\xi^{\cdot}(\,),\, \boldsymbol{x}] \quad \boldsymbol{x} \in V'\,. \tag{1.10}$$

Still, we shall use (1.8) as a symbolic notation indicating that, if a variation of $\xi_{,\alpha\boldsymbol{x}}^{\;a}[\ldots]$ is performed, the m $\delta\xi^b$'s ($\forall\, a, b$) are m constants $^b \equiv \delta\xi^b$ [(1.4a) or (1.4b) (footnote [30]))] in Equation (1.3). For the procedure of elimination, we need but the variations $\delta_{(c)} = \delta^{(1)}_{(c)}, \delta^{(2)}_{(c)}, \ldots, \delta^{(2k)}_{(c)}$ of a symbolic functional

$$\hat{F}[\xi^{\cdot}(\,)] \equiv F[\xi^1[\xi^{\cdot}(\,)], \ldots, \xi^m[\xi^{\cdot}(\,)], \xi^{\cdot}(\,)] \quad \boldsymbol{x} \in V' \tag{1.11}$$

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[30]) These formulae imply that all $G^a[\ldots]$'s depend on $\xi^1(\boldsymbol{x})$. If this is not true, we may define for these $G^a[\ldots]$'s

$$\delta\xi^2(\boldsymbol{x}) = \begin{Bmatrix} \delta\xi^b = \text{const}^b \\ \text{arbitrary} \end{Bmatrix} \text{ for } \boldsymbol{x} \in \begin{Bmatrix} V(b) \\ V' \end{Bmatrix} \tag{1.4b*}$$

etc. The exact notation of (1.4a), (1.4b), ... is rather complicated, but we have found one which leads effectively to (1.8). Let us remember that this case is generally realized, as shown in the example of the n.r. fluid (in Section 0).

[31]) The linear combination $c_a\, g^{a}_{,b}(\boldsymbol{x}) \neq 0$ unless all $c_a = 0$ and for all $\boldsymbol{x} \in V$ except an ensemble of measure zero.

[32]) See footnote [30]).

which has the property that the free variations $\delta^{(k)}$ of the symbolic $\hat{F}[\ldots]$ (functional on V') are equal to the variations $\delta^{(k)}_{(c)}$ of $F[\ldots]$ (functional on V), restricted by the m constraints $G^a[\ldots] = G'^a$. In formula:

$$\delta^{(k)}\, \hat{F}[\xi^{\cdot}(\,)]\ (\boldsymbol{x} \in V') = \delta^{(k)}_{(c)}\, F[\xi^{\cdot}(\,)]\ (\boldsymbol{x} \in V)\,. \tag{1.12$^{(k)}$}$$

2. The 1st Variation $\delta^{(1)}_{(c)} F(\ldots)$

We shall show here that the well-known method of LM's applies to the 1st variation, and that (if higher variations are considered), the ϑ_a's figuring both in the expression for $\Psi[\ldots]$ and for $\hat{F}[\ldots]$ are well defined intensive state variables [such as, in our fluid example, temperature $(\vartheta = \tau = -\,T^{-1})$ $(\boldsymbol{x})$, linear $(\vartheta^{-1}\,\zeta^i = v^i_0)$ $(\boldsymbol{x})$ and angular $(\vartheta^{-1}\,\omega^{ik} = \omega^{ik}_0)$ velocities and chemical potential at the point where $v^i_0 + x_k\,\omega^{ki}_0 = v^i(\boldsymbol{x})$ disappears $(\vartheta^{-1}\,\beta = \mu\,[\boldsymbol{v}(\boldsymbol{x}) = 0])$] symbolically written

$$\vartheta_a = \vartheta_a[\xi^1, \ldots, \xi^m]\,. \tag{2.1}$$

In higher variations $\delta^{(k)}\hat{F}[\ldots] = \delta^{(k-1)}(\delta^{(1)}\hat{F})\,[\ldots]$ (functional on V'), they will be considered as such, varying, state variables. However, if the variations are made for $\Psi\,[\ldots]$,(functional on V), the LM's have to be kept constant.

1) *Procedure of Elimination*

We write, in analogy to (1.5a), and eliminating $\delta\xi^b$ by (1.5b),

$$\begin{aligned}
\delta^{(1)}_{(c)} F[\ldots] &= \sum_b V(b)\, f,_b\, \delta\xi^b + \int_{V'} (dV\, f,_\alpha)\ (\boldsymbol{x})\ \delta\xi^\alpha(\boldsymbol{x}) \\
&= \int_{V'} dV(\boldsymbol{x})\ (f,_\alpha(\boldsymbol{x}) + (-\,f,_b\, g^{-1\,b}{}_{a})\ (\boldsymbol{x}_1 \ldots \boldsymbol{x}_m)\ g,^a_\alpha(\boldsymbol{x}))\ \delta\xi^\alpha(\boldsymbol{x}) \equiv \delta^{(1)}\,\hat{F}[\ldots] \\
&= \int_{V'} (dV\,\hat{f},_\alpha)\ (\boldsymbol{x})\ \delta\xi^\alpha(\boldsymbol{x}) \equiv \int_{V'} dV(\boldsymbol{x})\ \hat{F},_{\alpha\boldsymbol{x}}\, \delta\xi^{\alpha\boldsymbol{x}} \overset{\text{must}}{=} 0\,, \quad \boldsymbol{x} \in V'\,.
\end{aligned} \tag{2.2}$$

The $\omega\ \delta\xi^\alpha(\boldsymbol{x}) = \delta\xi^{\alpha\boldsymbol{x}}$ being arbitrary, ω equations follow:

$$\begin{aligned}
\hat{F},_{\alpha\boldsymbol{x}}[\ldots] &= \hat{f},_\alpha(\boldsymbol{x}) = \hat{f},_\alpha[\xi^{\cdot}(\boldsymbol{x}),\, \boldsymbol{x}] \\
&\equiv f,_\alpha(\boldsymbol{x}) + \vartheta_a(\boldsymbol{x}_1 \ldots \boldsymbol{x}_m)\ g,^a_\alpha(\boldsymbol{x}) \overset{\text{must}}{=} 0 \quad \boldsymbol{x} \in V'
\end{aligned} \tag{2.3}$$

with

$$\vartheta_a = \vartheta_a(\boldsymbol{x}_1 \ldots \boldsymbol{x}_m) \equiv -\,(f,_b\, g^{-1\,b}{}_{a})\ (\boldsymbol{x}_1 \ldots \boldsymbol{x}_m)\,. \tag{2.4}$$

We now compare this result with:

2) *Procedure of LM's*

It states:

$$\begin{aligned}
\delta^{(1)}\,\Psi[\xi^{\cdot}(\,)] &= \int_V dV(\boldsymbol{x})\ \Psi,_{\alpha\boldsymbol{x}}\, \delta\xi^{\alpha\boldsymbol{x}} = \int_V (dV\,\psi,_\alpha\, \delta\xi^\alpha)\ (\boldsymbol{x}) \\
&= \int_V (dV\ (f,_\alpha + \vartheta_a\, g,^a_\alpha)\ \delta\xi^\alpha)\ (\boldsymbol{x}) \overset{\text{must}}{=} 0 \quad \boldsymbol{x} \in V
\end{aligned} \tag{2.5}$$

for an extremum of the first order. For our particular variation (1.4a) and (1.4b), where the m $\delta\xi^{a}$'s are now *also* arbitrary, this gives

$$\delta^{(1)} \Psi[\ldots] = \sum_b V(b)\,(f,_b + \vartheta_a\, g,^{a}_{b})\,(\boldsymbol{x}_1 \ldots \boldsymbol{x}_m)\,\delta\xi^b + \int_{V'} (dV\,(f,_\alpha + \vartheta_a\, g,^{a}_{\alpha})\,\delta\xi^\alpha)\,(\boldsymbol{x}) \overset{\text{must}}{=} 0 \tag{2.6}$$

from which follows first

$$V(b)\,(f,_b + \vartheta_a\, g,^{a}_{b})\,(\boldsymbol{x}_b) = 0\,. \tag{2.7}$$

This equation leads to (2.4). For points $\boldsymbol{x} \in V'$, Equation (2.6) leads to (2.3). Thus the well-known theorem that stationarity (of the first order) can be found by the use of LM's is proved. We need this particular 'physical demonstration'[33]) in order to calculate the higher ($\delta^{(k)}_{(c)}F[\ldots]$) variations[34]).

3. The 2nd Variation $\delta^{(2)}_{(c)}F\,(\ldots)$

We form from (2.2) and (2.3)

$$\delta^{(2)}_{(c)}\, F[\xi^{\cdot}] = \delta^{(2)}\,\hat{F}[\xi^{\cdot}] = \int_{V'} (dV\,\hat{f},_{\alpha\beta}\,\delta\xi^\alpha\,\delta\xi^\beta)\,(\boldsymbol{x})$$
$$= \int_{V'} dV(\boldsymbol{x})\,(\psi,_{\alpha\beta}(\boldsymbol{x}) + \vartheta_{a,\,b}[\xi^1[\xi^{\cdot}], \ldots \xi^m[\xi^{\cdot}]]\,\xi,^{b}_{\beta}(\boldsymbol{x})\,g,^{a}_{\alpha}(\boldsymbol{x}))$$
$$\times\,(\delta\xi^\alpha\,\delta\xi^\beta)\,(\boldsymbol{x}) \overset{\text{must}}{\leqslant} 0\,;\quad \boldsymbol{x} \in V' \tag{3.1}$$

where the 1st term of the integral in the fourth member of Equation (3.1) is due to the fact that we have written

$$f,_{\alpha\beta} + \vartheta_a[\ldots \xi^c[\xi^{\cdot}] \ldots]\,g^a_{,\alpha\beta} \equiv \psi,_{\alpha\beta}(\boldsymbol{x}) \qquad \forall\ \boldsymbol{x} \in V' \tag{3.2}$$

leaving the LM's unvaried. Now, from (1.7a) or (1.7b*), there follows from

$$(g,^{a}_{c}\, g^{-1}{}^{c}_{b}),_b\,[\ldots \xi^c \ldots] = (\delta^a_b),_b = 0 \tag{3.3}$$

the relation

$$g^{-1}{}^{c}_{a,b} = -\,g^{-1}{}^{c}_{f}\,g,^{f}_{eb}\,g^{-1}{}^{e}_{a}\,. \tag{3.4}$$

Thus, from (2.4), and as $g,^{f}_{ab} = \sum_b g,^{f}_{bb}\,\delta_{ab}$

$$\vartheta_{a,b} = -\,\psi,_{bc}\,g^{-1}{}^{c}_{a} = -\sum_b \psi,_{bb}\,\delta_{bc}\,g^{-1}{}^{c}_{a} = -\sum_b \psi,_{bb}\,g^{-1}{}^{b}_{a}\,. \tag{3.5}$$

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33) We use this term 'physical demonstration' in contradistinction to a rigorous proof of our theorem, to be published shortly by our collaborator, the mathematician Dr. J. PONCET, who states the exact mathematical restrictions for the validity of this theorem.

34) It is easily seen [writing $F,_{\alpha\boldsymbol{x}}$ (respectively $F,_b$) for $f,_\alpha(\boldsymbol{x})$ (respectively $f,_b(\boldsymbol{x}_b)$) and $G,^{a}_{\alpha\boldsymbol{x}}$ (respectively $G,^{a}_{b}$) for $g,^{a}_{\alpha}(\boldsymbol{x}_b)$ (respectively $g,^{a}_{b}(\boldsymbol{x})$)] that the use of LM's is valid for $\delta^{(1)}_{(c)}F$ for all functionals. It includes, however, contributions from surface terms. (We actually try to free ourselves from the 'density type' functionals in order to include action at distance for higher variations.)

Inserting (3.5) into (3.1), we find

$$\delta^{(2)}_{(c)} F[\ldots] = \delta^{(2)} \hat{F}[\ldots] = \int\limits_{V'} (dV\, \psi,_{\alpha\beta}\, \delta\xi^\alpha\, \delta\xi^\beta)\, (\boldsymbol{x}) + \sum_b V(b)\, \psi,_{bb}\, (\delta\xi^b)^2$$

$$\overset{*}{=} \delta^{(2)}\, \Psi[\ldots] = \int\limits_{V} (dV\, \psi,_{\alpha\beta}\, \delta\xi^\alpha\, \delta\xi^\beta)\, (\boldsymbol{x}) \overset{\text{must}}{\leqslant} 0\,. \tag{3.6}$$

The $\overset{*}{=}$ equality restricts $\delta^{(2)}\, \Psi[\ldots]$ to our particular variation (1.4a) or (1.4b*). This is stated as follows:

1st Theorem of LM's

If the 2nd variation of $\Psi[\ldots]$, making use of constant LM's, is smaller or equal to zero, the 2nd variation of the functional $F[\ldots]$ submitted to m constraints $G^a[\ldots] = G'^a$ satisfies the same inequality.

Thus, $\delta^{(2)}\, \Psi[\ldots] \leqslant 0$ is a *sufficient* condition for $\delta^{(2)}_{(c)} F[\ldots] \leqslant 0$. But we shall see that it is also a *necessary* one.

In order to do that, we use a particular variation in V', $(\delta_{(p)}\, \xi^\alpha(\boldsymbol{x}),\ \boldsymbol{x} \in V')$, separating a small $V(0)$ out of V'

$$\delta_{(p)}\, \xi^\alpha(\boldsymbol{x}) = \begin{Bmatrix} \delta\eta^\alpha/V(0) = \text{const}^\alpha \\ 0 \end{Bmatrix} \text{ for } \boldsymbol{x} \begin{smallmatrix} \in \\ \notin \end{smallmatrix} V(0) \subset V'\,. \tag{3.7[35]}$$

This gives us, from (1.4) – or (1.4b*) – (1.5a) and (1.5b), after elimination with the inverse matrix $\{g^{-1}{}^a_b\}$:

$$\delta_{(p)}\, \xi^a = -\, V(a)^{-1}\, g^{-1}{}^a_b \int\limits_{V'} (dV\, g,^b_\alpha\, \delta_{(p)}\, \xi^\alpha)\, (\boldsymbol{x}) = -\, V(a)^{-1}\, g^{-1}{}^a_b\, g,^b_\alpha\, (\boldsymbol{x}_0)\, \delta\eta^\alpha\,. \tag{3.8}$$

So we arrive at:

$$\begin{aligned} \delta^{(2)}_{(c\,p)} F[\ldots] &= \delta^{(2)}_{(p)} \hat{F}[\ldots] \\ &= \frac{1}{V(0)}\, \psi,_{\alpha\beta}\, (\boldsymbol{x}_0)\, \delta\eta^\alpha\, \delta\eta^\beta + \sum_b \frac{1}{V(b)}\, \psi,_{bb}\, (\boldsymbol{x}_b)\, ((g^{-1}{}^b_c\, g,^c_\alpha)\, (\boldsymbol{x}_0\, \boldsymbol{x}_1 \ldots \boldsymbol{x}_m)\, \delta\eta^\alpha)^2 \\ &= \delta^{(2)}_{(p)}\, \Psi[\ldots] \overset{\text{must}}{\leqslant} 0\,. \end{aligned} \tag{3.9}$$

As the $m+1$ small regions $[V(0), V(1), \ldots, V(b), \ldots, V(m), \subset V]$ are all *arbitrary positive*, and as their $m+1$ 'locations' $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_b, \ldots, \boldsymbol{x}_m)$ are also $m+1$ *arbitrary points* $\in V$, we conclude that the following quadratic form must be negative definite

$$\psi,_{\alpha\beta}(\boldsymbol{x})\, \eta^\alpha\, \eta^\beta \leqslant 0 \qquad \{\psi,_{\alpha\beta}(\boldsymbol{x})\} \leqslant 0 \qquad \boldsymbol{x} \in V \tag{3.10}$$

(3.10) being sufficient and necessary, we can conclude, for any arbitrary variation $\delta\xi^{\cdot}\, (\boldsymbol{x})$

$$\begin{aligned} \delta^{(2)}_{(c)} F[\ldots] &= \delta^{(2)}\, \Psi[\ldots] \\ &= \int\limits_{V} dV(\boldsymbol{x})\, \psi,_{\alpha\beta}\, [\xi^1_0(\boldsymbol{x}) \ldots \xi^\alpha_0(\boldsymbol{x}) \ldots \xi^\omega_0(\boldsymbol{x})]\, (\delta\xi^\alpha\, \delta\xi^\beta)(\boldsymbol{x}) \overset{\text{must}}{\leqslant} 0 \end{aligned} \tag{3.11}$$

[35]) To $V(0)$ applies the same remark as to (1.1) and (1.2).

is negative, on account of the generality ($\forall\, \boldsymbol{x} \in V$) of (3.11), at the extremum $\xi_0^{\cdot}$.

Thus our theorem (0.75) is proved [if $<$ holds in (3.11) for $k = 1$ in (0.75)].

We consider now the arbitrary high ($k \geqslant 3$) variations $\delta^{(k)}_{(c)} F[\ldots]$.

4. Arbitrary High ($k \geqslant 3$) Variations $\delta^{(k)}_{(c)}\, F\,(\ldots)$

For $k = 3$, let us consider first

$$\begin{aligned}\hat{f},_{\alpha_1\alpha_2\alpha_3}(\boldsymbol{x}) &= \left(\psi,_{\alpha_1\alpha_2} + \vartheta_{a,b}\, \xi,^{b}_{\alpha_2}\, g,^{a}_{\alpha_1}\right),_{\alpha_3}(\boldsymbol{x}) \\ &= \psi,_{\alpha_1\alpha_2\alpha_3}(\boldsymbol{x}) + \left(\vartheta_{a,b}\left(\xi,^{b}_{\alpha_2}\, g,^{a}_{\alpha_1\alpha_3} + \xi,^{b}_{\alpha_2\alpha_3}\, g,^{a}_{\alpha_1}\right)\right. \\ &\quad \left. + \vartheta_{a,bc}\, \xi,^{c}_{\alpha_3}\, \xi,^{b}_{\alpha_2}\, g,^{a}_{\alpha_1}\right)(\boldsymbol{x}) \qquad \forall\, \boldsymbol{x} \in V'\end{aligned} \tag{4.1}$$

in which, on account of (3.2) and (3.5), ($\psi,_{bb} = 0$) the second term vanishes.

Further, we calculate $\vartheta_{a,\, bc}$

$$\vartheta_{a,bc} = -\,(\psi,_{bbc}\, \delta_{bd}\, g^{-1d}_{\ \ a} + \psi,_{bb}\, \delta_{bd}\, g^{-1d}_{\ \ a,c}) \tag{4.2}$$

in which, for the same reason, the second term vanishes too.

Thus, the third variation is given by:

$$\begin{aligned}\delta^{(3)}_{(c)} F[\ldots] = \delta^{(3)}\, \hat{F}[\ldots] &= \int_{V'} \left(dV\, \psi,_{\alpha_1\alpha_2\alpha_3}\, \delta\xi^{\alpha_1}\, \delta\xi^{\alpha_2}\, \delta\xi^{\alpha_3}\right)(\boldsymbol{x}) \\ &\quad + \sum_b V(b)\, \psi,_{bbb}(\delta\xi^b)^3 \overset{*}{=} \delta^{(3)}\, \Psi[\ldots] \overset{\text{must}}{=} 0\,.\end{aligned} \tag{4.3}$$

Therefore we are again led to the statement: for the extremum of the second order ($k = 2$), $\delta^{(3)}\, \Psi[\ldots] = \delta^{(2k-1)}\, \Psi[\ldots] = 0$ is a *sufficient condition*. In order to state that the condition is *necessary*, we proceed as in Section 3 (with $\delta^{(3)}_{(cp)}\, F[\ldots]$) and arrive at

$$\begin{aligned}\delta^{(3)}_{(cp)}\, F[\ldots] &= \frac{1}{V(0)^2}\, \psi,_{\alpha_1\alpha_2\alpha_3}(\boldsymbol{x}_0)\, \delta\eta^{\alpha_1}\, \delta\eta^{\alpha_2}\, \delta\eta^{\alpha_3} \\ &\quad + \sum_b \frac{1}{V(b)^2}\, \psi,_{bbb}(\boldsymbol{x}_b)\, ((g^{-1b}_{\ \ c}\, g,^{c}_{\alpha})\, (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)\, \delta\eta^{\alpha})^3 \\ &= \delta^{(3)}_{(p)}\, \Psi[\ldots] \overset{\text{must}}{=} 0\end{aligned} \tag{4.4}$$

where again the *arbitrary choice* of $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in V$ necessitates

$$\psi,_{\alpha_1\alpha_2\alpha_3}(\boldsymbol{x}) \overset{\text{must}}{=} 0 \qquad \forall\, \boldsymbol{x} \in V \tag{4.5}$$

for the extremum of the second order.

It is easy now to formulate a recursion formula. For any k, with the assumption

$$\delta^{(1)}\, \hat{F}[\ldots] = \delta^{(2)}\, \hat{F}[\ldots] = \ldots = \delta^{(k-1)}\, \hat{F}[\ldots] = 0 \tag{4.6}$$

one has

$$\vartheta_{a,\,(b)^{k-2}\,c} = -\, \psi,_{(b)^{k-1}}\, \delta_{bc}\, g^{-1c}_{\ \ a} \tag{4.7}$$

with the notation $\vartheta_{a,(b)^k c} = \vartheta_{a,b\ldots b c}$ (k times index b), and

$$\hat{f},_{\alpha_1\ldots\alpha_k}(\boldsymbol{x}) = \left(\psi,_{\alpha_1\ldots\alpha_k} + \vartheta_{a,(b)^{k-2}c}\, \xi,^{c}_{\alpha_k}\, \xi,^{b}_{\alpha_{k-1}} \ldots \times \ldots \xi,^{b}_{\alpha_2}\, g,^{a}_{\alpha_1}\right)(\boldsymbol{x}) \qquad \forall\, \boldsymbol{x} \in V' . \qquad (4.8)$$

Thus the $(k)^{th}$ variation is given by

$$\delta^{(k)}_{(c)} F[\ldots] = \delta^{(k)} \hat{F}[\ldots] = \int\limits_{V'} \left(dV\, \psi,_{\alpha_1\ldots\alpha_k}\, \delta\xi^{\alpha_1} \ldots \delta\xi^{\alpha_k}\right)(\boldsymbol{x})$$

$$+ \sum_b V(b)\, \psi,_{(b)^k}\, (\delta\xi^b)^k \overset{*}{=} \delta^{(k)}\, \Psi[\ldots]$$

$$\left\{\begin{matrix} \overset{\text{must}}{=} 0 \\ \\ \overset{\text{must}}{\leqslant} 0 \end{matrix}\right\} \text{ if } k = \left\{\begin{matrix} 2\,k' - 1 \\ \\ 2\,k' \end{matrix}\right\} \begin{matrix}\text{for a maximum} \\ \text{of order } k'\end{matrix} . \qquad (4.9)$$

The same construction remains valid for the necessary condition:

$$\delta^{(k)}_{(c\,p)} F[\ldots] = \frac{1}{V(0)^{k-1}}\, \psi,_{\alpha_1\ldots\alpha_k}(\boldsymbol{x}_0)\, \delta\eta^{\alpha_1} \ldots \delta\eta^{\alpha_k}$$

$$+ \sum_b \frac{1}{V(b)^{k-1}}\, \psi,_{(b)^k}(\boldsymbol{x}_b)\, \left(\left(g^{-1}{}^{b}_{c}\, g,^{c}_{\alpha}\right) (\boldsymbol{x}_0, \boldsymbol{x}_1 \ldots \boldsymbol{x}_m)\, \delta\eta^{\alpha}\right)^k . \qquad (4.10)$$

Acknowledgements

The writers thank to Dr. C. Piron and Dr. J. Poncet for many suggestions and to Prof. Ch. P. Enz, who was kind enough to discuss and clarify many points. Dr. J. Poncet shall publish a more strictly 'mathematical' proof of this theorem [7]. Our approach is a more empirical one, therefore we should like to call it a 'physical' proof.

References

[1] E. C. G. Stueckelberg de Breidenbach and G. Wanders, Helv. phys. Acta *26*, 307 (1953) (referred to as I).
[2] E. C. G. Stueckelberg de Breidenbach, Helv. phys. Acta *26*, 417 (1953) (referred to as II).
[3] E. C. G. Stueckelberg de Breidenbach, Helv. phys. Acta *35*, 568 (1962) (referred to as III).
[4] E. C. G. Stueckelberg de Breidenbach, Helv. phys. Acta *36*, 875 (1963) (referred to as IV).
[5] An exception is made by: I. P. Bazarow, *Thermodynamics* (Pergamon Press, London 1964).
[6] W. Pauli, *Relativitätstheorie* (Sonderabdruck aus der Encyclopädie der mathematischen Wissenschaften, Teubner, Leipzig und Berlin 1921). We give our citation from the English version, p. 134 (Pergamon Press, London 1958).
[7] E. C. G. Stueckelberg de Breidenbach and J. Poncet (in preparation).
[8] P. A. M. Dirac, Proc. Roy. Soc. *167*, 148 (1938).
[9] M. H. L. Pryce, Proc. Roy. Soc. *168*, 389 (1938).
[10] E. C. G. Stueckelberg de Breidenbach, Helv. phys. Acta *14*, 51 (1941).
[11] Ch. Gruber, Diploma Thesis, University of Lausanne (1963) (unpublished).