Feynman Functions Associated with the Sixth Order Ladder Diagram [1])

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Abstract. The analytic structure of those Feynman functions associated with the sixth order ladder diagram which do not have second type singularities is studied by the homological method. Previous calculations by P. FEDERBUSH are criticized. Some general conjectures concerning the structure of Feynman functions are put forward.

1. Introduction

The problem of determining the analytic structure of the function of several complex variables defined by Feynman integrals has been the subject of many investigations. The main aim of this work has been to secure a guide to the analytic structure to be expected of the S-matrix elements in a field theory or to be postulated in an axiomatic framework independent of field theory (S-matrix theory). At first sight this enterprise would appear ill-founded since the Feynman amplitudes are the terms of a perturbation series which is not expected to converge, and which must certainly be renormalized. However, it has been shown that the Landau equations which determine the location of a singularity admit a natural interpretation as the equations for the threshold of a multiparticle scattering process if the singularity occurs in the physical region [1], and it has been shown in outline that the whole set of Landau singularities in complex regions, located as in perturbation theory, must be present if the S-matrix is assumed to be unitary, to be analytic, to satisfy crossing symmetry and to have pole singularities corresponding to the existence of stable and unstable particles [2]. This gives support for the heuristic principle that those properties of Feynman functions which depend only on the location of singularities and the existence of integral relations such as are given by unitarity are shared by the S-matrix elements, provided that the masses are given their physical values.

In 1963, D. FOTIADI, M. FROISSART, J. LASCOUX, and F. PHAM introduced a new method for the study of the analytic structure of Feynman functions. This method, the homological method, was in part a rigorous mathematical presentation of the method of analytic continuation of a function defined by an integral representation

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[1]) The content of this paper forms part of a thesis submitted by the author to St. John's College, Cambridge, in December 1966 for the Fellowship competition to be held in March 1967.

by distorting of the contour of integration, which had been extensively used and developed to study Feynman functions, particularly by the Cambridge school [2]. However, it also showed that Feynman functions have a general property that had not previously been noted—the germs of a Feynman function with a given nonsingular point as center span a finite dimensional vector space. It is therefore possible to label the different sheets of the function by vectors in this space and to specify the effect of analytically continuing the germs along a given loop by the matrix of the corresponding linear transformation on the vector space. We will call the dimension of the vector space the sheet index of the function. This kind of description of analytic structure is familiar from the Fuchsian theory of differential equations—for example, the transformation of the Hankel functions corresponding to analytic continuation along a loop around the origin is well known to physicists. Moreover, O. PARASSIUK has pointed out that the finiteness of the dimension of the vector space implies that Feynman functions also satisfy ordinary differential equations in any one of their variables [3]. This property of Feynman functions makes it reasonable to attempt to give a very complete and explicit description of their structure.

FOTIADI and PHAM gave a complete analysis of a Feynman function without second-type singularities associated with the square diagram. However, it was not possible to use the homological method to analyse Feynman functions associated with diagrams having more than one loop because it was not known how to write the customary integral representations of the functions in the standard form required for the application of the FFLP theory. In the momentum space integral representation of a Feynman function the pole loci of the integrand are in general position (provided that the argument of the function is not on a Landau singularity). The problem is to find a compactification of the integration space such that the pole loci of the transformed integrand should continue to be in general position in the compactified space (at least for general values of the argument of the function). It was pointed out by D. FOTIADI in a lecture given at the end of 1964 that a method for constructing a suitable compactification exists[2]). This method consists in first making any compactification of the integration space and then changing the compactification by means of a series of elementary transformations called blowing-ups. That this process will always yield a compactification in which the general position condition is satisfied, is guaranteed by a deep theorem of H. Hironaka.

In view of the technical difficulty of giving a detailed analysis of Feynman functions it is necessary to press the question as to how far it is necessary for the physical theory to do so. The author can only offer an opinion on this subject. G. F. CHEW has put forward the proposal that an S-matrix theory for the strongly interacting particles may be based on a very small set of assumptions [4]. These assumptions should include the following:

1) The S-matrix is unitary.

2) The connected parts of the S-matrix elements, after the factoring out of δ-functions corresponding to momentum conservation, are boundary values of analytic functions, whose only singularities are simple poles corresponding to stable and un-

[2]) The author would like to thank Prof. A. S. WIGHTMAN for making available to him a copy of this lecture.

stable particles and those singularities which are then required by the remaining assumptions.

3) The connected parts for crossed processes are related to each other by analytic continuation along certain paths.

The problem of developing an S-matrix theory along these lines has been taken up by groups in Berkeley, Cambridge (England) and elsewhere. (See *e.g.* [2]). The different formulations have this in common—they begin by postulating the existence of analytic functions having the infinite set of singularities required by (2). Thus if the theory is to be realized it will be necessary eventually to exhibit such functions. The author does not understand how this is to be done unless by first exhibiting similar functions having only a finite number of the singularities and then adding them together in infinite uniformly convergent series. If this view is adopted then it is necessary not only to make a detailed analysis of Feynman functions, but to study how to synthesise functions having given singularities and sheet structure and to determine how restrictive is the requirement that a function have a given singularity structure.

The main result of this paper is to show that for those Feynman functions associated with the sixth order ladder diagram which do not have second-type singularities the analysis of their analytic structure by means of the homological method can be carried out without the necessity of explicitly constructing the compactification given by Hironaka's theorem. We require Hironaka's theorem only as an existence theorem which together with the FFLP ambiant isotopy theorem guarantees the possiblity of distorting the contour of integration as the argument of the Feynman function follows certain paths. Second-type singularities are singularities which can arise if the contour of integration has to be distorted to infinity in the integration space. They can be excluded by considering only differential forms as integrands which decrease sufficiently rapidly toward infinity. We give the conditions on the integrand of a Feynman function associated with the sixth order ladder diagram which are imposed by this requirement. The exclusion of second-type singularities is natural if Feynman functions are studied in order to gain insight into S-matrix theory since it is not clear whether second-type singularities must be present in the connected parts of S-matrix elements. It is also mathematically more difficult to show that a given Feynman function has second-type singularities when the simple necessary condition for their existence is satisfied, than it is to show that the function has Landau singularities.

Some calculations of homology groups relevant to the sixth order ladder diagram were carried out by P. FEDERBUSH in 1964 [5]. On the basis of these calculations FEDERBUSH gives an upper bound of 127 for the sheet index of a Feynman function associated with the sixth order ladder diagram, this being the rank of the homology group calculated by him. However, FEDERBUSH's paper does not attempt to decide whether the upper bound of 127 is actually attained, and so does not sustain the claim that the homology groups computed are the relevant ones. We find the sheet index to be 78. In making this criticism of the conclusion reached by FEDERBUSH, the author does not wish to detract from the importance of FEDERBUSH's paper, a preprint of which was very kindly made available to the author by Prof. FEDERBUSH. In particular the author would like to acknowledge that it was this paper which convinced him of the importance of Serre's spectral sequence for the calculation of the homology groups which arise in the study of Feynman functions.

In § 2 we give a brief review of the homological method and we emphasise particularly the problems which arise in making a close connection between the homology group and the structure of the function being studied. Then we focus attention on what we have called the layers of the homology group or of the function[3]). If the layers of a Feynman function are independent, we say that the function has a layer structure. There is then one layer associated with each intersection of pole loci of the integrand (and so with each contracted diagram) and the homology group is a direct sum of homology groups defined by each intersection. This decomposition, if it exists, is canonical and replaces in our work the Froissart decomposition which has the disadvantage of not being canonical. It is our conjecture that all Feynman functions have a layer structure. We have therefore stated this conjecture together with certain related conjectures concerning the form of the layers. Since the recent book by R. C. HWA and V. L. TEPLITZ [6] and the forthcoming book by F. PHAM [7] are certain to stimulate interest in the homological method, it seemed to the author that the development of the theory might be furthered by putting forward some definite conjectures which more competent mathematicians may be able to disprove by counterexample or establish generally. In this paper the main conjecture, that Feynman functions have a layer structure will be established for Feynman functions not having second-type singularities associated with the sixth order ladder diagram and also for functions which are defined by integrals of standard form in which the pole loci are complete intersections in the sense of I. FÁRY [8]. This result is not of immediate value for the study of Feynman functions because only the Feynman functions associated with the square diagram and not having second-type singularities fall into this class. However, the result does give some additional insight into this special case, and mathematically it is a natural first step. In particular it seems to the author that the key to the general proof of the conjecture may lie in giving an appropriate generalization of FÁRY's concept of the finite homology group of an algebraic manifold which is a complete intersection.

Throughout the general discussion of the homological method in § 2 we suppose the function to be studied written as an integral of standard form. Nevertheless we do not construct explicitly a standard form representation for Feynman functions associated with the sixth order ladder diagram. The homology groups associated with the intersections of pole loci, which in the general discussion are understood to be compactified so as to be in general position, can be calculated from the intersections obtained by means of the inversion compactification. In calculating these homology groups we exploit the fact that the differential form to be integrated vanishes on certain homology classes of the intersections either because it is holomorphic and the homology class contains a cycle whose support is contained in an analytic subvariety or because it is an iterated residue and the homology class has a vanishing iterated coboundary.

[3]) The name ,layer' was suggested by Prof. T. REGGE in a conversation with the author following the publication of FOTIADI and PHAM's analysis of the square diagram. The author would like to thank Prof. REGGE for the stimulus given by this conversation.

The purely topological results are presented in § 3. This section contains a generalization of the Leray coboundary sequence, a self-contained exposition of the results of Fáry [8] on the finite homology groups of complete intersections and the calculations of some homology groups. The detail of the analysis of Feynman functions associated with the sixth order ladder diagram is presented in § 4. This analysis is not complete; we show how to obtain a basis for the homology group in which the basis elements are Lefschetz classes and we prove that this homology group is the relevant one in the sense that its rank is equal to the sheet index. We do not carry out the task of expressing all the Lefschetz classes in terms of an explicitly defined basis. However, the methods of the FFLP theory suffice to do this.

2. The Homological Method

2.1. *Definition of the notion of a function having sheet structure of finite type*

Let $F(t)$ be a (multivalued) analytic function defined on a complex manifold T. Denote by N the singularity set of F and suppose that N is of complex codimension 1 in T. For every fixed point $t \varepsilon T - N$ the set of germs of F with center t span a vector space V_t over the rational field. Since N is of complex codimension 1 in T, $T - N$ is path connected. Let $c_{t_2 t_1}$ be any path from t_1 to t_2 in $T - N$. For each germ f_1 of F with center t_1 we define a germ $f_2 = C_{t_2 t_1} f_1$ of F with center t_2 by analytic continuation of f_1 along $c_{t_1 t_2}$. A linear dependence relation between germs is preserved under analytic continuation; $C_{t_1 t_2}$ can be extended to a linear map $C_{t_1 t_2}: V_1 \to V_2$. This map is non-singular, for it has inverse $C_{t_1 t_2}^{-1}$ defined by analytic continuation along $c_{t_1 t_2}$ from t_2 to t_1. The vector spaces (V_t) therefore form a local system on $T - N$. Note in particular that the dimension of V_t is independent of t.

If the dimension of V_t is finite and if the fundamental group of $T - N$ is finitely generated, then we say that the sheet structure of $F(t)$ is of finite type. This structure can be described as follows: choose a fixed point $t_0 \varepsilon T - N$. The germs of F with center t_0 span a vector space V_0 of dimension s, the sheet index of F. Let c_i, $1 \leqslant i \leqslant k$ be loops generating $\pi_1 (T - N; t_0)$. To each c_i corresponds a non-singular linear map $V_0 \to V_0$ which can be specified relative to a chosen basis for V_0 by a $s \times s$ matrix $[C_i]$.

2.2. *Definition of an integral of standard form*

An integral $I_0 = \int\limits_{\Gamma_0} \frac{\omega(t_0)}{S(t_0)}$ is said to be of standard form if

a) $S(t) = 0$ is a closed analytic subset of a compact complex analytic manifold Z^n of complex dimension n and depends analytically on a parameter $t \varepsilon T$, T a complex manifold.

b) $\omega(t)$ is an analytic n-form on Z^n, holomorphically dependent on $t \varepsilon T$.

c) Γ_0 is an oriented C^1 closed chain, of dimension n, in $Z^n - S(t_0)$.

d) $S(t_0)$ is in general position (G.P.) in Z^n. Further the set

$$G = \{t \varepsilon T : S(t) \text{ not in G.P.}\}$$

is an analytic set of complex codimension 1 in T.

The compactification of momentum space whose existence is guaranteed by the

Hironaka theorem does not in fact give an integral representation for a complete Feynman amplitude in which the contour of integration Γ_0 is closed. However, the boundary of Γ_0 does not have to be moved during the process of analytic continuation. There are no end point singularities associated with this boundary. For every discontinuity function the contour of integration is in fact closed. To ignore this boundary and speak of Γ_0 as closed is therefore a convenient legal fiction.

Let $I_0 = \int\limits_{\Gamma_0} \frac{\omega(t_0)}{S(t_0)}$ be an integral of standard form.

Let V be the open subset of T defined by $V = \{t \varepsilon T : \Gamma_0 \cap S(t) = \phi\}$. Then $t_0 \varepsilon V$. Let V_0 be the connected component of t_0 in V. For $t \varepsilon V_0$ we can define a function $I_0(t)$ by $I_0(t) = \int\limits_{\Gamma_0} \frac{\omega(t)}{S(t)}$. Since $\omega(t)$ and $S(t)$ are holomorphic in t and Γ_0 is compact, it follows that $I_0(t)$ is holomorphic in t in V_0. The analytic function $F(t)$ determined by the germ $I_0(t)$ is called a function having an integral representation of standard form.

From the FFLP ambiant isotopy theorem it follows

(i) that $F(t)$ is analytic in $T - G$

(ii) that if c is a loop in $T - G$ on t_0 the germ $CI_0(t)$ defined by analytic continuation of $I_0(t)$ along c has an integral representation

$$CI_0(t) = \int\limits_{C\Gamma_0} \frac{\omega(t)}{S(t)}$$

for t in some neighbourhood of t_0. Here $C\Gamma_0$ satisfies the same condition c) as Γ_0.

2.3. *The homology group*

Any element $v(t)$ of the vector space V_0 spanned by germs of $F(t)$ with center t_0 has a representation

$$v(t) = \sum_{i=1}^{r} a_i \int\limits_{\Gamma_i} \frac{\omega(t)}{S(t)} = \int\limits_{\Gamma} \frac{\omega(t)}{S(t)} . \tag{1}$$

Here the Γ_i are oriented C^1 closed chains of dimension n and Γ is defined as $\sum_{i=1}^{r} a_i \Gamma_i$.

Denote by W_0 the vector space over the rationals spanned by germs $w(t)$ with center t_0 defined by integral representations of the form (1). We may consider V_0 to be a subspace of W_0.

Since the differential form $\omega(t)/S(t)$ in (1) is closed, Stokes theorem implies that Γ can be replaced by any chain Γ' such that

$$\Gamma - \Gamma' = \partial R \tag{2}$$

and the germ defined will not be changed. R is an oriented C^1 $(n + 1)$ chain, ∂R its boundary. Two chains Γ, Γ' satisfying (2) are called homologous and are said to belong to the same (compact) homology class of $Z - S(t_0)$. The homology classes can be made into a vector space over the rationals $H_n^c (Z - S(t_0))$ by giving them the linear structure which they inherit from the chains. There is a natural homomorphism

$q: H_n^c(Z - S(t_0)) \to W_0$ which assigns to each homology class the germ with center t_0 defined by a chain Γ in the homology class.

It is known that the dimension of $H_n^c(Z - S(t_0))$ is finite. It follows that the dimension of V_0 is also finite.

It is not necessary that $V_0 = W_0$ or that q be an isomorphism.

2.4. *Feynman functions*

Denote by P an n-dimensional vector space over the complex numbers. Suppose a nonsingular scalar product given in P. We write this product of two vectors a, $b \varepsilon P$ as $a.b$ and abbreviate $a.a = a^2$. The vectors of P will be called momentum vectors.

Let H be a finite graph. Orient H and consider the set Q of sets of momentum vectors, one for each oriented line of H, which satisfy momentum conservation at each vertex of H. Q is a finite dimensional vector space and can be written as a direct sum of the space E of vectors assigned to the external lines of H and the space K of elements of Q for which the momenta of the external lines are zero. K may be written as a direct sum of the spaces K_j of elements of Q which assign zero momentum to all lines except those in a single loop j. The l loops which define this decomposition must be chosen to be independent cycles on H. An element of K_j is specified by giving the momentum k_j of some particular line in loop j. Thus K_j is n-dimensional. On K_j define a holomorphic differential form $d^n k_j = \underset{s}{\Lambda}\, d_s k_j$. ${}_s k_j$ is the s^{th} component of k_j with respect to some basis in K_j. Apart from a constant factor it is independent of the choice of a basis. Denote by ϕ the differential form on K defined by $\phi = \underset{j}{\Lambda}\, d^n k_j$.

Let M denote the space of N complex variables m_i^2 one for each internal line of H. Denote by T the complex manifold $M \times E$. An analytic function $F(t)$ defined on an open subset of T will be called a Feynman function associated with the graph H, if, for some $t_0 \varepsilon T$, a germ of $F(t)$ with center t_0 admits an integral representation

$$I(t) = \int_{\Gamma_0} \frac{P(k, t)\, \phi}{\prod_{i=1}^{N} (q_i^2 + m_i^2)^{n_i}} . \tag{1}$$

n (1) the q_i are momentumvectors assigned to the internal lines which together with the external momenta specified by t form an element of Q. They are understood to be expressed in terms of t and of the loop momenta k_j. $P(k, t)$ is a polynomial in the components of the loop vectors and of the external momenta. The n_i are positive integers. Γ_0 is an oriented C^1 n-chain in K which does not intersect any of the sets $P_i = \{(k) : q_i^2 + m_i^2 = 0\}$ for $t = t_0$. Γ_0 is either to be compact or may be a chain whose support is the set of loop momenta whose components for the chosen basis are real. In that case (1) is to be understood to mean

$$I(t) = \lim_{r \to \infty} \int_{\Gamma_r} \frac{P(k, t)\, \phi}{\prod_{i=1}^{N} (q_i^2 + m_i^2)^{n_i}} \tag{2}$$

where $\Gamma_r = \Gamma_0 \cap \{(k) : \|k\| \leqslant r\}$. We require the convergence in (2) to be uniform with respect to t in some neighbourhood of t_0.

2.5. *Compactification*

The space K is a space of nl complex variables. A compact analytic manifold $\bar{K}$ such that

a) there is a biholomorphic map $i: K \to \bar{K}$ of K into $\bar{K}$.

b) $\bar{K}$ is the topological closure of iK in $\bar{K}$, is called a compactification of K (as an analytic manifold). There are many possible choices for $\bar{K}$. Particularly useful for the study of Feynman functions is the inversion compactification (for the chosen set of basic loops). K is written as a topological product $K_1 x \ldots x K_l$ and each factor K_j is compactified by embedding it into the projective space P^{n+1} by means of the embedding $i: k_j \to (1, k_j, k_j^2)$ (projective coordinates). $\bar{K}_j$ is the closure in P^{n+1} of iK_j and $\bar{K} = \bar{K}_1 x \ldots x \bar{K}_1$.

If $\bar{K}$ is a compactification of K, the integral representation 2.4(2) which defines a germ $I(t)$ of $F(t)$ can be transformed into a representation of $I(t)$ as an integral on $\bar{K}$.

The pole sets P_i are known to be in G.P. in K for all $t \varepsilon T$ $t \bar{\varepsilon} L$ where L is a bunch of algebraic varieties in T, called the Landau singularities, which are defined implicitly by the equations expressing the failure of the G.P. condition to hold at some point of K. For $t \varepsilon L$ the set of points in K at which the G.P. condition does not hold is called the pinch.

By a Hironaka compactification of K for $F(t)$ we mean a compactification $\bar{K}$ of K as an analytic manifold such that the representation of $I(t)$ on $\bar{K}$ is an integral of standard form. Hironaka's theorem on the resolution of singularities guarantees the existence of such a compactification. When the transformed differential form for the integral representing $I(t)$ on $\bar{K}$ is computed it will in general have pole loci other than the closures in $\bar{K}$ of the images under i of the P_i. These additional poles we call effective. The set G of points $t \varepsilon T$ such that the pole loci of the standard form integral representation are not in G.P. will therefore contain in addition to the Landau singularities L, a further set of algebraic varieties S which are defined implicitly by the condition that the G.P. should fail at a point of $\bar{K}_\infty = \bar{K} - iK$. These are the second type singularities[4]). They correspond to pinches of the poles P_i which only occur at infinity or to pinches involving one or more of the effective poles. If there are any second-type singularities of the former type, they present a difficulty for the classification, and in the discussion of layers in 2.6 we class them with the Landau singularities. A further obstacle to a clear terminology is the fact that one (irreducible) algebraic variety in T may correspond to several types of pinch in $\bar{K}$ (*i.e.* for a given general point on the variety the pinch in $\bar{K}$ may have several algebraic components on which different sets of poles fail to be in G.P.).

For the standard form representation of a Feynman function the set G is a bunch of algebraic varieties in T because the equations which define it implicitly are poly-

[4]) Strictly speaking the Landau singularities as we have defined them here are not algebraic varieties but open sets contained in algebraic varieties. For a general point on a Landau singularity there is a pinch in K. For an exceptional set of points of complex codimension 1 there may be no pinch in K but a pinch at a point of K_∞. The varieties of S correspond to pinches which occur only at points of K_∞.

nomial equations from which the equation of G is obtained by elimination of the coordinates of a point of the pinch in $\bar{K}$. It is known that the fundamental group of the complement in a space of several complex variables of a bunch of algebraic varieties is finitely generated. Hence a Feynman function has sheet structure of finite type.

2.6. *Layers*

Consider a function $F(t)$ having an integral representation of standard form. We use the notation of 2.2 and suppose $S(t)$ the union of m manifolds $S_i(t)$ in G.P. in Z^n for $t \bar{\varepsilon} G$. Denote by J the set of subsets of $\{1, \ldots, m\}$ and for each $I \varepsilon J$ denote by S_I the intersection $Z \bigcap_{i \varepsilon I} S_i$, by S^I, the union $\bigcup_{i \varepsilon I} S_i$ if I is not empty, and by CI the complement of I in J.

Define the subspace $V_I(t)$ of the vector space of germs of $F(t)$ with center $t \bar{\varepsilon} G$ to be the space spanned by elements of $V(t)$ which can be represented by means of the Leray residue formula as integrals of cycles on $S_I - S^{CI}$ of the residue of $\omega(t)$ on the intersection S_I. If t_1, t_2 are two points and $c_{t_1 t_2}$ a path from t_1 to t_2, $C_{t_1 t_2} V_I(t_1) = V_I(t_2)$ so $\{V_I(t)\}$ is a local system of vector spaces on $T - G$. In particular the representation of $\pi_1 (T - G)$ on V_0 leaves $V_I(t_0)$ invariant. Then we define the layer M_I as the quotient space $V_I / \bigcup_{I_1 \supset I} V_{I_1}$. $F(t)$ will be said to have a layer structure if $V \simeq \bigoplus_{I \varepsilon J} M_I$ (1).

To clarify the meaning of (1) consider a case in which there are just two pole loci S_1 and S_2. Let $v_1 \varepsilon V_1$, $v_2 \varepsilon V_2$. Suppose that some linear combination $v_1 + v_2 \varepsilon V_{12}$. Then if $F(t)$ has a layer structure this implies $v_1 \varepsilon V_{12}$ and $v_2 \varepsilon V_{12}$.

By the singularity of the layer M_I we mean the set G_I of points $t \varepsilon T$ such that $S_I(t)$ is not a manifold.

Conjecture 1. Every Feynman function has a layer structure.

Conjecture 2. The representation of the fundamental group $\pi_1 (T - G_I)$ on the layer M_I is irreducible.

In the case of a Feynman function we distinguish Landau and second-type layers according to whether the set of poles $S_i(t)$ $i \varepsilon I$ consists entirely of the propagators P_i displayed already in the representation of the function as an integral in K, or whether it contains one or more effective poles which appear when the Hironaka compactification of K is made. The Landau layers may be labelled by the diagrams H' obtained from the diagram H of $F(t)$ by contracting the lines of H whose propagators do not appear in the corresponding intersection. We then write $G(H')$ rather than G_I. In this paper we will restrict ourselves to the case of Feynman functions whose standard form integral representation contains no effective poles. In the examples we have studied this can be assured by conditions whose effect is to require the integers n_i to be sufficiently large. Thus given any Feynman function it is necessary only to differentiate it with respect to certain of the variables m_i^2 a sufficient number of times to obtain a Feynman function without second-type layers.

The notion of layers can be introduced for any function of finite sheet type as follows. Consider the representation of $\pi_1 (T - N)$ on $V(t_0)$. Let

$$0 = V_0 \subset V_1 \subset \ldots \subset V_{k-1} \subset V_k = V \quad \text{(all inclusions proper)}$$

be a chain of subspaces invariant under $\pi_1 (T - N)$ and of maximal length. Then

$\pi_1 (T - N)$ is irreducible on $M_i = V_i/V_{i-1}$, $1 \leqslant i \leqslant k$. We call the M_i layers of $F(t)$. Conjecture 2 implies that the layers given by this definition coincide in the case of Feynman functions with the layers which are defined geometrically. $\pi_1 (T - G_I)$ appears in the conjecture rather than $\pi_1 (T - G)$ because it can be shown from the Picard-Lefschetz theory that loops around components of G not in G_I act trivially on M_I.

The Landau singularities of a Feynman function associated with the graph H are defined implicitly by the Landau equations

$$\alpha_i (q_i^2 + m_i^2) = 0 \text{ for each } i \sum_{i \varepsilon \text{loop} j} \alpha_i q_i = 0 \text{ for all } j. \tag{2}$$

In (2) the α_i are complex numbers not all zero. From these equations it is immediate that L is not an irreducible algebraic variety but is the union of components which can be defined by the equations

$$q_i^2 + m_i^2 = 0 \sum_{i \varepsilon \text{loop} j} \alpha_i q_i = 0 \text{ for all } j \tag{3}$$

where now the index i is to run over a subset I of the indices labelling the lines of H and all the α_i are required to be non-zero for a general point on the component in question. Equations (3) are the equations which define that component of the Landau singularities of the graph H', obtained from H by contracting the lines of H not in I, which has non-zero α's for every line. This is called the leading or principal Landau singularity of H'. The Landau singularity of H is thus the union of the principal Landau singularities of its contracted graphs. We denote the principal Landau singularity of a graph H by $L(H)$.

Conjecture 3. For every graph H, $L(H)$ is either empty or an irreducible algebraic variety of complex codimension 1 in T. If $H_1 \neq H_2$, $L(H_1) \neq L(H_2)$.

Note that the parameter space T contains the variables m_i^2 so that we do not consider the principal Landau singularity of the self-energy diagram to degenerate into normal and pseudothreshold curves. The parameter space T' for a contracted graph H' is not the same as that for H. However, it is obtained from that for H simply by deleting the variables m_i^2 for the contracted lines. Hence if $L(H')$ is irreducible in T' then it is irreducible in T.

Conjecture 3 has been proved by T. Regge and G. Barucchi [9] for graphs H which arise in the strip approximation and by G. Barucchi [10] for the crossed square graph. Thus Conjecture 3 is known to be true for the contractions of the sixth order ladder graph.

The sets of α_i which satisfy (3) for a given point $t_0 \varepsilon L(H')$ and for a given point (q_i) of the corresponding pinch in K form a finite dimensional vector space. If this space has dimension >1 then it must contain a set of α_i one or more of which is zero. Hence $t_0 \varepsilon L(H'')$ where H'' is some contraction of H'. If Conjecture 3 holds, $L(H'') \cap L(H')$ is of complex codimension 1 in $L(H')$. Hence for a general point $t_0 \varepsilon L(H')$ the set of α_i for each point of the pinch is unique up to a factor. Moreover if the pinch has dimension >0 or consists of more than one point the set of α_i is the same for each point of the pinch (up to a factor). These non-simple pinches occur for contracted diagrams which are anomalous—in this case the pinch is not simple because of the freedom to rotate the dual diagram in momentum space about the subspace spanned by the external momenta—or which are obtained by contracting all the lines of one or more loops—in this case the pinch is not simple because the loop momenta for the contracted loops may be given arbitrary values. Nevertheless, the dual diagram is defined up to a congruence so the α_i are up to a factor.

If the pinch in K corresponding to a general point of one of the components $L(H')$ of L has dimension 0, then $L(H')$ can appear as a component only of the singularity $G(H')$ of the layer of $F(t)$ corresponding to the contracted diagram H'. If the pinch has complex dimension $d > 0$ then $L(H')$ may appear among the components of the singularities $G(H'')$ of the layers of $F(t)$ corresponding to graphs H'' which can be contracted onto H' and which contain not more than d lines not in H. This has been emphasized by P. V. LANDSHOFF *et al.* [11]. It is illustrated in § 4.

We have pointed out that for a general point $t_0 \varepsilon L(H')$ the set of complex numbers (α_i) satisfying (3) are uniquely defined up to a factor. However, there appears no obvious reason why having found $(\alpha_i(t))$ satisfying (3) for all t in the neighbourhood of t_0 and chosen the normalization so that $\alpha_i(t)$ varies smoothly with t in the neighbourhood of t_0, one should not take any smooth function $\lambda(t)$ non-zero in the neighbourhood of t_0 and form a set $(\lambda(t)\,\alpha_i(t))$ which would do equally well. By contrast if the α_i are introduced by first representing the Feynman function as an integral over loop momenta and Feynman parameters α_i, the normalization of the set (α_i) is not arbitrary from point to point on $L(H')$ but rather there is a single overall normalization. That there is natural overall normalization can be proved without introducing the Feynman parametrization from a result proved by F. PHAM [12] and independently bay the author [13] according to which the normal 1-form to $L(H')$ at a nonsingular point t_0 is given by

$$\sum \alpha_i\, q_i\, dq_i\,. \tag{4}$$

In (4) the differentials dq_i are to be expressed in terms of the differentials dk_j of the loop vectors and the differentials of the external vectors. The dk_j cancel by virtue of the loop equations. If the α_i are normalized so that

$$dl = \sum \alpha_i\, q_i\, dq_i \tag{5}$$

where $l = 0$ is the polynomial equation of $L(H')$, then their normalization is arbitrary only up to an overall constant factor.

We continue the discussion of layers in 2.8 after further development of the homological method.

2.7. *The Picard-Lefschetz theorem. A review of references [14], [5]*

We continue the discussion begun in 2.3. of a function defined by an integral of standard form. The assumption made in 2.6. regarding $S(t)$ is imposed.

If it is possible to choose loops which generate $\pi_1\,(T-G)$ which are elementary loops for certain points on G, and if, for each loop, the pinch in Z which occurs for the point on G which the loop circles, is simple quadratic or non-simple quadratic, then the action of $\pi_1\,(T-G)$ on $H_n^c\,(Z-S(t_0))$ (and hence on W_0) can be determined. According to the Picard-Lefschetz theorem the transformation C which corresponds to an elementary loop c around a point on G corresponding to a simple pinch is given by

$$Ch = h + l(h)\, e \tag{1}$$

where $l(h)$ is a certain linear functional of h which is specified in the full statement of the theorem and e is an element of $H_n^c\,(Z - S(t_0))$ called the Lefschetz class for the loop. The definition of a nonsimple quadratic pinch and the corresponding generalization of the Picard-Lefschetz theorem are given in § 4. Formula (1) expresses the fact that the linear transformation $C - 1$ has rank 1. In the case of sixth order ladder diagram we find that this is also true for the transformations corresponding to the non-simple quadratic pinches. It will not hold in general (in particular not for the eighth order ladder diagram).

It is always possible to choose a finite set of elementary loops generating $\pi_1\,(T - G)$ if G is a bunch of algebraic varieties. Then for the purpose of the present discussion the Lefschetz classes for a loop c can be defined to be the elements in the image of $H_n^c\,(Z - S(t_0))$ under the map $C - 1$.

In 3.1. we define a filtration of $H_n^c\,(Z - S(t_0))$ with the help of Leray's coboundary operator. We have already used this filtration implicitly in the definition of layers. We refer to 3.1. for the notations used below.

Denote by E' the subspace of $H_n^c\,(Z - S(t_0))$ spanned by the homology class of Γ_0 and by the Lefschetz classes for the elementary loops generating $\pi_1\,(T - G)$. The filtration of $H_n^c\,(Z - S(t_0))$ induces a filtration of E'. We denote by GrE' the corresponding graded group and by E'_I, $I \varepsilon J$, the corresponding layers.

We expect that if $F(t)$ is a Feynman amplitude, so that Γ_0 is the compactification of the real part of the space K of internal momenta for the corresponding diagram, then $V_0 = qE'$—in other words that every pinch is singular for some sheet of $F(t)$. If $F(t)$ has only Landau singularities this is implied by the argument of Landau quoted by Cutkosky [15] and elaborated by him.

In [14] Fotiadi and Pham make a complete analysis of a Feynman amplitude for the square diagram which satisfies the condition for the absence of second-type singularities. This condition is satisfied in particular for the amplitude in a Lagrangian theory of spin 0 particles with cubic interaction, as also for the amplitude corresponding to the sixth order ladder diagram in this theory. We now review the main points of their calculation.

The inversion compactification gives an integral representation of $F(t)$ of standard form. Z is a complex quadric Q_4^4 and $S(t_0)$ the union of 4 complex quadrics Q_3^3 in G.P. in Z[5]). Single out one of the 4 pole loci, S_4 say, and introduce an affine coordinate system in the projective space $P^5 \supset Z$ in which S_4 is the intersection with Z of the prime at infinity. Then

$$H_4^c\left(Z - \bigcup_{i=1}^{4} S_i\right) = H_4^c\left(Z^F - \bigcup_{i=1}^{3} S_i^F\right) \tag{2}$$

where X^F denotes the finite part of the algebraic variety X. By the Froissart decomposition theorem

$$H_4^c\left(Z^F - \bigcup_{i=1}^{3} S_i^F\right) \simeq \bigoplus_{I \varepsilon J_4} \delta^{|I|}\, H_{4-|I|}^c\,((Z \cap S_I)^F) \tag{3}$$

[5]) See 3.3. for the notation we use for the complex quadrics and cones.

where J_4 denotes the set of subsets of 1, 2, 3 (including ϕ). The intersections $(Z \cap S_I)^F$ are affine quadrics $Q_{4-|I|}$ and the base point t_0 and the affine coordinate system may eb so chosen that each of these quadrics is retractible onto its real section which is a real sphere $S^{4-|I|}$.

Thus
$$H^c_{4-|I|}((Z \cap S_I)^F) \simeq R \tag{4}$$
and this group is generated by the homology class e_I of the real section of $Z \cap S_I$ with its natural orientation. The group which appears on the right hand side of (3) is a graded group associated with $H^c_4(Z - S(t_0))$ (with respect to a different filtration from that introduced generally in 3.1.) so that the choice of generators e_I of the homology groups in the direct sum does not uniquely define a basis for $H^c_4(Z - S(t_0))$. A unique basis may be defined by constructing for each $I \varepsilon J_4$ a cycle g_i in $Z \cap S_I$ whose support does not intersect S^{CI} and which is homologous in $H^c_{4-|I|}((Z \cap S_I)^F)$ to e_I. Then the homology classes $h, h_1, h_2, h_3, h_{12}, h_{23}, h_{31}, h_{123}$ of the cycles $\{\Gamma_0, \delta g_1, \delta g_2, \delta g_3, \delta^2 g_{12}, \delta^2 g_{23}, \delta^2 g_{31}, \delta^3 g_{123}\}$ form a basis for $H^c_4(Z - S(t_0))$. The Landau singularity is the union of irreducible algebraic manifolds L_I, one for each intersection S_I, and a general point on L_I corresponds to a simple quadratic pinch in Z on S_I. Fotiadi and Pham show that the g_I may be chosen to be Lefschetz cycles on the corresponding intersections S_I, $I \varepsilon J_4$ for certain loops on t_0. Hence in this case q is an isomorphism and $E' = H^c_4(Z - S(t_0))$. Further loops are defined for the intersections S_I, $I \bar{\varepsilon} J_4$ with corresponding Lefschetz classes h_I. These classes are then expressed as linear combinations of the elements of the chosen basis. On examining these linear relations, Fotiadi and Pham remark that the classes h_I with $|I|$ even form a symmetrical basis. This basis corresponds to the filtration of $H^c_4(Z - S(t_0))$ introduced generally in 3.1. They do not prove that their chosen loops generate the fundamental group but remark that the corresponding transformations generate the algebra of the representation of the fundamental group on $H^c_4(Z - S(t_0))$—in other words that nothing is lost by considering these loops. Their analysis also shows $V_0 = qE'$.

In [5] Federbush studies a Feynman amplitude associated with the sixth order ladder diagram. He supposes that the integrand does not have the effective poles associated with the two square subgraphs but he does not exclude the effective pole associated with the loop formed by the six lines of the diagram running around its edge (see 4.1.). The function studied by him may therefore have second-type singularities arising from pinches involving this pole. The integration space Z is the inversion compactification of the space of internal momenta for the two square subgraphs as basic loops. The representation of $F(t)$ as an integral on Z is not of standard form. For all $t \varepsilon T$ the pole locus $S_6(t)$ is singular, having a fixed set U of singular points[6]). It is this permanent pinch which appears as an additional effective pole in a Hironaka compactification. It can also be displayed explicitly by making the inversion compactification defined by taking one of the square loops and the six-line loop as basic loops. Federbush then makes two calculations: in the first calculation he introduces an enlarged parameter space T' such that $F(t)$ is the restriction to T of certain germs of a function $F'(t')$ which has a representation of standard form on Z and calculates $H^c_n(Z - S(t'_0))$ where t'_0 is a general point in T'; in the second calculation he calculates

[6]) The labelling of the lines which we use is defined in 4.1. 6 is the line common to the two square loops.

$H^c_n(Z - S(t_0))$. The use of the Froissart decomposition theorem requires two lines to be singled out, one from each loop, (say lines 2 and 4 in our notation and an affine coordinate system to be chosen in $P^5 \times P^5$ for which the two primes at infinity intersect Z in S_2, S_4. The calculation of the homology groups of the set of affine algebraic varieties defined in this coordinate system, $(Z \cap S_I)^F$, is not trivial but FEDERBUSH succeeds in calculating them using standard methods of algebraic topology (exact sequences, Serre's spectral sequence) and some geometrical lemmas. The result of the calculations is the determination of the ranks of the groups $H^c_{8-|I|}((Z \cap S_I)^F)$ and therefore in particular of the ranks of the groups $H^c_n(Z - S(t'_0))$, $H^c_n(Z - S(t_0))$. These numbers should be upper bounds for the sheet index of $F(t)$. However, since the homology classes obtained by FEDERBUSH as basis classes for the groups $H^c_{8-|I|}((Z \cap S_I)^F)$ are not Lefschetz classes, it is not possible from Federbush's results to decide whether the lower of his two bounds—127—is attained. In fact the results which we present in § 4 show that for a Feynman amplitude associated with the sixth order ladder diagram and having only Landau singularities, the sheet index is only 78. It is true that in comparing this result with that of FEDERBUSH it is necessary to recall that the function studied by him is allowed to have certain second-type singularities so that one should add to 78 the contribution of the second-type layers. This does not account for the difference (though we do not prove this assertion in this paper).

Although we prefer to use the symmetrical basis for $H^c_n(Z - S(t_0))$ defined by means of the filtration of 3.1. rather than that obtained by means of the Froissart decomposition theorem, it may be useful when completing the analysis which we give in § 4 by defining a complete set of Lefschetz classes and expressing them in the chosen basis to consider the implication of this theorem. As J. B. BOYLING [16] has emphasised, the Froissart decomposition theorem gives a basis for $H^c_n(Z - S(t_0))$ over the integers. The symmetrical basis which we use is a basis for $H^c_n(Z - S(t_0))$ only when rational coefficients are used.

2.8. *The structure of the layers*

We continue the discussion of 2.6.

The layer M_I will be said to be independent if

$$M_{|I|} = \bigcup_{I_1 : |I_1| = |I|} M_{I_1}$$

is a direct sum

$$M_{|I|} = M_I \oplus \bigcup_{I_1 : |I_1| = |I| \; I_1 \neq I} M_{I_1}.$$

$F(t)$ has a layer structure if and only if all its layers are independent.

Conjecture 4. $M_I \simeq E_I$ where E_I denotes the subgroup of $H_{n-|I|}(S_I)$ spanned by the Lefschetz classes for loops in T around the singularitiy G_I of M_I. Furthermore unless $E_I = 0$, the Kronecker index defines on E_I a nonsingular bilinear form.

Remark 1. It is trivial that $M_I \simeq qE'_I$ if it is known that $V_0 = qE'$. But it is not clear that $E'_I = \delta^{|I|} E_I$, only that $E'_I \supseteq \delta^{|I|} E_I$.

Remark 2. If $M_I \simeq qE_I$ and the second part of Conjecture 4 holds, then the Picard-Lefschetz theorem shows that $q \mid E_I$ is an isomorphism so the first part of Conjecture 4 holds.

Not wishing to introduce additional notation we have used q and $\delta^{|I|}$ in these remarks to denote maps closely related to their defined meaning, trusting that their meaning will be clear in context.

A form $\psi(t)$ on S_I will be called a separating form for S_I if

a) $\psi(t)$ is the iterated residue onto S_I of a form on Z satisfying the same conditions as $\omega(t)$ but having poles only on the S_i, $i \varepsilon I$ (so that $\psi(t)$ is regular on S_I);

b) for any Lefschetz class $e_I \neq 0$ for G_I the function

$$G(t) = \int_{e_I} \psi(t)$$

is not identically zero. $G(t)$ will be called a separating function for S_I.

Proposition 1. In order that the layer M_I be independent it is sufficient that S_I have a separating form, and the Conjecture 4 hold.

If M_I is not independent, we have some linear relation

$$\delta^{|I|} h_I = \sum_{I_1 : |I_1| = |I| \ I_1 \neq I} \delta^{|I_1|} h_{I_1} \tag{1}$$

where $\delta^{|I|} h_I$ defines a non-zero element of M_I. According to Conjecture 4, we can find a Lefschetz class f_I such that $h_I \cdot f_I \neq 0$.

Consider the function

$$H(t) = \int_{h_I} \psi(t)$$

Equation (1), condition a) on $\psi(t)$ and the Leray residue formula imply that $H(t)$ is identically zero. But according to the Picard-Lefschetz theorem the separating function

$$G(t) = \int_{f_I} \psi(t)$$

is a discontinuity of $H(t)$, and this contradicts condition b).

Now consider a Feynman function $F(t)$ for a graph H.

Let S_I be a Landau intersection, *i.e.* an intersection of propagators. $S_I = S(H')$ where H' is a contraction of H. The differential form

$$\mu(H') = \operatorname{res}^{|I|} \left(\frac{\phi}{\prod_{i \varepsilon I} (q_i^2 + m_i^2)} \right)$$

is called the phase space form of H'. It is holomorphic on the finite part of $S(H')$. A function defined by integrating $\mu(H')$ over a Lefschetz cycle $\underline{e}(H')$ for $G(H')$ will be called a phase space function. This is a reasonable terminology because the phase space functions for multiparticle scattering processes are of this form [12].

Conjecture 5. The differential form

$$\mu^{(r)}(H') = \operatorname{res}^{|I|} \left(\frac{\phi}{\prod_{i \varepsilon I} (q_i^2 + m_i^2)^{r_i}} \right)$$

is a separating form for $S(H')$ provided that the integers r_i are sufficiently large.

By this we mean that if we construct a standard form representation of $F(t)$ by making a Hironaka compactification of K then the transform of $\mu^{(r)}$ under the injection $i: K \to \overline{K}$ is the restriction to iK of a separating form for $S(H')$—in other words that $\mu^{(r)}$ has no effective poles at infinity, and that the integral of $\mu^{(r)}$ over any Lefschetz class $e(H') \neq 0$ for $G(H')$ does not vanish identically. Conjecture 5 can be expressed in terms of phase space functions: If the Lefschetz class $e(H')$ of the cycle $\underline{e}(H')$ defining a phase space function is non-zero, then by differentiating the function a sufficient number of times with respect to the squared internal masses, a function is obtained which is non-zero and has no second-type singularities. Conjecture 5 can thus be regarded as a conjecture regarding the dependence of the phase space functions on the internal masses.

We now come to the last of our conjectures which concerns the structure of the possible layers with a view to securing a classification of them.

The bilinear form defined on E_I by the Kronecker index is skew-symmetric or symmetric according as the complex dimension of S_I is odd or even. Conjecture 4 therefore implies for the odd case that the dimension of E_I is even and that a basis for E_I may be chosen so that the Kronecker index is expressed in the standard form for a symplectic product. In the even dimensional case it is natural to enquire about the signature of the product. Suppose first, to simplify the discussion, that the Lefschetz classes of E_I correspond to simple quadratic pinches on S_I. Then for each Lefschetz class e the loop transformation for the path used to define it is given by

$$T(h) = h - (-1)^k (e \cdot h) e \quad \text{where} \quad 2k = n - |I| = \dim_c S_I \tag{2}$$

and $e.e = 2\,(-1)^k$. In particular $e.e$ is positive or negative according as k is even or odd. Now suppose that $h.h \neq 0$ for all $h \in E_I$. Define a new scalar product by $(h.l)' = (-1)^k\,(h.l)$ if $h, l \in E_I$. Then E_I with this scalar product is a Euclidean space provided that we extend the base field from the rationals to the reals. This we do as to be able to take square roots and rewrite (2) as

$$T(h) = h - 2\,(\hat{e} \cdot h)\hat{e}\,. \tag{3}$$

In (3) we have dropped the prime on our new scalar product and introduced a unit vector $\hat{e} = e/\sqrt{2}$. T can therefore be interpreted as a reflection in the prime $\hat{e}.h = 0$.

Thus $\pi_1\,(T - G_I)$ is represented on E_I by a group R generated by reflections. Such a group R has a very special structure, COXETER [17]. We summarize: R is a direct product of irreducible subgroups. Each of these subgroups is a finite group having as fundamental region a spherical simplex. The group may be characterized by the angles between the walls of this simplex. The possible simplexes may be symbolically represented by graphs invented by COXETER for the purpose[7]). The meaning of the graphs is given by COXETER in Chapter XI of [17] and a list of them, together with the orders of the corresponding finite groups, appears on page 297, *loc. cit.*

We will suppose R is irreducible (Conjecture 2.).

It would appear from the above that a group R arising in this way could only be trigonal, *i.e.* for two vectors $\hat{e}_i$, $\hat{e}_j$ defining reflections of R

$$\hat{e}_i \cdot \hat{e}_j = 0 \quad \text{or} \quad \pm {}^1/_2 .$$

[7]) But as representations for the simple Lie groups they are universally known to physicists as Dynkin diagrams.

However, it must be remembered that we introduced the simplifying assumption that the Lefschetz classes of E_I correspond to simple quadratic pinches on S_I. For the nonsimple pinches which arise in the analysis of Feynman functions associated with the sixth order ladder diagram we find that the loop transformations can still be written in the form (3) but that $\hat{e}$ is not related to the Lefschetz class e by $\hat{e} = e/\sqrt{2}$. In general we expect that some of the loop transformations T will be products of reflections rather than single reflections. This possibility is realized for the leading layer of the 8th order ladder diagram. In that case our conjecture regarding the signature is confirmed and we find that R has the exceptional structure F_4.

Suppose that S_I has a separating form. Then a separating function $G(t)$ for S_I is obtained by integrating this over a Lefschetz class e. e (or $-e$) defines a reflection T in R. If U is any element of R, the homology class Ue is a Lefschetz class and defines the reflection $U^{-1}TU$ conjugate to T. Hence: $[G(t)]^2$ is an algebraic function. The germs of $[G(t)]^2$ over a fixed point t_0 may be set into one-one correspondence with a conjugate class of reflections in R. If the components of G_I are divided into sets according to the conjugacy class in R of the reflections defined by their Lefschetz classes, there is, for each set of components, a separating function $G(t)$, defined by the given separating form, having these components as square root branch curves and no others.

We summarize this discussion as

Conjecture 6. For each layer corresponding to an intersection of even complex dimension, the Kronecker index defines on E_I a scalar product which is positive definite or negative definite according as the dimension is divisible by 4 or not.

It would be interesting to know whether the truth of Conjecture 6 is necessary for the separating functions of S_I to be algebraic. We remark that a separating function for a layer of odd complex dimension cannot be algebraic for the Picard-Lefschetz theorem shows that in a neighbourhood of a point on G_I it behaves like a logarithm.

It is important to reduce the problem of computing the dimension of a layer (and hence the sheet index) to a problem in enumerative algebraic geometry. In this context the following considerations may prove useful.

Consider a function $F(t)$ of finite sheet type defined on the complex plane T. Denote by $N = \{t_1, \ldots, t_n\}$ the set of singular points of $F(t)$. Define a sheaf $\mathfrak{F}$ on T: if D is a closed set in T then $\mathfrak{F}(D)$, the section of $\mathfrak{F}$ over D, is the vector space of functions, defined by linear combinations of germs of $F(t)$ over some point in D, which are single-valued in D. $\mathfrak{F}$ is a locally constant sheaf on $T - N$ and the sheet index of $F(t)$ is the dimension n of $\mathfrak{F}(t)$, $t \bar{\varepsilon} N$. We suppose that for each singular point t_i, $\mathfrak{F}(t_i)$ has dimension $n - r_i$, $r_i > 0$. r_i will be called the rank of t_i. We call $F(t)$ irreducible if the representation of $\pi_1 (T - N; t_0)$ on $\mathfrak{F}(t_0)$ $(t_0 \bar{\varepsilon} N)$ is irreducible.

Proposition 2. If $F(t)$ is irreducible, $H_1 (T; \mathfrak{F})$ has rank $\sum_{j=1}^{k} r_j - 2n$.

In Figure 1 we exhibit a complex from which $H(T; \mathfrak{F})$ can be calculated. This complex has Euler-Poincaré characteristic

$$\chi = 2n - \sum_j n + \sum_j (n - r_j) = -\Big(\sum_j r_j - 2n\Big)$$

Denote by l a loop on t_0 around one of the singularities t_j and by T_l the corresponding loop transformation of $\mathfrak{F}(t_0)$. Then $\bigcup_l (T_l - 1)\ \mathfrak{F}(t_0) \neq 0$ is the subspace of

$\mathfrak{F}(t_0)$ spanned by discontinuities of $F(t)$ for different loops around t_j. This subspace is invariant under π_1 $(T - N; t_0)$ and so, by the supposition of irreducibility, $\bigcup_l (T_l - 1)$ $\mathfrak{F}(t_0) = \mathfrak{F}(t_0)$. All the elements of $\bigcup_l (T_l - 1)$ $\mathfrak{F}(t_0)$ are boundaries in the chain group which defines $H(T; \mathfrak{F})$. Hence $H_0(T; \mathfrak{F}) = 0$.

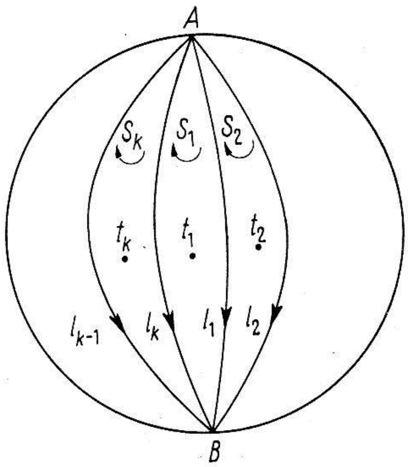


Figure 1

A complex for $H(T, \mathfrak{F})$ 0-complex $\mathfrak{F}(A) \oplus \mathfrak{F}(B)$
1-complex $\bigoplus_j \mathfrak{F}(l_j)$
2-complex $\bigoplus_j \mathfrak{F}(s_j)$

The elements of $H_2(T; \mathfrak{F})$ are defined by elements of $\mathfrak{F}(T)$, *i.e.* by elements of $\mathfrak{F}(t_0)$ which can be extended to the whole of T as single valued functions. The subspace spanned by such elements of $\mathfrak{F}(t_0)$ is invariant under π_1 $(T - N; t_0)$. It cannot be the whole of $\mathfrak{F}(t_0)$. Hence it must be empty. Thus $H_2(T; \mathfrak{F}) = 0$.

We now have rank $H_1(T; \mathfrak{F}) = -\chi = \sum_j r_j - 2n$

Corollary. If C is an irreducible algebraic curve of degree n having only ordinary multiple points and cusps its genus g is given by

$$g = \frac{m + \varkappa}{2} - n + 1 \qquad \text{(Plücker's formula)}$$

where m is the class and $\varkappa$ the number of cusps.

Consider the fibering of C by a generic pencil of sections. Denote by t a parameter for the pencil and by u a parameter for the fibers so that C is considered as embedded in the product of the t-plane and the u-plane. The function $u(t)$ defined by $(t, u(t)) \varepsilon C$ is n-valued and is a function of finite sheet type. Let $\mathfrak{U}$ denote the sheaf defined by $u(t)$.

$$H(C) \simeq H(T; \mathfrak{U})$$

$u(t)$ is not irreducible for $u_1(t) + \ldots + u_n(t)$ is invariant. We therefore introduce the function $F(t)$ whose values are $u_i(t) - u_j(t)$. $F(t)$ is of finite sheet type and the irreducibility of C implies that of $F(t)$. Moreover

$$\mathfrak{U} \simeq R \oplus \mathfrak{F} \quad \text{where } R \text{ is the constant sheaf.}$$

Hence $H_1(C) \simeq H_1(T; \mathfrak{J})$. By Proposition 2, $H_1(T; \mathfrak{J})$ has rank $(m + \varkappa) - 2(n - 1)$, which gives Plücker's formula.

Now consider the ladder diagram A_N with N rungs. The phase space function defined by integrating $\mu(A_N)$ over a Lefschetz class for the leading Landau singularity of A_N is a separating function G_N for $S(A_N)$. Choose the loop momenta $k_1, \ldots, k_{N-1}$ as indicated in Figure 2. If we regard the integral defining G_N as a repeated integral over $K_1, \ldots, K_{N-2}$ and then over K_{N-1}, we get a representation of G_N in terms of G_{N-1}

$$G_N(t_0) = \int G_{N-1}(t_0, u)\, du$$

where u denotes a complex parameter for the intersection defined in K_{N-1} by the last three propagators which is a conic C and so rational. We can take u to be

$$\sqrt{\frac{s_1 - s_1^{1)}}{s_1 - s_1^{2)}}}$$

where s_1 denotes the energy variable indicated in Figure 2. and $s_1^{1)}$, $s_1^{2)}$ are the two values of s_1 for which the points on C having this value of s_1 coincide. Now the Landau singularity $G(A_{n-1})$ for the $(N - 1)$ rung ladder diagram with s_1 as energy variable has as components the leading Landau singularity together with components independent of s_1. In [9] Regge and Barucchi show that the leading Landau singularity of A_{N-1} has degree 2^{N-2} in s_1 and hence 2^{N-1} in u. If we assume

a) that $G_{N-1}(u)$ is irreducible and has rank 1 singularities in u

b) that the homology group $H_1(U; \mathcal{G}_{N-1})$ is isomorphic with the vector space V_0 of germs of $G_N(t)$ over t_0 then it follows from Proposition 2 that

sheet index of $G_N = 2^{N-1} - 2\, x$ (sheet index of G_{N-1})

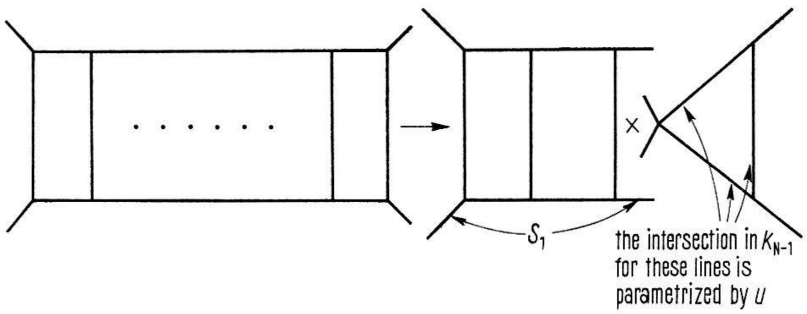


Figure 2

Now the sheet index of G_2 is 1, so this formula gives

sheet index of G_N = dimension of the layer $M(A_N) = 2^{N-2}$

This formula will be proved in this paper for $N = 3$ and it is also true for $N = 4$. The first part of assumption a) is a strong form of Conjecture 2, the second part is proved by the Picard-Lefschetz theorem, since the singularities in u correspond to simple quadratic pinches in the integration space for G_{N-1}. Assumption b) asserts that the elements of $H_1(U; \mathcal{G}_{N-1})$ which correspond to Lefschetz classes on $S(A_N)$ span this space and the operation q of integrating over these Lefschetz classes defines an isomorphism onto V_0. The first part of assumption b) can only be established by assuming Conjecture 4 for A_{N-1} which gives an isomorphism between $H_1(U; \mathcal{G}_{N-1})$ and a group which appears in the Serre spectral sequence which expresses geometrically the idea of representing a multiple integral as a repeated integral. Then it can be interpreted entirely in geometrical terms. The second part of b) is Conjecture 4 for A_N. The relation between the Serre spectral sequence and repeated integration will be illustrated in § 4.

2.9. *The proof of Conjectures 1–6 for the sixth order ladder diagram*

Let D denote the inversion compactification of the space K of internal momenta for the sixth order ladder diagram. We denote by D_I, $I \varepsilon J$, the set of subsets of $\{1, \ldots, 7\}$, the intersections in D of the pole loci for a Feynman function $F(t)$ associated with this graph and not having second-type singularities. In § 4 we prove that for every $I \varepsilon J$

$$H_{n-|I|}(D_I) = E_I \oplus A_I \tag{1}$$

where a) E_I is spanned by Lefschetz classes on D_I for loops around G_I

b) the Kronecker index defines on E_I a nondegenerate bilinear form, positive or negative definite if $|I|$ is even

c) on A_I the residue onto D_I of the differential form in the integral defining $F(t)$ vanishes.

Since S_I, the intersection in Z of the pole loci for a standard form representation of $F(t)$ is a modification of D_I, there is an algebraic variety X_I in S_I and an algebraic variety Y_I in D_I together with a biholomorphic map $i : S_I - X_I \to D_I - Y_I$. We may use this map to identify Lefschetz cycles on S_I for G_I with Lefschetz cycles on D_I for G_I, since the Lefschetz cycles have support not intersecting X_I, Y_I respectively. The only question which arises is whether the linear relations between the corresponding homology classes are the same on S_I as on D_I. This is guaranteed by b) according to which these relations are determined by computing intersection numbers and these are the same for corresponding pairs of cycles on $S_I - X_I$ and $D_I - Y_I$. We are therefore justified in identifying the E_I which appears in (1) with that defined in 2.8.

The first part of Conjecture 5, that for each contracted graph H', $\mu^{(r)}(H')$ has no effective poles is proved by the analysis which we summarise at the beginning of § 4. The second part that the functions obtained by integrating it over non-vanishing Lefschetz classes are not zero may be proved by computing these functions or by using the form of the Picard-Lefschetz theorem which gives the explicit local form of the function near a singular point corresponding to a simple quadratic pinch. Then the decomposition (1) with the identification established above proves Conjectures 4 and 6. The detailed description of the E_I also establishes Conjecture 2. Finally, Conjecture 1 follows from Proposition 1 of 2.8. We noted earlier that Regge and Barucchi have established Conjecture 3.

3. Some Topological Results

3.1. *A Spectral Sequence*

Let Z be a complex manifold, $S_{i(i=1,\ldots,m)}$ complex submanifolds of codimension 1 in Z, and in G.P. We consider the problem of relating the homology group $H^c\,(Z - \bigcup_{i=1}^{m} S_i)$ to the homology groups $H^c(S_I)$. Here $S_I = Z \bigcap_{i \varepsilon I} S_i$ and I runs through the set J of subsets of $\{1, \ldots, m\}$.

Consider first the case $m = 1$. In this case $H^c(Z - S_1)$, $H^c(Z)$, $H^c(S_1)$ are related by the exact sequence (J. LERAY [18]).

$$\begin{tikzcd} & H^c(Z - S_1) \arrow[dl, "i"'] & \\ H^c(Z) \arrow[rr, "\tilde{\omega}"] & & H^c(S_1) \arrow[ul, "\delta"'] \end{tikzcd} \tag{1}$$

where i is the homomorphism induced by the injection map $Z - S_1 \to Z$, $\tilde{\omega}$ is the homomorphism $a \to a.s_1$ where the symbol $a.s_1$ is used to denote the homology class on S_1 of the intersection with S_1 of a representative cycle for a in G.P. with respect to S_1. δ is the coboundary map of J. LERAY.

The situation for general m can be described by introducing a spectral sequence. Let a be a singular chain of $S_I - S^{CI}$, $I \varepsilon J$. Construct a Riemannian metric μ on Z in such a way that the S_i are orthogonal. Let $\varepsilon > 0$ be sufficiently small. Then we may construct a singular chain b of $Z - \bigcup_{i=1}^{m} S_i$ as follows:

Let $I = \{i_1 \ldots i_p\}$ $i_1 < \ldots < i_p$. Set $I_s = \{i_{s-1} \ldots i_p\}$ $1 \leqslant s \leqslant p$ and define for each s a singular chain a_s of $S_{I_s} - S^{CI_s}$ by induction on s. a_{s+1} is the boundary of a tubular neighbourhood of a_s of radius ε constructed by means of the metric μ (for this construction see J. LERAY [18]. Set $a_p = b$.

We call any chain which is a linear combination of chains such as b a tubular chain of order p. I may be any element of J with $|I| = p$ and the Riemannian metric μ and the radius $\varepsilon > 0$ may be chosen freely subject to the conditions stated above. Denote by C_p the group of tubular chains of order $m - p$. The boundary of a tubular chain of order p is also a tubular chain of order p. Hence

$$\phi = C_{-1} \subset C_0 \subset \ldots \subset C_m = C$$

is an admissible filtration of the group C of singular chains of $Z - \bigcup_{i=1}^{m} S_i$. This filtration gives rise to a spectral sequence E^k which terminates in a graded group associated with $H^c\left(Z - \bigcup_{i=1}^{m} S_i\right)$.

We will now describe the spectral sequence E^k.

There is a natural isomorphism between ${}_nE_p^0 = {}_nC_p / {}_nC_{p-1}$ (dimension index on the left, filtration index on the right) and $\bigoplus_{I \varepsilon J} {}_{n-m-p}T_I$ where the summation is over $I \varepsilon J$ with $|I| = m - p$ and T_I denotes the group of singular chains of $S_I - S^{CI}$ modulo tubular chains of $S_I - S^{CI}$.

Given a Riemannian metric μ as above we can define a homomorphism

$$\mu : C^*(S_I) \to T_I$$

where $C^*(S_I)$ is the group of singular chains of S_I in G.P. with respect to the S_j, $j \varepsilon CI$. If g is a singular simplex in $C^*(S_I)$ we can subtract from g a tubular neighbourhood $g\varepsilon$ of its intersection with S^{CI} of sufficiently small radius ε to give a chain of $S_I - S^{CI}$ and hence an element μg of T_I. The singular simplexes generate $C^*(S_I)$ freely so we can extend μ to a homomorphism. The boundary of $g - g\varepsilon$ differs from that of g by a tubular chain so μ is an admissible homomorphism. Also $C^*(S_I)$ is chain homotopic to $C(S_I)$ the full group of singular chains of S_I. Hence μ induces a homomorphism

$$\nu: H^c(S_I) \to H(T_I)\,.$$

Since every tubular cycle in $S_I - S^{CI}$ bounds in S_I, ν has kernel zero. Also the map $\mu: C^*(S_I) \to T_I$ is onto so ν is onto. Thus ν is an isomorphism.

Hence E_p^1 is isomorphic with $\bigoplus_{I \varepsilon J} H^c(S_I)$ (sum over I with $|I| = m - p$).

The differential d^1 is the immediate generalization of the homomorphism $\tilde{\omega}$ of (1)—that is if $g = \bigoplus_{I \varepsilon J |I| = m-p} g_I \,\varepsilon\, E_p^1$

$$d^1 g = \bigoplus_{I_1 \varepsilon J |I_1| = m-p+1} \left(\sum_{I;(I,k) = I_1} \varrho(I, k)\, g_I \cdot s_k \right) \tag{2}$$

In (2), $\varrho(I, k)$ denotes the sign of the permutation $(I, k) \to I_1$. This factor enters because of the skewsymmetry of LERAY's iterated coboundary [18].

Now focus attention on the homology groups in the middle dimensions of the spaces under consideration. The subgroup of $H_n^c(Z)$ which survives into the second term of the spectral sequence we call the layer corresponding to Z. The layer corresponding to an intersection S_I is defined in the same way by considering the spectral sequence which related $H^c\,(S_I - S^{CI})$ to the homology group $H^c(S_{I_1})$, $I_1 \supseteq I$. If the term ${}_nE_p^2$ of filtration p in the graded group associated with $H_n^c\left(Z - \bigcup_{i=1}^{m} S_i\right)$ by the spectral sequence is the direct sum of the layers for the intersections S_I with $|I| = m - p$ then we say that $H_n^c\left(Z - \bigcup_{i=1}^{m} S_i\right)$ has a layer structure.

The layer corresponding to Z may be identified with the quotient of $H_n^c\left(Z - \bigcup_{i=1}^{m} S_i\right)$ by the subgroup spanned by elements which are coboundaries of homology classes of one of the groups $H_{n-1}^c\left(S_j - \bigcup_{i \neq j} S_i\right)$.

3.2. *Finite Homology Groups of Complete Intersections*

In this section we introduce the notion of the finite homology groups of an algebraic variety in a projective space, following FÁRY [8], and present some striking results proved by FÁRY concerning these homology groups in the case in which the algebraic variety is a complete intersection (defined below). At the end of the section we point out that these results may be applied to the study of a function $F(t)$ defined by an integral representation of standard form in which the integration space Z and the intersections of pole loci S_I are complete intersections.

One of FÁRY's main purposes in [8] is to give a proof of the Lefschetz theorem on hyperplane sections. We prefer to quote the stronger form of this theorem due to ANDREOTTI and FRAENKEL, referring to MILNOR [19] for the elegant proof using Morse theory.

Theorem 1. A complex analytic manifold M of complex dimension k, bianalytically embedded as a closed subset of C^n has the homotopy type of a k-dimensional CW-complex.

COR 1. If $M \subset C^n$ is an algebraic manifold in complex n-space of real dimension $2k$ then

$$H_i(M; R) = 0 \quad \text{for} \quad i > k .$$

COR 2. With M as in COR 1, if $\mathfrak{J}$ is a locally constant sheaf (system of local coefficient groups) on M

$$H_i(M; \mathfrak{J}) = 0 \quad \text{for} \quad i > k .$$

Proposition 1. (Fáry *loc. cit.*, Chapter 1 3°). Let W^n be an irreducible algebraic variety without singularities in P^m. Let V^n be the affine part of W^n; $V^n = W^n - (W^n \cap P^{m-1})$ where P^{m-1} is not tangent to W^n. Consider the injections $i: H(V^n) \to H(W^n)$. The images of these injections are independent of V^n.

$$i_1\, H(V_1^n) = i_2 H(V_2^n) .$$

Definition 1. Let W^n be an irreducible algebraic variety without singularities in P^m, and let $i: H(V^n) \to H(W^n)$ be the injection. We define the finite homology group $H^F(W^n) = iH(V^n)$. By Proposition 1 the group so defined is independent of the particular affine coordinate system used in the definition.

We will also refer to the quotient group $H_p(V^n)/\delta H_{p-1}(W_\infty^n) \simeq H_p^F(W^n)$ as the finite homology group of V^n.

Proof of proposition 1.

If $h \,\varepsilon\, i_1 H(V_1^n)$, we can choose a representative cycle $\underline{h}$ of h with support in V_1^n. We have to prove that $\underline{h} \cdot V_2^{n-1}$ is homologous to zero in $W^n \cap P_2^{m-1}$. Here $V_\alpha^{n-1} = V_1^n \cap P_\alpha^{m-1}$, $\alpha \neq 1$.

Consider the pencil of hyperplane sections of W^n with base $P_1^{m-1} \cap P_2^{m-1}$ and complex parameter α. Since this pencil contains sections which are not singular, it contains only a finite number of singular sections. If we delete the corresponding values of α from the α-plane, the remaining points form a connected region R containing 1 and 2. Choose a simple arc λ in R from 2 to 1. If $\alpha \,\varepsilon\, \lambda$ is sufficiently close to 1, h, which has compact support in V_1^n, does not intersect V_α^{n-1}. Denote by 3 a value of α satisfying this condition and by Y the union of the V_α^{n-1} for all $\alpha \,\varepsilon\, \lambda$ from 2 to 3. Then Y is homeomorphic to a direct product of the interval 23 along λ and a typical section V_α^{n-1} so there are isomorphisms

$$H(V_2^{n-1}) \simeq H(Y) \simeq H(V_3^{n-1}) .$$

The fact that $h \cdot V_3^{n-1} = 0$ thus implies $h \cdot V_2^{n-1}$ is homologous to zero in V_2^{n-1} and *a fortiori* in $W^n \cap P_2^{m-1}$.

This proof is a rephrasing of Fáry's proof (*loc. cit.*, Chapter 4, 16°) in terms of homology rather than cohomology.

Definition 2. Let W^n be an algebraic variety in P^m, the cycles of W^n which have as support the intersections with W^n of linear subspaces P^k of P^m are called algebraic and the subgroup of $H(W^n)$ which they span is called the algebraic homology group of $H^A(W^n)$ of $H(W^n)$. (Cf. Fáry, *loc. cit.*, Chapter 1, 3°). Note that here we are using rational coefficients.)

Definition 3. (Fáry, *loc. cit.*, 4°) An algebraic variety W^n of P^m is a complete intersection if there is an increasing sequence

$$W^n \subset W^{n+1} \subset \ldots \subset W^{m-1} \subset W^m = P^m \tag{1}$$

of algebraic varieties of P^m having the following properties:

1° W^{n+1} is for all i, $0 \leqslant i \leqslant m-n$, irreducible and non-singular.

2° W^{n+i} belongs to a linear system $\{W_\xi^{n+i}\}$ of subvarieties of W^{n+i+1} whose base is in $W^{n+i+1} \cap P_\infty^{m-1}$, and such that

i) W_ξ^{n+i} has no singularities if $\xi \,\bar{\varepsilon}\, \Gamma_i$; Γ_i a finite set of points in the ξ-plane.

ii) W_ξ^{n+i} has for $\xi \varepsilon \Gamma_i$ a single quadratic simple pinch not on P_∞^{m-1}.

Theorem 2. Let W^n be a complete intersection.

$$H_p^A(W^n) = 0 \quad \text{or } R \text{ according as } p \text{ is odd or even.}$$

$$H_p^F(W^n) = 0 \quad \text{except for } p = 0, n.$$

$$H_p(W^n) = H_p^A(W^n) \oplus H_p^F(W^n) \quad \text{for all } p > 0.$$

(This is FÁRY's Theorem 2, Chapter 1, 4° with the simplification that results from the use of rational coefficients.)

We first prove by induction along the sequence (1) that the assertions of Theorem 2 hold for dimensions other than the middle dimension (FÁRY, *loc. cit.*, Chapter 4, 17°).

We have a Leray sequence (cf. 3.1.).

$$\begin{tikzcd} & H(V^{n+1}) & \\ H(W^n) \arrow[ur, "\delta"] & & H(W^{n+1}) \arrow[ul, "i"'] \arrow[ll, "\tilde{\omega}"'] \end{tikzcd} \qquad V^{n+1} = W^{n+1} - W^n$$

By Theorem 1, $H_p(V^{n+1}) = 0$ for $p > n+1$ so we obtain an isomorphism

$$H_p(W^n) \simeq H_{p+2}(W^{n+1}) \quad p > n\,.$$

The induction hypothesis, that the theorem is true except possibly in the middle dimension for W^{n+1}, then gives $H_p(W^n) \simeq H_p^A(W^n) = 0$ or R according as p is odd or even for $p > n$. The Poincaré duality theorem then establishes this result for $p < n$, so the theorem is true except possibly in the middle dimension for W^n.

Now suppose that P_∞^{m-1} is a prime in P^m such that $W'^n = W^n \cap P_\infty^{m-1}$ is non-singular. Then a simple geometrical lemma (FÁRY, *loc. cit.*, Chapter 2, 7°, Lemma 2) shows that W^n is a complete intersection.

From the segment of the Leray sequence

$$\to H_n(V^n) \to H_n(W^n) \to H_{n-2}(W_\infty^n) \to$$

for the pair (W^n, W_∞^n) and the isomorphism $H_{n-2}(W_\infty^n) \simeq H_{n-2}^A(W_\infty^n)$ if it follows that $H_n(W^n) \simeq H_n^F(W^n) \oplus H_n^A(W^n)$.

COR 1. Suppose Z, $S_{i\,(i=1\ldots m)}$ satisfy

a) Z is a complete intersection of complex dimension $n \geqslant m$.

b) Each intersection $Z \bigcap\limits_{i \varepsilon I} S_i$ is a complete intersection, I a subset of $\{1 \ldots m\}$ then

$$H_n^c\left(Z - \bigcup_{i=1}^m S_i\right) \simeq \bigoplus_I \delta^{|I|} H_{n-|I|}^F (Z \cap S_I)\,.$$

This corollary follows at once from Theorem 2 and the result of 3.1. Under the action of d^1, algebraic classes are mapped into algebraic classes and finite classes are killed. Thus $H_n^c\left(Z - \bigcup\limits_{i=1}^m S_i\right)$ has a layer structure.

COR 2. The finite homology group $H_n^F(W^n)$ of a complete intersection is spanned by Lefschetz classes for paths in the ξ-plane from ξ_0 to the points of Γ_0. (We use the notation of definition 3; ξ_0 is the value of the parameter ξ corresponding to W^n, ∞ is the value of ξ corresponding to W_∞^{n+1}).

Let $h \varepsilon H_n^F(W^n)$. Then we can write $h = ih_1$ where $h_1 \varepsilon H_n(V^n)$ and i is the injection map $H_n(V^n) \to H_n(W^n)$. We will prove that h_1 is not invariant under the action of π_1 (ξ-plane $- \Gamma_0$; ξ_0) on $H_n(V^n)$, so that by the Picard-Lefschetz theorem there is some Lefschetz class e for path from ξ_0 to a point of Γ_0 such that $h_1 \cdot e \neq 0$. It then follows from the Poincaré duality theorem that the group E generated by Lefschetz classes for paths in the ξ-plane from ξ_0 to the points of Γ_0 is equal to $H_n^F(W^n)$.

Suppose h_1 is invariant. We can then construct a homology class $b \varepsilon H_{n+2}(W^{n+1})$ such that $h_1 = \tilde{\omega} b$. But $H_{n+2}(W^{n+1}) = H_{n+2}^A(W^{n+1})$ so this implies that h_1 is an algebraic homology class and not finite. The construction of b can be indicated as follows: the invariance of h_1 implies that every section W^n (even the singular sections) has a homology class $h_1(\xi)$ which is obtained from h_1 by an ambiant isotropy for some path $\xi_0 \xi$ but is independent of the path. Geometrically it can be shown that each ξ_1 contains a disc neighbourhood $D(\xi_1)$ such that

$\bigcup_{\xi \in D(\xi_1)} V(\xi) = D(\xi_1) \times V(\xi_1)$ (possibly after excision of a neighbourhood of the pinch on $V(\xi_1)$ if $\xi_1 \varepsilon \Gamma_0$) so that a chain $\underline{b}(\xi)$ can be defined such that $\underline{b}(\xi)$. $V(\xi)$ is for all $\xi \in D(\xi_1)$ a representative of $h(\xi)$. By patching these chains together with the help of partition of unity, we get a cycle $\underline{b}$ whose homology class is b.

Suppose $F(t)$ is a function defined by an integral representation of standard form in which the integration space Z and the intersections S_I of pole loci are complete intersections. More precisely suppose that the sequence

$$S_I \subset S_{I_1} \subset \ldots \subset S_{I_r} = Z^n \subset Z^{n+1} \subset \ldots \subset Z^m = P^m$$

where $I \supset I_1 \supset \ldots \supset I_r = \phi$ is a sequence of the kind described in Definition 3 with the corresponding complex parameters ξ_i appearing among the set of parameters t. Then the results of this section show that

$$H_n^c\,(Z - S(t_0)) \simeq \bigoplus_{I \varepsilon J} \delta^{|I|}\, H_{n-|I|}^F\,(Z \cap S_I(t_0)) \tag{1}$$

and that $H_{n-|I|}^F\,(Z \cap S_I(t_0)$ is spanned by Lefschetz classes ($I \neq \phi$). The decomposition 1) implies that $V \simeq \bigoplus_{I \varepsilon J} V_I$, *i.e.* that $F(t)$ has a layer structure in the sense of 2.6. Moreover, the results of Fáry [8] reduce the problem of calculating the rank of $H_{n-|I|}^F\,(Z \cap S_I(t_0))$ to a problem in algebraic geometry.

Fáry's results thus show that the complete intersection case is as favorable for the application of the homological method as the particular complete intersection case studied by Fotiadi and Pham (*i.e.* the square diagram—we prove in 3.3. that complex quadrics are complete intersections in Fary's sense.)

It is remarkable that in the case of a complete intersection S one should be able to single out the subgroup of the homology group in middle dimension spanned by Lefschetz classes by a definition which does not refer to Lefschetz classes at all (Definition 1). This possibility suggests the following viewpoint: The Lefschetz classes are a certain set of elements of the homology group in middle dimension which are intrinsically singled out. They may be visualized as forming some kind of lattice in the homology group. The transformations of the homology group which may be realized as loop transformations by introducing a parameter space T and a family of varieties $S(t)$ of which S is a member are precisely the symmetries of this lattice. This viewpoint suggests itself particulary forcibly if S has even complex dimension and the Kronecker index defines a definite bilinear form on the subspace spanned by the Lefschetz classes (see 2.8.). For there the associated functions defined by integrating over the Lefschetz classes differential forms which are holomorphic on S are algebraic, and we have a candidate to replace the fundamental group in the theory, *viz.* the Galois group of these functions. This would be desirable because the fundamental group is a very intractable object. Also a theory in which one did not define the transformations on the vector space V_0 of germs of $F(t)$ over a fixed point t_0 by analytic continuation but as symmetries of the structure of V_0 would remove the problem of how large the parameter space T should be made in the case of Feynman functions, *i.e.* whether it should contain all the internal and external masses as we have taken it to do.

3.3. *Some Calculations of Homology Groups*

Define $Q^n_m = \{(z_1, \dots, z_{n+2}) : z_1^2 + \dots + z_{m+2}^2 = 0 \subset P^{n+1}\}$.
We pose the problem of calculating the singular homology groups of Q^n_m.
First consider Q^n_n.
The section of Q^n_n by a tangent hyperplane is a Q^{n-1}_{n-2}. $Q^n_n - Q^{n-1}_{n-2}$ is a paraboloid

$$\{(u_1, \dots, u_{n+1}) : u_1 = u_2^2 + \dots + u_{n+1}^2\} \subset C^{n+1}$$

and is retractible onto the point $(0, \dots, 0)$. The cone Q^{n-1}_{n-2} has vertex V and $Q^{n-1}_{n-2} - V$ is retractible onto Q^{n-2}_{n-2} (by projection from the vertex.)

Hence $H_p(Q^n_n) = Z \qquad p = 0, 2n$

$$H_p(Q^n_n) = H_{p-2}(Q^{n-2}_{n-2}) \qquad 2 \leqslant p \leqslant 2n-2$$

Q^0_0 is a point pair so $H_p(Q^0_0) = 2Z \quad p = 0$

$$H_p(Q^0_0) = 0 \qquad p > 0$$

Q^1_1 is a conic. The conic is a rational curve, so is homeomorphic to P^1.

$$H_p(Q^1_1) = Z \quad p = 0,2$$

$$H_p(Q^1_1) = 0 \quad p \neq 0,2.$$

It now follows by induction on n that

$$H_p(Q^n_n) = Z \qquad p \text{ even} \quad 0 \leqslant p \leqslant 2n \quad p \neq n$$

$$H_n(Q^n_n) = 2Z \quad \text{if } n \text{ is even}$$

$$H_p(Q^n_n) = 0 \qquad \text{otherwise}.$$

Now consider Q^n_m for arbitrary m.

The cone Q_m^n has a vertex V which is a projective space P^{n-m-1}. $Q_m^n - V$ is retractible onto Q_m^m. Hence

$$H_p(Q_m^n) = Z \qquad p \text{ even} \quad 0 \leqslant p \leqslant 2n \quad p \neq 2n - m$$

$$H_p(Q_m^n) = 2Z \qquad p = 2n - m,\ m \text{ even}$$

$$H_p(Q_m^n) = 0 \qquad \text{otherwise}.$$

More precisely we have shown that Q_m^n can be represented as a CW-complex with cells of even dimension only.

The quadrics Q_n^n may readily be shown to be complete intersections in the sense of FÁRY: consider the pencil $Q(\xi): u_1(\xi)x_1^2 + \ldots + u_{n+1}(\xi)x_{n+1}^2 = 0$, $\xi = (\xi_0, \xi_1) \in P^1$. $Q(\xi)$ is a Q_n^n unless $u_i(\xi) = 0$ for some i. If we choose $u_i(\xi) = \xi_0 a_i - \xi_1$ with the $a_{i\,(i=1\ldots n+1)}$ all different, then the pencil will contain $(n+1)$ singular sections Q_{n-1}^n and each of these has a simple quadratic pinch. If we compare the above results with FÁRY's general theorem on the homology of complete intersections (Theorem 2, 3.2.), we see that $H_n^F(Q_n^n) = Z$ or 0 according as n is even or odd. The fact that the finite homology groups of odd dimensional quadrics are zero gives rise for the Feynman amplitude for the square diagram to the relations between discontinuities found by FOTIADI and PHAM [14][8]). However, it is not to be expected that the dimension of an odd dimensional layer should be zero in general—the odd dimensional quadrics are in a sense exceptional (see the results of Leray cited by FÁRY [8] in a footnote in 3°).

Denote by e_n the generator of $H_n^F(Q_n^n)$ given by the above constructions in the case n even. We can prove that $e_n \cdot e_n = -e_{n-2} \cdot e_{n-2}$ and hence, since $e_0 \cdot e_0 = +2$ that $e_n \cdot e_n = 2(-1)^{n/2}$. Comparison with the Picard-Lefschetz theorem then shows that e_n is the Lefschetz class of Q_n^n.

Proof of the assertion that $e_n \cdot e_n = -e_{n-2} \cdot e_{n-2}$:

A generic pencil of primes P^n contains two primes tangent to Q_n^n. These intersect Q_n^n in quadric cones $Q_{n-2}^{n-1}\, C_1$, C_2 with vertices V_1, V_2. The base of the pencil intersects Q_n^n in a Q_{n-2}^{n-2}. According to the results given above, a generator for $H_n^F(Q_n^n)$ is obtained by taking the suspension with vertex V_1 or V_2 of a generator e_{n-2} of $H_{n-2}^F(Q_{n-2}^{n-2})$. In this way we obtain two generators $e_n^{1)}$, $e_n^{2)}$ of $H_n^F(Q_n^n)$. There is a symmetry of the configuration which leaves Q_n^n invariant and interchanges the two tangent primes. Under this symmetry $e_{n-2} \to -e_{n-2}$. It follows that $e_n^{1)} = -e_n^{2)}$. If we choose $e_n^{1)}$ as the generator e_n for $H_n^F(Q_n^n)$ we have

$$e_n \cdot e_n = e_n^{1)} \cdot e_n^{1)} = -e_n^{1)} \cdot e_n^{2)} = -e_{n-2} \cdot e_{n-2}.$$

Define $Q^n = \{(z_1, \ldots, z_{n+1}) : z_1^2 + \ldots + z_{n+1}^2 = 1\} \subset C^n$. $Q^n = Q_n^n - Q_{n-1}^{n-1}$. From the Leray coboundary sequence

$$H_n^c(Q^n) \simeq H_n^F(Q_n^n) \oplus \delta H_{n-1}^F(Q_{n-1}^{n-1})$$

where now we have reverted to rational coefficients. Thus $H_n^c(Q^n) \simeq R$ for all n. This

[8]) In the notation of 2.8. these relations have the form

$$h_1 = 1/2\,(h_{12} + h_{13} + h_{14}) - 1/4\,h_{1234} \qquad h_{234} = -1/2\,h_{1234}$$

together with the relations obtained by permuting indices.

result can also be proved by showing that Z^n is retractible onto its real section S^n. For the finite homology group of Q^n (see 3.2.) we have

$$H_n^F(Q^n) = R \text{ or } 0 \text{ according as } n \text{ is even or odd}.$$

Finally we calculate the homology group

$$H_n\,(Q_n^n - 2\,Q_{n-1}^{n-1}; \mathfrak{J})$$

of the complement in Q_n^n of a pair of non-singular sections Q_{n-1}^{n-1} in general position. The coefficient group $\mathfrak{J}$ is a locally constant sheaf quadratically branched around the sections. We set $Z = Q_n^n$ and denote by S_1 and S_2 the two sections.

Consider first of all the case $n = 1$. The $Z = Q_1^1 = P^1$ is a complex plane and S_1 and S_2 two point pairs $S_1 = (S_1^{1)}, S_1^{2)})$, $S_2 = (S_2^{1)}, S_2^{2)})$. Construct a cut $L = (L^{1)}, L^{2)})$ from S_1 to S_2 as shown in Fig. 3. On $Z - L$ and on $\hat{L}$, the interior of L, the sheaf $\mathfrak{J}$ is constant. $\hat{L}$ is retractible onto a point pair so

$$H_p(\hat{L}) = 2\,R \quad \text{for } p = 0 \text{ and } 0 \text{ otherwise}.$$

From the Leray coboundary sequence for the pair Z, $S_1 \cup S_2$ we find

$$H_p\,(Z - S_1 \cup S_2) = R \text{ for } p = 0, \quad 3\,R \text{ for } p = 1, \quad 0 \text{ for } p = 2.$$

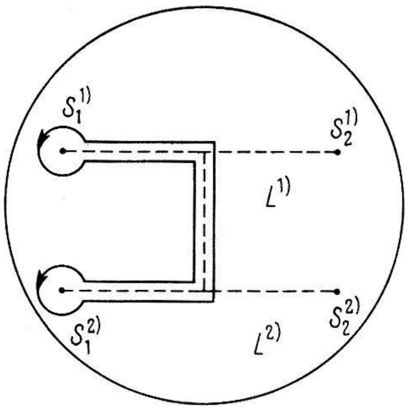


Figure 3

Then from the exact sequence for the pair $Z - S_1 \cup S_2$, $\hat{L}$ we have

$$H_1\,(Z - S_1 \cup S_2,\, \hat{L}) = 4R$$

As generators for this group we can take

$$\delta(S_1^{1)} - S_1^{2)}), \quad \delta\,(S_2^{1)} - S_2^{2)}), \quad \delta\,(S_1^{1)} - S_2^{1)}), \quad \partial^{-1}\,(L_0^{1)} - L_0^{2)})$$

where $L_0 = (L_0^{1)}, L_0^{2)})$ is a point pair onto which $\hat{L}$ is retractible. The exact sequence for the pair $Z - S_1 \cup S_2$, $\hat{L}$ taken with coefficients in $\mathfrak{J}$ differs from that with rational coefficients only in that the groups

$$H(Z - S_1 \cup S_2; \mathfrak{J}) \text{ and } H(Z - S_1 \cup S_2; R)$$

are not the same and that the boundary operator $\partial_{\mathfrak{J}}$ which connects

$$H(Z - S_1 \cup S_2,\, \hat{L}; \mathfrak{J}) \simeq H(Z - S_1 \cup S_2,\, \hat{L}; R) \quad \text{and} \quad H(\hat{L}; \mathfrak{J}) \simeq H(\hat{L}; R) \quad (1)$$

is not the same as the boundary operator ∂. In fact

$$\partial_{\mathfrak{J}}: H_1\,(Z - S_1 \cup S_2,\, \hat{L}) \to H_0(\hat{L})$$

is given by

$$\partial_{\mathfrak{J}}\,\delta(S_1^{1)} - S_1^{2)}) = 2\,(L_0^{1)} - L_0^{2)}) \quad \partial_{\mathfrak{J}}\,\delta(S_2^{1)} - S_2^{2)}) = -2\,(L_0^{1)} - L_0^{2)})$$

$$\partial_{\mathfrak{J}}\,\delta(S_1^{1)} - S_2^{1)}) = 0 \quad \partial_{\mathfrak{J}}\,\partial^{-1}(L_0^{1)} - L_0^{2)}) = L_0^{1)} + L_0^{2)}\,.$$

The signs are fixed by the choice of the isomorphisms (1). We have chosen them so that the positive generator of the section of $\mathfrak{J}$ over a point P in $Z - L$ goes over into the positive generator of the section of $\mathfrak{J}$ over a point Q in L if P approaches Q from below the cut in Figure 3. From these boundary relations we find

$$H_1\,(Z - S_1 \cup S_2;\, \mathfrak{J}) = 2\,R$$

and that generators of this group are defined by the cycles indicated in Figures 4a and 4b. Denote by e_1 and e_2 these generators. Then from the representative cycles we compute

$$e_1 \cdot e_1 = 0 \quad e_2 \cdot e_2 = 0 \quad e_1 \cdot e_2 = 4\,.$$

If it is required only to determine the rank of $H_1\,(Z - S_1 \cup S_2;\, \mathfrak{J})$ then the proof can be much shortened. Our comparison of the exact sequences of the pair $Z - S_1 \cup S_2,\, \hat{L}$ with coefficients $\mathfrak{J}$ and R shows that $H(Z - S_1 \cup S_2;\, \mathfrak{J})$ and $H(Z - S_1 \cup S_2;\, R)$ have the same Euler-Poincaré characteristic. But

$$\chi(H(Z - S_1 \cup S_2;\, R)) = \chi(Z) - \chi(S_1) - \chi(S_2) = -2$$

and $H_0(Z - S_1 \cup S_2;\, \mathfrak{J}) = 0$ (a 0-chain $P \otimes \mathsf{F}$ where F is an element of the section of $\mathfrak{J}$ over P bounds $1/2\; L \otimes \mathsf{F}$, where L denotes a loop on P around one of the quadratic branch points of $\mathfrak{J}$) and

$$H_2(Z - S_1 \cup S_2;\, \mathfrak{J}) = 0$$

(since $Z - S_1 \cup S_2$ is retractible onto its 1-dimensional skeleton—a very special case of Theorem 1, 3.2.) so

$$H_1(Z - S_1 \cup S_2;\, \mathfrak{J}) = 2\,R$$

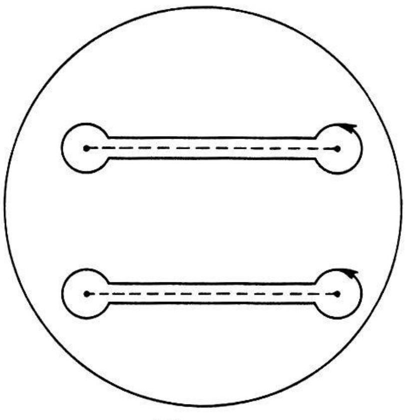


Figure 4a
A cycle representing
$i^{-1}\,\delta(S_1^{1)} - S_2^{1)})$

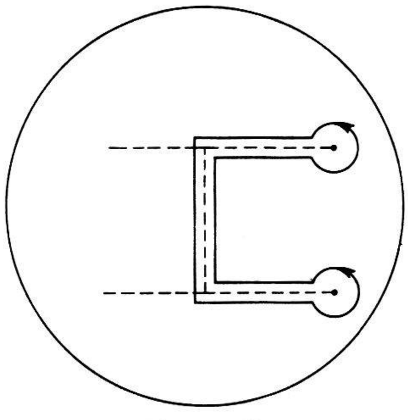


Figure 4b
A cycle representing
$i^{-1}\,(\delta(S_1^{1)} - S_1^{2)}) + \delta\,(S_2^{1)} - S_2^{2)}))$

The calculation for $n = 1$ is essentially the calculation of the homology of the Riemann surface of an algebraic function with four quadratic branch points. It is, of course, well-known that this surface is a torus, but we felt it worthwhile to give the argument in detail in order to make the argument for general n easier to follow.

Now consider the general case, $n > 1$.

Denote by $S(\alpha)$ a member of the pencil of sections of Z containing S_1 and S_2. Here α is a complex parameter. The pencil contains two singular sections; denote by α_3 and α_4 the corresponding values of α. Construct a path λ from α_1 to α_2 not containing α_3 and α_4 and denote by $L \bigcup_{\alpha \varepsilon \lambda} S(\alpha)$. On $Z - L$ the sheaf $\mathfrak{F}$ is constant, for $Z - L$ is simply connected (L is retractible in Z onto S_1 and $Z - S_1 = Q_n^n - Q_{n-1}^{n-1} = Q_n$ is simply connected for $n > 1$). $\hat{L}$, the interior of L, is a topological product $\hat{\lambda} \times (S(\alpha_0) - B)$ where $\hat{\lambda}$ denotes the interior of λ, $B = S_1 \cap S_2$ the base of the pencil, and α_0 some point in $\hat{\lambda}$.

$$L_0 = S(\alpha_0) - B = Q_{n-1}^{n-1} - Q_{n-2}^{n-2} = Q_{n-1} .$$

Compare the exact sequences for the pair $Z - S_1 \cup S_2, \hat{L}$ with coefficients in R and $\mathfrak{F}$

$$\begin{array}{ccccccccc}
0 & \xleftarrow{i_{\mathfrak{F}}} & H_{n-1}(\hat{L}; \mathfrak{F}) & \xleftarrow{\partial_{\mathfrak{F}}} & H_n(Z - S_1 \cup S_2, \hat{L}; \mathfrak{F}) & \xleftarrow{j_{\mathfrak{F}}} & H_n(Z - S_1 \cup S_2; \mathfrak{F}) & \longleftarrow & 0 \\
 & & \updownarrow i_1 & & \updownarrow i_2 & & & & \\
0 & \xleftarrow{i} & H_{n-1}(\hat{L}; R) & \xleftarrow{\partial} & H_n(Z - S_1 \cup S_2, \hat{L}; R) & \xleftarrow{j} & H_n(Z - S_1 \cup S_2; R) & \longleftarrow & 0 .
\end{array} \tag{2}$$

Here we have used the fact that $H_n(\hat{L}; \mathfrak{F}) = H_n(\hat{L}; R) = 0$ and that $i_{\mathfrak{F}} H_{n-1}(\hat{L}; \mathfrak{F}) = iH_{n-1}(\hat{L}, R) = 0$ where i denotes the injection $i: \hat{L} \to Z - S_1 \cup S_2$. The isomorphisms i_1 and i_2 are defined by choosing sections of $\mathfrak{F}$ over $\hat{L}$ and $Z - L$. We choose these sections so that the positive generator of the section of $\mathfrak{F}$ over a point $P \varepsilon S(\alpha) - B$, $\alpha \bar{\varepsilon} \lambda$ goes over into the positive generator of the section of $\mathfrak{F}$ over a point $Q \varepsilon \hat{L}$ if $P \to Q$ through sections $S(\beta) - B$ with β below λ (Figure 5).

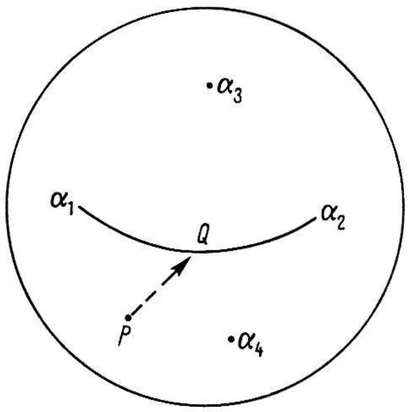


Figure 5
α-plane

By Theorem 2, COR 1., 3.2.,

$$H_n(Z - S_1 \cup S_2) = H_n^F(Z) \oplus \delta H_{n-1}^F(S_1) \oplus \delta H_{n-1}^F(S_2) \oplus \delta^2 H_{n-2}^F(S_{12}) .$$

If n is even the second and third of these groups are zero and the first and fourth are generated by Lefschetz classes e, e_{12}.

If n is odd, the first and fourth of these groups are zero and the second and third are generated by Lefschetz classes e_1, e_2. In any case $H_n(Z - S_1 \cup S_2) = 2R$ and from (2) we deduce that

$$H_n(Z - S_1 \cup S_2; \mathfrak{J}) = 2R .$$

To give the generators for this group explicitly, we must write down the action of $\partial_{\mathfrak{J}}$. Denote by $\hat{l}$ the generator of $H_{n-1}(L)$ which is defined by the cycle whose support is the real section of the quadric L_0. Then $H_n(Z - S_1 \cup S_2, \hat{L}; \mathfrak{J})$ is generated by $\partial^{-1}\hat{l}$ and e, $\delta^2 e_{12}$ or δe_1, δe_2 (strictly by the images of these elements under i_2^{-1} but we will identify isomorphic groups) and

$$\partial_{\mathfrak{J}}\, \partial^{-1}\hat{l} = \hat{l}$$

$$\partial_{\mathfrak{J}}\, \delta e_1 = 2\,\hat{l} \qquad \partial_{\mathfrak{J}}\, \delta e_2 = -2\,\hat{l} \qquad \text{for n odd}$$

$$\partial_{\mathfrak{J}}\, e = 0 \qquad \partial_{\mathfrak{J}}\, \delta^2 e_{12} = 4\,\hat{l} \qquad \text{for } n \text{ even} .$$

Then for n odd we have generators

$$f_1 = j^{-1}(\delta e_1 - 2\,\partial^{-1}\hat{l}) \qquad f_2 = j^{-1}(\delta e_2 + 2\,\partial^{-1}\hat{l})$$

and for n even we have generators

$$g_1 = j^{-1} e \qquad g_2 = 1/2\, j^{-1}(\delta^2 e_{12} - 4\hat{l})$$

(the factor $1/2$ in g_2 is included for convenience). Representative cycles are indicated in Fig. 6a, 6b. Fig. 6 shows the projections of the cycles under the map $Z - S_1 \cup S_2 \to$ (α-plane $- \alpha_1 \cup \alpha_2$) which assigns to each section $S(\alpha) - B$ the corresponding value of α. The fiber $S(\alpha) - B$ is for $\alpha \neq \alpha_3$, α_4 an affine quadric $Q_{n-1}(\alpha)$. The cycles f_1, f_2, g_1, g_2 are defined up to orientation by choosing over some point α_0 of their projection a spherical cycle S_{n-1}, retract of $Q_{n-1}(\alpha_0)$, and then moving this cycle along the path in a continuous way (*i.e.* by constructing an ambiant isotopy). In the case of the closed loops which appear in Figure 6a, the spherical cycle obtained by moving round the loop will not necessarily be the same as that initially chosen, but since each loop goes round two quadratic branch points (one for the sheaf $\mathfrak{J}$ and one for the spherical cycle—by the Picard-Lefschetz theorem) they will be homologous in Q_{n-1} (α_0). Adding in the chain which their difference bounds we get the cycle which was to be constructed. In the case of the paths of Figure 6b which bound on α_3 of α_4, the radius of the spherical cycle in $Q_{n-1}(\alpha)$ is to decrease to zero as α approaches α_3 or α_4.

From Figure 6a we read off the intersection numbers

$$f_1 \cdot f_2 = 4 \qquad f_1^2 = f_2^2 = 0 .$$

The cycle representing g_1 in Fig. 6b is a Lefschetz cycle so we have

$$g_1^2 = 2\,(-1)^{n/2} .$$

From this result we can then obtain, using Fig. 6b,

$$g_1 \cdot g_2 = 2 \qquad g_2^2 = 4\,(-1)^{n/2} .$$

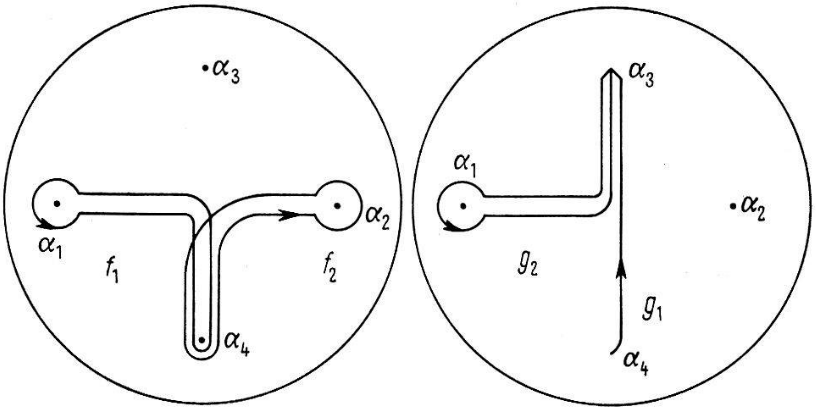


Figure 6a
Representatives for f_1, f_2

Figure 6b
Representatives for g_1, g_2

In the applications in § 4 the group $H_n(Z - S_1 \cup S_2; \mathfrak{F})$ arises in the following way: There is a space Y and a projection map $p: Y \to X$ such that $p^{-1}(z)$ is a Q^{2k}_{2k} for some k, provided $z \bar{\varepsilon} S_1 \cup S_2$ and $p^{-1}(z)$ is a Q^{2k}_{2k-1}, if $z \varepsilon S_1 \cup S_2$. If we denote by Y_1, Y_2, $p^{-1}(S_1)$, $p^{-1}(S_2)$; $Y - Y_1 \cup Y_2$ is a Serre fiber space with projection p, fiber Q^{2k}_{2k} and base $Z - S_1 \cup S_2$. If we calculate $H(Y - Y_1 \cup Y_2)$ by means of the Serre spectral sequence, we find in the second term of the spectral sequence the group $H_n\,(Z - S_1 \cup S_2; \mathfrak{F})$ where $\mathfrak{F}$ denotes the sheaf $\{H^F_{2k}(p^{-1}(z))\}$. By the Picard-Lefschetz theorem this sheaf is quadratically branched around S_1 and S_2. In this case we take as the generator of $H^F_{2k}(p^{-1}(z))$ its Lefschetz class $e(z)$. The differentials d_r, $r \geqslant 2$, act trivially on $H_n(Z - S_1 \cup S_2; \mathfrak{F})$ so the homology classes f_1, f_2 or g_1, g_2 give rise to homology classes F_1, F_2 or G_1, G_2 in $Y - Y_1 \cup Y_2$. Since $e(z) \cdot e(z) = 2\,(-1)^k$, the intersection numbers of F_1, F_2 or G_1, G_2 may be determined from those of f_1, f_2 or g_1, g_2 by multiplying by the factor $2\,(-1)^k$.

We shall also need in § 4 the fact that for $n > 1$, as for $n = 1$

$$H_p(Z - S_1 \cup S_2; \mathfrak{F}) = 0 \quad \text{for} \quad p \neq n\,.$$

This result is easily obtained from the exact sequences for the pair $Z - S_1 \cup S_2$, $\hat{L}$ with coefficients in R and $\mathfrak{F}$. Since $H_p(\hat{L}) = H_p(Z - S_1 \cup S_2) = 0$ for $p \neq 0, 1, n$, we have $H_p(Z - S_1 \cup S_2; \mathfrak{F}) = 0$ for $p = 0, 1, n$. By the same arguments as in the case $n = 1$, we have $H_0(Z - S_1 \cup S_2; \mathfrak{F}) = 0$ and

$$\chi(H(Z - S_1 \cup S_2; \mathfrak{F})) = \chi(H(Z - S_1 \cup S_2; R) = -2\,(-1)^n\,.$$

But we have shown that $H_n(Z - S_1 \cup S_2; \mathfrak{F}) = 2\,R$. It follows that

$$H_1(Z - S_1 \cup S_2; \mathfrak{F}) = 0\,.$$

4. The Sixth Order Ladder Diagram

4.1. *Preliminaries*

Figures 7a and 7b show the sixth order ladder diagram together with the numbering of its lines and the choice of loop momenta that we use in the following analysis. We take the dimension of the space P of momentum vectors to be 4. The function $F(t)$

to be studied is one defined by a germ $I(t)$ which admits an integral representation

$$I(t) = \int_{\Gamma_0} \frac{P(k, p) d^4 k_1 \Lambda \, d^4 k_2}{\prod_{i=1} (q_i^2 + m_i^2)^{n_i}} \tag{1}$$

for t in some neighbourhood of t_0. Γ_0 is the chain whose support is the set of loop momenta whose components are real with the orientation defined by ordering the coordinates of a point in K as they appear in $d^4k_1 \Lambda \, d^4k_2$. The n_i are positive integers.

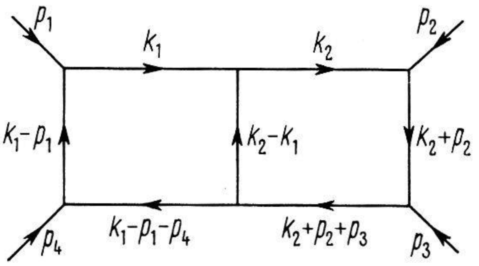


Figure 7a

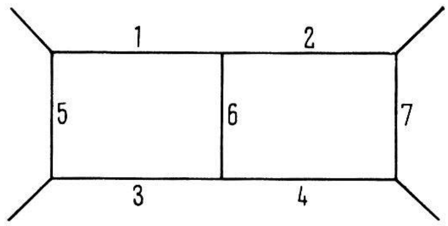


Figure 7b

If k_j is a loop momentum, the inversion compactification of K_j is defined by means of the embedding

$$i: (1, k_j, k_j^2) \to ({}_0a_j, a_j, {}_5a_j) \; \varepsilon \; P^5 . \tag{2}$$

Let $Q(k_j)$ be a polynomial in the components of k_j. Then under the transformation (2)

$$Q(k_j) \to Q(a_j/{}_0a_j) = ({}_0a_j)^{-s} \, Q'({}_0a_j, a_j, {}_5a_j)$$

where Q' is a homogeneous polynomial in its variables, whose restriction to $\overline{iK_j}$ is not divisible by ${}_0a_j$, and s is an integer. We call s the inversion degree of Q in k_j. Note that a propagator $q_i^2 + m_i^2$ has inversion degree 1 in any loop momentum k_j flowing through line i. The concept of inversion degree can be extended to differential forms in an evident way and in [14] it is shown that the differential form d^4k_j has inversion degree 4 in k_j.

If we make the inversion compactification of $K = K_1 \times K_2$ and write down the integral representation of $I(t)$ on $\overline{K}$ derived from (1) we will find that the transformed differential form has in addition to the poles displayed in (1) a pole of order $4 + s_1 - \sum_{i \varepsilon \mathrm{loop}\,1} n_i$ in ${}_0a_1$ and a pole of order $4 + s_2 - \sum_{i \varepsilon \mathrm{loop}\,2} n_i$ in ${}_0a_2$. These are effective poles and we wish to exclude them so we impose the conditions

$$\sum_{i \varepsilon \mathrm{loop}\,1} n_i - 4 - s_1 \geqslant 0 \tag{3}$$

$$\sum_{i \varepsilon \mathrm{loop}\,2} n_i - 4 - s_2 \geqslant 0 . \tag{4}$$

In (3), (4) s_1, s_2 denote the inversion degrees of $P(k, p)$ in k_1, k_2 respectively. There is a third effective pole which can be displayed by making the inversion compactification corresponding to the choice of loop momenta shown in Figure 8. It can be excluded by the condition

$$\sum_{i \varepsilon \mathrm{loop}\,3} n_i - 4 - s_3 \geqslant 0 . \tag{5}$$

A detailed analysis shows that these effective poles are the only ones which appear in a standard form representation of (1). Conditions (3), (4), (5) therefore suffice to exclude second-type layers for $F(t)$. We note also that they suffice to ensure the uniform convergence of (1) for t in some neighbourhood of t_0.

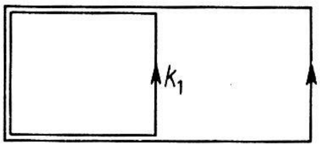


Figure 8

Table 1 enumerates the topologically distinct types of contractions of the sixth order ladder diagram. For each of these types a representative is given. If the contration has a leading Landau singularity this is stated and the complex dimension d of the corresponding pinch in the integration space is given. Each of these pinches can be proved to be quadratic for a general point on the corresponding Landau singularity. The singularity $G(H')$ for a given contracted graph H' is not given in the table but for the analysis of 4.3., it follows that $G(H')$ consists of the leading Landau singularity of H' together with the leading Landau singularity of each contracted graph H'', obtained by contracting c lines of H', which corresponds to a pinch in K of dimension $d > 0$ such that $d - c$ is even. For example G_{6124} is the union of L_{612} and L_{24}. If the layer corresponding to the graph has dimension > 0 then this dimension is given and multiplied by the number of graphs of that type to give the contribution of these graphs to the sheet index of $F(t)$.

4.2. *The Homology Groups*

The following convention will be useful in the presentation of these calculations:

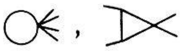
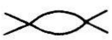
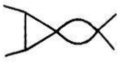
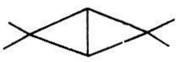
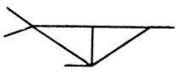
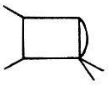
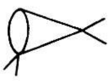
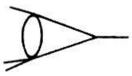
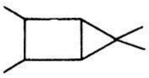
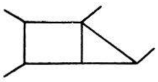
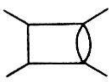
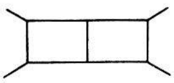
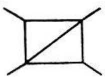
Let E, B be topological spaces; p, *a* continuous map of E onto B. Suppose that E and B admit decompositions into disjoint subsets E^v, B^v, $(1 \leqslant v \leqslant u)$, such that E^v is a (Serre) fiber space with projection p over B^v with fiber F^v. Then we say that E is a ‚near fiber space' over B with projection p and write

$$E = \sum_{v=1}^{m} F^v \,\bar{x}\, B^v .$$

If for each v, $1 \leqslant v \leqslant u$, E^v is a fiber bundle over B^V (that is a locally trivial fiber space) we say that E is a ‚near fiber bundle' over B. If for some v, E^v is a topological product of F^v with B^v with projection p we replace the term $F^v \,\bar{x}\, B^v$ in (1) by $F^v \, x \, B^v$.

The single loop cases in Table 1 can be dismissed at once. For each of these the corresponding intersection of propagators may be compactified as a topological product of two complex quadrics. In some cases, *e.g.* 1234, the compactification which does this is the inversion compactification defind by choosing k_1, k_2 as basic loop momenta. In some case, *e.g.* 6247, the compactification which does this is the inversion compactification defined by choosing k_1, k_3 as basic loop momenta. However, our method gives us the freedom to use different compactifications to study different intersections, if necessary. Then from the analysis of the single loop diagrams [14] we can complete this part of Table 1. For the single loop layers which have dimension 1, a separating function is equal to $\sqrt{G(H')}$ multiplied by a single-valued function (where $G(H')$ denotes the corresponding singularity).

Table 1
Contractions of 6th-order Ladder Diagram

Description	Contracted Graph	Example	Leading Landau Singularity	Dimension of Pinch	Contribution to sheet index
Γ_0	.	ϕ	No	–	$1 \times 1 \;=\; 1$
single loops, odd number of lines	 etc.	6, 135	Yes	4	0
single loops, even number of lines	 etc.	13	Yes	4	$1 \times 6 \;=\; 6$
bouquets, both loops even number of lines	 etc.	1324	No	–	$1 \times 9 \;=\; 9$
remaining bouquets	 etc.	13524	No	–	0
I		612	Yes	3	0
II		61234	Yes	2	0
III		61327	Yes	0	0
IV		61325	No	–	0
V		6124	No	–	$1 \times 8 \;=\; 8$
VI		6147	Yes	0	$2 \times 10 = 20$
VII		613524	No	–	$1 \times 2 \;=\; 2$
VIII		613527	Yes	0	$2 \times 4 \;=\; 8$
IX		614	Yes	0	$2 \times 7 \;=\; 14$
X		61357	Yes	0	$2 \times 2 \;=\; 4$
XI		6135247	Yes	0	$2 \times 1 \;=\; 2$
XII		61457	Yes	0	$2 \times 2 \;=\; 4$
					78

For the remaining double loop cases we use consistently the inversion compactification defined by k_1, k_2. We denote by $D_I \subseteq P^5 \times P^5$ the compactification of S_I. Denote by p_1, p_2 the natural projection of $P^5 \times P^5$ onto the first factor or second factor respectively. For each intersection D_I we choose as the projection p the restriction to D_I of p_1 or p_2. D_I is a near fiber bundle over pD_I with projection p. We use the corresponding decomposition (1) to calculate D_I. The decompositions can be written concisely using the notation of 3.3. It is understood that $t_0 \bar{\varepsilon} G$

I. $$p_1: D_{612} = Q_2^2 \bar{x} (Q_3^3 - Q_2^2) + Q_1^2 x Q_2^2$$

$$H(D_{612}): \frac{R \;\; 2R \;\; 2R \;\; 3R \;\; 3R \;\; R}{0 \;\; 2 \;\; 4 \;\; 6 \;\; 8 \;\; 10}$$

II. $$p_1: D_{61234} = Q_1^1 \bar{x} (Q_2^2 - Q_1^1) + Q_0^1 \bar{x} Q_1^1$$

$$H(D_{61234}): \frac{R \;\; 4R \;\; 4R \;\; R}{0 \;\; 2 \;\; 4 \;\; 6}$$

III. $$p_1: D_{61327} = Q_1^1 \bar{x} (Q_2^2 - Q_1^1 \cup Q_1^1) + Q_0^1 \bar{x} (Q_1^1 \cup Q_1^1)$$

$$H(D_{61326}): \frac{R \;\; 3R \;\; 5R \;\; R}{0 \;\; 2 \;\; 4 \;\; 6}$$

IV. $$p_1: D_{61352} = Q_2^2 \bar{x} (Q_1^1 - Q_0^0) + Q_1^2 \bar{x} \; Q_0^0$$

$$H(D_{61352}): \frac{R \;\; R \;\; 3R \;\; R}{0 \;\; 2 \;\; 4 \;\; 6}.$$

The above homology groups have been written down using the homology groups of the complex quadrics given in 3.3. We note that in cases I–IV the homology group in middle dimension is zero so the corresponding layers have dimension 0.

In case V we find it convenient not to make any compactification.

$$p_2: S_{6124} = Q_2 \bar{x} Q_2$$

$$H(S_{6124}): \frac{R \;\; 2R \;\; R}{0 \;\; 2 \;\; 4},$$

$H_4(S_{6124}) = R$ is generated by a Lefschetz class for L_{612} or L_{24}. A separating function is equal to $\sqrt{L_{612} L_{24}}$ multiplied by a single valued function. The layer has dimension 1.

The remaining cases, with the exception of VII, give layers of dimension 2. They fall into two classes according as the dimension of the intersection is odd or even. In the odd dimensional cases a separating function is the period function of an elliptic integral. For the leading intersection XI the intersection is actually a torus and this is immediate. In the even dimensional cases VI and VIII a separating function is algebraic in accord with Conjecture 6. These both give the same structure, viz. that symbolized by the Dynkin diagram B_2. One of these (VIII) we examine in detail in 4.4. We also give more detail for case VII, although it turns out simply to give a layer of dimension 1 because it caused us some confusion.

Even Cases

VI. $$p_2: D_{6147} = Q_2^2 \bar{x} (Q_2^2 - Q_1^1 \cup Q_1^1) + Q_2^2 \bar{x} (Q_1^1 \cup Q_1^1). \tag{2}$$

The second term of the Serre spectral sequence for the first fiber bundle in (2) contains

$$H_p(Q_2^2 - Q_1^1 \cup Q_1^1; \mathfrak{F}) \tag{3}$$

where $\mathfrak{F} = \{H_2^F(Q_2^2)\}$ is the sheaf formed by the finite homology groups of the fibers. As we anticipated at the end of 3.3. the sheaf $\mathfrak{F}$ is quadratically branched around the singular fibers. The calculation of 3.3. shows that the group (3) is zero unless $p = 2$. The remainder of the second term of the Serre spectral sequence for the first fiber bundle may be combined with the second term of the Serre spectral sequence for the second fiber bundle in (2) to give

$$H(Q_2^2; \mathcal{A})$$

where $\mathcal{A} = \{H^A(F)\}$ is the sheaf formed by the algebraic homology groups of the fibers, singular and nonsingular, and forms over the base space a constant sheaf. Both (3) and (4) contain no nonzero elements outside the even dimensions so the differentials d_r, $r \geqslant 2$, must act trivially. Moreover an analytic differential form on D_{6147} will vanish on cycles which define elements in (4), for its integral over a cycle corresponding to an element of (4) can be written as an integral first over the fiber of the cycle then over the base and the integral over the fiber will vanish since the fiber is algebraic. Thus we have only to consider the group (3).

In 3.3. we proved that $H_2(Q_2^2 - Q_1^1 \cup Q_1^1; \mathfrak{F})$ has dimension 2, being generated by homology classes G_1 and G_2. In the notation of 3.3., $n = 2$, $k = 1$ so

$$G_1^2 = 4 \qquad G_2^2 = 8 \qquad G_1 \cdot G_2 = -4\,.$$

Thus the Kronecker index defines a positive definite scalar product on $E = \{G_1, G_2\}$. G_1 and G_2 are (apart from a multiplicative factor) Lefschetz classes. We do not prove this but in 4.4. we prove the corresponding statement for the homology group which occurs in case VIII, and this has the same structure.

$$\text{VII.} \quad p_2 : D_{613524} = Q_0^0 \times (Q_2^2 - Q_1^1 \cup Q_1^1) + Q_{-1}^0 \times (Q_1^1 \cup Q_1^1)\,. \tag{5}$$

The same argument that we have used for the decomposition (2) leads in the case of (5) to the conclusion that the homology group of D_{613524} in middle dimension can be split into the sum of a group on whose elements an analytic differential form vanishes together with a group spanned by elements G_1, G_2 having the same intersection numbers as the G_1, G_2 defined for D_{6147}. Nevertheless we do not conclude that the intersection D_{613524} gives a layer of dimension 2 for the reasons set out in 4.4.

$$\text{VIII.} \quad p_2 : D_{613527} = Q_0^0 \times (Q_2^2 - Y) + Q_{-1}^0 \times Y\,. \tag{6}$$

(6) does not follow the pattern of (2) and (5), since the branch curve Y is an irreducible algebraic curve and not the union of two conics in G.P. in Q_2^2. We find, nevertheless, that this intersection has a homology group having the same structure as in the other even cases. We could have established this by calculating the sheaf homology group

$$H(Q_2^2 - Y; \{H_0^F(Q_0^0)\})$$

but we found it more convenient to use the decomposition of D_{613527} corresponding to the projection p_1

$$p_1 : D_{613527} = Q_1^1 \times (Q_1^1 - 2\, Q_0^0) + Q_0^1 \times 2\, Q_0^0\,. \tag{7}$$

The calculation of $H(D_{613527})$ from (7), and the detailed analysis of the layer for this intersection is given in 4.4.

Odd Cases

$$\text{IX.} \quad p_2 : D_{614} = Q_2^2 \times (Q_3^3 - Q_2^2 \cup Q_2^2) + Q_1^2 \times (Q_2^2 \cup Q_2^2)\,. \tag{8}$$

The second term of the spectral sequence for the first fiber bundle of (2) contains

$$H_p(Q_3^3 - Q_2^2 \cup Q_2^2; \mathfrak{F}) \tag{9}$$

where $\mathfrak{F} = \{H_2^F(Q_2^2)\}$. It has been shown in 3.3. that this group is zero for $p \neq 3$. The remaining part of the second term of the spectral sequence for the first fiber bundle of (8) combines with the second term of the spectral sequence for the second fiber bundle of (8) to give

$$H(Q_3^3;\, \mathcal{A}) \tag{10}$$

where $\mathcal{A}$ denotes the sheaf $\{H^A(F)\}$ formed by the algebraic homology groups of the fibers, singular or nonsingular. $\mathcal{A}$ is a constant sheaf and the differentials d^r, $r \geqslant 2$ act trivially on (10) and so also on (9). The argument we have given in the case of the similar group (4) shows that (10) may be discarded and we are left with $H_3(Q_3^3 - Q_2^2 \cup Q_2^2;\, \mathfrak{J}) = \{F_1, F_2)\}$ where F_1, F_2 are the homology classes defined in 3.3. In order to see that $F_{1/2}$ and $F_{2/2}$ are Lefschetz classes for L_{614}, it is necessary to pull tight the representatives for f_1 and f_2 over the points in the α-plane which they circle (Figure 9). Note that the Kronecker index defines on $\{F_1, F_2\}$ a nonsingular scalar product.

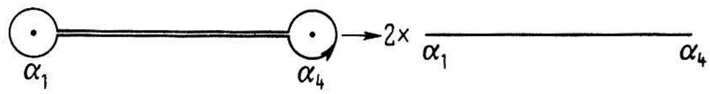


Figure 9

The following two cases X, XI can be discussed in exactly the same way as IX.

$$\text{X.} \quad p_1 : D_{61357} = Q_2^2 \,\bar{x}\, (Q_1^1 - 2\, Q_0^0) + Q_1^2 \,\bar{x}\, 2\, Q_0^0$$

$$\text{XI.} \quad p_1 : D_{6135247} = Q_0^0 \,\bar{x}\, (Q_1^1 - 2\, Q_0^0) + Q_-^0 \;\, \bar{x}\, 2\, Q_0^0 .$$

The last case does not follow exactly the same pattern.

$$\text{XII.} \quad p_2 : D_{61457} = Q_1^1 \,\bar{x}\, (Q_2^2 - Y) + Q_0^1 \,\bar{x}\, Y$$

where Y is an irreducible curve. From this decomposition we find

$$H(D_{61457}) : \frac{R \;\; 4R \;\; 2R \;\; 4R \;\; R}{0 \;\;\; 2 \;\;\; 3 \;\;\; 4 \;\;\; 6} .$$

Now D_{61456} is a manifold so by the Poincaré duality theorem it follows that the Kronecker index defines on $H_3(D_{61457})$ a nonsingular bilinear form. It can also be shown that this group is spanned by Lefschetz classes for L_{61457}.

We conclude this section with a remark on the meaning of the simplification of the Serre spectral sequence which we have noted in these calculations. The method of calculating the homology groups of intersections by representing them as near fiber bundles is the geometric counterpart of the method of studying multiple integrals by writing them as repeated integrals. The above calculations show that the connection is very close. The sheaf $\mathfrak{J}$ which appears for example in (3) is isomorphic with the sheaf defined by a separating function for the first loop, *i.e.* a square root function quadratically branched around $k_2^2 = (m_1 + m_6)^2$, $k_2^2 = (m_1 - m_6)^2$. The group $H_2\,(Q_2^2 - Q_1^1 \cup Q_1^1;\, \mathfrak{J})$ would arise directly if we applied the homological method to the study of the integral of this function over the second loop. This integral gives a separating function for the full intersection. These considerations indicate that if the homological method is generalized to deal with integrals in which the integrand is not single-valued, then rather than make a reduction to the case in which the integrand is single-valued by writing the integral as an integral on a covering space, one should introduce homology groups with coefficients is the sheaf defined by the integrand. Only in this way will be possible generally to obtain the isomorphism we have noted here between the homology group which classified the contours for the multiple integration and the homology group which classifies the contours for the integration of the many-valued function obtained by carrying out a partial integration.

4.3. *The Picard-Lefschetz Theorem for Non-simple Quadratic Pinches*

We consider a function $F(t)$ defined by an integral representation of standard form. The pole set $S(t)$ is assumed to be the union of m manifolds $S_i(t)$ in G.P. in Z for $t \,\bar{\varepsilon}\, G$. We denote by J the set of subsets of $1, \ldots, m$. If $I \varepsilon J$, $S_I = \bigcap_{i \varepsilon I} S_i$ $S^I = \bigcup_{i \varepsilon I} S_i$.

Suppose that for $t_1 \varepsilon T$ the intersection S_I is singular at points of an analytic set P of complex dimension $d > 0$, and that no intersection S_{I_1}, $I_1 \subset I$ is singular for t_1. Then we call P a non-simple pinch.

Index the poles so that $I = \{1, \ldots, r\}$. Denote by I_2, $\{1, \ldots, r-1\}$. If $z_0 \varepsilon S_I$, denote by f the restriction to S_{I_2} of a holomorphic function whose vanishing gives a local equation for S_r in the neighbourhood of z_0. We call P a non-simple quadratic pinch if

a) P, S^{CI} are in G.P. in Z.

b) For every $z \varepsilon P$ the Hessian $H(t)$ has rank $n - r + 1 - d$ so that local coordinates may be chosen in which the S_i, $i \varepsilon I$, have local equations

$$S_1 : z_1 = 0 \quad \ldots \quad S_{r-1} : z_{r-1} = 0$$

$$S_r : z_r^2 + \ldots + z_{n-d}^2 - q(t) = 0$$

where $q(t)$ is holomorphic in t and $q(t_1) = 0$. We suppose further that $q(t_1) \neq 0$.

c) P is simply connected.

Suppose that P is a non-simple quadratic pinch. Choose a reference point t_2 in the neighbourhood of t_1. Since a) holds we may construct on Z a Riemannian metric in which P, S_i, $i \varepsilon CI$ are orthogonal. Let $W(\varepsilon)$ be a tubular neighbourhood of P in Z constructed by means of this metric. There is a natural projection $p : W(\varepsilon) \cap S_I \to P$ which maps a point of $W(\varepsilon) \cap S_I$ onto the center of the polycylinder orthogonal to P on which it lies. With this projection $W(\varepsilon) \cap S_I$ is a fiber bundle with base P and fiber homeomorphic to an affine quadric of dimension $n - r - d$. Moreover for each $i \varepsilon CI$, $W(\varepsilon) \cap S_I \cap S_i$ is a subbundle of $W(\varepsilon) \cap S_I$ with base $P \cap S_i$. The homology groups $H^{F_e}_{n-r-d}(F \cap W)$ and $H^c_{n-r-d}(F)$ of one of the fibers are generated by elements f, $e \cdot e$ is the homology class defined by a spherical representative cycle e. f is the dual homology class of the dual group $H^{F_e}_{n-r-d}(F \cap W)$ (compare the FFLP proof of the Picard-Lefschetz theorem [6]. Note that we use the suffix F_e for 'closed' rather than F because we have already used F to indicate 'finite' in FÁRY's sense). Over P they form local systems of groups and since P is simply connected these local systems are constant. Hence there are isomorphisms,

$$H^{F_e}_{n-r}((S - S^{CI}) \cap W) \xleftarrow{\theta} H^c_d(P - S^{CI})$$

$$H_{n-r}((S_I, S^{CI}) \cap W) \xleftarrow{\phi} H_d(P, S^{CI})$$

defined by $\theta a = a \otimes f$, $\phi b = b \otimes e$. These isomorphisms preserve the Kronecker index, *i.e.* $a \cdot b = (\theta a) \cdot (\phi b)$.

The Picard-Lefschetz theorem gives the automorphism of the homology group $H_{n-r-d}^{F_e}(F \cap W)$ of each fiber corresponding to a loop on t_2 around t_1. Hence an element $h \varepsilon H_n^{F_e}(W - S(t_2))$ is transformed by the automorphism for this loop according to

$$h \to h + \sum_k (-1)^{\frac{(g+1)(g+2)}{2}} (e_k \cdot h) g_k \tag{1}$$

where $g = n - r - d$ and the e_k, g_k are vanishing classes defined as follows: Choose a basis $\{u_k)$ for $H_d^c(P - S^{CI})$. Denote by $\{v_k\}$ the dual basis for $H_d(P, S^{CI})$. Let $\{e_k\}$ be the basis for $H_n(W, S(t_2))$ obtained from v_k by following the isomorphisms ∂^{n-r}, ϕ in the diagram

$$H_n(W, S(t_2)) \xrightarrow{\partial^{n-r}} H_{n-r}(S_I \cap W, S^{CI}) \xleftarrow{\phi} H_d(P, S^{CI})$$

and let $\{g_k\}$ be the set of classes of $H_n^c(W - S(t_2))$ defined by $g_k = \delta^{n-r}(\theta u_k)$. Formula (1) also gives the action of the loop transformation on an element h of $H_n^c(Z - S(t_2))$ since this is determined by the trace of h in $H_n^{F_e}(W - S(t_2))$ according to the localization lemma of the FFLP theory.

For the non-simple quadratic pinches which occur in the analysis of functions associated with the sixth order ladder diagram P is a complex quadric of dimension 2, 3, or 4 (Table 1). The $S_i \cap P$, $i \varepsilon CI$ are complex quadrics in G.P. in P. By 3.2., Theorem 2, COR 1

$$H_d^c(P - S^{CI}) \simeq \bigoplus_{I'} \delta^c H_{d-c}^F(P \cap S_{I'}) \tag{2}$$

where the summation is over $I' \supseteq I$ such that $P \cap S_{I'} \neq \phi$ and c denotes the number of elements of I' not in I. Now $H_{d-c}^F(P \cap S_{I'}) = 0$ if $d - c$ is odd so that the decomposition (2) shows that only those intersections $S_{I'}$ having $d - c$ even have Lefschetz classes for the loop on t_2 around t_1. This justifies the assertion made in 4.1. concerning the determination of the G_I.

4.4. *Detail of cases VII, VIII*

VIII. The decomposition 4.2. (7) may be written more explicitly

$$p_1 : D_{613527} = D_{627} \,\overline{x}\, (D_{135} - L_{627}) + SD_{627} \,\overline{x}\, (L_{627} \cap D_{135}) \tag{1}$$

In (1) the fiber D_{627} is the inversion compactification of the intersection of propagators for the single loop 627, the momentum k_1 being fixed. This is nonsingular if $k_1 \bar{\varepsilon} L_{627}$ and has a simple quadratic pinch if $k_1 \varepsilon L_{627}$. If we define $\alpha = (k_1 - p_3)^2$ we see from the corresponding dual diagram that $L_{627} = 0$ determines two values of α. D_{135} intersects the closure of $k_1 : (k_1 - p_3)^2 = \alpha$ in general in two points. Hence $D_{135} \cap L_{627}$ consists of two points pairs $N_1^{1)}, N_1^{2)}; N_2^{1)}, N_2^{2)}$ as indicated in 4.2. (7).

From the decomposition 4.2. (7) we calculate

$$H(D_{613527}) : \frac{R \;\; 6R \;\; R}{0 \;\; 2 \;\; 4}.$$

Basic cycles for $H_2(D_{613527})$ may be defined as follows[9]:

a_1: the section of D_{613527} by a prime of $\overline{K_1}$

a_2: the section of D_{613527} by a prime of $\overline{K_2}$

$$b_1 = m\ x\ (n_1^1 + n_1^2) \qquad b_2 = m\ x\ (n_1^1 + n_2^2)$$

$$h_1 = m\ x\ (n_1^1 - n_1^2) \qquad h_2 = m\ x\ (n_1^1 - n_2^2)$$

Here $m = p - q$ denotes the difference of the two canonically oriented cycles which are the algebraic components of the singular section over the point which it multiplies, *e.g.* $m\ x\ (n_1^1 + n_1^2)$ is shorthand for $(p_1^1 - q_1^1) + (p_1^2 - q^2)$. Now $p \cdot q = 1$ and $(p + q)\ x\ n \sim a_1/2$, and $a_1^2 = 0$ so $((p - q)\ x\ n)^2 = ((p + q)\ x\ n)^2 - 4p \cdot q = -4$ Thus $b_1^2 = b_2^2 = h_1^2 = h_2^2 = -8$. Also all their cross products are zero and they have no intersection with the algebraic classes a_1, a_2.

Consider D_{61327} which contains D_{613527}. Corresponding to (1) this intersection has the decomposition

$$D_{61327} = D_{627}\ \bar{x}\ (D_{13} - L_{627}) + SD_{627}\ \bar{x}\ (D_{13} \cap L_{627})\,. \tag{2}$$

Denote by l_1, l_2 the cycles on D_{13} defined by the pair of conics $L_{627} \cap D_{13}$. They give rise to cycles $r_1 = m\ x\ l_1$, $r_2 = m\ x\ l_2$ on D_{61326} which intersect D_{613527} in n_1 and b_2. Thus $\delta\, b_1$ and $\delta\, b_2$ bound in D_{61327}. Thus of the six basic classes, four may be discarded because an analytic differential form which is an iterated residue onto D_{613527} will vanish on them—a_1, a_2 because they are algebraic, b_1 and b_2 because they have vanishing coboundaries in D_{61327}.

Define $e_1 = h_1/2$, $e_2 = h_2/2$. The leading Landau singularity L_{613527} corresponds to the coincidence of $N_1^{1)}$ with $N_1^{2)}$ or of $N_2^{1)}$ with $N_2^{2)}$. Hence e_1 and e_2 are Lefschetz classes for this singularity.

The remaining components of G_{613527} obtained from Table 1 using the criterion given in 4.1. are L_{27} and L_{612}. If $t \,\varepsilon\, L_{27}$ or L_{612} then the corresponding nonsimple quadratic pinch intersects L_{613527} in a point pair. In the representation (1) we have $N_1^{1)} = N_2^{1)}$ and $N_1^{2)} = N_2^{2)}$ or $N_1^{1)} = N_2^{2)}$ and $N_1^{2)} = N_2^{1)}$. The corresponding Lefschetz class is $1/2\ (e_1 - e_2)$ or $1/2\ (e_1 + e_2)$.

On $E = \{e_1, e_2)$ the Kronecker index defines a negative definite scalar product. We redefine the scalar product by introducing a minus sign. Then we can represent the Lefschetz classes by vectors in the Euclidean plane and the loop transformations by reflections (Figure 10). The group R of the general discussion of Conjecture 6 (2.8.) has order 8 and there are two conjugate classes of reflections.

The above analysis agrees with that given by Landshoff *et al.* [11] for the cone diagram which gives rise to the same structure. In particular of our Landau singularities are identified with the corresponding ones in their analysis as indicated in Figure 10, then the relation $\delta_{N'}[\partial_L(I)] = \delta_{N'}(I)$ to which they draw attention may be easily checked from the diagram.

[9]) We will not distinguish between cycles and the homology classes they define since no confusion is likely to arise.

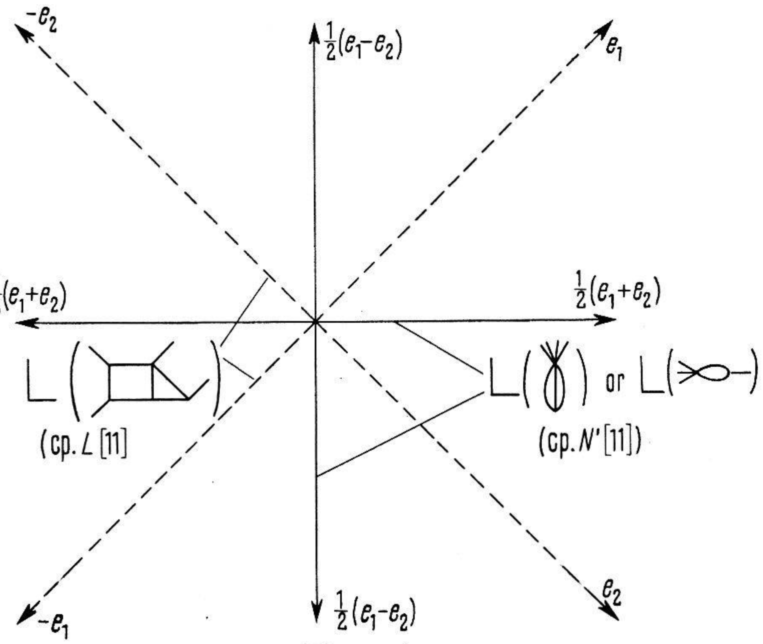


Figure 10
Lefschetz classes for the intersection D_{613527}.

VII. Denote by M the intersection D_{613524}. The decomposition 4.2. (5) gives

$$H_2(M) = G \oplus H_2^A(M) = 4R$$

where $H_2^A(M)$ denotes the subgroup of $H_2(M)$ generated by algebraic homology classes, (*i.e.* by the homology classes of sections of M by a prime in either projective space P^5 in whose product M is embedded, $M \subset \overline{K}_1 \, x \, \overline{K}_2 \subset P^5 \, x \, P^5$) and $G = \{G_1, G_2\}$ is spanned by homology classes defined in the analysis of 3.3.

M is not manifold. It has two permanent simple quadratic pinches $M^{1)}$, $M^{2)}$. This may readily be understood from 4.2. (5). M may be considered as the Riemann surface of a function defined over the base space Q_2^2 and having two quadratic branch curves Q_1^1. In the two points in which these intersect the function behaves like $\sqrt{z_1 z_2}$ and the Riemann surface M is nonuniformizable. If $M^{1)}$ and $M^{2)}$ are blownup this gives a manifold M which has $H_2(M) = 6R$, there being two additional homology classes defined by the cycles on M lying over $M^{1)}$, $M^{2)}$. M admits a decomposition which has exactly the same form as that given for D_{613527} (4.2. (7)) so that M and D_{613527} have the same intersection ring and it is possible to set up a correspondence between the classes e_1, e_2 used in the analysis of D_{613527} and the present G_1, G_2. We find

$$G_1 = -(e_1 - e_2) \qquad G_2 = -2\,e_1\,.$$

It is not surprising that such a correspondence should exist for Case VII is a specialization of VIII obtained by setting the external momentum for one line equal to zero.

M has no leading Landau singularity. According to the rule given in 4.1., the singularity of this intersection should consist of L_{24} and L_{613}, L_{624}. A Lefschetz class of L_{24} can be shown to be $1/2\,(e_1 - e_2)$. L_{613}, L_{624} are exceptional for although they correspond to pinches in K of dimension 3 and D_{613524} is defined by 3 additional poles the pinch in K corresponding to a point on L_{613} or L_{624} does not intersect M. Only in $\overline{K}$ is there an intersection—in the fixed points $M^{1)}$, $M^{2)}$. Thus a point on L_{613} or L_{624} must correspond on $\tilde{M}$ to a pair of simple quadratic pinches at points over $M^{1)}$, $M^{2)}$

respectively. $\tilde{M}$ is also singular if $G(p_1, p_2) = 0$. Then $M^{1)}$ and $M^{2)}$ coincide on M and $\tilde{M}$ acquires a pair of simple quadratic pinches. The situation whose possibility was noted in 2.5. is realized—we have some pinches on $\tilde{M}$ which occur only at infinity but which are not pinches of $\tilde{M}$ with additional effective poles.

$1/2\ (e_1 + e_2)$ can be shown to be a Lefschetz class for $G(p_1, p_2) = 0$. Now looking at Figure 10, we can see that the structure for case VIII appears to have degenerated in case VII to that symbolized by the Dynkin diagram $A_1 \times A_1$, since the absence of the leading Landau singularity means that there are no diagonal reflections which would carry $1/2\ (e_1 + e_2)$ into $1/2\ (e_1 - e_2)$. We can show, however, that the structure for case VII is just A_1, so nondegenerate, since the integral over $1/2\ (e_1 + e_2)$ of an analytic differential form which is an iterated residue onto the intersection must vanish. $1/2\ (e_1 + e_2)$ has zero coboundary in D_{1234}. The pinches corresponding to $G(p_1, p_2) = 0$ or $L_{613} = 0$ or $L_{624} = 0$ which occur only at infinity can therefore be disregarded. There must be similar pinches for other intersections which we have discarded without giving them a detailed examination so it is important to give a general reason why they should be dismissed. This gap leaves open the possibility that our enumeration of singularities is incomplete. The conclusion that $F(t)$ has a layer structure with the layers which we have determined is not weakened. What is left open is the question of whether some of the homology classes which we have identified as Lefschetz classes for Landau singularities should not also be Lefschetz classes for second-type singularities.

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