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A Note on L. D. Faddeev's Three-Particle Theory

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Abstract. An error in L. D. Faddeev's work [1] on quantum mechanical three-particle systems is pointed out and corrected.

Introduction

In his admirable analysis of quantum mechanical three-particle systems, L. D. Faddeev [1] employs a scale of Banach spaces $B_{\mu\theta}$ of Hölder-continuous functions on R^n which are characterized by their Hölder-index μ and by their behaviour at infinity (θ), such that $B_{\mu'\theta} \subset B_{\mu\theta}$ for $\mu' \geqslant \mu$. Without proof, he states that the union of all $B_{\mu'\theta}$ with $\mu' > \mu$ is *dense* in $B_{\mu\theta}$. However, this turns out to be false, as will be shown below by a counter example. Nevertheless, the Fredholm Alternative for the three-particle equations (which was based in part on this erroneous statement) can still be derived from Faddeev's estimates.

1. The counter example

Since we are concerned only with local properties of Hölder-continuous functions, we simply consider the spaces B_μ, $0 < \mu \leqslant 1$, of complex valued functions f on the interval $I = [0, 1]$ of R^1, satisfying the estimates

$$| f(x) | \leqslant C(f)$$

$$| f(x') - f(x) | \leqslant C(f) \, | x' - x |^\mu$$

for all $x, x' \in I$. B_μ, normed by

$$|| f ||_\mu = \sup_{\substack{x, x' \in I \\ x \neq x'}} \{| f(x) | + | x' - x |^{-\mu} \, | f(x') - f(x) |\} \tag{1}$$

is a Banach space, and clearly $B_{\mu'} \subset B_\mu$ for $\mu' \geqslant \mu$.

Theorem 1

$$\bigcup_{\mu'>\mu} B_{\mu'} \quad \text{is } \textit{not dense} \text{ in } B_\mu .$$

Proof

The function $f(x) = x^\mu$ is an element of B_μ according to the elementary inequality

$$| x'^\mu - x^\mu | \leqslant | x' - x |^\mu$$

for all $x', x \in I$. Now let $g \in B_{\mu'}$, $\mu' > \mu$.

g satisfies

$$| g(x) - g(0) | \leqslant C(g)\, x^{\mu'}$$

for all $x \in I$, therefore, by (1),

$$\begin{aligned} || f - g ||_\mu &\geqslant \sup_{0<x\leqslant 1} x^{-\mu} | (f - g)(x) - (f - g)(0) | \\ &= \sup_{0<x\leqslant 1} | 1 - x^{-\mu} (g(x) - g(0)) | \\ &\geqslant \sup_{0<x\leqslant 1} 1 - C(g)\, x^{\mu' - \mu} = 1 \end{aligned}$$

Therefore $\| f - g \|_\mu \geqslant 1$ for all $g \in B_{\mu'}$ if only $\mu' > \mu$ which proves the theorem. The extension to the spaces $B_{\mu'\theta}$ used by Faddeev is immediate.

2. The Fredholm Alternative

It is well known [2] that the Fredholm alternative applies to equations of the form $f = g + Af$ on a Banach space X, if the operator A is bounded and if some power A^N of A is compact. Our aim is to generalize this result to cases where A may be unbounded and need not even be densely defined. (This is precisely the situation in Faddeev's theory.)

Theorem 2

Let A be a linear operator on a Banach space X with domain $D(A)$, such that some power A^n has a compact extension K (from $D = D(A^n)$ to X), which maps X into D. Then

I) Either $f = Af$ has a nontrivial solution $f \in D(A)$

II) Or $f = g + Af$ has a unique solution $f \in D(A)$ for any $g \in D$.

Proof:

A) A and K map D into D and commute on D. Proof: $f \in D$ implies $A^n f = Kf \in D$, hence $f \in D\,(A^{n+1})$, or $Af \in D$. Therefore, $KAf = A^{n+1} f = AKf$.

B) We recall some results of the Riesz-Schauder theory [3]: there exists an integer $\nu \geqslant 0$ such that $X = M \oplus N$, where

$$M = \{f : f \in X, \quad (1 - K)^\nu f = 0\}$$

$$N = \{f : f = (1 - K)^\nu g,\ g \in X\}$$

M and N reduce K, M is of finite (possibly zero) dimension, and on N $(1\text{-}K)^{-1}$ exists and is bounded. Now we have

a) $M \subset D$

b) M is invariant under A

c) $N \cap D$ is invariant under A.

Proof: (a) follows from $M \subset R(K)$[1]) $\subset D$. Therefore, by (A), $A\,(1-K)^\nu f = (1-K)^\nu Af = 0$ for all $f \in M$, which proves (b). To show (c), let $f \in N \cap D$. $f \in N$ means $f = (1-K)^\nu g$, hence $f - g \in R(K) \subset D$. Therefore, $f \in D$ implies $g \in D$ and, by (A), $Af = (1-K)^\nu Ag$.

C) Let $g \in D$, and let $g = g_M + g_N$ be the (unique) decomposition of g with respect to M and N. By (B), $g_M \in D$, hence $g_N \in N \cap D$, and $A^k g_N \in N \cap D$ for all integers $k \geqslant 0$. This allows us to define f_N as the unique solution in N of the equation

$$f_N = g_N + Ag_N + \ldots + A^{n-1} g_N + Kf_N$$

By (B), $Kf_N \in N \cap D$, hence $f_N \in N \cap D$ and, using (A), we easily find

$$(1-K)\,[g_N - (1-A)\,f_N] = 0$$

and since, on N, $1-K$ is injective, we conclude that

$$f_N = g_N + A\,f_N\,.$$

D) In the finite dimensional subspace M the following alternative holds: Either $Af = f$ has a nontrivial solution $f \in M$ (case (I) of theorem 2), or $(1-A)^{-1}$ exists. In the second case, we define

$$f_M = (1-A)^{-1}\,g_M$$

and obtain, by (C),

$$f = f_N + f_M = g + Af$$

Then f is the *unique* solution to this equation, for $Ah = h$, $h \in D(A)$, implies $h = A^n h = Kh$. Consequently $h \in M$, hence $h = 0$. Therefore, the second alternative coincides with case (II) of theorem 2.

Remarks

1) Let $D(A^k)$ be normed by $\|f\|_k = \sum_{i=0}^{k} \|A^i f\|$. Then, in case (II), $(1-A)^{-1}$ is a bounded operator from $D(A^n)$ to $D(A)$.

Proof:

The mappings $g \to g_N$, $g \to g_M$ are bounded with respect to the norm $\|\;\|$. By (B), they map D into D and commute with A, hence they are bounded on D with respect to the norm $\|\;\|_n$. Now $(1-K)^{-1}$ and $(1-A)^{-1}$ are bounded operators on N and M, respectively, with respect to $\|\;\|$ – the latter because M is finite dimensional. Therefore the mappings $g_N \to f_N$, $g_M \to f_M$ are bounded operators from $D(A^n)$ to X, and the same

[1]) $R(K)$ denotes the range of K.

follows for the mapping $g \to f$. Finally, $Af = f - g$ implies that $g \to f$ is also bounded from $D(A^n)$ to $D(A)$.

2) Theorem 2 still holds if we relax the condition $R(K) \subset D(A^n)$ to $R(K) \subset D(A)$, if we add the consistency requirement that $AK = KA$ on $D(A)$.

3) Another method to establish the Fredholm alternative in FADDEEV's theory is the following: Define $B^{\mu\theta}$ as the inductive limit of the spaces $B_{\mu'\theta}$ with $\mu' > \mu$. The operator A treated by FADDEEV is then a continuous operator on $B^{\mu\theta}$ such that A^n is compact for $n \geqslant 5$, and one can apply the extension of Fredholm theory to separated, locally convex topological vector spaces [4].

References

[1] L. D. FADDEEV, *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory* (Translated from the Russian by the Israel Program for Scientific Translations, Jerusalem 1965).

[2] F. RIESZ and B. SZ. NAGY, *Vorlesungen über Funktionalanalysis*, Kap. XI, Nr. 152. (Deutscher Verlag der Wissenschaften, Berlin 1956).

[3] See [2], Kap. IV, Nr. 77.

[4] J. LERAY, Acta Sci. Math. Szeged *12* (1950); An account of this may be found in R. E. EDWARDS, *Functional Analysis*, chapter 9. (Holt, Rinehart and Winston, 1965).