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Determination of Phenomenological α-α-Potentials from Scattering Data

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Summary. By means of the Gelfand-Levitan solution of the inverse problem in the scattering theory α-α-interaction potentials were calculated from the experimental phase shifts.
The potentials obtained in this way reproduce exactly the given phases.

I. Introduction

The problem of determining unique interaction potentials from scattering phase-shifts, the so-called inverse problem, has attracted large interest in nuclear physics. The problem has been solved mathematically by the fundamental work of GELFAND and LEVITAN [1]. However to our knowledge this mathematical solution has never been applied to a real collision problem [2]. Potentials are usually determined by fitting parameters in a given analytic expression. This method of calculation may be convenient, but there is a serious drawback in that the results depend on the choice of the particular shape of the potential. Consequently each author gets his "own" potential. In contrast, if one uses the Gelfand-Levitan theory, it is possible to calculate unique potentials, which reproduce exactly the given phases.

In the following work this is demonstrated for the case of α-α-scattering. The theory to be used is the version of MARCHENKO [3]. Since the original theory is related to short range potentials a modification due to the Coulomb interaction is necessary. The method is described in the next section.

The α-α-problem is favourable for this analysis for several reasons: By the direct measurement of the ^{8}Be ground state [4] the last gap in the low energy elastic scattering phase shifts has been closed. Therefore fairly good experimental phase values over the whole interesting energy region are now available. Owing to the spin 0 of the α-particle only scalar potentials must be considered. Since low energy inelastic channels do not occur, the scattering at low energies may be described by the relative motion of the two α-particles without internal excitation. Then at larger distances the long range part of the nucleon-nucleon-interaction plays the most important role in the α-α-interaction. The potential model gives interesting information to this direct interaction. At shorter distances nucleon exchange between the two α-particles contributes strongly and makes the effective potentials l-dependent.

The theoretical background will be given in the second section. In the third section details of the calculation and the results are presented. The last section is devoted to a discussion of the results.

II. Theoretical Background

We start by writing down the main relations of the Gelfand, Levitan, Marchenko theory, which will be used later. For details of the theory we refer to [3, 5].

First let us consider S-scattering by a short range potential $V(r)$. Let $f(k,r)$ and $f(-k,r)$ be solutions of the radial Schrödinger equation

$$\left[\frac{d^2}{dr^2} - V(r) + k^2\right] \psi(r) = 0 \qquad \left(\frac{\hbar^2}{2m} = 1\right) \tag{1}$$

defined by the asymptotic conditions

$$\lim_{r\to\infty} e^{\pm ikr} f(\pm k, r) = 1 . \tag{2}$$

These solutions form a fundamental system; hence a regular solution $\varphi(k, r)$ satisfying

$$\varphi(k, 0) = 0$$

can be written as a linear combination

$$\varphi(k, r) = f(k, r) - \frac{f(k, 0)}{f(-k, 0)} f(-k, r) \qquad k \geqslant 0 .$$

From (2) we find the asymptotic behaviour

$$\varphi(k, r) \sim e^{-ikr} - \frac{f(k, 0)}{f(-k, 0)} e^{ikr} \qquad r \to \infty \tag{3}$$

and from this the connection with the phase shift $\delta(k)$ and the S-matrix $S(k)$ respectively:

$$\frac{f(k, 0)}{f(-k, 0)} = S(k) = e^{2i\delta(k)} . \tag{4}$$

It is a very important feature of the theory, that $f(k, r)$ possesses a representation

$$f(k, r) = e^{-ikr} + \int_r^\infty K(r, t)\, e^{-ikt}\, dt \tag{5}$$

by means of a so-called orthogonalizing kernel $K(r, t)$, which is independent of k. Inserting (5) into (1) one obtains by partial integrations among other things the relation

$$-2\frac{d\,K(r, r)}{dr} = V(r) . \tag{6}$$

The inverse problem is therefore reduced to the determination of $K(r, r)$.

For $K(r, t)$ various integral equations can be derived. The one suited best for our purpose is the Marchenko equation, which follows from Parseval's equation for the $\varphi(k, r)$:

$$K(r, t) = \int_r^\infty F(t + x)\, K(r, x)\, dx + F(t + r) \qquad 0 < r < t \tag{7}$$

with

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [S(k) - 1]\, e^{ikt}\, dk . \tag{8}$$

In this relation it is assumed that no bound states occur. Through (4) the scattering phase shift $\delta(k)$ appears directly in (8). Consequently equations (8), (7) and (6) form the solution of our problem. Equation (7) is a Fredholm integral equation for $K(r, t)$ as a function of t with r entering as a parameter. Hence for every point, where the potential is to be determined, an integral equation (7) has to be solved.

In case of the α-α-scattering the above relations cannot be used without modifications, since we deal with charged particles and therefore the long range Coulomb potential is superposed to the nuclear potential. Consequently it is not the phase $\delta(k)$ appearing in (4), which is experimentally measured, but rather phases $\delta_l^c(k)$ relative to Coulomb waves, which are defined as follows: Let $F_l(\eta, \varrho)$ be the regular and $G_l(\eta, \varrho)$ the irregular Coulomb function of angular momentum l, where $\varrho = k \cdot r$ and $\eta = 2\,e^2/k$. Then

$$u_l^{(\pm)}(\eta, \varrho) = e^{\pm i\sigma_l(\eta)}\,(G_l \pm i\,F_l) \tag{9}$$

is the incoming resp. outgoing Coulomb wave, where $\sigma_l(\eta)$ is the Coulomb phase. Now the regular scattering solution ψ_l behaves asymptotically as

$$\psi_l \sim u_l^{(-)} - e^{2i\delta_l^c} \cdot e^{2i\sigma_l} \cdot u_l^{(+)} = \psi_l^c(k, r)\ . \tag{10}$$

δ_l^c are the experimentally measured phases. The phase $\delta(k)$ to be used in (4) and (8) resp. can be obtained from the $\delta_l^c(k)$ for every l in the following way:

Let R be chosen so large, that for $r \geqslant R$ the nuclear potential can be completely neglected in comparison with the Coulomb potential. Now we calculate the phase $\delta(k)$, which would produce a potential being identically zero for $r > R$ and equal to the sum of Coulomb, centrifugal and nuclear potential for $r \leqslant R$. Matching the logarithmic derivatives of (10)

$$\frac{d}{dr}\log\psi_l^c(k, r) = \frac{\sin\delta_l^c \cdot G_l' + \cos\delta_l^c \cdot F_l'}{\sin\delta_l^c \cdot G_l + \cos\delta_l^c \cdot F_l} \tag{11}$$

and (3)

$$\frac{d}{dr}\log\psi(k, r) = k\,\mathrm{ctg}\,(\delta + kr)$$

at the point R, one obtains:

$$\mathrm{tg}\,(\delta(k) + kR) = \frac{1}{k} \cdot \frac{F_l' + \mathrm{tg}\,\delta_l^c\,G_l'}{F_l + \mathrm{tg}\,\delta_l^c\,G_l}\bigg|_R = \frac{\dot{F}_l + \mathrm{tg}\,\delta_l^c \cdot \dot{G}_l}{F_l + \mathrm{tg}\,\delta_l^c \cdot G_l}\bigg|_{\varrho = kR} \cdot \qquad \left(\cdot \equiv \frac{\partial}{\partial\varrho}\right) \tag{12}$$

The phase $\delta(k)$ so determined can be directly used in (8) for every angular momentum. We then calculate for $r \leqslant R$ the sum of the nuclear, centrifugal and Coulomb potential. Apart from the evident arithmetical convenience this procedure offers considerable advantages from the physical point of view. The inclusion of the Coulomb potential is desirable, since its calculation for shorter r is uncertain due to the finite size of the α-particle. The inclusion of the centrifugal potential for higher l values diminishes the strong attraction of the nuclear force. As a consequence the calculation becomes easier. If R is greater than the range of the nuclear force, the result may depend on R only by numerical inaccuracy.

The phases for all energies from zero to infinite enter into the Fourier transform (8). Therefore one has to find a physically acceptable way to extrapolate to high energies. The procedure is obvious looking at the curves of $\delta(k)$ calculated according to (12) (see Fig. 1). All phases for energies up to 35 MeV go over into well-defined hard-core phases. From the slopes of the tangents through the origin we find reasonable values of the hard-core radius R_c of about 1.2 fm. The comparison between the points of contact at 34–35 MeV with the lowest inelastic threshold at 34.73 MeV [^{4}He (α, p)

^{7}Li] shows, that we may justly consider the α-particle as not internal excited up to this energy. The hard-core phase

$$\delta_{core} = - k \cdot R_c$$

can be subtracted from the phases $\delta(k)$ and the potential is calculated only for the region $R_c < r \leqslant R$.

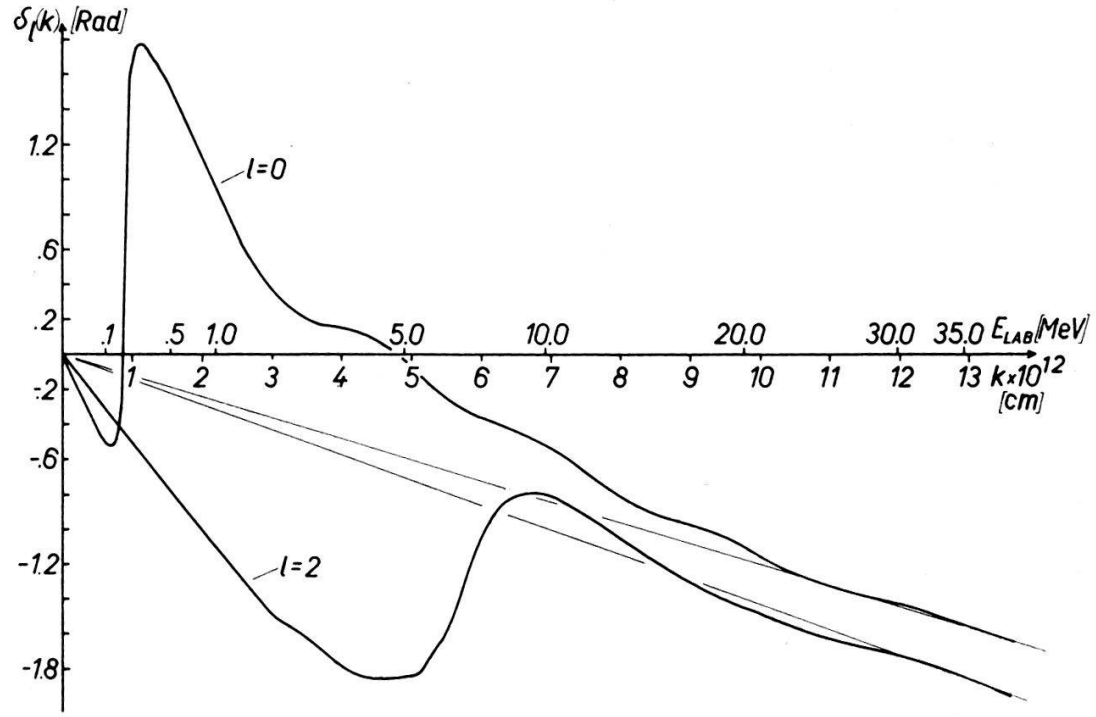

Figure 1

"Ideal" phases $\delta(k)$ related to the cut-off potentials for $l = 0,2$. The straight lines show the hard-core phases.

In case of the S-phase even the low energy values have to be treated with care. Below $E_{lab} = 400$ keV the very sharp Be^8 ground state resonance has to be considered. Unfortunately we do not know correct phase values in this region. We only can fit two parameters (position and width of the resonance) in an appropriate theoretical expression. For the purpose of extrapolating towards zero energy one might use the effective range theory [6]. But the validity of this theory in the neighbourhood of the sharp resonance is a priori doubtful. We therefore choose a different approach. We use a rigorous representation of the phase due to REGGE [7]. The S-phase shift of a potential of finite range R can be written in the following form:

$$\delta(k) = - kR - \sum_{n=1}^{N} \text{arc tg} \frac{k}{\varkappa_n} + \sum_{n=1}^{N'} \text{arc tg} \frac{k}{\varkappa'_n} + \sum_{n=1}^{\infty} \text{arc tg} \frac{2 k_n^{(1)} k}{|k_n|^2 - k^2} \cdot \qquad (13)$$

The sums run over all poles of the S-matrix, the first one over all bound states at energies $E_n = - \varkappa_n^2$, the second one over all virtually bound states at $E'_n = - \varkappa_n'^2$ and the third one over all resonances at $k_n = k_n^{(1)} + i\, k_n^{(2)}$; $k_n^{(1)}$, $k_n^{(2)} > 0$. In view of (13) we transform (12) as follows:

$$\delta(k) = - kR + \text{arc tg} \left(\frac{F}{\dot{F}} + \frac{1}{\dot{F}^2} \frac{1}{\text{ctg}\delta^c(k) + \dot{G}/\dot{F}} \right) = - kR + \text{arc tg} \frac{F}{\dot{F}} + \text{arc tg}\, H \qquad (14)$$

with

$$H = \frac{1}{\dot{F}^2 + F^2} \cdot \frac{1}{\operatorname{ctg}\delta^c + \dot{G}/\dot{F} + F/\dot{F}\,(\dot{F}^2 + F^2)} \,. \tag{15}$$

The first two terms in (14) are caused by the cut-off Coulomb potential alone, the third one represents the contribution of the nuclear potential. Since below 400 keV apart from the very sharp ^{8}Be ground state only pure Coulomb scattering has been found experimentally [8], we may assume, that in this region only one pole of the S-matrix contributes to H. We take its parameters from the ground state data E_0, Γ[4]. Inserting

$$\delta^c = \operatorname{arc\ tg} \frac{\Gamma/2}{E_0 - k^2}$$

into (15) and taking the Coulomb functions at the position of the ground state, we have

$$H = \frac{A \cdot \Gamma/2}{E_0 + B \cdot \Gamma/2 - k^2} \tag{16}$$

with

$$A = \frac{1}{\dot{F}^2 + F^2}\bigg|_{E_0,\,R} \qquad B = \frac{\dot{G}}{\dot{F}} + \frac{F}{\dot{F}\,(\dot{F}^2 + F^2)}\bigg|_{E_0,\,R} \,.$$

Hence by cutting off the Coulomb potential, the ground state shifts to lower energies (B is negativ)

$$E_0' = E_0 + B \cdot \frac{\Gamma}{2} \tag{17}$$

and becomes wider

$$\Gamma' = A \cdot \Gamma \,. \tag{18}$$

From (13), (14), (15) we now have the following expression for the S-phase

$$\delta(k) = -kR + \operatorname{arc\ tg} \frac{F}{\dot{F}} + \operatorname{arc\ tg} \frac{\Gamma' k/2\, k_0}{E_0' - k^2} \quad (E_0 = k_0^2) \,, \tag{19}$$

which we use below 400 keV.

III. Treatment of the Data, Results

The experimental data on which the calculation was based are taken from [4, 8 and 9]. We have independently determined the effective S-, D- and G-potential. The experimental points were always connected by smooth curves. In order to test the influence of the experimental errors we have chosen lower and upper limiting curves as inferred from the errors and have repeated the calculation with these datas. The calculations were made on the CDC 1604/1604-A computer of the ETH-Rechenzentrum, Zurich.

The evaluation of the formulae (6), (8), (12), (19) followed standart procedures. The Coulomb function program was composed by methods given in [10]. The integral equation (7) was transformed into a linear algebraic system, the solution of which is also possible on smaler computers, if one makes use of an iterative procedure. In this respect the Gauss-Seidel-method was successful, whereby one takes in the zeroth approximation the solution of the last equation stepping in r from larger to smaler distances. Concerning the solution of (7) it is convenient to choose the cutoff radius R as short as possible. We have taken $R = 7$ fm for the D- and G-potential. In case of the S-potential on the other hand the ground state width increases if R decreases according to the increase of A in equations (16), (18). This implies that the ground state pole moves to higher imaginary parts and therefore becomes less predominant compared with the other poles due to the nuclear potential. Under these circumstan-

ces our one-pole-approximation becomes quite bad and therefore R should not be chosen too short in the case of the S-phase. We have taken $R = 15$ fm.

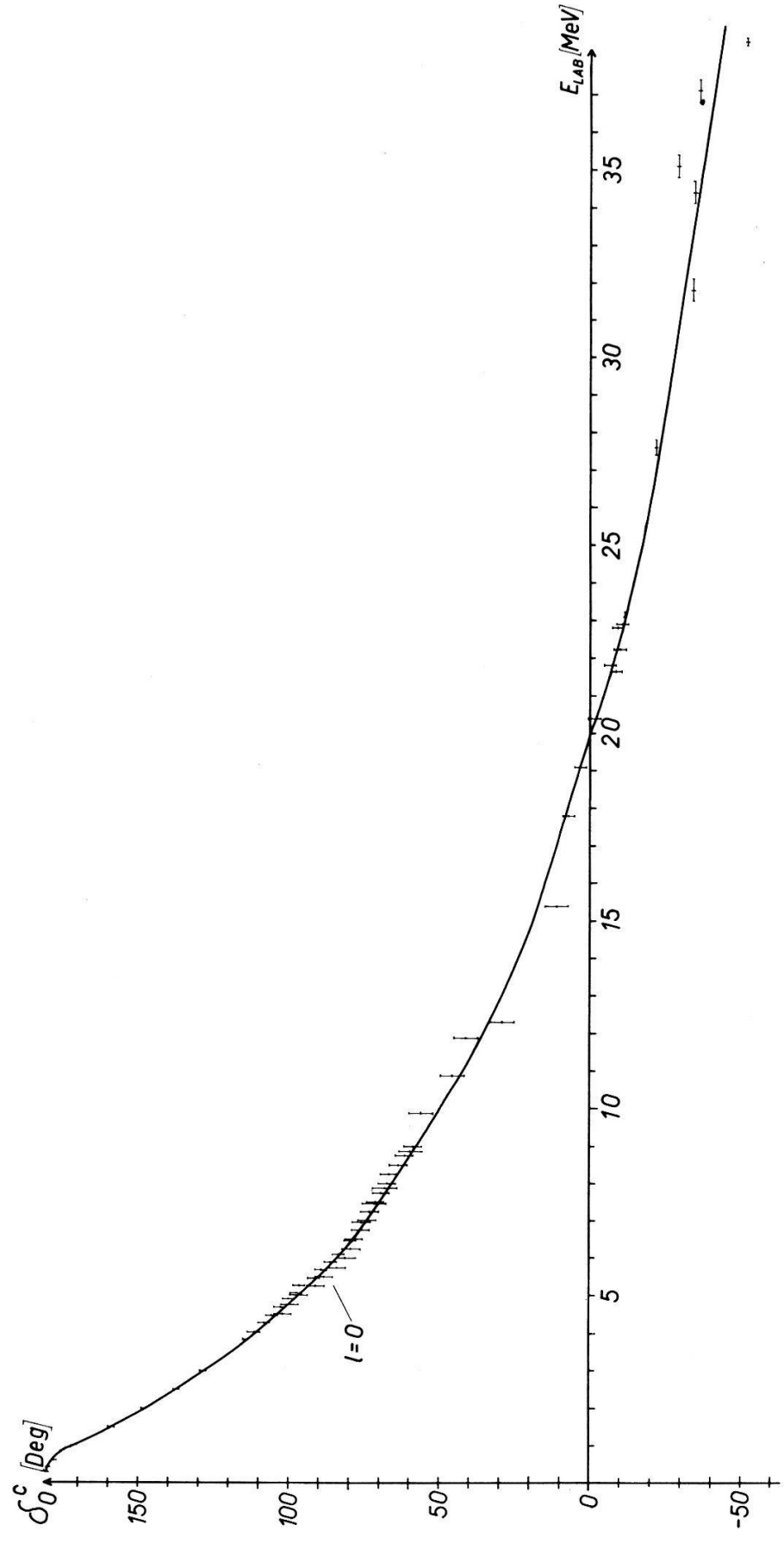

Figure 2
Experimental phase shifts (Ref. [8–9])for $l = 0$. The curve represents both the input data and the phase shifts calculated from our S-potential.

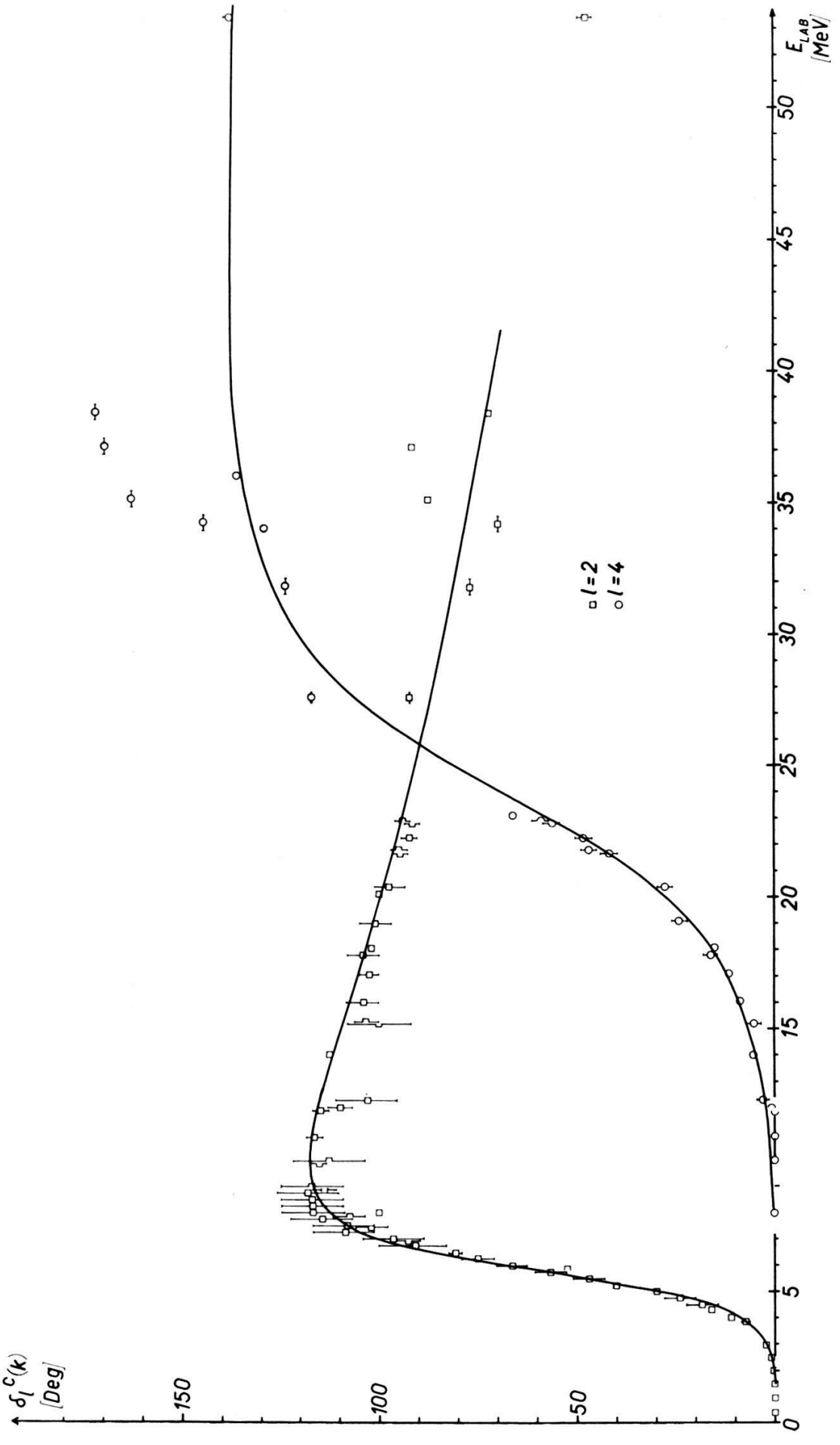

Figure 3

Experimental phase shifts for $l = 2,4$. The curves represent both the input data and the phase shifts calculated from the D- and G-potential.

All potentials were tested by numerical integration of the Schrödinger equation. The phase values obtained agree with the input data (see Fig. 2 and 3) up to an error less than 1.5 degrees (also in case of the G-potential!). The computation of the sharp ground state causes some difficulty. Only its position can be calculated directly. The calculation of its width Γ however would require a much greater arithmetical accuracy. Therefore we have determined Γ from equation (17) by looking for the passage of the last term in (19) through $\pi/2$. Our potential gives $E_0 = 92.08$ keV and $\Gamma = 7.18$ eV compared with the experimental values: $E_0 = 92.12 \pm 0.05$ keV and $\Gamma = 6.8 \pm 1.7$ eV [4].

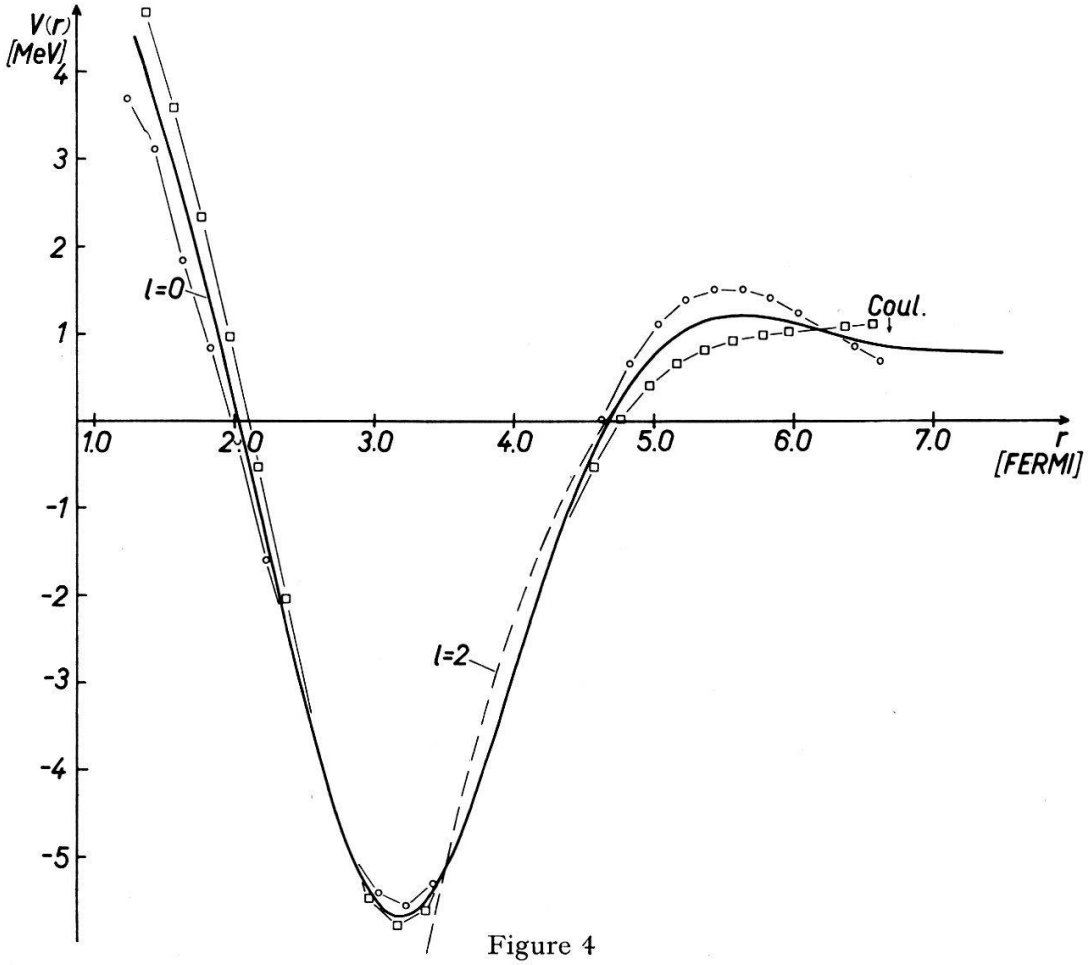

Figure 4

$l = 0$ potential together with "lower" (squares) and "upper" (cycles) values due to the errors of the experimental phases.

Numerical values of the potentials (see Fig. 4 and 5) are listed in Table 1. The Coulomb potential is always included, but the centrifugal potentials were substracted. The given errors are related to the experimental errors. They were found by calculation with the phase values from the limiting curves as inferred from the experimental errors. Particularly the outer part of the tail, the transition into the Coulomb potential is sensitive against small variation of the phases (see Fig. 4). At distances larger than 7 fm we found pure Coulomb force. For the G-potential no errors were calculated, because this potential is physically not very significant, although it is mathematically correct. We shall return to this point in the next section. The D-potential seems to show no smooth transition into the hard-core at 1.43 fm. But this is only due to the subtraction of the centrifugal barrier.

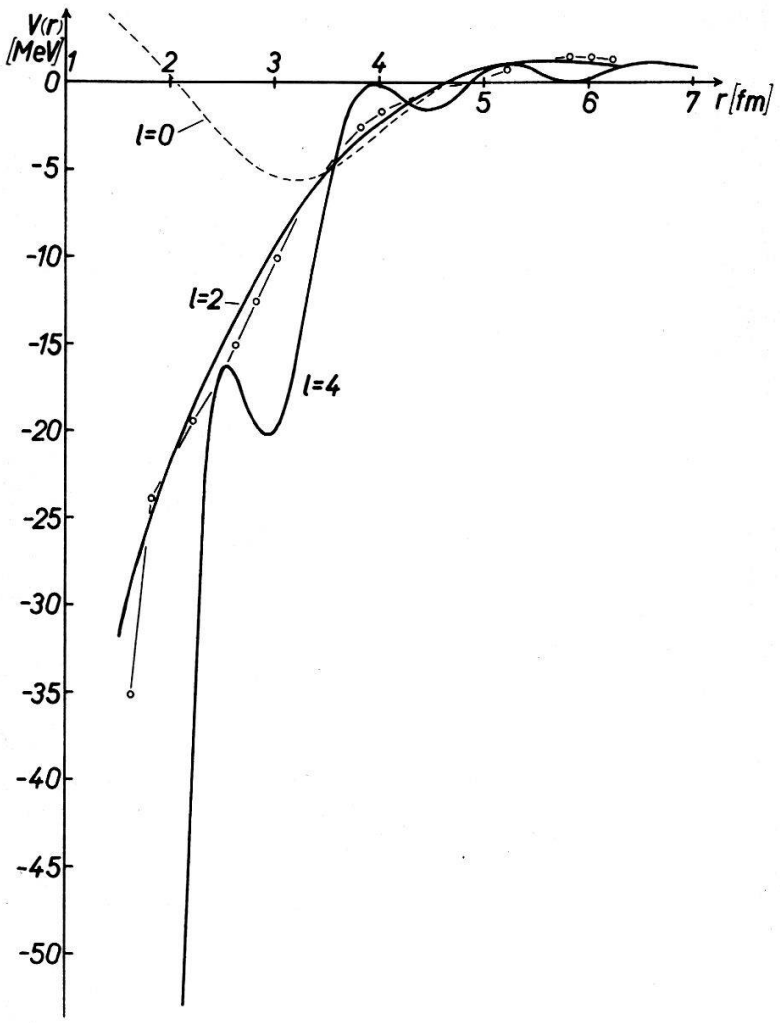

Figure 5

$l = 0, 2, 4$ potentials together with "lower" values of the $l = 2$ potential due to the errors of the experimental phases.

Table 1

S-Potential Radius r (fm)	$V(r)$ (MeV)	*D*-Potential Radius r (fm)	$V(r)$ (MeV)	*G*-Potential Radius r (fm)	$V(r)$ (MeV)
1.2037	$+\infty$				
1.3037	+4.329				
1.4037	3.862	1.4307	$+\infty$		
1.5037	3.334 ± .5	1.5307	−31.607 ± 5.0		
	2.769		−28.912	1.6049	$+\infty$
	2.177		−26.652	1.7049	−201.25
	1.564		−24.705	1.8049	−182.03
	0.9299		−22.978	1.9049	−147.23
2.0037	0.2736 ± .2	2.0307	−21.406 ± .5	2.0049	−107.08
	−0.4046		−19.939		− 70.322
	−1.103		−18.547		− 42.253
	−1.804		−17.208		− 26.335
	−2.491		−15.914		− 19.263

S-Potential Radius r (fm)	$V(r)$ (*MeV*)	*D*-Potential Radius r (fm)	$V(r)$ (*MeV*)	*G*-Potential Radius r (fm)	$V(r)$ (*MeV*)
2.5037	$-3.148 \pm .05$	2.5307	-14.660 ± 1.5	2.5049	− 16.391
	− 3.761		− 13.447		− 16.651
	− 4.314		− 12.278		− 18.027
	− 4.793		− 11.159		− 19.496
	− 5.180		− 10.092		− 20.031
3.0037	$-5.460 \pm .1$	3.0307	$-9.084 \pm .8$	3.0049	− 20.044
	− 5.626		− 8.135		− 18.547
	− 5.676		− 7.247		− 16.048
	− 5.611		− 6.421		− 12.913
	− 5.435		− 5.655		− 9.558
3.5037	$-5.158 \pm .04$	3.5307	$-4.947 \pm .4$	3.5049	− 6.454
	− 4.795		− 4.295		− 3.838
	− 4.369		− 3.693		− 1.924
	− 3.907		− 3.141		− 0,7177
	− 3.420		− 2.633		− 0.1677
4.0037	$-2.920 \pm .02$	4.0307	$-2.166 \pm .4$	4.0049	− 0.1318
	− 2.417		− 1.737		− 0.4280
	− 1.920		− 1.343		− 0.8742
	− 1.445		− 0.9816		− 1.285
	− 1.007		− 0.6510		− 1.547
4.5037	$-0.6110 \pm .1$	4.5307	$-0.3493 \pm .2$	4.5049	− 1.565
	− 0.2578		− 0.0753		− 1.347
	+ 0.0541		+ 0.1720		− 0.9254
	0.3286		0.3932		− 0.3894
	0.5620		0.5886		+ 0.1574
5.0037	$0.7512 \pm .3$	5.0307	$0.7585 \pm .5$	5.0049	+ 0.6301
	0.9012		0.9029		0.9461
	1.017		1.022		1.087
	1.103		1.115		1.045
	1.161		1.183		0.8681
5.5037	$1.194 \pm .3$	5.5307	$1.225 \pm .5$	5.5049	0.6149
	1.204		1.242		0.3540
	1.197		1.234		0.1555
	1.182		1.202		0.0547
	1.156		1.146		0.0804
6.0037	$1.121 \pm .1$	6.0307	$1.069 \pm .5$	6.0049	0.2131
	1.079		0.9711		0.4259
	1.034		0.9244		0.6713
	0.9889				0.8986
	0.9465				1.077
6.5037	$0.9098 \pm .3$			6.5049	1.177
	0.8806				1.207
	0.8591				1.143
	0.8465				1.058
					0.9300
				7.0049	0.8222

IV. Discussion

Comparing the S- and D-potential obtained (Fig. 4 and 5) the strong l-dependence at shorter distances is most striking. This was already found in earlier phenomenological studies [11], but without great quantitative certainty. The outer attractive part of the potentials depends only little on l, especially if one takes the experimental errors into account.

This situation can be understood with help of the cluster model [12]. In this model the α-particles are assumed to be in the ground state and unpolarized. The interaction results from the nucleon-nucleon-interaction which operates between nucleons of the two clusters. Apart from the Coulomb force the α-α-interaction then consists of a direct part and an ecxhange part. The direct interaction is independent of l, local attractive and has a wider range than the l-dependent non-local exchange interaction. If the former is identified with the tail of our potentials, one gets some informations about nuclear forces.

Starting with a simple nucleon-nucleon-potential of Gaussian radial shape

$$V(r) = V_0 \cdot e^{-ar^2}$$

and assuming a ground state wave function of Gaussian spatial part

$$\psi_\alpha \sim \exp\left(-\frac{\beta}{2}\sum_{i>j} r_{ij}^2\right), \quad r_{ij} = |\vec{r}_i - \vec{r}_j|, \quad i,j = 1, \ldots 4$$

for the α-particle, which is justified by electron scattering experiments [13], one can immediately calculate the direct α-α-potential. The result is again of a Gaussian form [14]:

$$V_{\alpha\alpha}(r) = 16\, V_0 \left(1 + \frac{4}{3}\varrho_\alpha^2 a\right)^{-3/2} \exp\left(\frac{ar^2}{1+4\,\varrho_\alpha^2 a/3}\right) = V_1\, e^{-br^2},$$

where $\varrho_\alpha = 3/4\,\sqrt{2\beta}$ is the r.m.s. radius of the α-particle. Fitting the tail of our D-potential by a Gaussian potential we obtain

$$b = 1.75 \cdot 10^{25}\ \mathrm{cm}^{-2} \qquad V_1 = -\,58{,}0645\ \mathrm{MeV}\,.$$

With a r.m.s. radius $\varrho_\alpha = 1.52$ fm [13] of the α-particle this gives for the N-N-potential

$$a = 3.8 \cdot 10^{25}\ \mathrm{cm}^{-2} \qquad V_0 = -\,11.6\ \mathrm{MeV}\,.$$

The agreement with the values used in [15] $a = 4.6 \times 10^{25}\ \mathrm{cm}^{-2}$, $V_0 = -\,72.98$ MeV is bad. It seems that a more accurate treatment of the nuclear force is necessary. This could be done using better phenomenological potentials.

The G-potential shows oscillations. Since this potential gives the correct phase shift, the calculation must be correct. Probably these oscillations are caused by the fact that the G-phase becomes different from zero only above 10 MeV and its main contributions come from energies between 25 and 45 MeV, where only few somewhat inconsistent experimental phase values exist. Because already inelastic processes occur in this region the application of the two-body-potential model becomes doubtful. This is also indicated by the fact, that the hard core approximation for the G-phase in contrast to the S- and D-phase is quite bad. The straight line of the hard core phase must intersect the ideal phase $\delta(k)$, which shows that the α-particle is no

longer tight. We have tried to calculate the G-potential assuming in the outer region the slope as given by the D-potential. This too gives no physically reasonable potential.

From the present work we get the impression, that the Gelfand-Levitan theory is quite appropriate for application of the potential model in a most general and rigorous manner. Since spin-dependent potentials can also be treated in case of particles with spin, the analysis of various other elastic processes seems to be possible in this way. Applications of the results to the C^{12}-problem are in progress.

Acknowledgments

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