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## A Note on L. D. Faddeev's Three-Particle Theory

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*Abstract.* An error in L. D. FADDEEV's work [1] on quantum mechanical three-particle systems is pointed out and corrected.

### Introduction

In his admirable analysis of quantum mechanical three-particle systems, L. D. FADDEEV [1] employs a scale of Banach spaces  $B_{\mu\theta}$  of Hölder-continuous functions on  $R^n$  which are characterized by their Hölder-index  $\mu$  and by their behaviour at infinity ( $\theta$ ), such that  $B_{\mu'\theta} \subset B_{\mu\theta}$  for  $\mu' \geq \mu$ . Without proof, he states that the union of all  $B_{\mu'\theta}$  with  $\mu' > \mu$  is *dense* in  $B_{\mu\theta}$ . However, this turns out to be false, as will be shown below by a counter example. Nevertheless, the Fredholm Alternative for the three-particle equations (which was based in part on this erroneous statement) can still be derived from FADDEEV's estimates.

### 1. The counter example

Since we are concerned only with local properties of Hölder-continuous functions, we simply consider the spaces  $B_\mu$ ,  $0 < \mu \leq 1$ , of complex valued functions  $f$  on the interval  $I = [0, 1]$  of  $R^1$ , satisfying the estimates

$$|f(x)| \leq C(f)$$

$$|f(x') - f(x)| \leq C(f) |x' - x|^\mu$$

for all  $x, x' \in I$ .  $B_\mu$ , normed by

$$\|f\|_\mu = \sup_{\substack{x, x' \in I \\ x \neq x'}} \{ |f(x)| + |x' - x|^{-\mu} |f(x') - f(x)| \} \quad (1)$$

is a Banach space, and clearly  $B_{\mu'} \subset B_\mu$  for  $\mu' \geq \mu$ .

*Theorem 1*

$$\bigcup_{\mu' > \mu} B_{\mu'} \text{ is not dense in } B_{\mu}.$$

*Proof*

The function  $f(x) = x^{\mu}$  is an element of  $B_{\mu}$  according to the elementary inequality

$$|x'^{\mu} - x^{\mu}| \leq |x' - x|^{\mu}$$

for all  $x', x \in I$ . Now let  $g \in B_{\mu'}$ ,  $\mu' > \mu$ .

$g$  satisfies

$$|g(x) - g(0)| \leq C(g) x^{\mu'}$$

for all  $x \in I$ , therefore, by (1),

$$\begin{aligned} \|f - g\|_{\mu} &\geq \sup_{0 < x \leq 1} x^{-\mu} |(f - g)(x) - (f - g)(0)| \\ &= \sup_{0 < x \leq 1} |1 - x^{-\mu} (g(x) - g(0))| \\ &\geq \sup_{0 < x \leq 1} 1 - C(g) x^{\mu' - \mu} = 1 \end{aligned}$$

Therefore  $\|f - g\|_{\mu} \geq 1$  for all  $g \in B_{\mu'}$  if only  $\mu' > \mu$  which proves the theorem. The extension to the spaces  $B_{\mu', \theta}$  used by FADDEEV is immediate.

**2. The Fredholm Alternative**

It is well known [2] that the Fredholm alternative applies to equations of the form  $f = g + Af$  on a Banach space  $X$ , if the operator  $A$  is bounded and if some power  $A^N$  of  $A$  is compact. Our aim is to generalize this result to cases where  $A$  may be unbounded and need not even be densely defined. (This is precisely the situation in FADDEEV's theory.)

*Theorem 2*

Let  $A$  be a linear operator on a Banach space  $X$  with domain  $D(A)$ , such that some power  $A^n$  has a compact extension  $K$  (from  $D = D(A^n)$  to  $X$ ), which maps  $X$  into  $D$ . Then

- I) Either  $f = Af$  has a nontrivial solution  $f \in D(A)$
- II) Or  $f = g + Af$  has a unique solution  $f \in D(A)$  for any  $g \in D$ .

*Proof:*

A)  $A$  and  $K$  map  $D$  into  $D$  and commute on  $D$ . Proof:  $f \in D$  implies  $A^n f = Kf \in D$ , hence  $f \in D(A^{n+1})$ , or  $Af \in D$ . Therefore,  $KAf = A^{n+1}f = AKf$ .

B) We recall some results of the RIESZ-SCHAUDER theory [3]: there exists an integer  $\nu \geq 0$  such that  $X = M \oplus N$ , where

$$\begin{aligned} M &= \{f : f \in X, (1 - K)^{\nu} f = 0\} \\ N &= \{f : f = (1 - K)^{\nu} g, g \in X\} \end{aligned}$$

$M$  and  $N$  reduce  $K$ ,  $M$  is of finite (possibly zero) dimension, and on  $N$   $(1-K)^{-1}$  exists and is bounded. Now we have

- a)  $M \subset D$
- b)  $M$  is invariant under  $A$
- c)  $N \cap D$  is invariant under  $A$ .

Proof: (a) follows from  $M \subset R(K)^1 \subset D$ . Therefore, by (A),  $A(1-K)^n f = (1-K)^n Af = 0$  for all  $f \in M$ , which proves (b). To show (c), let  $f \in N \cap D$ .  $f \in N$  means  $f = (1-K)^n g$ , hence  $f - g \in R(K) \subset D$ . Therefore,  $f \in D$  implies  $g \in D$  and, by (A),  $Af = (1-K)^n Ag$ .

C) Let  $g \in D$ , and let  $g = g_M + g_N$  be the (unique) decomposition of  $g$  with respect to  $M$  and  $N$ . By (B),  $g_M \in D$ , hence  $g_N \in N \cap D$ , and  $A^k g_N \in N \cap D$  for all integers  $k \geq 0$ . This allows us to define  $f_N$  as the unique solution in  $N$  of the equation

$$f_N = g_N + Ag_N + \dots + A^{n-1} g_N + Kf_N$$

By (B),  $Kf_N \in N \cap D$ , hence  $f_N \in N \cap D$  and, using (A), we easily find

$$(1-K)[g_N - (1-A)f_N] = 0$$

and since, on  $N$ ,  $1-K$  is injective, we conclude that

$$f_N = g_N + Af_N.$$

D) In the finite dimensional subspace  $M$  the following alternative holds: Either  $Af = f$  has a nontrivial solution  $f \in M$  (case (I) of theorem 2), or  $(1-A)^{-1}$  exists. In the second case, we define

$$f_M = (1-A)^{-1} g_M$$

and obtain, by (C),

$$f = f_N + f_M = g + Af$$

Then  $f$  is the *unique* solution to this equation, for  $Ah = h$ ,  $h \in D(A)$ , implies  $h = A^n h = Kh$ . Consequently  $h \in M$ , hence  $h = 0$ . Therefore, the second alternative coincides with case (II) of theorem 2.

### Remarks

1) Let  $D(A^k)$  be normed by  $\|f\|_k = \sum_{i=0}^k \|A^i f\|$ . Then, in case (II),  $(1-A)^{-1}$  is a bounded operator from  $D(A^n)$  to  $D(A)$ .

*Proof:*

The mappings  $g \rightarrow g_N$ ,  $g \rightarrow g_M$  are bounded with respect to the norm  $\|\cdot\|$ . By (B), they map  $D$  into  $D$  and commute with  $A$ , hence they are bounded on  $D$  with respect to the norm  $\|\cdot\|_n$ . Now  $(1-K)^{-1}$  and  $(1-A)^{-1}$  are bounded operators on  $N$  and  $M$ , respectively, with respect to  $\|\cdot\|$  – the latter because  $M$  is finite dimensional. Therefore the mappings  $g_N \rightarrow f_N$ ,  $g_M \rightarrow f_M$  are bounded operators from  $D(A^n)$  to  $X$ , and the same

<sup>1)</sup>  $R(K)$  denotes the range of  $K$ .

follows for the mapping  $g \rightarrow f$ . Finally,  $Af = f - g$  implies that  $g \rightarrow f$  is also bounded from  $D(A^n)$  to  $D(A)$ .

2) Theorem 2 still holds if we relax the condition  $R(K) \subset D(A^n)$  to  $R(K) \subset D(A)$ , if we add the consistency requirement that  $AK = KA$  on  $D(A)$ .

3) Another method to establish the Fredholm alternative in FADDEEV's theory is the following: Define  $B^{\mu\theta}$  as the inductive limit of the spaces  $B_{\mu'\theta}$  with  $\mu' > \mu$ . The operator  $A$  treated by FADDEEV is then a continuous operator on  $B^{\mu\theta}$  such that  $A^n$  is compact for  $n \geq 5$ , and one can apply the extension of Fredholm theory to separated, locally convex topological vector spaces [4].

### References

- [1] L. D. FADDEEV, *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory* (Translated from the Russian by the Israel Program for Scientific Translations, Jerusalem 1965).
- [2] F. RIESZ and B. SZ. NAGY, *Vorlesungen über Funktionalanalysis*, Kap. XI, Nr. 152. (Deutscher Verlag der Wissenschaften, Berlin 1956).
- [3] See [2], Kap. IV, Nr. 77.
- [4] J. LERAY, *Acta Sci. Math. Szeged* 12 (1950); An account of this may be found in R. E. EDWARDS, *Functional Analysis*, chapter 9. (Holt, Rinehart and Winston, 1965).