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The SU(6) Model and its Relativistic Generalizations

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(11. X. 66)

atque eadem magni refert primordia saepe
cum quibus et quali positura contineantur
et quos inter se dent motus accipiantque.
Titus Lucretius Carus, De rerum natura

Preface

This article is devoted to a critical discussion of the ideas involved in the SU(6) symmetry models. The main emphasis is laid upon the relativistic extensions of SU(6) symmetry. The reason for this is twofold. The theory of static SU(6) symmetry is to a certain extent closed and self-consistent. A deeper insight into the static symmetry can be gained only after the dynamics of elementary particles has been better understood. The application of the static symmetry is in addition limited to a very small number of problems. Both features make this model not very interesting for further theoretical investigations at the moment. The other reason is that there exists already an extensive literature dealing with the static SU(6) model and the techniques needed for computations in this model. In particular we refer the reader to the review article of A. Pais, Ref. [302].

Contrary to the static model, the relativistic extensions are aimed to apply to all phenomena of strong interaction physics (bound states excluded) and even to electromagnetic and weak interactions in first order after a proper definition of spurions. Since the discussion of SU(6) symmetries started, the belief has persistently been expressed that such relativistic models cannot be defined consistently. Indeed, the discussion of difficulties encountered in the relativistic models can be looked upon as one of the purposes of this article. Such difficulties are the conflict with unitarity or crossing symmetry of the S-matrix. But this criticism does not exhaust our presentation of these models. In our opinion the studies of relativistic extensions have well entailed a positive result: the invention and the exploration of the collinear and coplanar subgroup symmetries. These groups are a priori not in conflict with any known

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principle imposed on the S-matrix. At least as the collinear group S[U(3) $\otimes$ U(3)] is concerned the predictions are throughout in excellent agreement with experiment (with exception of the cases where SU(3) itself is already a bad symmetry).

The material is presented as follows. The article is divided into two parts and an Appendix. In the first part we develop the physical ideas and the mathematical basis involved in the different models. In the second part we apply the subgroup symmetries systematically to the most important physical problems. We intended to give the predictions in a form which is accessible even to experimental physicists not acquainted with the notions of unitary symmetry. We have therefore written the results in terms of observable amplitudes and not in terms of SU(3) invariants only. The Appendix contains some tables of decompositions of representations, which might be useful.

Some of our readers will perhaps regret that the current algebra approach has been skipped completely in our article. The current algebra approach is more restrictive than the group approach displayed here, because it identifies the generators of the algebra with specific observables, namely quark current densities and their space integrals. This gives predictions which one cannot get from the other approach. On the other hand, it is more general, because upon taking the matrix elements of the commutation relations, one uses intermediate states which belong to reducible representations of the algebra, as is done for example in the Adler-Weissberger relations. If one tries to saturate the algebra with irreducible representations, one gets of course the results of the group approach. If, however, one considers commutators of densities, one gets in many cases additional restrictions which may even be self-contradictory. For all these reasons the two approaches are not equivalent, despite the fact that in special cases the same predictions may result. Anyway it seems premature to include current algebras in a review of SU(6) symmetries.

We have also left out such work on SU(6) symmetries which is only concerned with an investigation of symmetries of interaction Lagrangians without giving a device to compute S-matrix elements which goes beyond a perturbative expansion (see for example Ref. [272] and succeeding papers of these authors). Such models are sometimes formally related to the models involving more than four momentum components discussed in Section 3. Nevertheless, the interpretation of both types of models is completely different and should not be confused.

The authors wish to express their gratitude to Prof. J. S. Bell for critical reading of the manuscript and for many constructive suggestions. They thank Dr. A. Kihlberg for looking through the mathematical parts of the article. Two of the authors (H. Ruegg and T. S. Santhanam) wish to thank Prof. Abdus Salam and the IAEA for their kind hospitality at the International Centre for Theoretical Physics in Trieste.

Review articles on SU(6): [110, 302, 354] in English and [231] in Russian.

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PART I

GENERAL THEORY

1. Static SU(6) symmetry

1.1 *The concept of static SU(6) symmetry*

1.1.1 Symmetries, approximate symmetries and related notions

We regard symmetries as groups which possess a representation by means of unitary operators in the physical Hilbert space. The physical relevance of symmetries lies in the fact that the group elements leave the dynamics unchanged. If a Hamiltonian exists, we can express this invariance by saying that the Hamiltonian commutes with the operators $U(g)$ which represent the group elements

$$U(g)\, H\, U(g)^{-1} = H\,.$$

If a scattering matrix S exists we can make the same statement for this operator.

Examples are known where this symmetry group is either of finite order or a Lie group. In the latter case we may either assume the global group or its infinitesimal part, the Lie algebra, as the structure which represents the physical symmetry. From the mathematical point of view the Lie algebras are a more general notion. But it is not clear today whether the algebras play the fundamental role in physics.

If a Hamiltonian exists, a symmetry algebra spanned by operators A_i satisfies

$$[H, A_i] = 0\,.$$

It is well known that this can be expressed as a conservation law

$$\frac{d}{dt} A_i = 0\,.$$

A subset of the operators A_i which corresponds to observables should be represented by self-adjoint operators.

The applications which we have in mind in this paper will mainly concern the S-matrix. Due to Schur's lemma the scattering matrix can be decomposed into a sum of projection operators with coefficients. These coefficients are called invariant functions, in special cases also form factors.

The distinction between kinematic and dynamic symmetries is, according to our present state of knowledge, only a matter of definition. The geometrical space-time invariance provides us with kinematic symmetries. The definition of dynamical symmetries is on the other hand always connected with some dynamical assumptions. All dynamical symmetries are valid only within some approximations, which in some cases mean the neglect of relatively weaker interactions. In the case of isospin the notion approximate seems clear. The deviations from physical reality of the predictions and premises of the symmetry is of the order of the fine structure constant. The unitary operators representing the group operators can only be defined on a certain subspace of the physical Hilbert space, in which physics is believed to be governed by the strong interactions. This subspace is defined in a Lorentz invariant way.

In the case of the unitary symmetry SU(3) it is much more difficult to say what the notion approximate means. The evidence for distinguishing between a semi-strong and a very strong interaction which breaks or preserves the symmetry is not very convincing. The current algebra approach which has recently been proposed[3]) and which is still being investigated may perhaps provide us with a deeper understanding of dynamical symmetries. For the reasons explained in the preface it is not our aim to discuss this method here.

We come already very close to the familiar interpretation of static SU(6) symmetry, if we consider the hydrogen atom with its fine structure neglected. This means skipping terms of order

$$\alpha \frac{v}{c} \sim \alpha^2$$

in the Hamiltonian. In this case the rotations of the spin of the electron form a group $SU(2)_\sigma$. This is an approximate symmetry to order α. Some authors (see e.g. Ref. [302]) see a fundamental difference between this symmetry and say isospin symmetry in the fact that the spin independence of the electromagnetic interaction is exhibited only by the particular system of the hydrogen atom whereas the isospin symmetry can be observed in all strongly interacting systems. In fact, it is known that for heavy atoms the fine structure cannot be neglected.

Nevertheless, in our opinion there must not necessarily exist a fundamental difference. It may well be that the applicability of isospin symmetry to all strongly interacting systems is due to an over-all consistency requirement, to which the strong interaction is submitted. Such an idea has been expressed in the technical form of bootstrap methods. This restriction on strongly interacting systems may possibly transmit the degeneracies from one system to another, say from the π's to the ϱ's and nucleons etc.

Due to the common interpretation of SU(6) as a static symmetry its analogy with the hydrogen atom symmetry $SU(2)_\sigma$ is intimate. Nevertheless the usual attitude is to apply it to all static systems and to see if it works.

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[3]) Because of the large number of publications on current algebras we give only two references of general importance: Refs. [167] and [168].

1.1.2 The definition of $SU(6)_\sigma$

We denote the group corresponding to static SU(6) symmetry by $SU(6)_\sigma$. By definition this symmetry applies to systems of particles which are all at rest [4]). It is sufficient to define the algebra of the group $SU(6)_\sigma$ in one representation, for which we choose the fundamental vector representation of dimension six. We have (see Refs. [185, 301, 353])

$$A_{i,\mu} = \lambda_i \, \sigma_\mu \,, \qquad i = 0, 1, 2 \ldots 8 \,, \qquad \mu = 0, 1, 2, 3, \;\; \mu = i = 0 \text{ excluded.}$$

λ_i are Gell-Mann's matrices, σ_μ are Pauli's matrices with σ_0 as the 2×2 unit matrix. The product is meant as a Kronecker product, i.e. the first matrix has to be inserted into the second. The operators $A_{0,k}$ apply to the spin variables, the $A_{i,0}$ operate on the variables of unitary symmetry. This definition is consistent since for static systems the spin of the systems is the sum of the individual spins.

If we assume that the elementary particles are bound states of objects which belong to single representations of $SU(6)_\sigma$, (see the literature on quark models, particularly Refs. [43, 66, 99, 100, 112, 157, 262]), static symmetries are not necessarily applicable to these bound states. From this point of view it seems rather natural to treat the physical particles as if they belonged to reducible representations. Such an ansatz is called supermultiplet mixing model (see Refs. [162, 290]). Since the main consequence of this mixing hypothesis is an introduction of additional parameters without providing us with new ideas about any problem involved in the SU(6) models we shall not make use of it. Instead we assume that the physical particles form multiplets. Historically the existence of the symmetry was suggested partly by some known multiplets of baryons and mesons, which fit quite nicely into $SU(6)_\sigma$ representations (see Section 1.3).

We shall see that the masses of one supermultiplet spread over a wide range. This is thought to be a consequence of the approximate nature of $SU(6)_\sigma$ symmetry. Since all generators of $SU(6)_\sigma$ commute with the parity operator, each supermultiplet has a common parity eigenvalue.

Contrary to the idea that the masses in a supermultiplet reflect symmetry breaking it has been suggested by several authors that a suitable chosen group could itself imply a mass formula (this mass breaking is denoted "intrinsic"). In particular the mathematical problem has been investigated whether a group G exists which contains the inhomogeneous Lorentz group and the internal symmetry group SU(3) or SU(6) as subgroups and in which the mass operator of the inhomogeneous Lorentz group has a discrete spectrum with more than one eigenvalue in at least one unitary irreducible representation of the group G. A preliminary negative answer has been given by O'Raifeartaigh (Refs. [308, 309]). Since the mathematical discussion is still going on, and its physical relevance is in addition not clear, we are not willing to give an exposition of the different arguments here.

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[4]) We must bear in mind the difference between the notion static and the notion of static models. In the static models only the source is static, i.e. at rest. The notions static and non-relativistic are also not synonymous. The common usage of non-relativistic is: inclusion of terms in v/c up to first order. The Lorentz force in electrodynamics is non-relativistic.

Further references on the general properties of static $SU(6)_\sigma$: [89, 186, 229, 248, 268, 283, 297, 381].

1.2 *Solutions for some mathematical problems involved in static SU(6) theory*

1.2.1 Tensors and Young tableaux

The group SU(6) is defined as the group consisting of unitary 6×6 matrices, the determinant of which is equal to one (unimodular matrices). It is a compact, simply connected Lie group. For compact Lie groups all unitary irreducible representations are of finite dimension and all finite dimensional representations are equivalent to unitary ones (see Ref. [284], theorem 2 p. 438 and theorem 4 p. 440). All unitary representations of SU(n) can be given in the form of tensor representations. By tensors we mean quantities

$$\psi_{i_1 i_2 \ldots i_k}, \qquad i_m = 1, 2 \ldots n,$$

which are transformed by the matrix T_g representing the element g of SU(n) as

$$\psi_{i_1 i_2 \ldots i_k} = T_{g\, i_1 i_2 \ldots i_k}{}^{j_1 j_2 \ldots j_k}\, \psi_{j_1 j_2 \ldots j_k}.$$

The matrix elements of T_g are polynomials of degree k in the matrix elements of the matrix g. The representation is irreducible if the indices satisfy certain symmetry conditions[5]). These are denoted by Young tableaux. Since in Section 4 we have to deal with unitary representations of $SL(n, C)$ we explain here a method to construct a basis for tensor representations which will be helpful for our later deductions.

Let ξ be a matrix of SU(n),

$$\xi = (\xi^{ij}),$$

and consider the linear vector space of polynomials $F(\xi)$ in the elements ξ^{ij} of this matrix. If we define the transformations

$$(T_g F)\,(\xi) = F(\xi\, g)$$

where ξg means the matrix product of the two matrices ξ and g of SU(n), this space becomes a representation space. We specify now certain subclasses of functions F which give us irreducible representations. First we introduce the variables

$$\Delta_{(k)}^{i_1 i_2 \ldots i_k} = \begin{vmatrix} \xi^{n-k+1, i_1} & \xi^{n-k+1, i_2} & \ldots & \xi^{n-k+1, i_k} \\ \xi^{n-k+2, i_1} & \xi^{n-k+2, i_2} & \ldots & \xi^{n-k+2, i_k} \\ \vdots & \vdots & & \vdots \\ \xi^{n, i_1} & \xi^{n, i_2} & \ldots & \xi^{n, i_k} \end{vmatrix}.$$

[5]) A short review of the properties of groups SU(n) is contained in Refs. [222, 190]. For a more extensive treatment we refer the reader to textbooks, such as Ref. [61], or to the original papers, e.g. WEYL's articles, Ref. [392]. WEYL's papers are more useful for the physicist's purposes than many modern treatises written by physicists themselves.

Then we construct homogeneous polynomials which are homogeneous of degree f_k in the variables $\Delta_{(k)}$. The corresponding Young tableaux are written

$f_{n-1} + f_{n-2} + \cdots + f_2 + f_1$ blocks
$f_{n-1} + f_{n-2} + \cdots + f_2$ blocks
f_{n-1} blocks

This construction of basis elements for an irreducible tensor representation can be found in Ref. [392]. Instead of giving the Young tableau it is obviously sufficient to know the $(n-1)$-tupel $(f_1, f_2, \ldots f_{n-1})$. Representations belonging to only one column are called fundamental representations. They correspond to linear functions in a single variable $\Delta_{(k)}$. For $k = 1$ we obtain

$$F(\xi) = \sum_i \psi_i \, \Delta^i_{(1)} = \sum_i \psi_i \, \xi^{n,i}$$

and

$$(T_g F)\,(\xi) = \sum_i \psi_i (\xi\, g)^{n,i} = \sum_{k,i} (g^i_k \, \psi_i)\, \xi^{n,k}$$

or

$$\psi_k \underset{g}{\rightarrow} g^i_k \, \psi_i \, .$$

For $k = n - 1$ we define

$$\Delta_i = \varepsilon_{i j_1 j_2 \ldots j_{n-1}} \, \Delta^{j_1 j_2 \ldots j_{n-1}}_{(n-1)}$$

and find

$$F(\xi) = \sum_i \psi^i \, \Delta_i \, , \quad \psi^k \underset{g}{\rightarrow} \psi^i (g^{-1})^k_i \, .$$

This representation ψ^k is contragredient to the representation ψ_k. Indeed, the expression

$$\psi'^k \, \psi_k$$

is invariant.

Since the notion of weights has become rather popular, we give the highest weights of the representations in terms of the homogeneities f_k:

$$M = \sum_k f_k \, M_k$$

where

$$M_k = \Big(\underbrace{1 - \frac{k}{n}, \; 1 - \frac{k}{n} \ldots,}_{k \text{ times}} \; \underbrace{- \frac{k}{n}, \; - \frac{k}{n} \ldots - \frac{k}{n}}_{(n-k) \text{ times}}\Big) .$$

The dimension of a representation is (see Ref. [392])

$$N = \frac{D(l_1, l_2 \ldots l_n)}{D(n-1, n-2 \ldots 1, 0)} \, .$$

Here D is Vandermonde's determinant,

$$D(x_1, x_2, x_3 \ldots x_n) = (x_1 - x_2)\,(x_1 - x_3)\,(x_1 - \ldots$$
$$(x_2 - x_3)\,(x_2 - \ldots$$
$$(x_3 -$$

Particularly this yields

$$D\,(n-1, n-2 \ldots, 1, 0) = 1!\,2!\,3! \ldots (n-1)!\,.$$

The arguments of the numerator determinant are

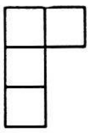
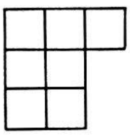
$$l_s = m_s + n - s\,, \quad l_n = 0\,,$$

where m_s is the length of the s^{th} row of the tableau

$$m_s = \sum_{k=s}^{n-1} f_k\,.$$

Instead of the parameters f_k we can often make convenient use of the set of parameters m_k. Since these parameters are ordered in a decreasing sequence the notation of tableaux can be shortened. We need to give only the number of times each non-vanishing m_k is contained in the $(n-1)$-tupel.

Examples

 $= (21^2)\,,$  $= (32^2)\,.$

The highest weight of a representation is written in terms of m_k

$$M = \sum_{k=1}^{n-1} m_k\,N_k\,,$$

with $N_1 = M_1$, $N_k = M_k - M_{k-1}$ for $k > 1$,

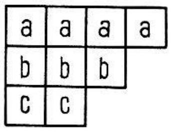
$$N_k = \left(-\frac{1}{n}, -\frac{1}{n} \ldots \underset{\substack{\uparrow \\ k^{th}\ \text{place}}}{1 - \frac{1}{n}}, -\frac{1}{n} \ldots -\frac{1}{n}\right).$$

From now on we shall throughout make use of the parameters m_k in denoting Young tableaux.

1.2.2 Decomposition of tensor products

The tensorial product of two irreducible representations belonging to the tableaux m_k and m_k' is again a representation of SU(n), but is in general reducible. This faces us with the problem of giving the irreducible representations contained in this product. A solution of this problem (see Refs. [222, 61]) can be formulated as the following tableau multiplication technique.

We take one Young tableau as fixed and add to it the boxes from the second tableau. Before doing this we label the boxes of the second tableau according to the rows



If we add now the boxes of the first row labelled 'a' to the first tableau in all possible ways, we must do this in such a manner that again a tableau results. To all the tableaux thus gained we add the boxes of the second row, the third row etc., in each step requiring that the resulting pattern has the form of a tableau (i.e. $m_k > m_{k+1}$).

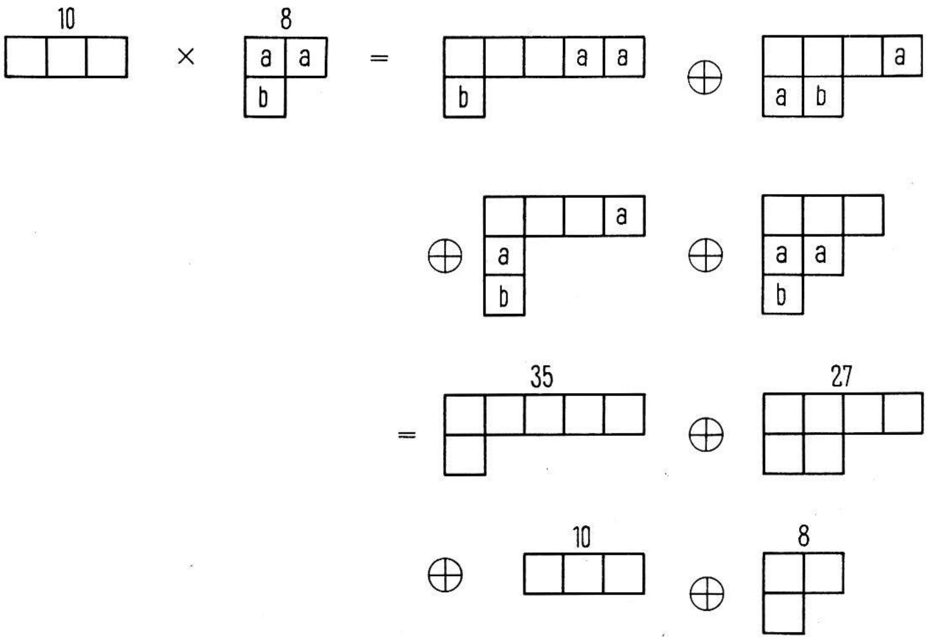
From this set of tableaux thus obtained we eliminate many patterns

1) Those which contain equal labels appearing in a column;
2) Those with a number of rows bigger than n. We drop the columns of length n from the tableaux.
3) We order the boxes of the tableaux. We start with the first row and take the boxes in the order from right to left. Then we run through the second row from right to left and so row after row through the whole tableau. This ordered sequence contains boxes with labels and empty boxes. If we cut this sequence at any point, the number of labels b must not exceed the number of a's, the number of c's the number of b's etc. counted from the start till the cut.

The resulting tableaux, which differ in form or in the places of the labels, each corresponds to an irreducible representation contained in the tensor product.

Example

$$\mathrm{SU}(3),\ (m_1, m_2) = (3, 0) \equiv (3),\ (m_1', m_2') = (2, 1) \equiv (21).$$



1.2.3 Representations of the group SU(mn) reduced into representations of the subgroup SU(m) ⊗ SU(n)

The methods of reducing representations of SU(mn) and SU($m+n$) into irreducible representations of SU(m) ⊗ SU(n) are quite important for the application of SU(6) theories. For the first problem this can already be inspected from the definition of $SU(6)_\sigma$. Splitting a representation of $SU(6)_\sigma$ into parts which are irreducible with respect to the subgroup SU(3) ⊗ $SU(2)_\sigma$ means a separation of spin and unitary spin. This reduction is therefore a part of nearly every calculation performed with the $SU(6)_\sigma$ group. On the other hand, the reduction of a representation of $SU(6)_\sigma$ into representation of SU(3) ⊗ SU(3) is needed in connection with the collinear subgroup (see Sections 2, 3 and Part II).

We start with the discussion of how the SU(m) ⊗ SU(n) content of a representation of SU(mn) can be extracted; the other problem will be dealt with in Subsection 1.2.4. If we write the tensor indices of SU(mn) as index pairs, with the first index of the pair corresponding to SU(m) and the other to SU(n), the problem can be expressed in the following manner. How can a tensor with a given simultaneous symmetry in the pairs be decomposed into tensors which satisfy certain symmetries in the first and second index of the pairs separately? We give the answer in a way that, given a representation of SU(m) ⊗ SU(n), we can decide whether or not it is contained in the representation of SU(mn).

First we note that, if the number of boxes of the representation of SU(mn) is f,

$$f=\sum_{k=1}^{n-1} m_k$$

the number of boxes of the representations of SU(m) or SU(n) can only be

$$f(m) \cong f \bmod m, \quad f(m) \leqq f, \qquad f(n) \cong f \bmod n, \quad f(n) \leqq f.$$

We have therefore to fill up the two tableaux for SU(m) and SU(n) adding columns of length m respectively n until they possess f boxes. Then we form the Kronecker product of the corresponding tensors which can be written as a tensor bearing index pairs of number f. The indices in the pairs have still separate symmetries. We decompose this product into irreducible representations of SU(mn). Our question will be answered by yes or no if we know whether or not the given representation of SU(mn) is in this decomposition series.

This problem can be solved by applying the results for the Clebsch-Gordan decomposition of the tensor product of two representations of the symmetric group $\Sigma(f)$. Indeed, if we operate with the permutations of $\Sigma(f)$ on the indices of a tensor of rank f corresponding to a Young tableau with f blocks, we obtain an irreducible representation of the permutation group $\Sigma(f)$. Therefore the first and second indices of the pairs give each an irreducible representation of $\Sigma(f)$. We have to find the decomposition series of the product of these representations. This reduction of the problem to a product decomposition problem of the symmetric group $\Sigma(f)$ makes sense only if tables for the latter decomposition are available. Some tables (up to $f=8$) can be found in Ref. [222]. They are not complete enough to cover all interesting cases as the representation of dimension 405 of the group $SU(6)_\sigma$, which has twelve boxes. We think it therefore to be most convenient, if we give some tables at the end, which contain the interesting cases and which are calculated recursively by straightforward methods (see the Appendix).

1.2.4 Representations of the group SU$(m + n)$ reduced into representations of the subgroup SU$(m) \otimes$ SU(n)

In this case we consider a tensor for the group SU$(m + n)$ whose indices run from 1 to $m + n$. We split the vector space of $m + n$ dimensions into a direct sum of a space with m and a space with n dimensions, and let the first m values for the indices correspond to the first vector space and the remaining n values to the second space. We have to decompose a tensor of a given symmetry into tensors which are separately symmetric in the first and second space.

Again we pose the problem in such a manner that we want to decide whether a given representation of SU$(m) \otimes$ SU(n) is contained in the representation of SU$(m + n)$ or not. As can be easily seen the answer to this problem is as follows. The tableau of the representation of SU(m) can be arbitrarily enlarged by adding columns of length m. Each such tableau with m rows is then considered as a tableau for the group SU$(m + n)$. We treat the tableau for the representation of SU(n) similarly. We take then the tensorial product of these two representations of SU$(m + n)$ and decompose it in the fashion explained under Section 1.2.2. The representation of SU$(m) \otimes$ SU(n) considered is contained as often in a given representation of SU$(m + n)$ as it appears in any of the resulting tensor decomposition series.

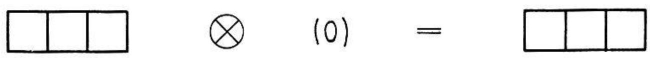
Example

SU(6), SU(3) ⊗ SU(3), representation (3) of SU(6).

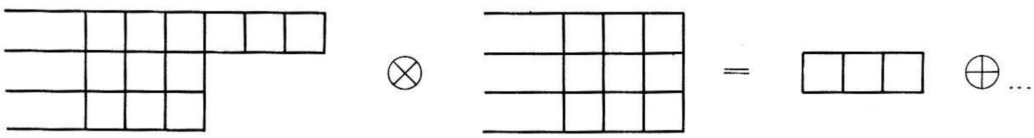
$$(3) = (3) \otimes (0) \oplus (2) \otimes (1) \oplus (1) \otimes (2) \oplus (0) \otimes (3) \, .$$

Indeed, we try

$$(3) \otimes (0)$$

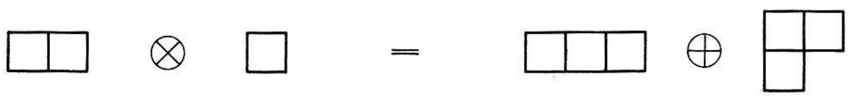


The only other possibilities are



But such examples where the effect of the newly introduced columns is to compensate each other are not counted as independent.

Let us try now $(2) \otimes (1)$:



There are again no further independent possibilities. The remaining two representations are treated analogously. A check of dimensions proves that the list is complete.

1.2.5 Final remarks

We have thus seen that only the determination of the $SU(m) \otimes SU(n)$ content of a representation of the group $SU(mn)$ is not quite simple. All the three problems can also be solved applying the theory of characters.

Another problem which is of more academic interest is the problem how to identify states in representations as eigenstates of certain operators. Since the physical role of the twenty operators necessary in the case of SU(6) is not clear, and because the representations of $SU(6)_\sigma$ which are needed in physical applications have a relatively low dimension, it is not very helpful to identify states completely in this way.

Further references on Clebsch-Gordan coefficients for SU(6): [75, 97, 366).

1.3 Known supermultiplets for SU(6)

1.3.1 Baryons and antibaryons

If we decompose the representation (3) of $SU(6)_\sigma$ with respect to $SU(3) \otimes SU(2)_\sigma$, we obtain

$$56 = (10, 4) \oplus (8, 2)$$

(see Table 1 in the Appendix) where the figures in the square bracket denote SU(3) and $SU(2)_\sigma$ multiplicities. If we assume the eigenparity to be positive, we can fit the known eight baryons of spin $1/2^+$ and the baryon resonances of spin $3/2^+$ which form a decuplet in SU(3) into this representation.

The tensors split according to

$$B_{ABC} = \chi_{\alpha\beta\gamma} D_{abc} + 18^{-1/2} (\varepsilon_{abd}\, \varepsilon_{\alpha\beta}\, N_c^d\, \chi_\gamma + \varepsilon_{bcd}\, \varepsilon_{\beta\gamma}\, N_a^d\, \chi_\alpha + \varepsilon_{cad}\, \varepsilon_{\gamma\alpha}\, N_b^d\, \chi_\beta)\,.$$

Here $\chi_{\alpha\beta\gamma}$ is the Pauli spinor for spin $3/2$ for the group $SU(2)_\sigma$ and χ_α correspondingly the Pauli spinor for spin $1/2$. In general we define (totally symmetric) $SU(2)_\sigma$ spinors $\chi_{\alpha_1\alpha_2 \cdots \alpha_k}$ for spin $S = k/2$, which are connected with spherical harmonics as

$$\hat{\chi}_{S,S_3} = \binom{2S}{S+S_3}^{1/2} \chi_{\underbrace{11\ldots1}_{S+S_3 \text{ times}}\underbrace{22\ldots2}_{S-S_3 \text{ times}}}$$

D_{abc} and N_a^b are SU(3) tensors for decuplet and octet and are identified with physical particles in the following way (see Ref. [169])

$$\begin{pmatrix} D_{111}\, D_{112}\, D_{122}\, D_{222} \\ D_{113}\, D_{123}\, D_{223} \\ D_{133}\, D_{233} \\ D_{333} \end{pmatrix} = \begin{pmatrix} N^{*++},\, 3^{-1/2}\, N^{*+},\, 3^{-1/2}\, N^{*0},\, N^{*-} \\ 3^{-1/2}\, Y^{*+},\, 6^{-1/2}\, Y^{*0},\, 3^{-1/2}\, Y^{*-} \\ 3^{-1/2}\, \Xi^{*0},\, 3^{-1/2}\, \Xi^{*-} \\ \Omega^- \end{pmatrix}$$

and

$$\begin{pmatrix} N_1^1 & N_1^2 & N_1^3 \\ N_2^1 & N_2^2 & N_2^3 \\ N_3^1 & N_3^2 & N_3^3 \end{pmatrix} = \begin{pmatrix} 2^{-1/2}\, \Sigma^0 + 6^{-1/2}\, \Lambda & \Sigma^+ & p \\ \Sigma^- & -2^{-1/2}\, \Sigma^0 + 6^{-1/2}\, \Lambda & n \\ \Xi^- & \Xi^0 & -2\, 6^{-1/2}\, \Lambda \end{pmatrix}.$$

The physical states like p and n, etc., are always normalized to one. While the SU(3) $\otimes$ SU(2)$_\sigma$ content of representations of SU(6)$_\sigma$ can be determined by the general methods discussed in Section 1.2.3, the normalization constants of the different parts have to be fixed independently (the most interesting examples are given in the Appendix Table 1). This can simply be achieved by computing the norm of the states explicitly. We require

$$\bar{B}^{ABC} B_{ABC} = \bar{\chi}^{\alpha\beta\gamma} \chi_{\alpha\beta\gamma} \bar{D}^{abc} D_{abc} + \bar{\chi}^{\alpha} \chi_{\alpha} \bar{N}^b_a N^a_b .$$

Performing the calculation we find the value $18^{-1/2}$ in front of the spin $^1/_2$ states which was given above, the value 1 in front of the spin $^3/_2$ states is obvious.

We assign antibaryons to the contragredient representation (3^5).

1.3.2 Mesons with negative parity

The adjoint representation [this means in general the representation (21^{n-2})] has dimension 35. It splits into the parts

$$35 = (8,3) \oplus (1,3) \oplus (8,1) .$$

We assume that the eigenparity is negative. We can then assign the parts

(8,3) $\otimes$ (1,3) to the nonet of vector resonances,

(8,1) to the octet of pseudoscalar mesons.

The tensor will have the form

$$M^B_A = \frac{1}{\sqrt{2}} \left(\delta^\beta_\alpha P^b_a + \sum_{k=1,2,3} \sigma^\beta_{k,\alpha} V^b_{k,a} \right) .$$

V and P correspond to the vector nonet and pseudoscalar octet and are identified with the physical particles as

$$\begin{pmatrix} V^1_1 & V^2_1 & V^3_1 \\ V^1_2 & V^2_2 & V^3_2 \\ V^1_3 & V^2_3 & V^3_3 \end{pmatrix} =$$

$$\begin{pmatrix} 2^{-1/2} \varrho^0 + 6^{-1/2} \omega_0 + 3^{-1/2} \varphi_0 & \varrho^+ & K^{*+} \\ \varrho^- & -2^{-1/2} \varrho^0 + 6^{-1/2} \omega_0 + 3^{-1/2} \varphi_0 & K^{*0} \\ \bar{K}^{*-} & \bar{K}^{*0} & -2 6^{-1/2} \omega_0 + 3^{-1/2} \varphi_0 \end{pmatrix},$$

$$\begin{pmatrix} P^1_1 & P^2_1 & P^3_1 \\ P^1_2 & P^2_2 & P^3_2 \\ P^1_3 & P^2_3 & P^3_3 \end{pmatrix} = \begin{pmatrix} 2^{-1/2} \pi^0 + 6^{-1/2} \eta & \pi^+ & K^+ \\ \pi^- & -2^{-1/2} \pi^0 + 6^{-1/2} \eta & K^0 \\ \bar{K}^- & \bar{K}^0 & -2\, 6^{-1/2} \eta \end{pmatrix}.$$

A question concerning the X_0 meson resonance of mass 958 MeV (sometimes it is also called η') remains open. This resonance lies well within the mass range of the other mesons of the 35^--plet and has similar properties as the η particle. It suggests that a reducible SU(6)$_\sigma$ multiplet of dimension 36, namely

$$35^- \oplus 1^-$$

plays a specific role in the SU(6) symmetry approach. Such representations result from certain dynamical models or from higher symmetry groups of the static type which generalize static $SU(6)_\sigma$, like

$$S[U(6) \otimes U(6)] .$$

The latter group has a representation

$$(1) \otimes (1^5)$$

of dimension 36 which decomposes into 35 and 1 under $SU(6)_\sigma$ if this group is defined as a particular subgroup (see Section 3).

1.3.3 Residual baryon resonances

There are baryon resonances of spin $^1/_2{}^-$, $^3/_2{}^-$ and $^5/_2{}^+$ which are not members of the 56-plet. It has been tried rather often to assign them to $SU(6)_\sigma$ supermultiplets. Particularly the $^3/_2{}^-$ has been studied. In connection with the $^3/_2{}^-$ resonances the 70^--plet has been investigated. This representation possesses the content (see Table 1 in the Appendix)

$$70 = (10,2) \oplus (8,4) \oplus (8,2) \oplus (1,2) .$$

Therefore $^3/_2{}^-$ resonances fit into this representation if they form an SU(3) octet (see Refs. [188, 301]. Such an octet has been proposed a long time ago as the γ-octet (see Ref. [175]). This γ-octet consists of

$$N(1518),\ Y_0^*(1520),\ Y_1^*(1650),\ \Xi^*(1816) .$$

But the situation is still rather unclear. The 70^--plet contains also a singlet of spin $^1/_2{}^-$ which could be identified with the resonance $Y_0^*(1405)$. (This resonance was denoted β-singlet in Ref. [175]).

We emphasize that evidence for assigning a certain set of phenomenological resonances to an $SU(6)_\sigma$ supermultiplet comes not only from the spin-parity properties of these resonances but mainly from the decays, partly also from mass relations (see Section 1.4 below). Since for higher resonances the spins are in general also big ($^3/_2$, $^5/_2$, etc.) two-particle final states of stable particles as the baryon octet and the pseudoscalar octet particles involve P-waves and higher orbital angular momenta. These decays are therefore not accessible to static $SU(6)_\sigma$ theory. Only the $^1/_2{}^-$ resonance of the 70^--plet can decay into an S-wave state formed of Σ and π. We shall come back to the vertex of the 70 decay in Section 1.5.

1.3.4 Residual meson resonances

Meson resonances of spin 0^-, 0^+, 1^+, 2^+ are known which do not belong to the 35^--plet discussed above. The X_0 meson of spin 0^- has already been treated in Section 1.3.3. It is usually thought to be a singlet of $SU(6)_\sigma$[6]. The 0^+ resonances are still quite obscure objects. Only the $\varkappa(730)$ is well defined, although recently some doubts arose concerning its existence. Another object with $I = 0$ has been observed in the

[6]) The X_0 may mix with the η, see Part II, Section 5.1.

missing mass spectrum of the neutron in a $\pi^- p$ reaction at about 700 MeV, see Refs. [117, 133]. Nevertheless the $I = 1$ part is missing and the $SU(6)_\sigma$ assignment is therefore still impossible. The situation with the 1^+ resonances is somewhat better. There exists with rather good evidence

$$D \text{ with } I = 0 \text{ at } 1285 \text{ MeV and}$$
$$A_1 \text{ with } I = 1 \text{ at } 1080 \text{ MeV.}$$

But there is still no complete SU(3) representation.

Finally we have the nonet of spin 2^+ resonances which was defined and investigated in Ref. [177]. This nonet consists of the resonances

$$\begin{array}{lll} f & \text{with } I = 0 & \text{at } 1254 \text{ MeV,} \\ f' & \text{with } I = 0 & \text{at } 1500 \text{ MeV,} \\ A_2 & \text{with } I = 1 & \text{at } 1290 \text{ MeV,} \\ K^{**} & \text{with } I = 1/2 & \text{at } 1405 \text{ MeV.} \end{array}$$

(The masses are taken from Ref. [325]).
There are some doubts concerning the uniqueness of A_2, but at least part of it seems to have the quantum numbers given here.

Two representations exist with a 2^+ nonet in the $SU(3) \otimes SU(2)_\sigma$ decomposition which have been considered as possible candidates: the representations of dimension 189 and 405. Their content is (see Table 1 in the Appendix)

$$189 = (8 \oplus 1{,}5) \oplus (8 \oplus 8 \oplus 10 \oplus \overline{10}{,}3) \oplus (27 \oplus 8 \oplus 1{,}1)\,,$$
$$405 = (27 \oplus 8 \oplus 1{,}5) \oplus (27 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10}{,}3) \oplus (27 \oplus 8 \oplus 1{,}1)\,.$$

They are both contained in the tensor decomposition of two 35^--plets

$$35 \times 35 = 1 \oplus 35 \oplus 35 \oplus 189 \oplus 280 \oplus \overline{280} \oplus 405$$

but only the representation 405 is contained in the baryon-antibaryon product

$$56 \times \overline{56} = 1 \oplus 35 \oplus 405 \oplus 2695\,.$$

The content of both representations differs only in the 27-plets of SU(3) and these are indeed the critical resonances. A resonance with $Y = 2$ which could be member of a 27-plet has so far been observed only once (see Ref. [136]). Provided it exists and belongs to 27 of SU(3) it could not distinguish between 189 and 405 since it has even spin and possibly spin 0. Therefore we have to discuss decays to get further information. But with respect to decays the same holds that we mentioned in the case of baryons: since the decays of the 2^+ resonances proceed mainly into states of two pseudoscalar mesons or one pseudoscalar and a vector meson, the final states are D-waves which we cannot handle with static $SU(6)_\sigma$ symmetry. (The problem has been discussed by means of the collinear group $SU(6)_W$ in Ref. [215]).

Further references on the SU(6) classification of particles and resonances: [66, 77, 88, 122, 126, 134, 183, 200, 320].

1.4 *Mass relations*

1.4.1 General problems in mass relations for $SU(6)_\sigma$

The motivation for setting up mass relations is the success which these relations had in the unitary symmetry scheme. Nevertheless, these mass relations of SU(3) have never been completely understood. The puzzling problem was why the additive terms in the mass operator which break the symmetry need only be taken to first order in a perturbative treatment, whereas the phenomenological mass breaking is so big that the validity of any perturbative treatment seems to be doubtful.

Mass relations are an implication of symmetry breaking. By definition the $SU(6)_\sigma$ symmetry is restricted to static systems. Breaking of $SU(6)_\sigma$ symmetry can therefore be due to two effects: one which shows up even in static systems and another one which reflects the fact that the systems is intrinsically non-static. An effect of the first kind can be accounted for by an additive term in the mass operator which is scalar with respect to $SU(2)_\sigma$.

Let us for the moment assume that the known strongly interacting particles and resonances can be considered as non-relativistic bound states of fundamental objects, quarks, which are themselves degenerate multiplets of $SU(6)_\sigma$. The $SU(6)_\sigma$ symmetry may be broken in the second manner if the orbital motion in this bound state does not vanish, and moreover, if a spin-orbit coupling is present in which the orbital angular momentum is treated as a scalar under $SU(6)_\sigma$. Such transformation property of the orbital angular momentum is in the spirit of Wigner's supermultiplet theory (Ref. [393]). The symmetry breaking term in the mass operator is then a vector under $SU(2)_\sigma$, the mass formula contains the invariants $J\ (J + 1)$, $L\ (L + 1)$ and $S\ (S + 1)$. We emphasize that such derivations of mass formulas are strongly model dependent, since only the parameter J is directly observable.

The extension of $SU(6)_\sigma$ to non-static systems in analogy to Wigner's model is not the only one possible. It may well be that the orbital angular momentum is itself transformed as a vector under $SU(2)_\sigma$. In such a case the spin-orbit coupling would lead to a scalar spurion. The kinetic energy part of the Hamiltonian would then also become a scalar spurion.

1.4.2 Mass relations for baryons and mesons

A systematic approach of dealing with the masses in $SU(6)_\sigma$ has been developed in Ref. [44]. Since we want a correspondence of the $SU(6)_\sigma$ mass relations with the analogous relations of SU(3), the symmetry breaking operators must have transformation properties under SU(3) of the trivial or the $T = Y = 0$ component of the adjoint representation. On the other hand they are ad hoc assumed to be scalar with respect to $SU(2)_\sigma$. Therefore we have either (8,1) or (1,1).

If a particle representation is given, the first order term in the perturbation series for the mass is simply the expectation value of the mass operator between the states of the representation and its complex conjugate. All those terms are included in the mass operator which are self-adjoint and have transformation properties that permit us to couple it to the product of the particle representation and its complex conjugate. For the baryons of the representation 56 there are contributions from

$$35,\ (8,1)\ .$$
$$405,\ (1,1) \oplus (8,1)$$

and from 2695, which have been neglected without justification (see Ref. [44]). Doing this we obtain the relation

$$M = M_0 + M_1\, S\,(S+1) + M_2\, Y + M_3 \left(I\,(I+1) - \frac{1}{4}\, Y^2\right).$$

The term involving $I\,(I+1)$ which is necessary for cancelling the degeneracy of Σ and Λ, comes from the part $(8,1)_{405}$. Besides the Gell-Mann-Okubo relations this equation yields one additional prediction: the distances in the decuplet (~ 145 MeV) can be predicted to be identical with those of the octet (~ 130 MeV). In the case of 35^--mesons there are contributions to the mass operator from

$$\begin{array}{ll} \text{two 35's}, & (8,1)\ , \\ 189, & (1,1) \oplus (8,1)\ , \\ 405, & (1,1) \oplus (8,1)\ . \end{array}$$

The argument for neglecting the representations 280 and $\overline{280}$ is that they are not self-adjoint.

The formula obtained with this ansatz is too general to give any useful information besides the Gell-Mann-Okubo relation for the ps-mesons. If we drop the octet term of 189 we get one additional relation which is well satisfied (see Ref. [44]). But it is impossible to drop in addition the singlet of 189.

The beautiful relation (see Ref. [301])

$$M(K^*)^2 - M(\varrho)^2 = M(K)^2 - M(\pi)^2$$

can be obtained assuming that the octets of 189 and 405 do not contribute, see Ref. [194].

1.4.3 Remarks

An attempt to obtain information about the 70^- by means of mass relations has also been made, see Ref. [45]. To get the mass relations with the ansatz explained in Section 1.4.2 methods are used which involve some information about Casimir operators of unphysical subgroups of $SU(6)_\sigma$. These operators have no simple physical meaning, (see the remark in section 1.2.5).

Further references on mass relations: [1, 81, 189, 203, 246].

References on the electromagnetic mass splitting: [79, 114, 121, 176, 247, 352, 383].

1.5 *Application of static $SU(6)_\sigma$ symmetry to S-matrix elements*

1.5.1 The technique to construct invariant forms

Due to the definition of $SU(6)_\sigma$ symmetry we can apply it only to a very limited number of problems:

a) Strong vertices which couple a fixed number of particles that are all at rest in an invariant manner; these invariant vertices imply relations only for the threshold of amplitudes.

b) Current matrix elements for electromagnetic and weak interactions, taken between single particle states of equal mass at rest.

In case b) a definite transformation property (spurion) of these currents has to be assumed. As a leading principle we can rely on similar assumptions made in the unitary symmetry scheme and on dynamic models (see the example of the electromagnetic current discussed extensively below). The necessity for introducing such spurions into the symmetry model arises, because the electromagnetic and the weak interactions violate the symmetry.

As an example of how invariants are constructed from representations of $SU(6)_\sigma$ we quote the vertex for the decay of the 70^--plet into 56^+ and 35^-. The product of 56^+ and 35^- decomposes as

$$56 \times 35 = 56 \oplus 70 \oplus 700 \oplus 1134$$

Since the 70-plet (tableau (21)) appears just once, the vertex is uniquely determined. In tensor notation we obtain

$$f \cdot \overline{G}^{[A\,B]\,C}\, B_{A\,C\,D}\, M^D_B\,,$$

where f is the coupling constant. This invariant can be used to find relations between the decay amplitudes for the processes

$$\frac{3^-}{2} \to \frac{1^+}{2} + 1^- \qquad S\text{-wave},$$

$$\frac{3^-}{2} \to \frac{3^+}{2} + 0^- \;(\text{or } 1^-) \quad S\text{-wave},$$

$$\frac{1^-}{2} \to \frac{1^+}{2} + 0^- \;(\text{or } 1^-) \quad S\text{-wave}.$$

The amplitudes must be taken in the limit where the velocity of the final particles is zero.

1.5.2 Strong interaction vertices, example: the $\overline{\text{B}}$BM vertex

Let us consider the decay of a spin $^3/_2{}^+$ resonance into spin $^1/_2{}^+$ baryons and 0^- mesons. Parity conservation implies that only P-waves may occur. The threshold behaviour is therefore trivial and we do not get any result. The same argument applies to the process [7])

$$\frac{1^+}{2} \to \frac{1^+}{2} + 0^- \;(\text{or } 1^-)$$

if we regard both 56-representations as different so that the masses can be chosen appropriately (positively real).

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[7]) See the discussion of this vertex in Section 3.2.3. We shall explain there how the well-known results for the D/F ratios at $q^2 = 0$ can be obtained. Here we state only that they are neither a consequence of static $SU(6)_\sigma$ nor of the collinear group $S[U(3) \otimes U(3)]$ in the strict sense. If we do not introduce spurions we must continue analytically in the representations.

The situation changes if we consider the annihilation channel $B\overline{B} \to M$. The meson mass must then be twice the baryon mass. The result can be expressed in terms of the vector coupling form factors a^F and a^D for the *ps*-octet and the charge and magnetic form factors of the Sachs type for the vector mesons a^F_m and a^D_m, a^F_c and a^D_c. At the threshold we have

$$a^F_c = a^F_m, \quad a^D_c = a^D_m$$

as a consequence of the definition of a_c and a_m. $SU(6)_\sigma$ implies

$$\begin{aligned} a^D &= 0, \\ 2\,a^D_c - 3\,a^F_c &= 0, \\ a^F + a^D_c &= 0. \end{aligned}$$

In Section 3.2.3 and Part II, Section 5.2 we will find that the second relation is independent of the meson mass.

1.5.3 Current matrix elements; example: the electromagnetic behaviour of baryons

The electromagnetic current is a very illustrative example to explain how $SU(6)_\sigma$ works and runs into troubles. The notion static is somewhat ambiguous for vertices which involve an external field. The magnetic moment of a nucleon is certainly a static property in the sense that it determines the energy of a nucleon at rest in a static homogeneous magnetic field. Nevertheless, we shall see that the definition of the spurion will bring in the non-static properties.

1.5.3.1 *The charge.* Only the static charge of the baryons is accessible to our treatment. In SU(3) the charge operator is taken proportional to the generator

$$\frac{1}{2}\,(\lambda_3 + 3^{-1/2}\,\lambda_8)$$

of the group SU(3). On the other hand, the charge distribution is assumed to transform as the $U = 1$, $Y = 0$ component of the octet. The reason for using the generators themselves for the charge instead of a tensor operator as for the charge distribution is that the conservation of the generators of the symmetry group implies automatically the conservation of charge. This argument suggests that we should take the charge operator proportional to the generator of the $SU(6)_\sigma$ group

$$\sigma_0\,\frac{1}{2}\,(\lambda_3 + 3^{-1/2}\,\lambda_8)\,.$$

Doing this we find a pure F-type coupling for the charge of the nucleons. This must be viewed upon simply as a proof of the consistency of our definitions.

1.5.3.2 *The magnetic moments.* In SU(3) the magnetic moments are defined to transform as a tensor operator belonging to the $U = 1$, $Y = 0$ component of the octet. This implies that the moments are composed of an F- and D-type part. They are not proportional to the charge.

To get a better feeling for what transformation properties could be expected in $SU(6)_\sigma$, we regard the physical particles again as bound states of some spin $^1/_2$

particles, quarks, with similar properties as the nuclei. Then we obtain a magnetic moment operator consisting of two parts, the spin contribution

$$M_s = \sum_k \frac{e_k\, g_k}{2\, M_k} \frac{1}{2} \boldsymbol{\sigma}_k ,$$

where e_k is an eigenvalue of the SU(3) charge operator, and a contribution from the convection

$$M_c = \frac{1}{2} \sum_k \boldsymbol{x}_k \times \boldsymbol{j}_{c,k} = \sum_k \frac{e_k}{2\, M_k} \boldsymbol{L}_k .$$

The term $\boldsymbol{j}_{c,k}$ is the contribution of particle k to the convection current. If SU(3) is good for quarks the Landé factors g_k should all be equal. The spin contribution reduces then to the operator

$$M_s = \frac{g}{2\, M} \left\{\frac{1}{2}\, \boldsymbol{\sigma}\, \frac{1}{2}\, (\lambda_3 + 3^{-1/2}\, \lambda_8)\right\} \text{ in the given representation}$$

which is one of the generators of $SU(6)_\sigma$.

If we neglect the convection term M_c and take only the spin contribution, we have an operator for the moment which is somewhat more restricted than the operator used in Ref. [46]. These authors assume that the magnetic moment is only a tensor operator of the representation 35 corresponding to the same component. Since for the baryons this makes no difference, because the representation 35 is coupled only once, all the results of Ref. [46] remain valid. Among these is the ratio for the magnetic moment of proton and neutron

$$\frac{\mu(p)}{\mu(n)} = -\frac{3}{2} .$$

The convection term is intrinsically non-static. Gell-Mann (Ref. [170]) has suggested that it should transform like the tensor product of the charge operator contained in 35 and the orbital angular momentum operator treated as the part (1,3) of 35. He proved that out of this product only the part (8,3) of 35 can influence the 56 representations[8]). The magnetic moment operator taken between two representations of dimension 56 has therefore the structure of the $U = 1$, $Y = 0$ component of (8,3) of 35, as it was assumed in Ref. [46] in general.

We emphasize again, that the treatment of the orbital angular momentum as a spurion in the static symmetry scheme is a new concept (in particular it deviates

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[8]) Gell-Mann (Ref. [170]) considers in fact not the convection term but the total magnetic moment and writes it

$$\boldsymbol{M} = \frac{1}{2} \int d^3x\, \boldsymbol{x} \times \boldsymbol{j}(\boldsymbol{x}) .$$

In a non-relativistic model the current can be decomposed into a convection term and a contribution from the magnetization

$$\boldsymbol{j}(x) = \boldsymbol{j}_c(x) + \boldsymbol{j}_m(x) .$$
$$\boldsymbol{j}_m(x) = \nabla \times \boldsymbol{M}(x) .$$

Introducing point particles and integrating by parts we obtain the expression for the magnetic moment used above. The tensor operator used by the authors of Ref. [46] is therefore the special case of Gell-Mann's operator if the convection can be neglected.

from the interpretation of $SU(6)_\sigma$ along the lines of Wigner's supermultiplet model) and means that we extend the domain of the static symmetry $SU(6)_\sigma$ to situations where it was not defined in the beginning.

1.5.4 Concluding remarks

The semileptonic vertex can be treated quite similarly. With similar simplifying assumptions as in Ref. [46] the authors of Ref. [47] obtain the weak D/F ratio

$$D/F = 3/2$$

and the value

$$G_A/G_V = -5/3 .$$

Further references on the static electromagnetic properties of baryons: [6, 58, 83, 199, 228, 236, 254, 352].

2. Extensions of static $SU(6)_\sigma$ symmetry

2.1 *Models defining an invariance against rotations of the spin of individual particles*

2.1.1 Introduction

We emphasize that there is no a priori need to extend static $SU(6)_\sigma$ symmetry to apply to non-static situations. It could well be that physics exhibits only the static symmetry. But even if experimental evidence were against a non-static extension, which can certainly only be decided after we know how an extension looks and what it predicts, it would be desirable to know how in non-static situations $SU(6)_\sigma$ symmetry is violated, this means: which spurions must be introduced. This twofold aim of investigating extensions of the static symmetry reflects itself in the models we are going to discuss. Indeed, we shall recognize later that some of the models (those dealt with in Section 3) reduce the symmetry group to a subgroup if instead of only static processes relative motions are permitted. Such models can be interpreted as broken symmetries with a recipe of handling this breaking by means of spurions. On the other hand the models discussed in Section 4 are real extensions, where the symmetry group does not depend on the kinematics. As was claimed in the original papers Refs. [185, 301, 353] static $SU(6)_\sigma$ is a group which couples spin independence with unitary symmetry. We could therefore think and this was the intention of Ref. [185], that relativistic notions of spin will supply us in a very simple manner with relativistic extensions of SU(6) symmetry (see Ref. [187]). In principle this is true, the difficulty arises, however, if we want to draw physically reasonable conclusions from such models. Before we go into details we shall repeat how spin can be defined relativistically.

2.1.2 Spin

From the point of view of Lorentz invariance the definition of spin is unique. The theory of representations of the Lorentz group (Ref. [394]), tells us that the spin operators are identical with the generators of the little group. They are contained in the four vector

$$W_\mu = \frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} p^\nu M^{\lambda\sigma} .$$

In this expression p^ν are the infinitesimal translation operators, $M^{\lambda\sigma}$ are the generators of the homogeneous Lorentz group. On the space spanned by state functions with positive timelike momenta we can define the unitary operator $U[\Lambda(p)]$ which represents the rotation free Lorentz transformation $\Lambda(p)$, $U(\Lambda(p)) \mid 0 > = \mid \boldsymbol{p} >$. Applying it to the vector operator W_μ, we obtain

$$U(\Lambda^{-1}(p)) W_\mu U(\Lambda(p)) = a^\nu_\mu(\Lambda^{-1}) W_\nu$$
$$= S_\mu m$$

with $S_0 = 0$ and

$$[S_i, S_k] = i \varepsilon_{ikl} S_l .$$

If we couple these operators S_i, $i = 1, 2, 3$, with the generators of SU(3) we obtain the Lie algebra of $SU(6)_\sigma$[9]).

If in addition to Lorentz invariance we know that a field exists which satisfies a certain field equation, we have additional choices for the definition of spin operators. From now on we refer to a spin-$^1/_2$ field which obeys the Dirac equation.

The Dirac equation deals simultaneously with states of positive and negative energy. This allows us to define spin rotation groups with different correlations of positive and negative energy contributions.

The study of these different possibilities of defining spin operators in Dirac theory serves as an introduction to the spin rotation group $SU(2)_W$ appearing later in the context of the collinear subgroups and clarifies the relation of this group to Foldy's definition of spin, $SU(2)_F$ (Ref. [140]). In addition all these groups $SU(2)_\sigma$, $SU(2)_W$ and $SU(2)_F$ can be used to define different models of spin independence of the type discussed in Section 2.1.3. In particular the use of the spin group $SU(2)_W$ in this context goes back to Gürsey (Ref. [187]).

Customarily one requires that the spin operators commute with the Hamiltonian of the Dirac field

$$H = \alpha_k p_k + \gamma_0 m , \quad \alpha_k = \gamma_0 \gamma_k .$$

This implies commutativity with the Lagrangian

$$L = \gamma_\mu p^\mu - \varepsilon m .$$

The discussion is simplified after a Foldy-Wouthuysen transformation has brought the Hamiltonian into diagonal form H'

$$H' = E \gamma_0 , \quad H = e^{iS} H' e^{-iS} , \quad E > 0 .$$

In this system the spin operators can be chosen out of the set of matrices commuting with γ_0,

$$\frac{1}{2} \sigma_k , \quad \frac{1}{2} \gamma_0 \sigma_k , \quad k = 1, 2, 3 .$$

[9]) We adopt this notation of the static symmetry group for this relativistically invariant group. This is only a matter of convention.

Out of this set of six operators we form two algebras SU(2), namely

$$\boldsymbol{\vartheta}_F^{(0)} = \boldsymbol{\sigma}_{/\!/} + \boldsymbol{\sigma}_{\perp}\,, \quad \text{called } \mathrm{SU}(2)_F \text{ (see Refs. [140, 371])}$$

and

$$\boldsymbol{\vartheta}_W^{(0)} = \boldsymbol{\sigma}_{/\!/} + \gamma_0\,\boldsymbol{\sigma}_{\perp}\,, \quad \text{called } \mathrm{SU}(2)_W \text{ (see Ref. [326])}\,.$$

Parallel and orthogonal refers here to the momentum. Since the FW-transform is not uniquely determined by the condition $H \underset{FW}{\rightarrow} H'$, we normalize it to be rotation free[10])

$$e^{-iS} = [2\,E\,(E+m)]^{-1/2}\,(E + m + \gamma_k\,p_k)\,.$$

Transforming back with this matrix into the system where the Hamiltonian was H, we obtain the algebras $\mathrm{SU}(2)_F$ and $\mathrm{SU}(2)_W$ in the form of vector operators $^1/_2\,\boldsymbol{\vartheta}$,

$$\boldsymbol{\vartheta}_F = \boldsymbol{\sigma}_{/\!/} + \varepsilon\,\gamma_0\,\boldsymbol{\sigma}_{\perp}\,, \quad \boldsymbol{\vartheta}_W = \boldsymbol{\sigma}_{/\!/} + \gamma_0\,\boldsymbol{\sigma}_{\perp}\,.$$

Thus we recognize that Foldy's spin operators are the Foldy-Wouthuysen transforms of operators $^1/_2\,\sigma_k$, whereas Wigner's operators are Lorentz transforms of the operators $^1/_2\,\sigma_k$. If applied to particles at rest both sets coincide. The W spin, however, is different from Wigner's and Foldy's spin even in the rest system: the transversal operators appear with a different sign in the case of particles and antiparticles. The group $\mathrm{SU}(2)_W$ has recently been rediscovered in investigations of relativistic SU(6) models, see Section 2.3 and Refs. [38, 261]).

2.1.3 Field theoretic models using the groups $\mathrm{SU}(6)_\sigma$, $\mathrm{SU}(6)_F$ and $\mathrm{SU}(6)_W$

The idea underlying these models was first put forward by GÜRSEY, Ref. [187]. We formulate it as follows.

Let us consider a given system of N particles which bear indices $\alpha = 1, 2, \ldots, N$, and have momenta p_α. We introduce spin rotation groups $\mathrm{SU}(2)_\sigma^{\alpha, p_\alpha}$ for each particle α and a "mother" group SU(2). Each of the N groups $\mathrm{SU}(2)_\sigma^{\alpha, p_\alpha}$ is connected with the mother by an isomorphism $I(\alpha, p_\alpha)$,

$$\mathrm{SU}(2) \underset{I(\alpha, p_\alpha)}{\longrightarrow} \mathrm{SU}(2)^{\alpha, p_\alpha}\,,$$

such that a rotation belonging to the mother group induces an N-tupel of in general different rotations in the daughter groups. Such N-tupels of rotations are assumed to leave the dynamics unchanged. Obviously we can define an SU(6) symmetry in an analogous fashion.

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[10]) Let the Dirac equation be

$$(p_\mu\,\gamma^\mu - \varepsilon\,m)\,\psi = 0,\ p_0 = E > 0,\ \varepsilon = \frac{H}{E}\,.$$

Then we can define three different but related types of matrices:

1) the projection operator P, $P^2 = P$, $P^+ = \gamma_0\,P\,\gamma_0$,

$$P = (2\,m)^{-1}\,(p_\mu\,\gamma^\mu + \varepsilon\,m)\,,$$

2) the Hermitian, rotation free Lorentz transformations $\Lambda(\boldsymbol{p})$, $\Lambda(\boldsymbol{p})^{-1} = \Lambda(-\boldsymbol{p})$,

$$\Lambda = [2\,m\,(E+m)]^{-1/2}\,(p_\mu\,\gamma^\mu\,\gamma_0 + m)\,,$$

3) the unitary, rotation free FW-transformations

$$e^{-iS} = [2\,m\,(E+m)]^{-1/2}\,(E + m + \gamma_k\,p_k)\,.$$

Because such models make a principal distinction between spin and orbital angular momentum, the definition of the symmetry has to specify what is to be considered as an elementary particle bearing a spin and what as a system of particles carrying a total angular momentum. This is done by basing the models on the theory of interacting quantized fields. A set of quark fields is introduced which interact strongly, the physical particles and resonances are interpreted as bound states formed out of quarks. If the quarks are Dirac particles we may construct different models depending on whether we use the spin group $SU(2)_\sigma$ or the groups $SU(2)_F$ and $SU(2)_W$. Indeed, most of the authors prefer the latter groups (Refs. [187, 269]).

It is obvious that no definite predictions on coupling constants etc. are at present possible in such models because of the lack of appropriate computation techniques. Our further arguments in this Section are therefore limited to a criticism of the models as a whole.

Let us make the following points which we shall discuss below.

i) The isomorphisms $I(\alpha, p_\alpha)$ have to be defined in a manner compatible with Lorentz invariance.

ii) The model is defined only for a system of particles which all have time-like momenta.

iii) The symmetry has been defined on separate subspaces $H_{n,\bar{n}}$ of the physical Hilbert space which belong to fixed quark number n and antiquark number $\bar{n}$.

Compatibility with Lorentz invariance means that the isomorphisms $I(\alpha, p_\alpha)$ are submitted to the following minimal condition. We may eliminate the mother group and obtain isomorphisms between each pair of daughter groups:

$$SU(2)^{\alpha, p_\alpha} \xrightarrow[I(\beta, p_\beta)\, I(\alpha, p_\alpha)^{-1}]{} SU(2)^{\beta, p_\beta} .$$

Such an isomorphism is then required to be independent of the Lorentz frame

$$I(\alpha, p_\alpha)\, I(\beta, p_\beta)^{-1} = I(\alpha, p'_\alpha)\, I(\beta, p'_\beta)^{-1} ,$$

where p_α, p_β go into p'_α, p'_β via the same Lorentz transformation (rotations included).

Such a condition is satisfied if we generate the daughter groups out of the mother group by Lorentz transformations: (in formal notation)

$$SU(2)^{\alpha, p_\alpha} = \Lambda_{p'_\alpha \to p_\alpha} (SU(2) \text{ mother}) \text{ for all } \alpha .$$

Here $\Lambda_{p'_\alpha \to p_\alpha}$ denotes the pure Lorentz transformation which transforms the momentum p'_α into p_α, and p'_α is the momentum of the particle α in the centre of momentum frame of the whole N-particle system.

Such a definition violates, however, the condition of "separability". The notion "separability" is the same as known from any theory involving clusters of particles: two subsystems which are separated by a spacelike distance which tends to infinity do not influence each other. Such a property cannot hold for clusters the dynamics of which are coupled to the centre-of-mass momentum through the symmetry. Compatibility with Lorentz invariance as formulated above is therefore in contradiction with separability.

If the compatibility condition is not satisfied (most of the models proposed are of this type) the spin group depends on the frame of reference in which it is defined. In order to maintain Lorentz invariance we have then to postulate invariance against the whole continuous manifold of spin groups. As proved in Ref. [227] this implies that all S-matrix elements taken between quark states the momenta of which are non-collinear (in all Lorentz frames) must vanish identically.

Statement ii) and iii) imply that the field theory (if it exists at all) will have some very unusual properties. The number of quarks and antiquarks must be independently conserved by any symmetric operator. This implies that the field theory is non-local (as was first observed in Refs. [269, 316]). Virtual particles do not appear.

Further reference on field models: [307].

2.2 *Classification of groups which combine Lorentz invariance with internal symmetry*

2.2.1 Introduction

The rotational subgroup SO(3) of the homogeneous Lorentz group L operates on a particle at rest exactly in the same way as the spin rotation group $SU(2)_\sigma$ contained in the group $SU(6)_\sigma$. If the particle moves with momentum p, the operator

$$\Lambda(p')\, R\, \Lambda^{-1}(p)\,, \quad R \in SU(2)_\sigma\,,$$

(where $\Lambda(p)$ transforms a particle at rest into a state with four momentum p) still coincides with WIGNER's operator

$$\Lambda(p')\, \tilde{R}\, \Lambda^{-1}(p)\,, \quad \tilde{R} \in SO(3) \subset L\,, \quad R \cong \tilde{R}\,,$$

which belongs to the homogeneous Lorentz group as long as we apply it to this one-particle state of momentum p only. The problem arises of investigating whether it is possible to ascribe sense to products of operators of the inhomogeneous Lorentz group and the group $SU(6)_\sigma$ in general. In other words, if a group G exists satisfying the following conditions:

a) it contains the inhomogeneous Lorentz group P as a subgroup,

b) it includes as a subgroup the group SU(6), which plays the role of an internal symmetry group,

c) the subgroup SU(2) of SU(6) if applied to systems of particles which are all at rest coincides with the group of rotations SO(3) which is a subgroup of the homogeneous Lorentz group.

Properties a) and b) concern the group structure of G; condition c) guarantees the correspondence with static $SU(6)_\sigma$ symmetry, it involves a physical interpretation of the representations of the group G.

We discuss in the following paragraphs groups, which satisfy conditions a) and b). We shall prove in Section 2.3 and Sections 3 and 4 that condition c) is fulfilled. Historically, the problem of finding an enlargement of the inhomogeneous Lorentz group which includes a general semi-simple Lie group as internal symmetry group is older. We treat first this more general problem and specify the internal symmetry group later. We shall refer always to Lie algebras instead of Lie groups but use the same symbols for them[11]). These algebras are restricted to be of finite dimension. We prefer the language of Lie algebras in this context, because we are going to quote some theorems contained in textbooks about Lie algebras. No physical reason is known which favours algebras (see Section 1.1.1).

Let us formulate the problem once more. We investigate Lie algebras G which contain the algebra P of the inhomogeneous Lorentz group and any semi-simple algebra S as subalgebras. We try to classify all these algebras G. The results we shall obtain are due to O'Raifeartaigh, Refs. [308, 309], they involve some earlier propositions made by Mc. Glinn, Michel and others, Refs. [94, 139, 181, 191, 274, 275, 278, 300, 373]. At the end we introduce the algebra SU(6) for S and give some definite examples for the algebra G which will be discussed thoroughly in the sequel.

2.2.2 Some mathematical results

Following the lines of Ref. [309] we make use of some theorems about Lie algebras which can be taken from the mathematical literature (see Ref. [224]).

Levi's theorem:

If G is a finite dimensional Lie algebra with radical R then there exists a semi-simple subalgebra F of G such that

$$G = F \times R .$$

(see the proof in Ref. [224], page 86).

Let us explain what this theorem means and where the problem of the proof lies. For an invariant subalgebra ("ideal") H of G we can form the chain of commutator subalgebras ("derived series"). Each member in the series is an ideal of G and H. If the series terminates at a fixed n, i.e. $H^{(n)} = 0$, H is called a solvable ideal, in this case $H^{(n-1)}$ must be abelian. The intersection and the sum of two solvable ideals give again a solvable ideal. The latter property allows us to take the union of all solvable ideals, this is a maximal solvable ideal and is denoted radical. If the radical of a Lie algebra vanishes, this Lie algebra is called semi-simple. The radical is obviously uniquely determined.

The factor algebra with respect to the radical

$$\frac{G}{R} = \bar{G}$$

is semi-simple by construction. It is in general not a subalgebra of G. Indeed, we decompose the vector space of G into the direct sum (which is non-unique)

$$G = F_\sigma \oplus R .$$

[11]) We use the following product signs for groups and correspondingly for Lie algebras,

$A \otimes B$:	direct product,	$[A, B] = 0$,
$A \times B$:	semidirect product,	$[A, B] \subset B$,

and the sum signs for vector spaces without reference to additional properties of these spaces as Lie algebras: $A \oplus B$ direct sum.

This decomposition induces a mapping σ of $\bar{G}$ onto F_σ with the property that, if

$$\bar{g} \in \bar{G}, \quad \bar{g} \to \sigma(\bar{g}) \in F_\sigma ,$$

we obtain after the mapping by the factor homomorphism

$$\overline{\sigma(g)} = \bar{g} .$$

This yields in general

$$S(\bar{g}_1, \bar{g}_2) \equiv [\sigma(\bar{g}_1), \sigma(\bar{g}_2)] - \sigma([\bar{g}_1, \bar{g}_2]) \in R .$$

The problem is to find a space F_σ with the property that $S(\bar{g}_1, \bar{g}_2) \equiv 0$. This is then equivalent to $F = F_\sigma$ being a subalgebra.

Levi's theorem states that F, the Levi factor, can be found. This factor F is, however, not unique. Indeed, F will be changed in general if we submit G to an inner automorphism. The radical R remains unchanged (as any ideal). If any semi-simple subalgebra $\tilde{F} < G$ is given, we can always find an inner automorphism such that F goes over into F' which contains $\tilde{F}$. This is the statement (somewhat weakened) of the theorem of Malcev and Harish-Chandra (see Ref. [224], p. 92).

2.2.3 The classification of groups

The homogeneous Lorentz group L is semi-simple. With the help of the theorem of Malcev and Harish-Chandra we bring its algebra into the Levi factor F. The radical R is invariant with respect to F, this implies its invariance with respect to L. The translations T_4 are also invariant with respect to L. The intersection of R and T_4, $R \cap T_4$, is invariant under L. But T_4 is irreducible under transformations induced by L. Thus we have either

$$R \cap T_4 = T_4 \quad \text{or} \quad R \cap T_4 = 0 .$$

We can therefore distinguish between the four possibilities (see Ref. [309])

$$\text{a)}\ L \subset F , \quad R = T_4 , \quad \text{b)}\ L \subset F , \quad R \supsetneqq T_4 , \quad R \text{ abelian} ,$$

$$\text{c)}\ L \subset F , \quad R \supset T_4 , \quad R \text{ non-abelian} , \quad \text{d)}\ L \subset F , \quad R \cap T_4 = 0 .$$

We discuss these four cases one after the other.

In case a) the four-dimensional algebra of translations is invariant with respect to transformations of F. It induces a fourdimensional representation of F, which is the direct sum in the algebra sense of simple algebras. Only those simple algebras are allowed among which one has a four-dimensional representation. Starting from this argument it has been proved in Ref. [309] that the Levi factor F is a direct sum

$$F = F_0 \oplus F_r ,$$

where the complex extension $\hat{F}_0$ of F_0 is either identical with Cartan's algebras $A_1 \otimes A_1$, A_3 or B_2. The factor F_r commutes with the translations, $[F_r, T_4] = 0$. The algebra of the physical Lorentz group L must then either belong to F_0 alone or have orthogonal projections on both F_0 and F_r,

$$P_{F_0}(L) = L_0 , \quad P_{F_r}(L) = L_r .$$

Here both components L_0 and L_r are separately isomorphic to the algebra L. Therefore F_0 can be equal to SO(3,1), SO(4,1), SO(3,2), SU(2,2), SU(3,1), SO(5,1), SO(3,3). If it is bigger than SO(3,1) it will be difficult to interpret. Let us assume therefore that

$$F_0 = L_0 .$$

We shall later sometimes call the part L_r, which commutes with the translations, the spin rotation part of L.

If L_r is empty we have obtained an algebra of the trivial type (concerning physical applications)

$$(F_0 \times T_4) \otimes F_r = P \otimes F_r .$$

In general we have, however, $L_r \neq 0$.

We now bring the algebra SU(6) into play. It can only be contained in F_r. According to the postulate that the spin rotation subalgebra SU(2) of SU(6) should be related to the subalgebra SO(3) of L, we identify SU(2) with the part SO(3) of L_r. This poses the problem of determining an algebra F_r, which satisfies

$$\left.\begin{array}{l} F_r \supset \mathrm{SU}(6) \quad \supset \mathrm{SU}(2) \\ F_r \supset \mathrm{SL}(2, \mathrm{C}) \supset \mathrm{SU}(2) \end{array}\right\} \text{identical} .$$

In addition we assume now that F_r is simple. The smallest candidate for such an algebra is

$$F_r = \mathrm{SL}(6, C)$$

other possibilities are

$$F_r = \mathrm{Sp}(6, 6) , \quad F_r = \mathrm{SU}(6, 6) \text{ etc.}$$

With these solutions for F_r the complete algebra looks

$$(L_0 \times T_4) \otimes \mathrm{SL}(6, \mathrm{C}) \equiv (L \times \mathrm{SL}(6, \mathrm{C})) \times T_4 = P \times \mathrm{SL}(6, \mathrm{C}) ,$$

and correspondingly for the other algebras F_r. Such models which have first been proposed in Ref. [73] will be dealt with in Section 4.

In case b) the radical R is an abelian Lie algebra of more than four dimensions. We may assume that it is irreducible under transformations of F. Otherwise we take that part which contains T_4 and is irreducible, and we are back in case a) or have an irreducible space belonging to case b). The Levi factor F is a direct sum

$$F = \prod_k \otimes F_k$$

and the question is how L and SU(6) are distributed over these constituents F_k. In general we have orthogonal projections

$$P_{F_k}(L) = L_k , \quad P_{F_k}(\mathrm{SU}(6)) = [\mathrm{SU}(6)]_k ,$$

where some of the L_k and $[\mathrm{SU}(6)]_k$ may be empty. In any case there must be one index k, such that L_k and $[\mathrm{SU}(6)]_k$ are not empty, since otherwise SU(6) would commute with L. As a simple argument shows, this implies that the dimension of R is at least 36 (see Ref. [333]). The simplest assumption we can make is that F itself is simple. Then we identify

$$F \supset SU(6) \supset SU(2) \equiv SO(3) \subset L .$$

Examples for such algebras are

$$F = SL(6, C) , \quad F = Sp(6, 6) , \quad F = SU(6, 6) ,$$
$$R = T_{36} \text{ or } T_{400} , \quad R = T_{78} , \quad R = T_{143} .$$

These models are treated in Section 3. They have been proposed by many authors, (Refs. [27, 69, 152, 208, 230, 286, 338] on SL(6, C) and Refs. [50, 51, 355, 356, 357, 358] on SU(6,6)). The additional translations introduced are difficult to interpret physically. We must restrict them by subsidiary conditions imposed on their spectra. These conditions are straightforward in the physical picture. They lie, however, outside the group approach and may lead to contradictions with physical principles (see the discussion in Section 3).

No use has so far been made of models belonging to class c). The theory of representations for such algebras is already quite complicated. Even in the mathematical literature few explicit examples for representations of such groups are known. (Unitary representations of R are necessarily infinite dimensional!)

Case d) is used in so-called dynamical group models. They need also restrictions on the spectra of the operators. No model is at present so far developed that definite predictions could be made. The difficulties seem also to be technical. These dynamical groups lie outside the scope of this article. An example for a simple algebra which contains the inhomogeneous Lorentz algebra and the algebra SU(6) is SU(6,6). This algebra contains even SL(6, C) $\times$ T_{36} as a subalgebra.

Further references on the combination of Poincaré invariance with internal symmetry groups: [20, 42, 56, 96, 155, 174, 240, 259, 279, 280, 340].

2.3 *Subgroup chains for SU(6) and S[U(6) ⊗ U(6)]*

2.3.1 Formulation of the problem

The examples of group models belonging to the classes a) and b) to which we shall refer solely from now on, have some common features. First they all involve groups of the type SL(6, C), Sp(6,6) or SU(6,6). For historical reasons we shall neglect the group Sp(6,6), it can be treated quite analogously. We begin with a parametrization of these groups.

The group SU(6,6) can be generated by a set of 143 matrices which can be thought to be Kronecker products of Dirac's matrices Γ_n, $n = 1, 2, \ldots 16$ the phases of which are adjusted to satisfy the symmetry requirement

$$\gamma_0 \Gamma_n \gamma_0 = \Gamma_n^{\dagger} ,$$

and of the nine Hermitian matrices λ_i, $i = 0, 1, \ldots, 8$ known to generate U(3). We write

$$\Gamma_A = \Gamma_n \lambda_i , \quad A = 1, 2 \ldots 143 , \qquad \Gamma_{144} = 1 \lambda_0 \text{ is left out} .$$

We use the symbols λ_i also for the 12×12 matrices $\Gamma_A = 1 \cdot \lambda_i$ and the symbols γ_μ for $\Gamma_A = \gamma_\mu \cdot 1$. If we refer to a representation of these matrices Γ_A we have always (if no different statement is made) WEYL's representation in mind.

$$\gamma_0 = \begin{pmatrix} 0 & 1_{(6)} \\ 1_{(6)} & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} i\,1_{(6)} & 0 \\ 0 & -i\,1_{(6)} \end{pmatrix}, \qquad \gamma_k = \begin{pmatrix} 0 & \sigma_k 1_{(3)} \\ -\sigma_k 1_{(3)} & 0 \end{pmatrix}$$

and the familiar representation of the matrices λ_i.

The elements of the group SU(6,6) satisfy then the pseudounitarity condition

$$U^\dagger = \gamma_0\, U^{-1}\, \gamma_0 \,.$$

We can define subgroups of SU(6,6) by imposing certain additional conditions on the generators Γ_A. Let us require

$$\Gamma_A\, \gamma_5 = \gamma_5\, \Gamma_A \,,$$

or in terms of the group elements

$$U\, \gamma_5 = \gamma_5\, U \,.$$

Out of the set of Dirac matrices Γ_n only the following matrices remain

$$1, \gamma_5 \quad \text{and} \quad \sigma_{\mu\nu} = \frac{1}{2i}\, [\gamma_\mu, \gamma_\nu] \,.$$

The matrices $\sigma_{\mu\nu}$ span the algebra of SL(2,C). In matrix notation we infer from

$$U = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix},$$

$$\begin{pmatrix} i\,1_{(6)} & 0 \\ 0 & -i\,1_{(6)} \end{pmatrix} \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix} = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix} \begin{pmatrix} i\,1_{(6)} & 0 \\ 0 & -i\,1_{(6)} \end{pmatrix}$$

that

$$\mu_{12} = \mu_{21} = 0 \,.$$

The pseudo-unitarity of U implies in addition

$$\mu_{11}^\dagger = \mu_{22}^{-1} \,.$$

The matrix U has therefore the final form

$$U = \begin{pmatrix} S & 0 \\ 0 & (S^\dagger)^{-1} \end{pmatrix}.$$

These matrices constitute the group GL(6,C)/U(1)[12]). The algebra of SL(6,C) can therefore be spanned by the Kronecker products

$$s^t_{\mu\nu,i} = \sigma_{\mu\nu}\, \lambda_i \,, \; i = 0, 1 \ldots 8 \,,$$

$$\left.\begin{array}{l} s_i^p = \gamma_5\, \lambda_i \\ s_i^s = \lambda_i \end{array}\right| \; i = 1, 2 \ldots 8 \,.$$

[12]) From $\det U = 1 = \det S \cdot (\det S^+)^{-1}$ it follows that $\det S$ is a real number.

Let the generators of the Lorentz group be denoted as $M_{\mu\nu}$. Then for all the models the following commutation relations hold

$$[M_{\mu\nu}, s_i^s] = [M_{\mu\nu}, s_i^p] = 0 .$$
$$[M_{\mu\nu}, s_{\varkappa\varrho,i}^t] = - i \, (g_{\mu\varkappa} \, s_{\nu\varrho,i}^t + g_{\nu\varrho} \, s_{\mu\varkappa,i}^t - g_{\mu\varrho} \, s_{\nu\varkappa,i}^t - g_{\nu\varkappa} \, s_{\mu\varrho,i}^t) .$$

These relations express that the generators of SL(6, C) transform as scalars or tensors of rank two under the Lorentz group. In the class b) the operators $M_{\mu\nu}$ are moreover identical up to a factor with $s_{\mu\nu,0}^t$. But we abstract from this property and consider only these commutation relations in this Section 2.3. If the models make use of the group SU(6,6), we have in addition one operator s_0^p which is proportional to γ_5 and seventy-two operators

$$s_{\mu,i}^v = \gamma_\mu \, \lambda_i , \qquad s_{\mu,i}^a = i \, \gamma_5 \, \gamma_\mu \, \lambda_i ,$$

where the subscript i runs from 0 to 8. These transform like vectors under the Lorentz group.

We are going to investigate in this Section 2.3 the consequences of these commutation relations. We shall find that the tensor operator properties, which are obviously more general than the detailed structure of the groups, have far reaching implications. This forces us on the other hand to motivate these transformation properties with more physical arguments and independently of the formal deductions of Section 2.2. Indeed, in these abstract deductions the coupling of SU(6) to the Lorentz group seemed to be based on some ad hoc assumptions. Our aim is also to gain a deeper understanding of what we did there.

2.3.2 The spin operator algebra

In the static $SU(6)_\sigma$ theory the spin operators are identified with the generators of $SU(2)_\sigma$ (see Section 1.1.2)

$$s_k = \frac{1}{2} \, \sigma_k , \quad k = 1, 2, 3 .$$

Under rotations of the subgroup SO(3) of the homogeneous Lorentz group L they transform as a three-vector operator

$$[M_k, s_l] = i \, \varepsilon_{klm} \, s_m .$$

How can this transformation behaviour be generalized to relativistic situations? Spin should transform then as a Lorentz tensor (see Ref. [73]). We require in addition that the spin operators themselves form a Lie algebra. This is characteristic for the approach discussed here which is opposite to models based on fundamental fields as dealt with in Section 2.1. This requirement rules out such vector operators as W_μ, since its components do not form a closed algebra. Most natural is the use of the algebra SL(2, C) with the commutation relations

$$[s_{\mu\nu}, s_{\sigma\varrho}] = - i \, (g_{\mu\sigma} \, s_{\nu\varrho} + g_{\nu\varrho} \, s_{\mu\sigma} - g_{\mu\varrho} \, s_{\nu\sigma} - g_{\nu\sigma} \, s_{\mu\varrho}) .$$

They form a tensor of rank two under transformations of the homogeneous Lorentz group

$$[M_{\mu\nu}, s_{\sigma\varrho}] = - i \, (g_{\mu\sigma} \, s_{\nu\varrho} + g_{\nu\varrho} \, s_{\mu\sigma} - g_{\mu\varrho} \, s_{\nu\sigma} - g_{\nu\sigma} \, s_{\mu\varrho}) .$$

We neglect the translation behaviour completely, because we want to deal with later theories with subsidiary conditions.

If we couple now the generators of SL(2,C) with the generators of unitary symmetry, by extending the compact part SU(2) of SU(6) into the complex domain, we obtain the algebra of SL(6,C) with the operators

$$s^t_{\mu\nu,i},\ s^s_i,\ s^p_i .$$

Their transformation properties are expressed by the commutation relations of the type given in Section 2.3.1.

2.3.3 Implications of the tensor operator behaviour of the SL(6,C) algebra

In this paragraph we shall use some heuristic notions and arguments which have a physical background; we assume that particles with the familiar invariant properties (mass and spin) exist and that by a pure Lorentz transformation moving particles with nonvanishing mass can be brought to rest. We should be able to prove that these notions are reasonable if a definite group model is considered. Let us consider a particle at rest. This state of motion is unchanged, if we apply to it the elements of the subgroup SO(3) of the homogeneous Lorentz group. The algebra spanned by

$$s^t_{kl,i}, \quad k, l = 1, 2, 3, \quad i = 0, 1 \ldots 8, \quad s^s_i, \quad i = 1, 2 \ldots 8$$

is invariant under transformations of SO(3) and is itself a closed Lie algebra belonging to SU(6). We can therefore define a symmetry of the type $SU(6)_\sigma$ by means of this algebra, if we assume that the particles form degenerate SU(6) multiplets, that their state of motion is unchanged if these compact operators of SL(6,C) are applied, and that the multiplets span representation spaces for the compact algebra SU(6). The dynamics can then be submitted to an $SU(6)_\sigma$ symmetry as long as the condition is fulfilled that all particles participating in the process are at rest.

Having fixed the meaning of the compact part of SL(6,C) in this fashion, the remaining arguments are straightforward. Let us assume that we have a kinematical situation with all particles moving in the direction of one spatial axis, say the third axis, with arbitrary velocities. We can create these moving states by applying finite transformations generated by the element M_{03}. Those generators of $SU(6)_\sigma$ which commute with M_{03} will still operate between states of the multiplets without changing the state of motion. They form a subalgebra of $SU(6)_\sigma$ as can be inspected from the commutation relations. It can be used to deduce restrictions on S-matrix elements taken between states which are eigenstates of the momentum operator and for which the momenta are collinear in one Lorentz frame. This subalgebra can be spanned by

$$s^t_{12,i}, \quad s^s_i .$$

If we use the linear combinations

$$\frac{1}{2}\,(s^s_i \pm s^t_{12,i}), \quad i = 1, 2 \ldots 8$$

and

$$s^t_{12,0}$$

we see that this algebra creates the group product

$$SU(3)_+ \otimes SU(3)_- \otimes U(1) \text{ helicity}$$ [13])

or in other terms

$$S\,[U(3)_+ \otimes U(3)_-]\,.$$

If the particles move in a plane, or are even more generally moving, only the eight generators

$$s_i^s$$

are left; they generate SU(3), the unitary symmetry group. The three groups $SU(6)_\sigma$, $S[U(3)_+ \otimes U(3)_-]$ and SU(3) are denoted the static, collinear and coplanar subgroup of SU(6); all together they form the so-called subgroup chain. The decomposition of the representation of $SU(6)_\sigma$ into those of the collinear and coplanar group can be performed with familiar techniques (see Section 1.2.4 and the Tables given in Appendix). The collinear subgroup was first discussed in Ref. [384].

We emphasize that invariance of the dynamics under the collinear and coplanar subgroups does not follow in general from the invariance under the static group, but must be postulated independently. This seems to be obvious for physical reasons.

2.3.4 Embedding SL(6,C) into SU(6,6)

We consider now SL(6,C) as the subgroup of SU(6,6), as was obtained in Section 2.3.1. The subgroup chains are different for SL(6,C) and SU(6,6). We start with the compact subgroup $S[U(6)_+ \otimes U(6)_-]$ of SU(6,6). We assume that physical particles can be ordered into degenerate multiplets belonging to this group. For this we need the invariance of this compact subgroup against transformations of the rotation group. Let the generators of $S[U(6)_+ \otimes U(6)_-]$ be 12×12 matrices. They are determined then by the condition

$$\gamma_0\,\Gamma_A = \Gamma_A\,\gamma_0\,.$$

They are[14])

$$\left.\begin{array}{ll} \sigma_{kl}\,\lambda_i \quad \text{and} & \gamma_0\,\sigma_{kl}\,\lambda_i\,,\ \ k,l = 1,2,3 \\ \quad\gamma_0\,\lambda_i\,, & \end{array}\right\}\ i = 0, 1, \dots 8\,,$$

$$\lambda_i\,,\ \ i = 1, 2, \dots 8\,.$$

In the form

$$\frac{1}{2}\,(1 \pm \gamma_0)\,\sigma_{kl}\,\lambda_i\,,\ \ i = 0, 1, \dots 8\,,$$

$$\frac{1}{2}\,(1 \pm \gamma_0)\,\lambda_i\,,\qquad i = 1, 2, \dots 8\,,$$

$$\gamma_0\,\lambda_0\,,$$

they generate the direct product of the groups $SU(6)_+$, $SU(6)_-$ and $U(1)_{\gamma 0}$. They are indeed invariant against rotations.

---

[13]) In this article we also use the notion "helicity" for the sake of brevity, in cases where we correctly should use "component of angular momentum in a fixed direction".

[14]) For the sake of simplicity we drop normalizing factors in front of the generators throughout this Section 2.3.

The collinear subgroup is obtained as that part of this algebra which commutes with M_{03}. From the commutation relations we find as the set which satisfies this requirement

$$\sigma_{12}\lambda_i, \ \gamma_0\sigma_{31}\lambda_i, \ \gamma_0\sigma_{23}\lambda_i, \ i = 0, 1, \ldots 8,$$
$$\lambda_i, \ i = 1, 2 \ldots 8.$$

They create a group of the structure SU(6). The SU(2) subgroup of this group is seen to coincide with the spin group $SU(2)_W$ discussed in Section 2.1.2. The collinear group is accordingly named $SU(6)_W$.

Reactions in a plane spanned by the first and third axis can be submitted to a symmetry algebra which commutes with M_{03} and M_{01}. Its generators can be expressed by

$$\gamma_0\sigma_{31}\lambda_i, \ i = 0, 1 \ldots 8,$$
$$\lambda_i, \ i = 1, 2 \ldots 8.$$

The group has the structure $S[U(3)_+ \otimes U(3)_-]$ where the $SU(3)_+$ parts are created by the linear combinations

$$\frac{1}{2}(1 \pm \gamma_0\sigma_{31})\lambda_i, \ i = 1, 2 \ldots 8.$$

For general kinematics we obtain again the unitary symmetry. At least processes with five particles are necessary for non-coplanar kinematics. The theory of representations for this chain is no longer trivial. We shall discuss it in the following paragraph. The collinear group as a subgroup of SU(6,6) was first introduced in Ref. [261], see also Ref. [264], the coplanar group in Ref. [103].

2.3.5 Representations of the group $S[U(6)_+ \otimes U(6)_-]$ and their reduction into representations of the collinear group

The representations of $S[U(6)_+ \otimes U(6)_-]$ are tensor products of representations of the groups $SU(6)_+$, $SU(6)_-$ and $U(1)_{\gamma 0}$. They can therefore be described by two Young tableaux with blocks $\boxed{+}$ and $\boxed{-}$ and by an integral number N. The fundamental (quark) representations are usually denoted as (N is still arbitrary, see below)

quark, (6,1) or $\boxed{+}$,

pseudoquark, (1,6) or $\boxed{-}$,

antipseudoquark, $(\bar{6},1)$ or $\begin{array}{|c|}\hline + \\ \hline + \\ \hline + \\ \hline + \\ \hline + \\ \hline \end{array}$,

antiquark, $(1,\bar{6})$ or $\begin{array}{|c|}\hline - \\ \hline - \\ \hline - \\ \hline - \\ \hline - \\ \hline \end{array}$.

Let us study their relations. In general the transition from one representation to its contragredient one is mediated by the substitution of the generators

$$A \to -A^T .$$

In this way we can obtain the representation $(\bar{6},1)$,-N from the representation (6,1),N:

$$\left.\begin{array}{l} \frac{1}{2}(1+\gamma_0)\,\sigma_{kl}\,\lambda_i \\ \frac{1}{2}(1+\gamma_0)\,\lambda_i \\ \gamma_0 \end{array}\right\} \to \begin{array}{l} -\frac{1}{2}(1+\gamma_0^T)\,\sigma_{kl}^T\,\lambda_i^T \\ -\frac{1}{2}(1+\gamma_0^T)\,\lambda_i^T) \\ -\gamma_0^T \end{array} .$$

If we apply a similarity transformation $B = C\,\gamma_5$ which has the property

$$B\,\gamma_\mu\,B^{-1} = \gamma_\mu^T , \quad C\,\gamma_\mu\,C^{-1} = -\gamma_\mu^T ,$$

to the right-hand side set of operators we obtain the generators for the representation $(\bar{6},1)$ in the form

$$+\frac{1}{2}(1+\gamma_0)\,\sigma_{kl}\,\lambda_i^T ,$$
$$-\frac{1}{2}(1+\gamma_0)\,\lambda_i^T ,$$
$$-\gamma_0 .$$

The matrix γ_5 and analogously the matrix C induces an outer automorphism of the group $S[U(6)_+ \otimes U(6)_-]$ which exchanges the role of $U(6)_+$ and $U(6)_-$,

$$\gamma_5\,\frac{1}{2}(1+\gamma_0)\,\sigma_{kl}\,\lambda_i\,\gamma_5^{-1} = \frac{1}{2}(1-\gamma_0)\,\sigma_{kl}\,\lambda_i ,$$
$$\gamma_5\,\frac{1}{2}(1+\gamma_0)\,\lambda_i\,\gamma_5^{-1} = \frac{1}{2}(1-\gamma_0)\,\lambda_i ,$$
$$\gamma_5\,\gamma_0\,\gamma_5^{-1} = -\gamma_0 .$$

We have thus proved: a quark representation (6,1),N goes under γ_5 into a pseudo-quark representation (1,6),-N.

Making use of these results we introduce an important new concept. So far the matrix form in which we gave the generators of the subgroups $S[U(6)_+ \otimes U(6)_-]$ and $SU(6)_W$ was fixed by the defining representation of the group SU(6,6). In this form the representations of the group $SU(6)_-$ belong to "negative energy" eigenvalues, i.e. to eigenvalues –1 of the matrix γ_0 (see below). In physics only positive energies appear. We search therefore for another matrix form for the generators such that quark and antiquark representations take symmetric places in the formalism and belong both to positive eigenvalues of γ_0. The matrix substitution we use is known to accompany the charge conjugation transformation in the theory of the quantized Dirac field.

Let u be the antiquark representation $(1,\bar{6})$ (from now on we consider only quarks and antiquarks). The substitution

$$u_c = C(\bar{u}\,\gamma_5) , \quad C^{-1}\,\gamma_\mu\,C = -\gamma_\mu^T$$

brings the generators of $SU(6)_-$ into the form

$$\left.\begin{array}{l} +\frac{1}{2}(1-\gamma_0)\,\sigma_{kl}\,\lambda_i \\ +\frac{1}{2}(1-\gamma_0)\,\gamma_i \end{array}\right\} \rightarrow \begin{array}{l} +\frac{1}{2}(1+\gamma_0)\,\sigma_{kl}\,\lambda_i^T \\ -\frac{1}{2}(1+\gamma_0)\,\lambda_i^T \end{array}.$$

It relates the representation $(1,\bar{6})$ to the representation $(6,1)$. We give generators for antiquark representations now throughout in this u_c-basis. In particular we are interested in the form of the $SU(6)_W$ algebra in this basis. We find from

$$+\sigma_{12}\,\lambda_i\,, \quad +\gamma_0\,\sigma_{31}\,\lambda_i\,, \quad +\gamma_0\,\sigma_{23}\,\lambda_i\,, \quad +\lambda_i$$

the algebra in the u_c-basis

$$+\sigma_{12}\,\lambda_i^T\,, \quad -\gamma_0\,\sigma_{31}\,\lambda_i^T\,, \quad -\gamma_0\,\sigma_{23}\,\lambda_i^T\,, \quad -\lambda_i^T\,.$$

The $SU(6)_W$ algebra was given in this form in Ref. [261].

We come now to the eigenvalues of the matrix γ_0. The constraint $\gamma_0^2 = 1$ is alien to a Lie algebra approach and can only be satisfied by certain representations. We require that it is fulfilled on the four quark representations. Since γ_0 is not contained in the collinear group, its eigenvalues are conserved quantum numbers only in static transitions. The eigenvalues belonging to the quark representations are ± 1. If we ascribe an independent meaning to the factor γ_0 in the generators $\gamma_0\,\sigma_{31}\,\lambda_i$ etc., these eigenvalues can only be $+1$ for quark and antiquark (see the entry N in the table below).

	N	Q	$P = e^{i/2\pi(N-Q)}$
quark	$+1$	$+1$	$+1$
antiquark	$+1$	-1	-1

2.3.6 Parity and an example

The parity operation is as usual connected with the matrix γ_0. This matrix belongs to the algebra of $S[U(6)_+ \otimes U(6)_-]$ but not to $SU(6)_W$ since it changes the direction of motion in three space. On general representations the parity is not proportional to the eigenvalue of the generator γ_0 since this eigenvalue would be the sum and not the product of the eigenvalues of the factors in a tensor product. We must proceed in a different fashion.

We enlarge the algebra of $S[U(6)_+ \otimes U(6)_-]$ by adding the element "1" getting the algebra $U(6)_+ \otimes U(6)_-$ and assume that the eigenvalue Q of this operator is $+1$ on quarks and -1 on antiquark representations. Parity can then be defined as

$$P = e^{i/2\pi(\gamma_0 - 1)}$$

which applied to a representation with eigenvalues N and Q reduces to

$$P = e^{i/2\pi(N-Q)}\,.$$

On the mesons $(6,\bar{6})$ we have $N = 2$, $Q = 0$ implying $P = -1$, for the baryons $(56,1)$ we find $N = Q = 3$, $P = +1$, and for the antibaryons $(1,\overline{56})$ we obtain finally $N = +3$, $Q = -3$, $P = -1$.

Let us give an example for the reduction of a representation of $S[U(6)_+ \otimes U(6)_-]$ into representations of $SU(6)_\sigma$ on the one hand and $SU(6)_W$ on the other hand. For baryons this decomposition is trivial since both groups coincide. A simple non-trivial example are the mesons $(6,\bar{6})$. We find in both cases

$$6 \times \bar{6} = 35 \oplus 1$$

and

$$35 = (8,3) \oplus (8,1) \oplus (1,3)\,,$$
$$1 = (1,1)\,.$$

The different assignments of these parts to physical states are

$SU(6)_\sigma$	$SU(6)_W$	
$V^8_\pm$, V^8_0	$V^8_\pm$, P^8	(8,3)
P^8	V^8_0,	(8,1)

and similarly for the singlets. V and P mean vector and pseudoscalar particles, respectively.

The reduction problem is very complicated in general. Only the simplification for representations of $S[U(2)_+ \otimes U(2)_-]$, $SU(2)_\sigma$ and $SU(2)_W$ has been dealt with recently, Ref. [204]. Fortunately the general problem is of no great practical importance. We emphasize that Part II of this work is devoted to a systematic study of the implications of these subgroup chain symmetries.

Further references on static $S[U(6) \otimes U(6)]$: (There are two versions of this group appearing in the literature, a "chiral" group and a "non-chiral" group. Only the latter has been dealt with in this article.)

non-chiral group [8, 84, 130, 289, 386, 387]
chiral group [2, 32, 33, 34, 137, 168, 193, 310, 315, 364].

3. Relativistic SU(6) theories with finite multiplets (Models with subsidiary conditions)

3.1 *Representations for the inhomogeneous groups $SL(6,C) \times T_{72}$ and $SU(6,6) \times T_{143}$*

3.1.1 The homogeneous groups SL(6,C) and SU(6,6) and their finite dimensional representations

The groups SU(6,6) and SL(6,C) have been introduced in Section 2.3.1 in matrix form. We shall refer to the notations defined there. We study now the finite dimensional representations of these groups.

Representations of the group SU(6,6) can be obtained as for the compact group SU(12) and any other group SU(n) in the form of tensors (see Section 1.2.1). The only difference, which is, however, nonessential for our purposes here, lies in the fact that these finite dimensional representations turn out to be non-unitary. Representations for the group SL(6, C) can be obtained by similar techniques with the important modification, that we have to distinguish between two different tensor bases. The one is constructed by a matrix ξ (see Section 1.2.1)

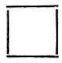
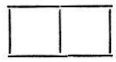
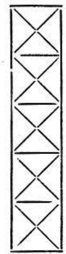
$$\xi' = \xi\, g\,, \quad \Delta_{(k)}^{i_1 i_2 \ldots i_k}(\xi)\,, \quad \xi, g \in SL(6, C)\,.$$

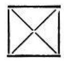
This basis is the analytic continuation of the basis used for the representations of SU(6) (see Section 1.2.1). The other basis uses the conjugate matrix $\bar{\xi}$,

$$\bar{\xi}' = \bar{\xi}\, \bar{g}\,, \quad \Delta_{(k)}^{i_1 i_2 \ldots i_k}(\bar{\xi})\,,$$

and can be regarded as the anti-holomorphic continuation of the SU(6) basis. Correspondingly our tensors will bear two types of indices, undotted and dotted ones. Their symmetry can be defined by two Young tableaux.

Examples:

$\psi_a \rightarrow$ , $\psi_{ab}^{\dot{c}}$, $\psi_{ab}^{\dot{c}} = \psi_{ba}^{\dot{c}} \rightarrow$  ,

$\psi_{\dot{a}} \rightarrow$ ,

As was shown in Section 2.3.1 the fundamental representation of SU(6,6) decomposes into two representations of SL(6, C)

$$U = \begin{pmatrix} S & O \\ O & (S^{\dagger})^{-1} \end{pmatrix}$$

or

$$\psi_A = \begin{pmatrix} \psi_a \\ \psi^{\dot{a}} \end{pmatrix}.$$

If a tensor of SU(6,6)

$$\psi_{A_1 A_2 \ldots A_k}$$

with a certain symmetry of its indices is given, then the quantity

$$\tilde{\psi}^{B_1 B_2 \ldots B_k} = \overline{\psi}_{A_1 A_2 \ldots A_k} (\gamma_0)^{A_1 B_1} \ldots (\gamma_0)^{A_k B_k}$$

transforms contragrediently:

$$\tilde{\psi}' = T_{(U^{-1})^T} \tilde{\psi}\,.$$

The expression

$$\tilde{\psi}^{A_1 A_2 \ldots A_k} \psi_{A_1 A_2 \ldots A_k}$$

is therefore invariant.

Let us consider then a representation of SL(6, C), say ψ_a. It is impossible to construct an invariant form of this quantity $\overline{\psi}_a$ and its conjugate ψ_a. We learn, however, what we must do if we split the SU(6,6) invariant form into parts which are still SL(6,C) invariant, namely

$$\psi'^A \psi_A = \psi'^a \psi_a + \psi'_{\dot a} \psi^{\dot a} .$$

We must therefore in general double the representation of SL(6,C) to find an invariant form, e.g.

$$\psi_a \to \begin{pmatrix} \psi_a \\ \psi^{\dot a} \end{pmatrix} .$$

After this doubling we contract the indices by means of a metric matrix which can be identified with γ_0:

$$(\overline{\psi}_{\dot a}, \overline{\psi}^a) \begin{pmatrix} 0 & \delta^{\dot a}_{\dot b} \\ \delta^b_a & 0 \end{pmatrix} \begin{pmatrix} \psi_b \\ \psi^{\dot b} \end{pmatrix} = \overline{\psi}_{\dot a} \psi^{\dot a} + \overline{\psi}^b \psi_b .$$

Another possibility of contracting the indices is to use the metric matrix $\gamma_0\gamma_5$. Both types of invariants are real.

This doubling does not necessarily lead to a representation of SU(6,6) as can be seen from inspection of the examples

$$\psi_{a\dot b} \to \begin{pmatrix} 0 & \psi_{a\dot b} \\ \psi^{\dot a b} & 0 \end{pmatrix}, \quad \psi^b_a \to \begin{pmatrix} \psi^b_a & 0 \\ 0 & \psi^{\dot a}_{\dot b} \end{pmatrix},$$

since only the sum of both matrices can be considered as a representation of SU(6,6):

$$\psi^B_A \to \begin{pmatrix} \psi^b_a & \psi_{a\dot b} \\ \psi^{\dot a b} & \psi^{\dot a}_{\dot b} \end{pmatrix} .$$

We can destroy the invariance of a given invariant form with respect to SU(6,6), say

$$\psi'^A \psi_B \phi^B_A$$

and maintain its SL(6,C) invariance by inserting an arbitrary number of γ_5 matrices, e.g.

$$\psi'^A (\gamma_5)^B_A \psi_C \phi^C_B .$$

This corresponds to the substitution of the metric $\gamma_0 \gamma_5$ for γ_0. This way of writing SL(6,C) invariants allows us to apply the techniques of Dirac matrices also to the SL(6,C) group. The Pauli matrices are, however, not more difficult to manage.

The construction of invariant forms is intimately connected with the parity transformation. For a fundamental representation of SU(6,6) we define the parity operator by the generator γ_0. By this definition we guarantee that it coincides with the correct operator on the representations of the subgroup SU(2,2). We recognize that doubling the representation of SL(6,C) is just what is necessary to make an application of the γ_0 operator possible to SL(6,C) representations. Therefore, in general we take γ_0 as the parity operator. In order to maintain the parity invariance of a form which is invariant under SU(6,6) if we reduce the symmetry to SL(6,C), the γ_5 matrices must always be introduced pairwise.

3.1.2 The group of translations

Let us consider the familiar four momentum p_μ. In the representation of Dirac spinors it takes the form

$$p = (p_\mu \gamma^\mu) .$$

We assume that the momentum satisfies the following conditions: The four components of p_μ are real,

$$p_\mu p^\mu = m^2 > 0 , \quad p_0 > 0 .$$

A Lorentz transformation

$$p'_\mu = a^\nu_\mu p_\nu$$

can be written in matrix form

$$p' = L^{-1} p L .$$

The Lorentz group is a subgroup of SU(6,6). We can re-interpret all the quantities γ_μ, L, p, just introduced in terms of 12×12 matrices. If instead of L we transform with an arbitrary transformation $U \in$ SU(6,6), p' will in general not be given as a product of p_μ with the four matrices γ^μ, but will be a linear combination of all the 143 matrices Γ_A (see Ref. [53])

$$p' = \sum_{A=1}^{143} p_A \Gamma_A .$$

In this manner we obtain a manifold of momenta starting from the four components of physical momentum, which can be represented in a vector space of 143 dimensions. Let us try to characterize this manifold. The property of pseudo-hermiticity

$$p^\dagger \gamma_0 = \gamma_0 p \qquad \text{i)}$$

is certainly invariant under transformations $U \in$ SU(6,6). Since the physical momenta satisfy this requirement the complete manifold does. Another property is

$$p^A_A = 0 , \qquad \text{ii)}$$

which is obviously also invariant and valid for the four-momenta. Further invariant properties of the manifold are

$$p^B_A \, p^C_B = m^2 \, \delta^C_A , \qquad \text{iii)}$$

iv) $\gamma_0 p$ is a positive definite matrix.
Let i), ii), iii) and iv) be satisfied. It can then be proved that a matrix $U \in$ SU(6,6) exists which transforms p "to rest",

$$U^{-1} p \, U = m \gamma_0 , \text{ with } m > 0 ,$$

(see Ref. [341]). This teaches us that the manifold is in a certain sense completely characterized by these four conditions.

The proof is simple. Consider a 12-dimensional vector space spanned by basis vectors $e_1, e_2 \dots e_{12}$ with the pseudo-norms

$$\bar{e}_1 \gamma_0 e_1 = +1, \dots \quad \bar{e}_6 \gamma_0 e_6 = +1,$$
$$\bar{e}_7 \gamma_0 e_7 = -1, \dots \quad \bar{e}_{12} \gamma_0 e_{12} = -1,$$
$$\bar{e}_k \gamma_0 e_m = 0 , \quad k \neq m .$$

If x is an eigenvector of p

$$p\,x = \lambda\,x\,,$$

x has a non-vanishing real pseudo-norm

$$0 < \bar{x}\,\gamma_0\,p\,x = \lambda\,\bar{x}\,\gamma_0\,x \text{ implies } \bar{x}\,\gamma_0\,x \neq 0 \text{ and real.}$$

The eigenvalue λ is also real. We normalize x to ± 1. Then we rotate the co-ordinate system by means of a matrix $U \in \mathrm{SU}(6,6)$

$$e'_k = U\,e_k\,,$$
$$\bar{e}'_k\,\gamma_0\,e'_m = \bar{e}_k\,\gamma_0\,e_m\,,$$

so that x goes into e'_1 or e'_{12} depending on whether the pseudo-norm of x is $+1$ or -1. In any case we obtain a block matrix for

$$U^{-1}\,p\,U$$

with respect to the basis e_k which has a form

$$\left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|c} & 0 \\ \hline 0 & \lambda \end{array}\right)$$

because $\gamma_0\,U^{-1}\,p\,U$ is Hermitian. By induction we proceed until we have obtained either six eigenvectors of positive (and some of negative) pseudo-norm or six eigenvectors with negative (and some with positive) pseudo-norm. The remaining block matrix is multiplied by γ_0 and becomes then Hermitian. It can be diagonalized by a unitary transformation U which necessarily commutes with γ_0. The matrix U therefore also belongs to SU(6,6).

We have thus obtained a diagonal matrix D,

$$D = U^{-1}\,p\,U\,,\quad U \in \mathrm{SU}(6,6)$$

with the properties

$$D^A_A = 0\,,$$
$$D^B_A\,D^C_B = m^2\,\delta^C_A\,,$$
$$\gamma_0\,D \text{ is positive definite}\,.$$

They imply

$$D = m\,\gamma_0\,,\quad m > 0\,.$$

Let us now turn to SL(6,C). We write the matrix p^B_A as

$$p^B_A = \begin{pmatrix} p^b_a & p_{a\dot{b}} \\ p^{\dot{a}b} & p^{\dot{a}}_{\dot{b}} \end{pmatrix}.$$

We know that $p_\mu\,\gamma^\mu$ is antidiagonal and anticommutes with γ_5. These properties are invariant under transformations of SL(6,C). If we therefore restrict the SU(6,6) manifold of momenta to antidiagonal matrices we obtain a manifold for SL(6,C). It consists of a set of pairs of 6×6 matrices $p_{a\dot{b}}$, $\hat{p}^{\dot{a}b}$) and can be represented in a space of 72 dimensions. These matrices are both Hermitian and positive definite and satisfy (we normalize $\det p = m^6$)

$$p_{a\dot{b}}\,\hat{p}^{\dot{b}c} = m^2\,\delta_a^c\,,$$
$$\hat{p}^{\dot{a}b}\,p_{b\dot{c}} = m^2\,\delta_{\dot{c}}^{\dot{a}}\,.$$

Under a parity transformation they change their place.

These properties of the matrix pairs can again be shown to characterize the manifold completely. By a transformation of SL(6,C) both matrices in the pair can simultaneously be brought to a form where they are proportional to the identity:

$$(S^{-1})_a^b\,p_{b\dot{c}}\,((S^\dagger)^{-1})_{\dot{d}}^{\dot{c}} = m\,\delta_{a\dot{d}}\,,$$
$$(S^\dagger)_{\dot{b}}^{\dot{a}}\,\hat{p}^{\dot{b}c}\,S_c^d = m\,\delta^{\dot{a}d}\,.$$

We need only take $S = S^\dagger \equiv V_p^{-1} = (p/m)^{1/2} = (\hat{p}/m)^{-1/2}$ defined as the positive definite roots.

We have so far studied manifolds which have been generated by positive-timelike four-momenta. These manifolds are special cases of orbits which in general can be defined for SU(6,6) and the space of dimension 143 as manifolds of momentum matrices satisfying the following three conditions:

i) vanishing trace, $p_A^A = 0$,
ii) hermiticity, $\gamma_0\,p^\dagger = p\,\gamma_0$,
iii) transitivity in the sense that if p lies in the manifold, then $U\,p\,U^{-1}$ lies also in the manifold for any $U \in$ SU(6,6), and if p and p' lie in the manifold then a matrix U exists such that $p' = U\,p\,U^{-1}$.

Condition iii) was satisfied in a trivial manner by the manifolds just studied. They are therefore orbits. We call them positive-timelike orbits in the sequel. We are not interested here in the problem of characterizing all possible orbits in the 143-space. We only remind the reader that all other vector spaces corresponding to self-conjugate representations of SU(6,6), which contain an invariant four-space under the Lorentz group, are as well suited to carry the orbits and to construct on it a relativistic SU(6) model. In the case of the SL(6,C) group the representation of dimension 400 has attracted some interest (see Refs. [27, 232, 233]).

3.1.3 Unitary representations of the inhomogeneous groups $SL(6,C) \times T_{72}$ and $SU(6,6) \times T_{143}$

The inhomogeneous groups involve translations in vector spaces of 72 or 143 dimensions. We can apply Wigner's theory of representations (see Ref. [394]) for the inhomogeneous Lorentz group to both cases.

The irreducible representations of the subgroups of translations are one-dimensional. Let x be an element of T_{143}. Then the representation characterized by the matrix p is

$$T_x\,\psi(p) = e^{i\,Tr(x\,p)}\,\psi(p)\,.$$

We construct the representations of the inhomogeneous groups on functions $\psi(p)$ where p runs over an orbit. Let us assume that the orbit under consideration is positive-timelike. We introduce a scalar product in the function space by defining an invariant measure on the orbit. For the case of the inhomogeneous Lorentz group this scalar product looks as follows

$$\int |\psi(p)|^2 \delta(p^2 - m^2)\, d^4p$$

where the invariant measure is

$$\delta(p^2 - m^2)\, d^4p .$$

The representation space is then a Hilbert space of square integrable functions with respect to this measure. In the case of SL(6,C) and SU(6,6) and positive-timelike orbits this measure can be defined as

$$\delta(\det p - m^6)\, \delta^{(36)}(\hat{p}^{a\dot{b}}\, p_{\dot{b}c} - m^2\, \delta^{a}_{c})\, d^{72}p .$$

respectively

$$\delta^{(71)}(p_A^B\, p_B^C - m^2\, \delta_A^C)\, d^{143}p .$$

The functions $\psi(p)$ bear additional indices, which describe their properties as elements of a finite vector space. In the case of the Lorentz group these quantities can either be spinorial, of the Dirac or Weyl type, or canonical; that means the indices describe spin projections on a fixed axis in the rest system. The norm in these finite dimensional spaces depends on the choice of spinors. We obtain

$$\overline{\psi}\,\gamma_0\,\psi\,,\ \overline{\psi}_{\dot{a}}\,\frac{\hat{p}^{\dot{a}b}}{m}\,\psi_b\,,\ \sum_{S_3}\overline{\psi}_{S_3}\,\psi_{S_3}$$

respectively.

We introduce now the notion of little groups. We may define them here as those subgroups of the homogeneous groups which leave a given momentum unchanged (they are therefore also called "stationary subgroups"). The structure of the little group is independent of which momentum we choose out of one orbit. Indeed, if U is an element of the little group belonging to p,

$$U\,p\,U^{-1} = p\,,$$

another momentum on the same orbit can by definition be written as

$$p' = V\,p\,V^{-1}\,,\quad V \in \mathrm{SU}(6,6)\,.$$

Then

$$W = V\,U\,V^{-1}$$

is an element of the little group belonging to p'.

We consider now only positive timelike orbits and take the momentum $p = m\,\gamma_0$. The condition defining the little group is then

$$U\,\gamma_0\,U^{-1} = \gamma_0$$

which implies

$$U^{-1} = U^{\dagger}\,.$$

The little group is therefore compact and has the structure $\mathrm{S}[\mathrm{U}(6)_+ \otimes \mathrm{U}(6)_-]$. Let U be an arbitrary element of SU(6,6) which transforms a given momentum p into p':

$$p' = U^{-1}\,p\,U\,.$$

We define V_p as the Hermitian matrix with transforms p to rest in a manner which is "free of rotations" (see Ref. [341])

$$p = m\, V_p^{-1}\, \gamma_0\, V_p = m\, \gamma_0\, V_p^2\,, \quad V_p = \left(\frac{\gamma_0\, p}{m}\right)^{1/2}. \text{ [15]}$$

By means of this matrix V_p we construct

$$W(p', p, U) = V_p\, U\, V_p^{-1}\,,$$

and find that W lies in the group $S[U(6)_+ \otimes U(6)_-]$.

Let $\chi(p)$ be a square integrable function on a fixed orbit which corresponds to a unitary representation R of the little group. We can define a unitary representation of the inhomogeneous group $SU(6,6) \times T_{143}$ by

$$T(U, x)\, \chi(p) = e^{i\, Tr(x p)}\, R_{W(p', p, U)}\, \chi(p')\,.$$

This basis $\chi(p)$ corresponds to what Wigner calls "canonical basis" in the case of the Poincaré group.

In a similar manner we obtain unitary representations for the inhomogeneous group $SL(6, C) \times T_{72}$. The little group belonging to the pair $(p_{a\dot{b}}, \hat{p}^{a\dot{b}})$ consists of matrices S which satisfy simultaneously

$$S\, p\, S^\dagger = p\,, \quad (S^\dagger)^{-1}\, \hat{p}\, S^{-1} = \hat{p}\,.$$

These conditions are equivalent because of $p\,\hat{p} = m^2$ and imply

$$(V_p\, S\, V_p^{-1})\, (V_p\, S\, V_p^{-1})^\dagger = 1\,,$$

where V_p is the "rotation free' transformation of p to rest introduced in Section 3.1.2.

$$V_p = \left(\frac{p}{m}\right)^{-1/2} = \left(\frac{\hat{p}}{m}\right)^{+1/2}.$$

The little groups are therefore isomorphic to SU(6).

Let R be a representation of SU(6) operating on the indices of the function $\chi(p)$. Then the element $(S, x, \hat{x})$ of $SL(6,6) \times (T_{36} \otimes T_{36})$ is represented by

$$T(S, x, \hat{x})\, \chi(p) = e^{i\, Tr(x p + \hat{x} \hat{p})}\, R_{W(p', p, S)}\, \chi(p')\,.$$

Here p' and W mean

$$p' = S^{-1}\, p\, (S^+)^{-1}\,, \quad W(p', p, S) = V_p\, S\, V_p^{-1} \in SU(6)\,.$$

3.1.4 Spinor bases and Bargmann-Wigner equations

In order to introduce spinorial bases we must make use of quite different methods for the two groups. We shall discuss this problem separately for both cases and start with the simpler group SL(6, C). The notion of Bargmann-Wigner equations has been chosen in accordance with similar equations known to exist for the inhomogeneous Lorentz group, see Ref. [37].

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[15]) It remains to be proved that the matrix V_p is in SU(6, 6) if we define it as the positive definite square root of $\gamma_0\, p/m$. See Ref. [341].

Let $\chi(p)$ be a spinor (synonymously: tensor) of SU(6)

$$\chi_{i_1 i_2 \ldots i_k}(p) ,$$

with a symmetry of the indices corresponding to a certain representation R of SU(6). If we continue the representation T analytically we are able to define R_S for general $S \in$ SL(6,C). We can then ascribe independent meaning to each factor of W and obtain

$$R_W = R_{V_p} \, R_S \, R_{V_{p'}^{-1}} .$$

We denote

$$\psi_{a_1 a_2 \ldots a_k}(p) = (R_{V_p^{-1}} \, \chi)_{a_1 a_2 \ldots a_k} .$$

The indices "a" have necessarily the same symmetry as had the indices "i", but transform now under SL(6,C). We obtain a transformation formula

$$T(S, x, \hat{x}) \, \psi(p) = e^{i \, Tr(x p + \hat{x} \hat{p})} \, R_S \, \psi(p') .$$

Another possibility of introducing a spinor of SL(6,C) consists in the analytic continuation of the conjugate representation of R.

$$\begin{aligned} \overline{R}_W = \overline{R}_{\overline{W}} &= \overline{R}_{(W^{-1})^T} \\ &= \overline{R}_{(V_p^{-1})^T \, (S^{-1})^T \, V_{p'}^T} \\ &= \overline{R}_{(V_p^{-1})^T} \, \overline{R}_{(S^{-1})^T} \, \overline{R}_{V_{p'}^T} . \end{aligned}$$

Conjugating again after this continuation we obtain a spinor (note $V_p^\dagger = V_p$)

$$\psi^{\dot{a}_1 \dot{a}_2 \ldots \dot{a}_k}(p) = (R_{V_p} \, \chi(p))^{\dot{a}_1 \dot{a}_2 \ldots \dot{a}_k}$$

This spinor has the same symmetry as the quantity $\chi(p)$ and transforms as

$$T(S, x, \hat{x}) \, \psi(p) = e^{i \, Tr(x p + \hat{x} \hat{p})} \, R_{(S^\dagger)^{-1}} \, \psi(p') .$$

Both spinors are connected by

$$\psi^{\dot{a}_1 \dot{a}_2 \ldots \dot{a}_k}(p) = \frac{\hat{p}^{\dot{a}_1 b_1} \, \hat{p}^{\dot{a}_2 b_2}}{m \;\; m} \cdots \frac{\hat{p}^{\dot{a}_k b_k}}{m} \, \psi_{b_1 b_2 \ldots b_k} .$$

These equations together with their inversions and all intermediate steps, which involve spinors with dotted and undotted indices simultaneously, are called the Bargmann-Wigner equations.

In the case of SU(6,6) it is obviously impossible to obtain a representation of SU(6,6) by continuing a representation of $S[U(6)_+ \otimes U(6)_-]$. We must therefore proceed differently.

Let

$$\chi_{i_1 i_2 \ldots i_k ; j_1 j_2 \ldots j_l}(p)$$

denote a representation of the little group, in which the first k indices transform under the group $SU(6)_+$ and the remaining l indices under $SU(6)_-$ (in the u-basis, see Section

2.3.5). We know that (see Section 2.3.5) the following equations are satisfied by this representation

$$(1-\gamma_0)^{i_1}_{n}\, \chi_{i_1 i_2 \ldots i_k\,;\, j_1 j_2 \ldots j_l} = 0\,,$$

analogously for all indices "i", and

$$(1+\gamma_0)^{j_1}_{n}\, \chi_{i_1 i_2 \ldots i_k\,;\, j_1 j_2 \ldots j_l} = 0\,,$$

etc. If we insert into these equations

$$m\,\gamma_0 = V_p\, p\, V_p^{-1}\,,$$

we obtain

$$(m-p)^B_A\, \chi_{B\, i_2 i_3 \ldots i_k\,;\, j_1 j_2 \ldots j_l} = 0 \text{ etc.,} \qquad (m+p)^B_A\, \chi_{i_1 i_2 \ldots i_k\,;\, B j_2 j_3 \ldots j_l} = 0 \text{ etc.,}$$

where

$$\chi_{B\, i_2 i_3 \ldots i_k\,;\, j_1 j_2 \ldots j_l} = (V_p^{-1})^{i_1}_B\, \chi_{i_1 i_2 \ldots i_k\,;\, j_1 j_2 \ldots j_l}\,.$$

If we multiply $\chi(p)$ with $k+1$ factors V_p^{-1} corresponding to each index, we obtain spinorial quantities

$$\psi_{A_1 A_2 \ldots A_k\,;\, B_1 B_2 \ldots B_l}\,,$$

which satisfy Bargmann-Wigner equations for each index. We can introduce contragredient quantities in an analogous manner; the Bargmann-Wigner equations look then as follows:

$$\tilde{\psi}^{A_1 A_2 \ldots A_k\,;\, B_1 B_2 \ldots B_l}\,(m-p)^C_{A_1} = 0\,, \qquad \tilde{\psi}^{A_1 A_2 A_k\,;\, B_1 B_2 \ldots B_l}\,(m+p)^C_{B_1} = 0 \text{ etc.}$$

Certainly we can also form all mixed quantities.

If now the representation R has the property that by adding other representations R', R'' ... of the little group a finite representation Z of the group SU(6,6) can be built up, we may again split the matrix W of the little group

$$Z_{W(p',\,p,\,U)} = Z_{V_p}\, Z_U\, Z_{V_{p'}^{-1}}\,.$$

We call the direct sums of the spinors ψ again ψ and obtain the following transformation formula for the reducible representation $\sum_i \oplus T^i$

$$\Big(\sum_i \oplus T^i\Big)\,(U,x)\,\psi(p) = e^{i Tr(x p)}\, Z_U\, \psi(p')\,.$$

If we are interested in one irreducible part, we must project it out with the help of the Bargmann-Wigner equations.

Example:

A quark representation (6,1) of $S[U(6)_+ \otimes U(6)_-]$ can be completed by adding the pseudoquark representation (1,6). It leads to a spinor ψ_A of SU(6,6). The quark part can later be projected out again by the condition

$$(p-m)^B_A\, \psi_B = 0\,.$$

Since scalar products and invariants of all kinds can be constructed more easily in terms of spinor quantities, this completion-projection technique is, practically, quite important. It was developed first in Ref. [358]. The advantage of the group theoretical method over the treatment of Ref. [358] lies only in the fact, that it allows us to derive the Bargmann-Wigner equations. In addition we shall find that the group theoretical procedure implies unambiguous recipes for constructing invariant forms in the most general form.

Further references on the representations of the inhomogeneous SL(6,C) and SU(6,6) groups: [86, 342].

The Bargmann-Wigner equations have been used to define a mass splitting. For the SL(6,C) model see Refs. [153, 210, 375]; and for the SU(6,6) model: [319, 395].

3.2 *The inhomogeneous groups as physical symmetry groups with subsidiary conditions*

3.2.1 Little groups, subgroup chains and subsidiary conditions

In order to construct a relation between unitary representations of the inhomogeneous groups and physical multiplets, we must overcome in any way the difficulties of the additional components of the momenta. We do this by requiring that the momenta of observable particles ("physical momenta") have the form

$$p_A^B = (p^\mu \gamma_\mu)_A^B \text{ respectively } p_{a\dot{b}} = (\sigma_{\mu\,\alpha\dot{\beta}}\, \delta_i^j)\, p^\mu .$$

We take them from the positive timelike orbits in order to be sure that the physical momenta belong to the positive timelike orbits of the inhomogeneous Lorentz group.

We may visualize the content of this definition in the following way. The Minkowski space can be considered as a subspace of the 143-space, respectively as a subspace of both 36-spaces. The intersection of the orbits with this subspace is the orbit of the Lorentz group (indeed, a pair of equivalent orbits in the case of SL(6,C)). If we consider the Hilbert space of square integrable functions on the 143-orbits (or 72-orbits) it is impossible to define a relation with the Hilbert space of square integrable functions on the Lorentz orbits in terms of a continuous mapping. Indeed, the continuous functions of the 143-orbits create, in a natural fashion, continuous functions on the Lorentz orbits. By additional restrictions on the original functions we could even manage to obtain square integrable functions of the Lorentz orbits. However, this mapping of a dense set of one Hilbert space on a dense set of another Hilbert space is not continuous, since the Minkowski space is of measure zero in the higher dimensional spaces. Arbitrarily small changes of the functions on the 143-orbits in the neighbourhood of the Lorentz orbit may introduce arbitrarily big alterations in the functions on the Lorentz orbits. The restriction of the bigger spaces on the smaller one is therefore not describable as a linear projection operation. Nevertheless, the restriction on physical momenta can be defined uniquely. We can for example refer to plane wave states (and will do so always!), which depend analytically on the momentum and apply the definition of physical momenta to them. They lie outside the Hilbert spaces but this does certainly not encounter any difficulties.

The physical meaning of the subsidiary conditions imposed on the momenta in this manner is straightforward. First it excludes orbital contributions to isospin and hypercharge, which are known not to exist in strong interaction physics. In addition they imply that the sum of two physical momenta lies again on the same type of 143-orbit (72-orbit), namely the positive timelike one. This is not true in general, particularly the condition

$$p_A^B \, p_B^C = m^2 \, \delta_A^C$$

does not necessarily hold for a sum of momenta if it holds for each constituent of the sum. The little groups can be defined as those subgroups which leave a given momentum matrix unchanged (see Section 3.1.3). For positive timelike momenta these little groups are equivalent to the maximal compact subgroups of the homogeneous groups. We recognize, that the requirement imposed on the maximal compact subgroups in Section 2.3.3 is satisfied: Due to the definition of the little group its generators do not change the state of motion. We can, therefore, assign physical multiplets to the representations of the little groups. As a consequence we see that the subgroup chain symmetries apply to any system of particles the momenta of which are positive timelike and physical.

For the sake of our later discussion we introduce a few additional notions. We can extend the definition of the little group to "generalized" little groups, which are stationary subgroups belonging to a set of momenta. In general we expect two cases to occur. The generalized little groups contain only the identity element of the homogeneous group or they are non-trivial subgroups. In the latter case we call the sets of momenta "degenerate".

For the inhomogeneous Lorentz group we can distinguish between two cases of degenerate manifolds of momenta:

i) A set of positive timelike momenta which spans a straight line in Minkowski space. We denote such a set as "static kinematics". The little group has the structure SU(2).

ii) A set of positive timelike momenta spanning a two-dimensional plane in Minkowski space. We call this situation "collinear kinematics". In a special frame of reference the corresponding three-momenta are parallel to a fixed axis. The little group consists of the rotations around this common direction. It has the structure U(1).

If the set of positive timelike momenta spans a three-dimensional plane (coplanar kinematics) or the whole four-dimensional space (general kinematics) the little groups are trivial.

We realize that for the groups SL(6,C) and SU(6,6) there are many different classes of systems of degenerate momenta. All sets composed only of physical momenta are degenerate since the generalized little groups contain SU(3) as a subgroup. It turns out that the subgroup chains introduced in Section 2.3 coincide just with the generalized little groups corresponding to static (SU(6) respectively S[U(6) $\otimes$ U(6)]), collinear (S[U(3) $\otimes$ U(3)]) respectively $SU(6)_W$) and coplanar kinematics (SU(3) respectively S[U(3) $\otimes$ U(3)]).

3.2.2 The construction of invariant S-matrix elements

As a matter of principle we apply all the symmetry models discussed in this article only to the S-matrix. The notion of the Lagrangian is still so obscure in field theories of strong interactions that we see no chance to derive any prediction from the symmetry of a Lagrangian.

If we require that the S-matrix is invariant with respect to a symmetry group this anticipates that the representations of the symmetry group operate in the physical Hilbert space. The physical Hilbert space must decompose into a direct sum of spaces which are irreducible with respect to the symmetry. In the case of inhomogeneous groups with more than four dimensions of the translation space this is obviously impossible. We have therefore to redefine what we mean by symmetry in this case (see Ref. [343]).

First we consider the scattering matrix in a plane wave basis. We construct a symmetric matrix in this basis which has many unphysical elements,

$$T_u \Sigma T_u^{-1} = \Sigma ,$$

where "u" is an element either of $SL(6,C) \times T_{72}$ or of $SU(6,6) \times T_{143}$. Due to Schur's lemma the matrix Σ can be decomposed into contributions from different irreducible representations. The number of these representations fixes the number of invariant functions appearing. These invariant functions are multiplied with certain projection operators which project onto the corresponding irreducible representations. The elements of these operators in the plane wave basis are restricted finally to physical momenta.

The methods of constructing such covariant operators have never been worked out explicitly for the groups SL(6,C) and SU(6,6). Instead, the technical device used is taken over from what is known to hold for the inhomogeneous Lorentz group.

Indeed, the recipe looks as follows. We take the spinors (Weyl's type) and the momentum matrices of all the particles involved in the process and contract their indices. We obtain then forms which are linear in the spinors of the particles and which are index invariant[16]) against transformations of the homogeneous group. The momenta add to zero because of translational invariance. In general one multiplet can be described by different spinor quantities, which are, however, linearly connected by Bargmann-Wigner equations (see Section 3.1.4). We have then to account for all invariants which can be built out of all possible spinors and momentum matrices and which are linearly independent as far as Bargmann-Wigner equations are concerned.

This procedure raises two questions:

i) Is the set of invariants thus obtained independent in a more general sense (the problem of group theoretical independence)?

ii) Is the set of projection operators thus constructed complete (problem of completeness)?

As we can see from inspection of the inhomogeneous Lorentz group, the answer to the first question is negative. This problem seems, however, not very serious as far as applications are concerned; the predictions derived from an independent set of invariants are identical with those implied by a dependent set. The answer to the completeness problem which we shall reach below is in the affirmative. Nevertheless, we believe that this problem is not quite trivial.

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[16]) By this we mean that the functional dependence on the momenta must be neglected. The homogeneous group acts only on the indices.

In fact we can imagine other possibilities of constructing invariant forms than those contained in the prescription given above. We have matrices Γ_A in the case of the group SU(6,6) and similar matrices σ_A [17]) for the group SL(6,C), which can be used to build invariant forms. The task consists of showing that these additional invariant forms can always be reduced to the earlier ones, i.e. to those consisting only of spinors and momentum matrices. We shall do this now, thus establishing the completeness, not by investigating the matrix algebras Γ_A or σ_A but using arguments which are more familiar to physicists. We start with an exposition of the same problem for the inhomogeneous Lorentz group.

It is convenient to fix first the Lorentz frame; we choose the centre of momentum system (c.m.s.). The spins of all particles involved in the process can be coupled together and give a finite set of integral spin representations for the group of rotations in the c.m.s. If the four-momenta of the particles are non-degenerate (this notion has been defined in Section 3.2.1) their space components together with one cross-product of two three-momenta span a basis in the three-dimensional space and can be used to construct bases for all integral spin representations. Both the spin and the momenta can then be contracted yielding invariant forms. These invariant expressions can finally be rewritten in Lorentz covariant, spinorial form. We obtain spinorial invariants composed only of spinors and momentum matrices.

The case of degenerate momenta is very important for our later discussion (Section 3.2.3). When the momenta are degenerate, the spins can only be contracted with the three-momenta if some restrictions on the invariant amplitudes are satisfied. These restrictions can be understood as the predictions to be deduced from the generalized little groups, which as we know are by definition connected with degenerate systems of momenta. Let us assume, for example, that the momenta are collinear. Then the generalized little group of the inhomogeneous Lorentz group consists of rotations around the common axis of motion; the corresponding component of the total angular momentum must be conserved and induce restrictions on the invariant amplitudes.

The situation is quite similar for the inhomogeneous SL(6,C) and SU(6,6) groups. We refer again to the c.m.s. and limit our discussion to the SL(6,C) model. If we couple the SU(6) representations of all the participating particles together we can arrange the indices such that we obtain tensors with just the same number of quark (contragredient) indices as antiquark (congredient) indices.

If the momenta were non-degenerate, their 35-dimensional parts (corresponding to the adjoint representation of SU(6)) could again be used to contract these indices. But physical momenta are known to be degenerate, since the generalized little groups contain at least the unitary symmetry group SU(3). The invariant amplitudes must, therefore, satisfy restrictions which fall together with the predictions of the subgroup symmetries.

In this manner we cannot only convince ourselves that the set of spinorial invariants is complete, but can also develop a method of deducing the predictions of

[17]) We make use of Weyl's representation for the matrices Γ_A (see Section 2.3.1). Then we may define the matrices σ_A as the upper right or lower left 6×6 matrices contained in the anti-diagonal Γ_A's:

$$\left(\begin{array}{c|c} 0 & \sigma_A \\ \hline & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|c} 0 & \\ \hline \sigma_A & 0 \end{array}\right).$$

the models based on inhomogeneous groups, which is equivalent but independent of the method which starts with spinorial invariants. This formalism for the SL(6,C) model has been displayed in Ref. [329]. The following features are specific for such an approach: a particular frame of reference must be chosen; the approach is only implicitly covariant and only representations of SU(6) (or S[U(6) ⊗ U(6)]) are involved.

The most efficient technique of deriving the predictions, however, makes use of the generalized little groups, the chain of subgroups, themselves.

3.2.3 Predictions of the symmetry models based on inhomogeneous groups

Since the subgroup chains define symmetries and give us definite predictions the only problem we have still to investigate is whether additional predictions can be derived. Indeed, we know a prediction of the SL(6,C) model which can neither be obtained with the collinear nor with the static subgroup.

Let the form factors of the meson multiplet 35^- coupled to the nucleon octet part of 56 be denoted by a^F, a^D for the pseudoscalar octet (vector coupling form factors) and a_c^F, a_c^D, a_c^S, a_m^F, a_m^D, a_m^S for the vector nonet (Sachs type form factors)[18]). We can then express the results for the meson-nucleon vertex due to collinear S[U(3) ⊗ U(3)] symmetry in the following way (see Refs. [348, 384] and the discussion in Part II of this article)

$$a_m^D : a_m^F : a_m^S = 3:2:1\,, \qquad a^F = -\frac{2}{3}\,a_c^F - \frac{5}{9}\,a_c^D\,, \qquad a^D = -\,a_c^F + \frac{2}{3}\,a_c^D\,.$$

These equations hold identically in the meson mass μ^2. In addition the static symmetry $SU(6)_\sigma$ gives

$$a^D = 0$$

at the point $\mu^2 = 4\,M^2$. The well-known result of the inhomogeneous SL(6,C) group (and of some other recipes, see below)

$$a^F : a^D = 2:3 \ \text{at} \ \mu^2 = 0$$

cannot be obtained from the static or collinear subgroup but by the method of using spinor invariants (see Ref. [339]). Since the ratio of the charge form factors $a_c^D : a_c^F$ can be expressed by the ratio $a^D : a^F$ through the collinear subgroup, we can fix the value for $a^D : a^F$ by requiring "current conservation", i.e. $a_c^D = 0$ at $\mu^2 = 0$. But such conditions which make sense predominantly in the case of electromagnetic or weak currents should not be mixed with purely group theoretical predictions for the meson-baryon vertex. We note further that the symmetry models in Section 4 which possess the same subgroup chains lead to other results for the ratio $a^D : a^F$, see Section 4.2.5.

The number of independent invariant functions involved in a process results simply from counting the independent helicity amplitudes. If the system is non-degenerate, the number obtained is the product of the dimensions of all multiplets divided by some fixed number which is due to additional discrete symmetries (e.g. parity reflection invariance). The fundamental statement we make is that this number of amplitudes is independent of whether the symmetry is valid or not!

[18]) The notation is identical with the one introduced in Section 1.5.2.

Indeed, if we assume for example that we have invariance under the inhomogeneous rotation group $[SO(2) \times T_3] \otimes T_1$ and that we know by any *a priori* information that the particles possess a spin degree of freedom which corresponds exactly to their relativistic spin, the number of invariant functions in the rotation invariant theory is identical to the number of invariant functions in the Lorentz invariant theory provided we consider the same group of processes. We admit, however, that the invariant amplitudes, which in both cases depend implicitly on the same three-momentum and energy variables, are explicit functions of different sets of invariants arguments formed out of these variables.

If we express the invariant functions of a theory symmetric with respect to $SL(6,C) \times T_{72}$ or $SU(6,6) \times T_{143}$ by means of invariant functions of a theory symmetric only under the Lorentz group and unitary symmetry SU(3) [note that the momenta are still assumed non-degenerate, in particular unphysical; they transform non-trivially under SU(3)] no identity between the invariant amplitudes of the smaller symmetry will result and we cannot make any prediction. The fact that the invariant functions of the bigger symmetry depend on a smaller number of invariant arguments, can perhaps lead to some predictions, but has never been made use of in all the applications contained in the literature.

The situation changes, however, if the momenta are degenerate. We obtain relations which are the consequence of the subgroup chain symmetries. These arguments prove that the subgroup chains give us really all predictions. But how can we explain the result for the $a^D : a^F$ ratio? The condition $\mu^2 = 0$ shows that this result has been obtained by continuing the representations analytically. In fact, the spinor invariants depend on the momenta in a simple manner so that the analytic continuation is straightforward. If we know in addition that the invariant functions standing in front of the spinor forms have a definite analytic behaviour at the point we want to investigate (say regular behaviour at $\mu^2 = 0$), we are able to derive such predictions. It is obvious that an analytic continuation of this kind is impossible for the subgroups since their representations do not depend explicitly on parameters of the kind of a mass[19]).

As a last illustration of these remarks concerning implications of analytically continued representations we remind the reader of Van Hove's method to derive the ratio $a^D : a^F$ in "static" SU(6) theory, Ref. [382].

In the non-relativistic limit the 0^- and 1^- mesons couple to ${}^1/{}_2{}^+$ baryons through

$$\frac{p'_k - p_k}{2M} \varphi \bar{\chi} \sigma_k \chi \quad \text{respectively} \quad V_0 \bar{\chi} \chi ,$$

where $\bar{\chi}$, χ are Pauli spinors. These forms suggest that the gradient of the pseudoscalar field couples like a spin-one particle whereas the zeroth component of the vector field couples like a scalar field. Redefining the octets of the 35-plet in this manner and coupling this 35-plet to the baryons in the static limit leads to the value ${}^3/{}_2$ for the ratio $a^D : a^F$.

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[19]) The question whether the collinear symmetry together with the substitution rule in the limit of vanishing four momentum of the meson may lead to a prediction for $\mu^2 = 0$ has not yet been investigated (remark due to Prof. J. S. Bell).

We expect that in each theory in which the representations have a similar analytic structure as that of Dirac spinors $^3/_2$ will result. Models involving inhomogeneous groups imply such a structure, this is the explanation of why this ratio has been found in the model using the inhomogeneous SL(6,C) group (see Ref. [339]). Models based on infinite representations of the homogeneous groups yield a completely different analytic structure. This is the reason why in Section 4 we shall obtain a new value for the ratio $a^D : a^F$.

3.2.4 The unitarity of the S-matrix

Let $| p, \omega\rangle$ denote a one-particle state with physical momentum p and let ω fix the SU(6) state at rest, respectively S[U(6) ⊗ U(6)] state at rest. We normalize this state by

$$\langle p', \omega' \mid p, \omega\rangle = 2\, p^0\, \delta_{\omega' \omega}\, \delta^{(3)}\, (p' - p)\ .$$

In the sum over projection operators onto a complete set of states ("completeness sum") we extend the summation only over physical momenta and one SU(6) multiplet (respectively one S[U(6) ⊗ U(6)] multiplet),

$$J_{ph} = \int d^4p\, \delta\, (p^2 - m^2)\, \theta(p^0) \sum_{\omega} | p, \omega\rangle \langle p, \omega |\ .$$

This expression is used in the unitarity condition of the physical scattering matrix and similarly in computations of total transition probabilities. By Schur's lemma an invariant operator must necessarily reduce to the unit operator on each irreducible representation space. Contrary to this the operator I_{ph} is the unit operator only over physical states. We emphasize again that it is not a projection operator in the familiar sense, see the discussion in Section 3.2.1. The problem of unitarity of the S-matrix is concerned with the consequences of the substitution

$$\text{unit matrix} \to I_{ph}$$

on a fixed representation of the inhomogeneous group.

Let us consider an illustrating example (see Ref. [53]). We assume that two meson representations exist, the one forming an SU(6) singlet, the other an SU(6) 35^--plet. The masses are to satisfy

$$3\, m(1) > m(35) > 2\, m(1)\ .$$

All other particles are assumed to have such a mass that only the decay of the 35-plet into two singlet particles is allowed kinematically. In general we assume that the masses within one multiplet are degenerate.

In this case the SU(3) singlet vector particle of the 35^--plet decays into P-wave states whereas the octets are stable as a consequence of unitary symmetry. The total widths of the particles are

$$\Gamma_{\text{singlet}} > 0\ , \qquad \Gamma_{\text{octets}} = 0\ .$$

By the unitarity condition these widths can be connected with the imaginary parts of the masses; the masses cannot therefore be degenerate. Of course, the difficulty arises because we ascribe certain SU(6) transformation properties to the orbital angular momentum but do not on the other hand permit that the orbital angular momentum form complete SU(6) multiplets.

A way out of this inconsistency is to assume that the over-all coupling constant for this decay vertex vanishes. This is a customary attitude in the discussion of the unitarity problem: if an inconsistency arises one tries to adjust the invariant functions such that the trouble cancels. In some cases this may force us to put a certain set of amplitudes equal to zero. The additional requirement of crossing symmetry may lead to further restrictions on the amplitudes. In this manner the problem of unitarity has been split into two completely different questions:

i) Is the unitarity of the S-matrix compatible with the symmetry if the invariant functions may take arbitrary values (the problem of violation of unitarity "in principle") ? The answer is "no".

ii) We may ask whether it is possible to adjust the invariant functions such that the S-matrix becomes unitary (the problem of "effective" violation of unitarity)? The answer to this question depends on what S-matrix we are willing to accept.

The unrealistic example of quark-quark scattering has been investigated in Ref. [15]. In order to eliminate production processes the energy in the s-channel was assumed to be sufficiently small. To prevent the quark-antiquark pair from annihilating into two mesons in the t-channel the quark mass had to be taken very small. The authors could then prove that unitarity and crossing-symmetry taken together forbid any scattering in the SU(6,6) model, whereas no statement could be made for the SL(6,C) model. Indeed, the SL(6,C) model is always in a better position concerning this problem since predictions can only be made for collinear kinematics.

The interpretation of a result like this is that even effective unitarity and crossing symmetry cannot be achieved for the SU(6,6) model, because absence of any scattering is thought to be unphysical. In more realistic cases we expect that the restrictions due to unitarity and crossing symmetry allow some amplitudes to take non-zero values. The only criterion for deciding whether unitarity is violated or not is then the consistency of the theoretical description in terms of these allowed amplitudes with phenomenology. Because of the complexity of the algebra on the one hand and of the phenomenological analysis of strong interactions on the other hand, no final answer can be given at present.

Further references on the general properties of the SL(6, C) model: [207, 213, 245, 287, 343, 403].

Further references on the general properties of the SU(6,6) model: [48, 78, 104, 105, 110, 128, 129, 144, 145, 159, 160, 184, 196, 202, 260, 291, 292, 294, 351, 359, 360, 390).

Further references on the connection of collinear $SU(6)_W$ with the SU(6,6) model: [5, 39, 64, 216, 263].

Further references on the unitarity problem[20]): [31, 49, 52, 60, 62, 142, 211, 317]. A possible dynamical background of these symmetries has been studied in references: [99, 112].

[20]) The following remark is perhaps necessary to avoid confusion. In part of the original literature the invariants formed out of spinors have been divided into two classes: "regular" invariants formed out of spinors only, and "irregular" invariants which in addition contain momentum matrices. These matrices bear names like "kinetic spurions" or "kinetons". Instead of "irregular" invariants one can also find the notations "irregular couplings" or "derivative couplings". If the S-matrix elements are assumed to consist only of regular invariants, this proves to be in conflict with the unitarity requirement, see Refs. [132, 343]. Most of the references on the unitarity problem given above are concerned only with this "trivial" problem. In a group theoretical presentation of the models as given in this article, a distinction between regular and irregular invariants makes no sense.

A model related to the SU(6,6) model has been studied in: [321, 322, 323, 324].

A perturbation treatment of strong interactions with vertices symmetric under collinear groups is dealt with in: [161, 197, 318].

4. Relativistic SU(6) theories with infinite multiplets

4.1 *The mathematical structure of the infinite multiplet theories*

4.1.1 Introduction

In this section we consider models of the type which have been assigned class a) in Section 2.2.3. The group we are mainly interested in is

$$G = L \times \big(\mathrm{SL}(6,\mathrm{C}) \otimes T_4\big)\,.$$

Models of such type have first been suggested to be of physical interest in Ref. [148]. Because of the difficulties involved in the theory of representations we exclude the analogous group

$$G = L \times \big(\mathrm{SU}(6,6) \otimes T_4\big)$$

from our discussion.

If we split the homogeneous Lorentz group into its orbital component L_0 and its spin component L_r which is a subgroup of SL(6,C) (we hope that this notation which has been introduced in Section 2.2.3 will become clearer below: the notation "orbital" has to be taken with care, since the generators of the "orbital" group are not identical with the familiar orbital angular momentum operators), we obtain the group G in the form

$$G = (L_0 \times T_4) \otimes \mathrm{SL}(6,\mathrm{C})\,.$$

Writing the group G in this form enables us to solve the representation problem immediately: first we find the representations for the group $P' = L_0 \times T_4$ which has the structure of the inhomogeneous Lorentz group; then we construct unitary representations for the group SL(6,C); and finally, we build the tensor products of both representations.

Representations of the group P' can be characterized by the invariants M, the mass, and S', the "orbital spin". The states of the representations spaces depend on the four-momentum p_μ; let us denote them by

$$|\, S_3', \boldsymbol{p} \rangle\,.$$

The representation theory for groups SL(n,C) has been dealt with in a wide mathematical literature; an introduction is given in Ref. [166]. The unitary representations of SL(6,C) are infinite dimensional. They can be reduced into an infinite direct sum of finite dimensional representations of the compact subgroup SU(6). A short account of the representation theory in a form which is most convenient for the applications we have in mind is contained in Sections 4.1.2 and 4.1.3. Let τ comprise all of the invariants necessary to characterize a unitary representation of SL(6,C), ν the invariants of a definite SU(6) representation contained in τ, and ω a state of this SU(6) representation space. Then we may denote a state of the representation space of the representation τ by

$$|\, \tau, \nu, \omega \rangle\,.$$

The representation space for a representation of the group G can then be spanned by the products

$$| S_3' \boldsymbol{p} \rangle \; | \tau, \nu, \omega \rangle .$$

For simplicity's sake we assume that $S' = 0$. We shall come back to this assumption at the end of the introduction.

Now we try to define the connection between these states of a representation space of G with a physical particle state. Let the particle have momentum p_μ and in its rest system belong to the component ω of the SU(6) multiplet ν, which is of course idealized so as to have a degenerate mass M. We write this state

$$| \nu, \omega, \boldsymbol{p} \rangle = \Lambda(p) \; | \nu, \omega, 0 \rangle ,$$

where $\Lambda(p)$ is a rotation-free Lorentz transformation, $\Lambda(p) \in L$. We assume that we know by any principle the representation τ of SL(6,C) which we have to select from the big set of representations of SL(6,C) containing the SU(6) representation ν as a representation of the compact subgroup. Then we define

$$| \nu, \omega, 0 \rangle = | \, 0 \, \rangle \, | \tau, \nu, \omega \rangle .$$

Physical SU(6) multiplets have therefore been identified with the representations of the compact subgroup SU(6) of SL(6,C) when they are at rest. The definition of states for particles in motion is straightforward. We apply the "booster" $\Lambda(p)$ to both sides of the defining relation, but note that on the right-hand side we can split the Lorentz transformation into the product of commuting operators

$$\Lambda(p) = \Lambda_0(p) \, \Sigma(p)$$

where $\Lambda_0(p)$ belongs to the orbital part L_0 and $\Sigma(p)$ lies in SL(6,C). Thus we obtain

$$| \nu, \omega, \boldsymbol{p} \rangle = | \, \boldsymbol{p} \rangle \, (\Sigma(p) \; | \tau, \nu, \omega \rangle) .$$

By definition an element U of the compact subgroup SU(6) applies to a state $| \tau, \nu, \omega \rangle$ as

$$U \; | \tau, \nu, \omega \rangle = \sum_{\omega'} U_{\omega' \omega} \; | \tau, \nu, \omega' \rangle .$$

An element V of SL(6,C) which is not in the compact subgroup will, however, bring in different SU(6) multiplets

$$V \; | \tau, \nu, \omega \rangle = \sum_{\nu' \omega'} V_{\nu' \omega', \nu \omega} \; | \tau, \nu', \omega' \rangle .$$

This holds in particular for the "booster" $\Sigma(p)$. A particle in motion therefore possesses components in different representations of the compact subgroup.

In Section 2.3.3 we saw that the subgroup chain symmetries can be derived from a model which satisfies the following conditions:

i) The group SL(6,C) is part of the symmetry and transforms under the Lorentz group in a particular manner.

ii) The compact subgroup SU(6) of SL(6,C) operates on a physical SU(6) multiplet at rest as a representation and does not change the state of motion of this multiplet.

Since both conditions are fulfilled by our model we can state:

If physical SU(6) multiplets ν_i are given each one of which is contained in a representation τ_i of SL(6,C), and if these representations τ_i are known to couple to at least one invariant form, the predictions of the subgroup chain symmetries are valid for these SU(6) multiplets.

The same statement can obviously be made for a model built on the group SU(6,6), see Ref. [361]. The existence of the subgroup chain symmetries was first proved in Ref. [350]. The role played by the subgroup symmetries in such a model will become quite clear in the following discussion.

Consider a particle of a physical multiplet ν moving with momentum p. Those elements of SL(6,C) which bring this state into other states of the same multiplet ν with the same momentum p, have the form

$$\Sigma(p)\, U\, \Sigma(p)^{-1}$$

with U from the compact subgroup SU(6) and $\Sigma(p)$ as defined above. These elements form a subgroup $SU(6)_p$. The intersection of several such subgroups yields the groups of the subgroup chain of SU(6). In this manner we can prove the following statement:

Let a set of particles of SU(6) multiplets ν_i with momenta p_i be given. The maximal subgroup of SL(6,C) whose elements transform each particle i into a particle of the same multiplet ν_i with momentum p_i is that group of the subgroup chain of SU(6) which belongs to the set of momenta p_i (see Ref. [330][21]). In a two-particle decay process above threshold the maximal subgroup of the spin-independence group SL(2,C) is the helicity group U(1). The model studied here therefore does not encounter a spin-conservation which would forbid processes like

$$\varrho \to 2\pi \quad \text{and} \quad N^*_{3/2\;3/2} \to N + \pi\,.$$

It is straightforward to introduce classical fields for this model. Fourier-transforming the states of the representation space spanned by

$$|\,\mathbf{p}\rangle\,|\,\tau, \nu, \omega\rangle$$

we obtain quantities which we write $\Phi(x, z)$. The first variable x denotes a four-vector in Minkowski space, the second variable z describes the degree of freedom needed for the application of elements of SL(6,C). The exact meaning of this variable will be explained in Sections 4.1.2 and 4.1.3.

The field $\Phi(x, z)$ satisfies the Klein-Gordon equation

$$(\Box_x - M^2)\, \Phi(x, z) = 0\,.$$

The orbital component L_0 of the homogeneous Lorentz group operates only on x, whereas SL(6,C) and its subgroup L_r apply to the variable z. This is the origin of the notation "orbital" and "spin" parts of the Lorentz group.

Let us now say some words about the assumption made above that the orbital spin S' vanishes. This assumption is introduced for the sake of convenience. If S'

[21]) The further erroneous statement made in that paper that all predictions which can be made for a given set of SU(6) multiplets ω_i, each one belonging to a representation τ_i, are due to the subgroup chain symmetries, is based on wrong arguments.

would be non-zero, the spin of a particle at rest would consist of contributions from both the groups P' and SL(6,C). Take for example a 35-plet. For $S' = 0$ the spin content is

$$\{S' = 0{,}35\} = (8{,}1) \oplus (8{,}3) \oplus (1{,}3)$$

whereas for $S' = 1$ we obtain

$$\{S' = 1{,}35\} = (8{,}3) \oplus (8{,}5 \oplus 3 \oplus 1) \oplus (1{,}5 \oplus 3 \oplus 1)\ .$$

Since this bears some resemblance to the l-excitation of elementary particles (see Refs. [119, 157, 159, 160]) we may adopt this notion for theories with $S' \neq 0$. On the other hand, we denote the excitation of particle multiplets implied by the non-compactness of SL(6,C) "relativistic SU(6)" excitation. The idea that non-compact groups could be used to generate infinite series of multiplets with or without non trivial (that means non-perturbative) mass formulae, goes back to Ref. [41].

4.1.2 Unitary representations for SL(2,C)

There are two reasons for studying unitary representations for the group SL(2,C) in this context. First, this group is a simple example for SL(n,C) groups and allows us to display general methods of constructing representations and to work with them. Besides this technical reason, this group may well be of physical importance as a model defining relativistic spin independence. It is a subgroup of SL(6,C) and can be built into a symmetry of the type

$$\{L \times (\mathrm{SL}(2,\,\mathrm{C}) \otimes T_4)\} \otimes \mathrm{SU}(3)$$

which is simply a subgroup of the group we are studying in this section. We shall come back to this model later. Here we only note that the reduction of a unitary representation of SL(6,C) into unitary representations of the subgroup SL(2,C) $\otimes$ SU(3) raises problems connected with the normalization of states which are characteristic for the reduction of representations of non-compact groups into representations of non-compact subgroups.

Finite dimensional representations of SL(2,C) can be given in terms of spinors. Let ξ_1, ξ_2 be two variables transforming as

$$\xi' = \xi\, g\ ,\quad \xi = (\xi^1, \xi^2)\ ,\quad g \in \mathrm{SL}(2,\,\mathrm{C})\ .$$

Homogeneous polynomials $F(\xi)$ of degree λ in ξ and of degree μ in the conjugate variables $\bar{\xi}$ define a carrier space for an irreducible finite representation SL(2,C), if we transform these polynomials as

$$T_g\, F(\xi) = F(\xi\, g)\ .$$

If we expand $F(\xi)$,

$$F(\xi) = \sum_{\alpha,\beta} \chi_{\alpha_1\alpha_2\ldots\alpha_\lambda,\dot\beta_1\dot\beta_2\ldots\dot\beta_\mu}\, \xi^{\alpha_1}\,\xi^{\alpha_2}\ldots\,\xi^{\alpha_\lambda}\,\bar\xi^{\dot\beta_1}\ldots\bar\xi^{\dot\beta_\mu}\ ,$$

we obtain a set of coefficients $\chi_{\alpha_1\ldots\alpha_\lambda,\dot\beta_1\ldots\dot\beta_\mu}$ which are necessarily symmetric in the indices α and β. They define an algebraic quantity which is usually called spinor (synonymously: tensor). If the polynomial F corresponds to the spinor $\chi_{\alpha_1 \ldots \alpha_\lambda, \dot\beta_1 \ldots \dot\beta_\mu}$ the transformed polynomial is mapped on the spinor

$$\left(\prod_{\varkappa=1}^{\lambda} g_{\alpha_\varkappa}^{\gamma_\varkappa}\right)\left(\prod_{\sigma=1}^{\mu} \bar g_{\dot\beta_\sigma}^{\dot\delta_\sigma}\right) \chi_{\gamma_1 \gamma_2 \ldots \gamma_\lambda, \dot\delta_1 \dot\delta_2 \ldots \dot\delta_\mu} .$$

Another way of writing the vectors of the representation space is to introduce functions of homogeneity degree zero:

$$F(\xi) = (\xi^2)^\lambda \, (\bar\xi^2)^\mu \, f(z) ,$$

where

$$z = \frac{\xi^1}{\xi^2} , \qquad \bar z = \frac{\bar\xi^1}{\bar\xi^2} .$$

These functions $f(z)$ transform as

$$T_g \, f(z) = \alpha(g, z) \, f(z') .$$

Let

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} ,$$

then the transformed variable z' can be written as

$$z' = \frac{(\xi^1)'}{(\xi^2)'} = \frac{(\xi\, g)^1}{(\xi\, g)^2} = \frac{g_{11}\, z + g_{21}}{g_{12}\, z + g_{22}} .$$

The factor $\alpha(g, z)$ is the so-called multiplier which has the form

$$\alpha(g, z) = \frac{[(\xi\, g)^2]^\lambda}{[\xi^2]^\lambda} \, \frac{[(\bar\xi\, \bar g)^2]^\mu}{[\bar\xi^2]^\mu} = (g_{12}\, z + g_{22})^\lambda \, (\bar g_{12}\, \bar z + \bar g_{22})^\mu .$$

The basic idea of the theory of GELFAND and NEUMARK (see Ref. [166], for the particular case of SL(2, C) also Ref. [285]) consists in generalizing the function space $f(z)$ allowing general complex numbers λ, μ. The corresponding homogeneous functions $F(\xi)$ have been called generalized tensors (see Ref. [149]). In order to obtain unitary representations we must metricize the function space. We do this by means of an ansatz

$$(f, h) = \int M(x, y) \, \overline{f(x)} \, h(y) \, dx \, dy ,$$

where $x = x_1 + i x_2$, $y = y_1 + i y_2$, $dx = dx_1 dx_2$, $dy = dy_1 dy_2$.

The kernel $M(x, y)$ which will be specified later as a function or a distribution must be symmetric

$$M(x, y) = \overline{M(y, x)} ,$$

positive definite, and its support must be the closure of a transitive set of pairs (x, y). The first condition guarantees symmetry of the scalar products

$$(f, h) = \overline{(h, f)} ,$$

the second condition means:

$$(f, f) = 0 \quad \text{implies} \quad f \equiv 0 ,$$

and the condition of "transitive support" has the following origin.

Transforming the scalar product yields

$$(T_g f, T_g h) = \int M(x, y)\, \overline{\alpha(g, x)}\, \alpha(g, y)\, \overline{f(x')}\, h(y')\, dx\, dy$$
$$= \int M(x, y)\, \overline{\alpha(g, x)}\, \alpha(g, y)\, \overline{f(x')}\, h(y')\, J(x)\, J(y)\, dx'\, dy' .$$

Invariance of the scalar product can therefore be achieved if the metric kernel satisfies the functional equation

$$M(x', y') = M(x, y)\, \overline{\alpha(g, x)}\, \alpha(g, y)\, J(x)\, J(y) .$$

As can be easily seen, the Jacobians are

$$J(x) = | g_{12}\, x + g_{22} |^4 .$$

The first implication of the functional equation is that if (x, y) belongs to the support of M, then so does the transformed pair (x', y'). This is the first requirement of transitivity (see Section 3.1.2 and the definition of transitive manifolds used there). On the other hand, let two pairs (x, y) and $(\hat{x}, \hat{y})$ belong to the interior of the support of M. If no element $g \in \mathrm{SL}(2, \mathrm{C})$ exists such that

$$x \underset{g}{\rightarrow} \hat{x} , \quad y \underset{g}{\rightarrow} \hat{y} ,$$

we can split M into

$$M = M_1 \cup M_2 ,$$

where we choose M_1 such that its support is the closure of the set (x', y') which are created by the pair (x, y), and M_2 is zero on the interior of the support of M_1. The representation splits correspondingly and is reducible. The second requirement of transitivity which is necessary for the representation to be irreducible states that two pairs (x, y), $(\hat{x}, \hat{y})$ can always be transformed into each other by a group element g. Let us study sets of pairs which are transitive in this sense.

If (x, y) is in the set, we can always find an element g which transforms x to zero. We can obviously characterize a transitive set by the pairs $(0, y)$ it contains. This subset of pairs $(0, y) \equiv (y)$ is invariant against transformations $k \in \mathrm{SL}(2, \mathrm{C})$ which have upper triangular form,

$$k = \begin{pmatrix} k_{11} & k_{12} \\ 0 & k_{22} \end{pmatrix} , \quad k_{11}\, k_{22} = 1 , \quad k_{11} \neq 0 .$$

We find

$$(y) \underset{k}{\rightarrow} (y')$$

with

$$y' = \frac{k_{11}\, y}{k_{12}\, y + k_{22}} .$$

This formula tells us that the set (y) either consists only of the element (0) or is equal to the complete complex plane with exception of the point 0. In the first case the transitive set contains the pairs (x, x), in the second case all pairs (x, y), $x \neq y$.

In the case of the pairs (x, x) we obtain the principal series of representations. We choose the kernel M as a delta function (in the two-dimensional sense)

$$M(x, y) = \delta(x - y),$$

which satisfies the requirements of symmetry and positive definiteness. The representation space consists of square integrable functions $f(z)$

$$(f, f) = \int |f(z)|^2 \, dz < \infty .$$

From

$$(x' - y') = (x - y)(g_{12} x + g_{22})^{-1} (g_{12} y + g_{22})^{-1}$$

follows

$$\delta(x' - y') = \delta(x - y) \, | g_{12} x + g_{22} |^2 \, | g_{12} y + g_{22} |^2 .$$

The functional equation on M implies then

$$| g_{12} x + g_{22} |^4 \, | \alpha(g, x) |^2 = 1$$

or

$$\lambda + \bar{\mu} + 2 = 0 .$$

We write then

$$\lambda = -\frac{1}{2} m + \frac{i}{2} \varrho - 1 , \quad \mu = +\frac{1}{2} m + \frac{i}{2} \varrho - 1 ,$$

where m and ϱ are real.

Now we turn our attention to the compact subgroup SU(2) of SL(2,C). The homogeneous function $F(\xi)$ which corresponds to the vector of highest weight for any unitary representation of SU(2) contained in the representation of SL(2,C) has necessarily the form

$$[\xi^1]^{n_1} [\bar{\xi}^2]^{n_2} (\xi^1 \bar{\xi}^1 + \xi^2 \bar{\xi}^2)^{\gamma}$$

where n_1 and n_2 are non-negative integers, γ is arbitrary complex. This proves that

$$\lambda - \mu = (n_1 + \gamma) - (n_2 + \gamma) = n_1 - n_2 = -m$$

is integer. The spin of the SU(2) multiplet is $1/2\,(n_1 + n_2)$. It increases from the smallest value k_0,

$$k_0 = \frac{1}{2} \min (n_1 + n_2) = \left| \frac{1}{2} m \right|$$

in integer steps, and each spin is just contained once in the infinite multiplet. These arguments are not restricted to the principal series. If we use the parametrization for λ and μ in terms of m and ϱ, with ϱ as an arbitrary complex number, m must always be integer.

The case $x \neq y$ leads to the supplementary series of representations. The functional equation on M expresses that an arbitrary transformation changes the kernel only by a factor. We call functions with this property of transformation "quasi-invariant functions" (see Ref. [347]). We construct a set of "elementary" quasi-invariants in the following way. We use spinor variables ξ^i, η^i and construct SL(2,C) invariants out of them. There is only one,

$$\xi^1 \eta^2 - \xi^2 \eta^1 .$$

We translate this quantity back into a function of Gelfand variables and obtain

$$x - y.$$

We can prove that the kernel M is a product of powers of elementary quasi-invariants (in the general case we may have also delta functions instead of powers, as in the case of the principal series):

$$M(x, y) = |x - y|^{-A} \left(\frac{x-y}{\bar{x}-\bar{y}}\right)^{-B}.$$

Symmetry requires that A is real and B is purely imaginary. Further we have

$$\frac{M(x', y')}{M(x, y)} = |g_{12} x + g_{22}|^A \, |g_{12} y + g_{22}|^A \left(\frac{g_{12} x + g_{22}}{\bar{g}_{12} \bar{x} + \bar{g}_{22}}\right)^B \left(\frac{g_{12} y + g_{22}}{\bar{g}_{12} \bar{y} + \bar{g}_{22}}\right)^B.$$

We compare this with the functional equation for M and obtain

$$A = \lambda + \mu + 4 = i\varrho + 2, \quad B = -m.$$

Since B is imaginary and m is integer we find

$$B = m = 0.$$

If we introduce $\varrho = i\sigma$, we can write the kernel M finally

$$M(x, y) = |x - y|^{-2+\sigma}.$$

It can be shown that the condition of positive definiteness restricts the parameter σ to

$$0 < \sigma < 2.$$

The arguments used here to derive unitary representations of the group SL(2, C) have a form which enables us to generalize them in several directions. In the next paragraph we will introduce spinor variables for the group SL(6, C). The construction of representations belonging to the principal series or to the supplementary series can again be based on an investigation of transitive pairs of variables (see Ref. [166]).

In a manner similar to that applied when building the metric kernel M out of elementary quasi-invariants, we can find generalized vector coupling coefficients which are needed to construct invariant forms for an arbitrary number n of representations. Doing this we must first study the transitive domains in the manifold of n-tupels of variables, then find all elementary quasi-invariants, and finally solve a power ansatz. If the domains have lower dimension than the maximal possible one, we can substitute delta functions for some of the powers. This programme was performed in Ref. [347].

We note finally that the transformation of the Gelfand variable z,

$$z \underset{g}{\rightarrow} z',$$

can be rewritten in terms of matrices:

$$\mathsf{z} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \quad \mathsf{z}\, g = k\, \mathsf{z}' = \begin{pmatrix} k_{11} & k_{12} \\ 0 & k_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z' & 1 \end{pmatrix}.$$

In the product zg we split off a triangular matrix k. z' is then uniquely determined. We can interpret the representations therefore as linear operators in Hilbert spaces of functions over the right cosets of the subgroup K consisting of upper triangular matrices k.

4.1.3 Unitary representations for SL(6, C)

In Section 1.2.1 we introduced tensor representations for the group SU(6) using a vector space of homogeneous polynomials $F(\xi)$ depending in variables $\Delta_{(k)}^{i_1 i_2 \ldots i_k}(\xi)$ with degrees of homogeneity f_k, $k = 1, 2, \ldots 5$. Similarly we obtain finite-dimensional, non-unitary representations of SL(6, C) if we use the enlarged set of spinorial variables $\Delta_{(k)}^{i_1 i_2 \ldots i_k}(\xi)$ and $\Delta_{(k)}^{i_1 i_2 \ldots i_k}(\bar{\xi})$, and take ξ as a matrix of SL(6, C). The homogeneities for $\Delta_{(k)}(\xi)$ and $\Delta_{(k)}(\bar{\xi})$ will be λ_k and μ_k, $k = 1, 2, \ldots 5$, respectively.

As for SL(2, C), we introduce in addition polynomials of homogeneity zero by extracting powers

$$F(\xi) = \prod_{k=1}^{5} [\Delta_{(k)}^{6-k+1,\,6-k+2,\ldots,6}(\xi)]^{\lambda_k} \, [\Delta_{(k)}^{6-k+1,\,6-k+2,\ldots,6}(\bar{\xi})]^{\mu_k} f(z) \, .$$

We have to specify on which variables z the functions f depend.

We know that the variables Δ are in general not independent, e.g. the following identity must hold:

$$\Delta_{(2)}^{12}\, \Delta_{(1)}^{3} + \Delta_{(2)}^{23}\, \Delta_{(1)}^{1} + \Delta_{(2)}^{31}\, \Delta_{(1)}^{2} = 0 \, .$$

However, we want f to depend only on independent variables z. Therefore we have to eliminate the superficial variables. This can be done in the following manner.

We consider first non-degenerate representations, that means

$$|\lambda_k|^2 + |\mu_k|^2 > 0$$

for all $k = 1, 2, \ldots 5$. We start with the variables $\Delta_{(1)}$ and introduce the notation

$$z_{6k} = \frac{\Delta_{(1)}^{k}(\xi)}{\Delta_{(1)}^{6}(\xi)}, \qquad k = 1, 2, \ldots, 5 \, .$$

Next we consider $\Delta_{(2)}$ and eliminate all but $\Delta_{(2)}^{i6}$ by means of identities such as the one written above. Then we define

$$z_{5k} = \frac{\Delta_{(2)}^{k6}(\xi)}{\Delta_{(2)}^{56}(\xi)}, \qquad k = 1, 2, \ldots, 4 \, .$$

Proceeding in this manner we get the independent variables

$$z_{pq} = \frac{\Delta_{(6-p+1)}^{q,\,p+1,\,p+2,\ldots,6}(\xi)}{\Delta_{(6-p+1)}^{p,\,p+1,\,p+2,\ldots,6}(\xi)}, \qquad 2 \leqslant p \leqslant 6, \; 1 \leqslant q < p \, .$$

If $F(\xi)$ was a homogeneous polynomial, we obtain in this manner certain rational functions $f(z)$.

It can be proved by simple algebraic calculation that the variables z fit into a lower triangular matrix

$$z = \begin{pmatrix} 1 & & & & & \\ z_{21} & 1 & & & 0 & \\ z_{31} & z_{32} & 1 & & & \\ z_{41} & z_{42} & z_{43} & 1 & & \\ z_{51} & z_{52} & z_{53} & z_{54} & 1 & \\ z_{61} & z_{62} & z_{63} & z_{64} & z_{65} & 1 \end{pmatrix},$$

which can be obtained from the matrix ξ by splitting an upper triangular matrix $k \in SL(6, C)$:

$$\xi = k\, z\,,$$

$$k = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ & & k_{33} & k_{34} & k_{35} & k_{36} \\ & 0 & & k_{44} & k_{45} & k_{46} \\ & & & & k_{55} & k_{56} \\ & & & & & k_{66} \end{pmatrix}.$$

The matrix k is uniquely determined.

If we transform the matrix ξ,

$$\xi' = \xi\, g\,,$$

this induces a transformation of the matrix z of the following kind

$$k\, z\, g = k_1\, z' \quad \text{or} \quad z\, g = (k^{-1}\, k_1)\, z'\,.$$

We have to multiply z by g on the right and split a triangular matrix k' on the left, $k' = k^{-1}\, k_1$. We recognize that the factor k of ξ may be chosen arbitrarily without changing the functions $f(z)$.

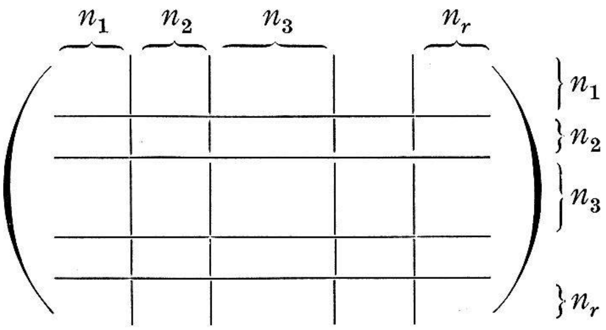
The case of degenerate representations is a bit more complicated. Let

$$|\lambda_{k_s}|^2 + |\mu_{k_s}|^2 > 0\,, \qquad s = 2, 3, \ldots, r\,,$$

whereas

$$|\lambda_k|^2 + |\mu_k|^2 = 0$$

for all other subscripts k. We bring the subscripts k_s into decreasing order. We divide the matrix ξ into blocks

$\xi =$ 

of length $n_1, n_2, \ldots n_r$,

$$\sum_{i=1}^{v} n_i = 6 ,$$

in such a manner that the subscripts k_s are given by

$$k_s = \sum_{t=s}^{v} n_t \quad \text{or} \quad k_s - k_{s+1} = n_s .$$

It can again be shown that the elimination of superficial variables can be performed in such a way that variables z_{pq} result which fit into a lower triangular matrix z of the same block structure as has been introduced for the matrix ξ with unit matrices in the diagonal blocks. We can again split

$$\xi = k\, z$$

with the upper triangular matrix k with arbitrary blocks on the diagonal. This splitting is unique. The formulae connecting z_{pq} and the variables $\varDelta$ can be found in Ref. [344].

In any case we have obtained rational functions $f(z)$ where z may represent all the matrix elements of a lower triangular matrix. These functions transform as

$$T_g\, f(z) = \alpha(z, g)\, f(z') .$$

The multiplier is obviously given by

$$\alpha(z, g) = \prod_{k=1}^{5} \left[\frac{\varDelta_{(k)}^{6-k+1,\, 6-k+2, \ldots, 6}\, (\xi')}{\varDelta_{(k)}^{6-k+1,\, 6-k+2, \ldots, 6}\, (\xi)} \right]^{\lambda_k} \left[\frac{\varDelta_{(k)}^{6-k+1,\, 6-k+2, \ldots, 6}\, (\bar{\xi}')}{\varDelta_{(k)}^{6-k+1,\, 6-k+2, \ldots, 6}\, (\bar{\xi})} \right]^{\mu_k} .$$

According to the theory of Gelfand and Neumark we generalize the functions $f(z)$ and allow λ_k and μ_k to take arbitrary complex values. In order to metricize the resulting space of functions, we make an ansatz

$$(f, h) = \int M(x, y)\, \overline{f(x)}\, h(y)\, dx\, dy ,$$

where dx is the product of all differential of the real and imaginary parts of the matrix elements of x. The kernel $M(x, y) = \delta\,(x - y)$ (in the multidimensional sense) leads to the principal series of representations. Kernels corresponding to other transitive domains of the manifold of matrix pairs (x, y) give rise to different supplementary series (compare Ref. [166]). The homogeneities λ_k and μ_k are submitted to certain conditions which guarantee the invariance of the scalar product (consequently the unitarity of the representation). For the principal series we find

$$\lambda_{k_s} = -\frac{1}{2}\,(m_s - m_{s-1}) + \frac{i}{2}\,(\varrho_s - \varrho_{s-1}) - \frac{1}{2}\,(n_s + n_{s-1}) ,$$
$$\mu_{k_s} = +\frac{1}{2}\,(m_s - m_{s-1}) + \frac{i}{2}\,(\varrho_s - \varrho_{s-1}) - \frac{1}{2}\,(n_s + n_{s-1}) .$$

We have used here the same symbols k_s and n_s as above. m_s are integer, ϱ_s are real. We may normalize m_1 and ϱ_1 to zero.

The order of the invariants m and ϱ and the parameters n characterizing the degeneracy, is unessential. Indeed, the following theorem of equivalence can be proved (see Ref. [166]):

Let a representation be given by m_s, ϱ_s, n_s, $s = 1, 2, \ldots r$. Any permutation of the subscripts s leads to an equivalent representation.

If we apply this theorem to a representation of SL(2,C) we recognize that the pairs m, ϱ and $-m$, $-\varrho$ belong to equivalent representations[22]).

Another corollary says that we can always bring the parameters m into a decreasing order,

$$m_s \geqq m_{s'}\,, \quad \text{if} \quad s < s'\,, \qquad m_6 = 0\,.$$

As in the case of SL(2,C), the integers m are intimately connected with the representations of the compact subgroup SU(6) which are contained in the SL(6,C) representation. The following content theorem holds, the proof of which is again contained in Ref. [166], (we give it only for the non-degenerate case):

We assume that the m_k are ordered in a decreasing sequence, $m_6 = 0$. We construct by means of these parameters m_k the weight

$$M = \sum_{k=1}^{5}{}' \, m_k \, N_k$$

as in Section 1.2.1. If this weight is q-fold degenerate in a given representation of SU(6) ($q = 0$ included), this representation of SU(6) is contained in the SL(6,C) representation q times.

As a corollary we find that the SU(6) representation, the highest weight of which is just M, is contained in the SL(6,C) representation exactly once. SU(6) representations with highest weight lower than M cannot be contained. We can therefore call M the "lowest highest weight" of all SU(6) representations contained in the SL(6,C) representation. For the degenerate case see Ref. [166].

4.2 *Physical applications of the infinite multiplet models*

4.2.1 Quantized global fields and the spin-statistics theorem

As usual we apply the symmetry of strong interaction to the scattering matrix and not to a Lagrangian of any field theory whatsoever. The scattering matrix is the operator which transforms scattering states of the "in"-type into scattering states of the "out"-type. Within a quantized field theory these in- and out-states can be described by asymptotic fields which behave as free fields.

In Section 4.1.1 we introduced fields $\Phi(x, z)$ which can be interpreted as "global free fields" in the sense that the different SU(6) multiplets are described globally by one unitary representation of the group SL(6,C). The argument x denotes the point in Minkowski space, the variable z is Gelfand's SL(6,C) variable, and the orbital spin is assumed to be zero. In spite of the difficulties involved in a theory of interacting quantized fields, the properties of such quantized global free fields have recently

[22]) In this case $m_1 = \varrho_1 = 0$, $m_2 = m$, $\varrho_2 = \varrho$. After a transposition and new normalization $m_1 = \varrho_1 = 0$ we obtain the result quoted.

attracted some interest. Let us discuss them briefly. We start with the classical Lagrangian. Invariance under the group $G = L \times [\mathrm{SL}(6,\mathrm{C}) \otimes T_4)$ requires that the Lagrangian has the following form

$$\mathfrak{L} = \int \left\{ \frac{\partial}{\partial x_\mu} \bar{\Phi}(x, z_1) \frac{\partial}{\partial x^\mu} \Phi(x, z_2) - \mu^2 \bar{\Phi}(x, z_1) \Phi(x, z_2) \right\} M(z_1, z_2)\, dz_1\, dz_2\, d^4x \,,$$

where M is the metric defining the scalar product in the representation space for SL(6,C) (see Section 4.1.2 and 4.1.3). Invariance with respect to the orbital group forces us to contract the derivatives $\partial/\partial x_\mu$ with themselves; this makes the Lagrangian unique. Only when the fields bear an additional orbital spin is it possible to construct a Lagrangian of first order in the derivatives. The Hamiltonian is similarly

$$\mathcal{H} = \int \left\{ \frac{\partial}{\partial x_0} \bar{\Phi} \frac{\partial}{\partial x_0} \Phi + \Delta\bar{\Phi}\, \Delta\Phi + \mu^2 \bar{\Phi}\, \bar{\Phi} \right\} M(z_1, z_2)\, dz_1\, dz_2\, d^3x \,.$$

If we require in addition invariance of the model against parity reflections, we must distinguish between the following two cases. A parity transformation may map the SL(6,C) representation onto itself or onto a different inequivalent representation (see the discussion in Section 4.2.3). In the former case the expressions for the Lagrangian and Hamiltonian given above are already invariant, since the parity transformation is represented by a unitary operator. In the second case we must introduce two fields which are parity transforms of each other

$$\Phi_1, \Phi_2, \quad \Phi_2 = P\,\Phi_1 \,.$$

The Lagrangian is additive in both fields

$$\mathfrak{L} = \mathfrak{L}(\Phi_1) + \mathfrak{L}(\Phi_2) \,,$$

$$\mathcal{H} = \mathcal{H}(\Phi_1) + \mathcal{H}(\Phi_2) \,.$$

Now we quantize these fields such that a local field theory results. By this we mean: the Lagrangian and Hamiltonian are integrals (over four, respectively three, dimensions) of densities which themselves are normal products of field operators $\psi(x, z)$, their adjoints, or their parity transforms. These operators belong to representations of the group G of the same type as the classical fields $\Phi(x, z)$.

Using arguments like Pauli's (see Ref. [306]) we can deduce from the form of the Hamiltonian that the quantized fields are boson fields. The same result can be obtained introducing the operator fields in WEINBERG'S manner (Ref. [389], see also Ref. [131]), which we sketch briefly.

We consider the physical particles in states $|\, \nu, \omega, \boldsymbol{p}\rangle$ (this notation has been introduced in Section 4.1.1) and define the creation and annihilation operators for such states. They commute or anticommute in the canonical manner. Then we go over to annihilation operators for states in the product representation $|\, \boldsymbol{p}\rangle \cdot |\, \tau, \nu, \omega\rangle$ and apply a Fourier transformation. The result is the positive frequency part of a field operator, $\psi^{(+)}(x, z)$. In the same way we introduce a positive frequency operator $[\psi^{(-)}(x, z)]^\dagger$ which annihilates antiparticles.

The antiparticles must necessarily belong to the conjugate representation of SL(6,C). Taking the adjoint of the latter operator and adding both fields we obtain the local field

$$\psi(x, z) = \xi\, \psi^{(+)}(x, z) + \eta\, \psi^{(-)}(x, z)\,,$$

with two arbitrary complex parameters ξ, η. We note the formula

$$[\psi^{(+)}(x, z_1), (\psi^{(+)}(y, z_2))^\dagger]_\pm = i\, \Delta^{(+)}(x - y)\, M^{-1}(z_1, z_2)\,,$$

where M^{-1} is an "inverse" of the kernel M. Since this field operator ψ satisfies the Klein-Gordon equation, the positive and negative frequency parts both transform as irreducible representations of the group G. In the case of two fields Φ_1, Φ_2 we may consider these as dynamically independent. We need then two sorts of particles and antiparticles.

The fact that the Hamiltonian is diagonal in the Fock space with positive eigenvalues, forces us to substitute the adjoint field operator $\psi^\dagger$ for the conjugate classical field. This is not trivial. Indeed, we could think that $\psi^\dagger$ is substituted for the parity transformed conjugate field $P\overline{\Phi}$, see Ref. [151] which contains the same SU(6) multiplets as $\overline{\Phi}$. The Hamiltonian would take the form

$$\mathcal{H} = :\int \psi^\dagger(x, z_1) \left[\frac{\overleftarrow{\partial}}{\partial x_0} \frac{\overrightarrow{\partial}}{\partial x_0} + \overleftarrow{\Delta}\, \overrightarrow{\Delta} + \mu^2\right] \psi(x, z_2)\, P\, M(z_1, z_2)\, dz_1\, dz_2\, d^3x:,$$

with P as a number matrix. States from SU(6) multiplets with different eigenparities would give positive as well as negative contributions to the energy. We shall see in Section 4.2.3 that such SU(6) representations exist always in one SL(6,C) representation. This argument rules out this choice for $\psi^\dagger$. The Lagrangian is therefore of the following operator form:

$$\mathcal{L} = :\int \psi^\dagger(x, z_1) \left[\frac{\overleftarrow{\partial}}{\partial x_\mu} \frac{\overrightarrow{\partial}}{\partial x^\mu} - \mu^2\right] \psi(x, z_2)\, M(z_1, z_2)\, dz_1\, dz_2\, d^4x:$$

In the case of two independent fields, the same argument together with the requirement that the Hamiltonian is diagonal in Fock space yields as for unquantized fields

$$\mathcal{L} = \mathcal{L}(\psi_1) + \mathcal{L}(\psi_2)\,.$$

To assure that

$$[\mathcal{H}(x), \mathcal{H}(y)]_- = 0$$

for space-like distances x–y, where $\mathcal{H}(x)$ is the Hamiltonian density, we require

$$[\psi(x, z_1), \psi^\dagger(y, z_2)]_\pm = 0\,,$$
$$[\psi_\alpha(x, z_1), \psi_\alpha^\dagger(y, z_2)]_\pm = 0\,, \quad \alpha = 1, 2$$

for space-like distances. The brackets on the left-hand side yield

$$+\, i\, M^{-1}(z_1, z_2)\, [|\,\xi\,|^2\, \Delta^{(+)}(x - y) \pm |\,\eta\,|^2\, \Delta^{(+)}(y - x)]$$

(respectively M_α^{-1}). Only for $|\,\xi\,|^2 = |\,\eta\,|^2$ and the lower sign can our requirements be satisfied. This result was first established in Ref. [132].

For fields with half-integral orbital spin we can similarly assert that we must quantize them like fermions. It is, however, in no way possible to obtain a fermion field which contains a 56-plet of SU(6) with half-integral total spin.

It seems necessary to add a remark about the fields introduced by GELFAND and JAGLOM (see the presentation given in Ref. [285], Chapter IV; the references to the original literature can also be found there). These authors use unitary and non-unitary infinite dimensional representations of SL(2,C), which group is identified with the homogeneous Lorentz group. The theory is based on a covariant field equation of first order in the derivatives, a generalization of the Dirac equation. Instead of matrices γ_μ which couple the two-dimensional Weyl spinors, they use infinite-dimensional matrices coupling sets of infinite representations of the homogeneous Lorentz group. A generalization of this approach to groups SL(6,C) or SU(6,6) would bring in the 72 or 143 momenta.

The Gelfand-Jaglom fields themselves are not required to satisfy the Klein-Gordon equation: they split into one or more infinite sequences of irreducible representations of the Poincaré group. The masses of these irreducible components depend on the spin, however, in a completely uninteresting manner; the masses decrease monotonically for increasing spin and accumulate at zero. Since there is no direct connection between the Gelfand-Jaglom fields and the fields introduced by us above in particular the results concerning the connection between spin and statistics cannot be transferred from one model to the other.

The conclusion we draw from this discussion, namely that in a quantized theory of global fields there is no room for a fermion field with half-integral spin fitting into a 56-plet of SU(6), can be interpreted such that only the familiar fields with one SU(6) multiplet in them have a meaning as asymptotic fields which can be submitted to second quantization. The non-compact symmetry SL(6,C) is used only to group these fields together and to restrict the form of the scattering matrix.

We should always keep in mind that even if we start from a local, quantized theory of global interacting fields, putting the baryons say in a 189-plet with orbital spin $^1/_2$, we could expect that the S-matrix comes out unitary and crossing symmetric in the unitary basis used for the asymptotic fields. If we translate this matrix into the spinorial basis, for which the Mandelstam postulates are usually formulated, at least crossing symmetry will be lost. The latter point will be discussed in Section 4.2.7.

4.2.2 The assignment of particles to states of a representation of SL(6,C)

The problem how to assign the known resonances to states of a unitary representation of SL(6,C) involves two separate tasks:

i) Let an SU(6) multiplet be given. How can we decide into which representation of SL(6,C) we must embed it? A minor restriction on the set of possible candidates is due to the postulate that parity acts as a unitary operator in the irreducible representation space for one representation of SL(6,C). We discuss this requirement in Section 4.2.3. Out of the continously infinite number of representations a still infinite subset is left by the parity postulate. The general answer which representation to choose out of this infinite subset is not known. The familiar attitude in such cases is to try starting with the simplest representations and look what the physical predictions are.

ii) Assume the representation of SL(6,C) to be known. How can the Gelfand-Neumark function be constructed which corresponds to a certain state of a physical particle? This technical problem has a simple solution (see Ref. [346]). We start in the rest system of the particle and construct the homogeneous function (see Section 4.1.3) corresponding to the particular component of the SU(6) tensor. We translate this homogeneous function into a Gelfand-Neumark function in the manner discussed in Section 4.1.3. Finally, we apply the pure Lorentz transformation which brings the particle into the state of motion desired. As an illustration, we discuss the most important examples: the baryons 56^+ and the mesons 35^-.

The baryon multiplet 56 can be considered to lie in all those representations of SL(6,C) for which the invariants m_k define a weight of the representation 56 of SU(6) (see the theorem stated at the end of Section 4.1.3; for degenerate representations we have to account for some modifications). The invariants ϱ_k can be chosen arbitrarily, but as we shall see in the discussion of the parity requirement it is natural to take them equal to zero. As we shall recognize later, one of the most serious problems in the infinite multiplet models is to assign all the infinite states of the multiplets to known resonances and particles. To reduce this problem we consider primarily representations with maximal possible degeneracy. They contain the smallest but still an infinite number of SU(6) multiplets. For the baryons we may choose the degeneracy (see Section 4.1.3)

$$n_1 = 5\,, \quad n_2 = 1\,.$$

This implies that the homogeneous functions are constructed only by means of the quantities

$$\Delta_{(1)}(\xi)\, \Delta_{(1)}(\bar{\xi})\,.$$

As a basis for the baryon tensor at rest we may then take (see Refs. [148, 346])

$$F^{A_1 A_2 A_3} = \Delta_{(1)}^{A_1}(\xi)\, \Delta_{(1)}^{A_2}(\xi)\, \Delta_{(1)}^{A_3}(\xi) \left[\sum_{C=1}^{6} \Delta_{(1)}^{C}(\xi)\, \Delta_{(1)}^{C}(\bar{\xi})\right]^{-9/2}$$

which corresponds to

$$m_1 = 0\,, \quad m_2 = -3\,, \quad \varrho_1 = \varrho_2 = 0\,.$$

This representation starts with a multiplet 56^+, the next representations are 700^- and 4536^+. Each representation appears just once. The parity assignments will turn out in the next paragraph.

The case of the mesons is a bit more intricate. Since we want to couple the mesons to the baryons, the degeneracy of the mesons cannot be chosen freely but must be $n_1 = 1$, $n_2 = 4$, $n_3 = 1$ or even a refinement thereof, which means: n_2 is split into further parts. The proof for this statement can be found in the results of Refs. [346, 347]. Let us take $n_2 = 4$. We have then to deal with variables $\Delta_{(1)}(\xi)$, $\Delta_{(1)}(\bar{\xi})$, $\Delta_{(5)}(\xi)$, $\Delta_{(5)}(\bar{\xi})$. With these variables we may obtain different representations:

1) We take the invariants to be

$$m_1 = 0\,, \quad m_2 = -1\,, \quad m_3 = -2\,, \quad \varrho_1 = \varrho_2 = \varrho_3 = 0\,.$$

This representation belongs to the principal series and starts with the representation 35^-. It does not contain a singlet. The basis for the meson tensor at rest is

$$F_A^B(\xi) = \Delta_A(\xi)\,\Delta^B(\xi)\left[\sum_{C=1}^{6}\Delta_C(\xi)\,\Delta_C(\bar{\xi})\right]^{-3}\left[\sum_{D=1}^{6}\Delta^D(\xi)\,\Delta^D(\bar{\xi})\right]^{-3},$$

$$\Delta^A(\xi) \equiv \Delta_{(1)}^A(\xi)\,,\quad \Delta_A(\xi) \equiv \frac{1}{5\,!}\,\varepsilon_{A B_1 B_2 \ldots B_5}\,\Delta_{(5)}^{B_1 B_2 \ldots B_5}(\xi)\,.$$

2) We take the invariants as

$$m_1 = m_2 = m_3 = 0\,,\quad \varrho_1 = \varrho_2 = \varrho_3 = 0\,.$$

This representation still belongs to the principal series.

3) Let us now introduce a representation of the supplementary series, which has the same degeneracy as the representations discussed before, namely $n_1 = 1$, $n_2 = 4$, $n_3 = 1$. With the invariants

$$m_1 = m_2 = m_3 = 0\,,\quad \varrho_1 = 0\,,\quad \varrho_2 = \varrho_3 - \varrho_2 = \sigma\, i\,,\quad 0 < \sigma < 1\,,$$

the content concerning SU(6) multiplets is the same as for the representation (2). There is one singlet (singlets are always uniquely determined), two 35-plets of opposite parity, etc.

The metric in the space of the Gelfand-Neumark functions of representation (3) is defined as

$$|f|^2 = \int f(z_1)\,\overline{f(z_2)}\,M(z_1, z_2)\,dz_1\,dz_2\,,$$

with

$$M(z_1, z_2) = |\,u\,v\,|^{-5+\sigma}\,,$$

$$u(z_1, z_2) = (z_1)_{61} - (z_2)_{61} + \sum_{k=2}^{5}[(z_1)_{6k} - (z_2)_{6k}]\,(z_1)_{k1}\,,$$

$$v(z_1, z_2) = u(z_2, z_1)\,.$$

The scalar product integral has to be regularized obviously. This is the reason why GELFAND and NEUMARK (Ref. [166]) give these degenerate supplementary series only in such a form that the blocks $n_s = 1$ are grouped together. In that case the scalar product can be defined without regularization.

Let us give the tensor basis for the 35-mesons in the rest system; we obtain

$$F_{(\pm)A}{}^{B}(\xi) = \frac{1}{\sqrt{2}\cdot N(\xi)}\left\{\frac{\Delta_A(\xi)\,\Delta_B(\xi) - 1/6\sum_{C=1}^{6}\Delta_C(\xi)\,\Delta_C(\bar{\xi})\,\delta_A^B}{\sum_{D=1}^{6}\Delta_D(\xi)\,\Delta_D(\bar{\xi})} \pm \frac{\Delta^A(\bar{\xi})\,\Delta^B(\xi) - 1/6\sum_{C=1}^{6}\Delta^C(\bar{\xi})\,\Delta^C(\xi)\,\delta_A^B}{\sum_{D=1}^{6}\Delta^D(\xi)\,\Delta^D(\bar{\xi})}\right\},$$

where the (+)-sign multiplet has the same parity eigenvalue as the singlet. The denominators $N(\xi)$ are defined as

$$N(\xi) = \Big[\sum_C \Delta_C(\xi)\, \Delta_C(\bar{\xi})\Big]^{5/2} \Big[\sum_D \Delta^D(\bar{\xi})\, \Delta^D(\xi)\Big]^{5/2}$$

in case of representation (2), and

$$N(\xi) = \Big[\sum_C \Delta_C(\xi)\, \Delta_C(\bar{\xi})\Big]^{5/2+\sigma/2} \Big[\sum_D \Delta^D(\bar{\xi})\, \Delta^D(\xi)\Big]^{5/2+\sigma/2}$$

in case of representation (3).

4.2.3 Parity

We adjoin the parity operator to the algebra of the group SL(6, C) by requiring that the parity operator commutes with the generators of the maximal compact subgroup

$$[P, s^t_{ij,\,k}]_- = 0\,, \quad i, j = 1, 2, 3,\ k = 0, 1, \dots 8\,,$$

$$[P, s^s_k]_- = 0\,,\ k = 1, 2, \dots 8\,.$$

Since the non-compact generators involve a polar vector, the momentum, we require that parity anticommutes with these generators

$$[P, s^t_{0j,\,k}]_+ = 0\,, \quad j = 1, 2, 3,\ k = 0, 1, \dots 8\,,$$

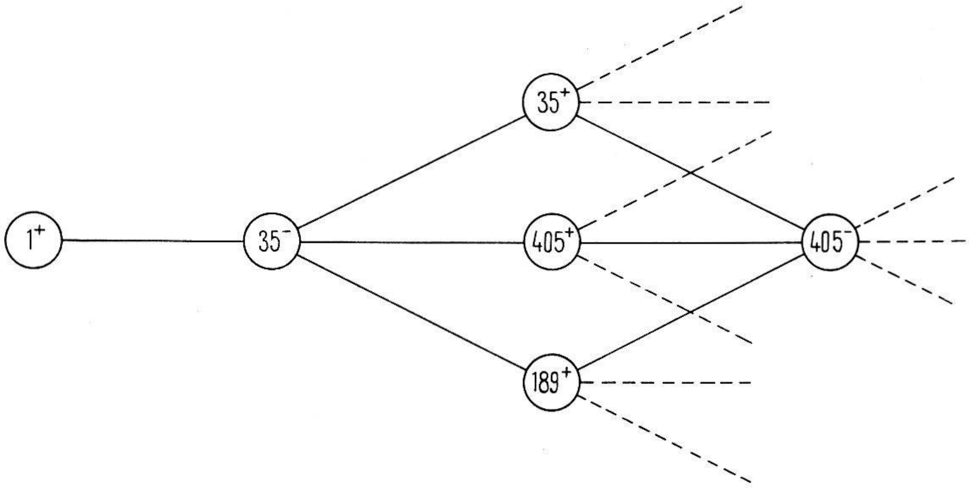
$$[P, s^p_k]_+ = 0\,,\ k = 1, 2, \dots 8\,.$$

Let $D(S)$ be a representation for the matrix $S \in$ SL(6, C). We define $D'(S) = D[(S^\dagger)^{-1}]$ as the conjugate contragredient representation. It can be shown quite easily (see Ref. [345]) that P maps the representation D on the representation D':

$$D'(S) = P\, D(S)\, P^{-1}\,.$$

Let us consider then an infinite multiplet at rest. Since P has been defined such that it commutes with the generators of SU(6), each SU(6) multiplet can be chosen as to possess an intrinsic parity eigenvalue. The parity operation maps then the SU(6) multiplets of D one to one on the multiplets of D'. If D and D' were inequivalent we would have to put both representations together in order to be able to define the parity operation as a linear operator in one space. The physical multiplets would then be linear combinations of multiplets of D and D': each multiplet would appear pairwise with opposite parity eigenvalues. Such pairs have not been observed in nature. We require therefore that D and D' are equivalent. P can then be chosen as a unitary operator in the irreducible representation space of D. As a corollary of the definition of the parity operation, we may deduce that each pair of states which can be connected by a non-vanishing matrix element of one of the non-compact generators, must possess different eigenparities. If we imagine a network formed out of vertices and lines connecting these, where the vertices are SU(6) multiplets of one infinite representation of SL(6, C) and the lines represent nonvanishing matrix elements of non-compact generators, we recognize that this net contains only loops with an even number of vertices.

We consider the following example which corresponds to the meson representations (2) and (3):



1st generation, 2nd generation, 3rd generation

There is no direct line between 1^+ and 35^+ and between 35^+ and 405^+ due to the parity postulate. We observe that the multiplets are ordered into "generations" in a natural manner. Since the role of a positive parity singlet is quite obscure (vacuum ?), we may reverse all the signs of the parity eigenvalues. The 35^--mesons belong then to the third generation together with a 405^-- and 189^--plet, and the 35^+-plet plays a certain key role.

The mathematical solution of the requirement imposed on the representations D and D' has been discussed in Ref. [345]. The unitary operator has been explicitly constructed there. The result may be formulated as follows:

Let m_s, ϱ_s, and n_s fix the unitary representation D[23]). D is equivalent to D' if and only if a permutation $s \to \pi(s)$ exists, for all $s = 1, 2, \ldots r$, such that

$$m_s = + m_{\pi(s)} ,$$

$$\varrho_s = - \varrho_{\pi(s)} ,$$

$$n_s = + n_{\pi(s)} .$$

4.2.4 Masses and general properties of vertices

In a systematic study we could proceed discussing n-point functions starting with $n = 2$ and letting n increase. But for such a presentation the information available today is still too limited.

For $n = 2$ we deal with Green's functions which if submitted to the strong symmetry depend only on the degenerate mass. Mass breaking can be introduced with the help of a perturbative ansatz of the kind

$$\Delta m^2 = \text{Gell-Mann-Okubo term} + \alpha W_\mu W^\mu .$$

where

$$W_\mu = \frac{1}{2} \varepsilon_\mu^{\nu\lambda\sigma} p_\nu s^t_{\lambda\sigma, 0} .$$

[23]) The result is even true for non-unitary representations, i.e. general complex parameters ϱ_s.

Since the momentum commutes with the generators of SL(6,C) this ansatz breaks the symmetry in a relativistically invariant manner. We obtain an additive term to the Gell-Mann-Okubo relation of the form

$$+ \alpha M^2 S (S + 1)$$

which has first been proposed by Pais for static SU(6) symmetry (see Section 1.4.2). In the perturbative approach we may certainly find a more general ansatz. However, no systematic investigation has been performed until now. The validity of the perturbative treatment is necessarily restricted to the lowest SU(6) multiplets.

In the case of three-point functions, we are faced with all the technical and interpretational difficulties of the model. We know that the collinear subgroup symmetry holds. The question arises whether there are additional predictions beyond those of the collinear subgroup. There are certainly those new results which involve different SU(6) multiplets from one ladder (predictions of second kind). They will be discussed in Section 4.2.6. But restrictions on the S-matrix might also occur which involve only one SU(6) multiplet from each ladder and which are not obtainable with the collinear subgroup (predictions of the first kind). Whether such predictions are possible depends on the number of invariant functions which can be constructed out of the particular representations involved. Two extreme cases are known: the baryon-meson vertex with the representations discussed in Section 4.2.2 which allow exactly one vertex function, and the meson-meson-meson vertex with the representation (1) of Section 4.2.2 which possesses a continuous set of invariants (these have been given in Ref. [347]). In the case of a four-point function we meet the continuous case always. Such infinite numbers of invariants must even be considered as natural for infinite dimensional unitary representations. A finite set of invariants can be achieved only if the representations used are sufficiently degenerate. We know that three representations of SL(2,C) always couple uniquely, if at all. Maximally degenerate representations (that means $r = 2$, $n_1 + n_2 = 6$) bear an intimate resemblance to representations of SL(2,C). The baryons have been chosen maximally degenerate. The mesons, however, are not degenerate enough to couple to themselves with a discrete number of invariants. We note that at present it is still unknown whether continuous sets of invariants give any predictions of both kinds beyond those of the collinear subgroup. We do not go into the details of the construction of the invariant vertices. A principal remark about the technique in the Gelfand-Neumark scheme was made in Section 4.1.2; a more general discussion of the mathematical problem can be found in Refs. [347, 348].

4.2.5 Predictions of the first kind

Fronsdal (see Ref. [150]) has discussed the vertex $\overline{B(56)}\, B(56)\, M(35)$ with the particular meson representation he uses (representation (3) in Section 4.2.2). Besides the results of the collinear subgroup, which for the mesonic form factors of the nucleon octet can be written as

$$a^F = -\frac{2}{3}\, a_C^F - \frac{5}{9}\, a_C^D ,$$

$$a^D = -\, a_C^F + \frac{2}{3}\, a_C^D ,$$

$$a_m^D : a_m^F : a :{}_m^S = 3 : 2 : 1 ,$$

(the notations are the same as in Section 1.5.2, μ and M are meson mass and baryon mass respectively) and the prediction of the static subgroup SU(6) in the annihilation channel at threshold $\mu^2 = 4\,M^2$,

$$a^D = 0\,.$$

(see Ref. [348]), he obtains one additional number in the static limit where the mass of the outgoing baryon M' is of the magnitude $M + \mu$:

$$a^D/a^F = 9/5\,.$$

Due to the collinear symmetry this corresponds to

$$a^D_c/a^F_c = -3/25\,.$$

These figures are independent of the Casimir invariant σ which was left open in Section 4.2.2. No functional dependences on μ/M, nor the interesting number

$$a^F_m/a^F_c \quad \text{at} \quad \mu^2 = 0$$

are known up to now[24]).

The fact that the $\bar{B}BM$ vertex is unique could lead us to the suggestion that the electromagnetic current is also uniquely determined. However, this is not true. Mesons and photons are treated quite differently in this model with infinite multiplets. Mesons fit into a representation which is unitary and infinite-dimensional; photons are treated in the familiar manner as belonging to a non-unitary four-vector representation which is in addition submitted to a subsidiary condition.

The electromagnetic current of baryons has been dealt with in Refs. [93, 257]. The current is composed of different contributions:

i) The convection current is of the structure

$$\left\langle 56, \omega', p' \left| \frac{1}{2} \left(s^s_3 + \frac{1}{\sqrt{3}}\, s^s_8 \right) \right| 56, \omega, p \right\rangle \frac{(p'+p)_\mu}{2\,M}\,.$$

It has first been investigated in Ref. [93]. It gives rise to a charge form factor of the Sachs type G^F_c; all the other form factors G^D_c, G^F_m, G^D_m vanish identically.

ii) The magnetization current which is constructed as the divergence of the magnetization density

$$\left\langle 56, \omega', p' \left| \frac{1}{2} \left(s^t_{\mu\nu,3} + \frac{1}{\sqrt{3}}\, s^t_{\mu\nu,8} \right) \right| 56, \omega, p \right\rangle.$$

Its contribution to the form factors is (see Ref. [257])

$$G^F_c : G^D_c : G^F_m : G^D_m = -\frac{5}{6}\,\frac{t}{4\,M^2} : +\frac{1}{10}\,\frac{t}{4\,M^2} : \frac{2}{3} : 1\,.$$

iii) Contributions which are either of higher order in the momentum or the generators of the group SL(6,C). In particular, we can introduce arbitrary polynomials in the Lorentz scalar

$$s^t_{\mu\nu,i}\,(p' + p)^\mu\,(p' - p)^\nu\,,$$

[24]) Fronsdal's method did not yet allow him to derive any functional dependences on the masses. In principle, the expressions given in Ref. [348] contain the complete information. A many-dimensional integral has still to be evaluated. The solution for this technical problem has been found recently, Ref. [257], so all the form factors will soon be known.

which are submitted to the only condition that the SU(3) indices build up a charge operator.

In any case the absolute value of the magnetic moments is free even if we only allow for contributions of type i) and ii).

A similar treatment applies to the semileptonic interactions. In that case, the weakly interacting currents can be composed of different expectation values of SL(6,C) operators, e.g.:

vector current, $\Delta I = 1$,

$$\frac{(p'+p)_\mu}{2\,M}\;\frac{1}{2}\;(s_1^s + i\,s_2^s)$$

and

$$\frac{(p'-p)^\nu}{2\,M}\;\frac{1}{2}\;(s^t_{\mu\nu,1} + i\,s^t_{\mu\nu,2})\,,$$

axial vector current, $\Delta I = 1$,

$$\frac{(p'-p)_\mu}{2\,M}\;\frac{1}{2}\;(s_1^p + i\,s_2^p)$$

and

$$\frac{(p'+p)^\nu}{2\,M}\;\frac{1}{2}\;(s^t_{\varkappa\lambda,1} + i\,s^t_{\varkappa\lambda,2})\;\varepsilon^{\varkappa\lambda}{}_{\mu\nu}\,.$$

4.2.6 Predictions of the second kind

Relations which connect different SU(6) multiplets contained in one infinite representation make sense only if the existence of such multiplets is proved in nature. So far no example for a recurrence of this type is observed. The multiplets 405^+ and 189^+ have been investigated quite thoroughly, but no candidate which might belong to a 27-plet (see Section 1.3.4) has yet been found. For baryons the situation is at least as obscure as it is for mesons.

Up to now only model calculations have been performed for the subgroup SL(2,C) $\otimes$ SU(3) of SL(6,C), Ref. [349]. For subgroups of compact symmetry groups, it is well known that the predictions of the subgroups are involved in the predictions of the enclosing group. For non-compact subgroups of non-compact symmetry groups the situation is not so simple. The reduction of a unitary representation of SL(6,C), into representations of SL(2,C) $\otimes$ SU(3) is a mathematically non-trivial problem. In order to get an idea of the problem we make use of the notion of characters which has been developed for the unitary representations of groups SL(n,C) in Gelfand's book (Ref. [166], page 93–123).

Let g be an element of SL(2,C) $\otimes$ SU(3), $g = a \cdot u$, $a \in$ SL(2,C), $u \in$ SU(3). The trace $S_{m_s \varrho_s n_s}(g)$ for a unitary representation of SL(6,C), defined by the invariants m_s and ϱ_s and the degeneracy n_s, can be decomposed in the manner[25])

$$S_{m_s \varrho_s n_s}(g) = \int\limits_{-\infty}^{+\infty} dR \sum_{M \geqslant 0}^{\infty} \sum_{\lambda_1 \lambda_2} K(R, M, \lambda_1, \lambda_2 \,|\, m_s\, \varrho_s\, n_s)\; S_{M,R}(a)\; \chi_{\lambda_1 \lambda_2}(u)\,.$$

[25]) This decomposition has not been performed explicitly. So the following facts are unproved: that we need only the principal series of representations of SL(2,C) and that the kernel K exists at all as a certain distribution. It is straightforward to include also the supplementary series in this decomposition, if this would turn out to be necessary.

$S_{M,R}(a)$ is the corresponding trace for a representation of SL(2,C) defined by the invariants M and R, $\chi_{\lambda_1 \lambda_2}(u)$ is the character of a representation of SU(3). Since the functions S and χ are known, the kernel K can in principle be determined. We expect that the support of this kernel with respect to the argument R includes at least one interval.

Take a certain SU(6) multiplet out of the infinite representation of SL(6,C), and decompose it into irreducible representations of SU(2) ⊗ SU(3). Each one of the latter representations then possesses, in general, projections onto a continuous set of irreducible representations of SL(2,C) ⊗ SU(3). On the other hand, states of one irreducible representation of SL(2,C) ⊗ SU(3) lie in general outside the Hilbert space of the representation of SL(6,C), their norm is infinite. This implies that it makes no sense to ascribe physical particles simultaneously to states of unitary irreducible representations of SL(6,C) and SL(2,C) ⊗ SU(3). It is in the scope of this article to believe that SL(6,C) is the correct group to generate the infinite multiplets. Nevertheless, an investigation of the subgroup can help us to understand better the features of models with infinite multiplets and, if it turns out that the subgroup does not generate the sequence of physical resonances, supports our belief that the group SL(6,C) is more suitable.

Infinite representations of SL(2,C) containing integral spins can be classified into π-ladders with intrinsic parity $(-1)^{k_0+1}$ and σ-ladders with intrinsic parity $(-1)^{k_0}$. The notation k_0 was defined in Section 4.1.2 as the lowest spin in the ladder. Since we are interested in the decay

$$\sigma \rightarrow \pi + \pi$$

we couple the σ-ladder to two π-ladders taking $k_0(\pi) = 0$. We find that a final state consisting of two 0^--particles has a definite signature $(-1)^{k_0(\sigma)}$. From this result (see Ref. (349]) we deduce that the physical 1^- and 2^+ octets do not fit into one σ-ladder, since in each octet at least one component decays strongly into two pseudoscalar mesons.

Another result is of more fundamental importance. If we compute the partial widths for the decays of hypothetical 2^+ and 0^+ mesons into two pseudoscalar mesons, which is possible if we put them together in a σ-ladder with $k_0 = 0$, we obtain a ratio (see Ref. [349])

$$\frac{\Gamma_{2^+ \rightarrow 0^- + 0^-}}{\Gamma_{0^+ \rightarrow 0^- + 0^-}} = 0(10^{-3})\,.$$

We now introduce some convenient notations. We call the invariant functions of the symmetry group G [or of the subgroup containing only SL(2,C)]D_i, and assume that only a finite number of them exists. The invariant functions of the inhomogeneous Lorentz group and SU(3) symmetry are denoted as A_j. Then the implications of the bigger symmetry are relations

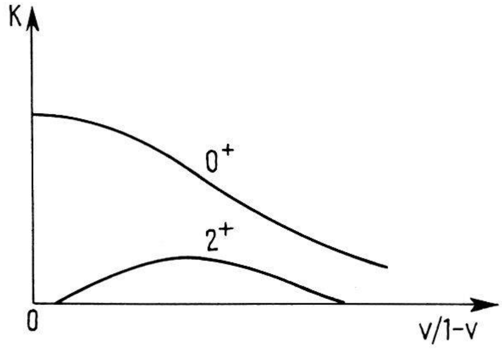
$$A_j = \sum_i K_{j\,i}\, D_i\,,$$

where the K_{ji} will be referred to as kinematical functions. The functions K_{ji} are the quantities of main interest.

In the case of the $\sigma\pi\pi$-vertex, there is only one coupling constant D. The variation of the functions K with the invariant velocity v,

$$v = \left[1 - \frac{4\,M(\pi)^2}{M(\sigma)^2}\right]^{1/2}$$

is shown in the following figure:



Qualitative behaviour of the kinematical functions K for the decay $\sigma \to \pi + \pi$.

The kinematic functions take account of the threshold behaviour:

$$\left.\begin{aligned} K_{0^+ \to 0^- + 0^-} &= 0(1)\,, \\ K_{2^+ \to 0^- + 0^-} &= 0(v^2)\,, \end{aligned}\right\} \quad \text{for } v \to 0\,.$$

The decrease of the kinematical functions for increasing mass $M(\sigma)$ is due to the fact that for increasing mass $M(\sigma)$ final states with higher and higher orbital angular momentum become accessible. The number of possible spins which can couple with this orbital angular momentum to the fixed total angular momentum grows correspondingly. The partial width for one final channel (in our case two spin-0 particles) must go down. Explicit asymptotic expansions for some kinematic functions are given in Ref. [349].

The K-function for spin-2 decay is damped away before it can reach the value of the kinematic function for spin-O decay. This explains the small ratio for the decay widths. Even if we break the symmetry and allow the masses of the spin-0 and spin-2 particles to vary independently of each other as arguments of the kinematic functions and the phase space factors up to several GeV, the order of this ratio does not change much (see Ref. [349] for exact numbers).

4.2.7 Unitarity, crossing symmetry and the substitution rule

Among the reasons to study such models with infinite multiplets was the desire to gain a deeper understanding of the unitarity problem which arose in the models discussed in Section 3. We remind the reader that the collinear and static subgroup symmetries cannot yet run into conflict with unitarity. They require only that the particles must fit into SU(6) multiplets but make no definite statement whether in a completeness sum we have to sum over one or more such SU(6) multiplets. The theories dealt with in Section 3 violate unitarity in principle. This can be understood to be a consequence of summing only over one SU(6) multiplet in the completeness relation. In a certain manner the unphysical momenta may represent the higher SU(6) multiplets which are cut off by the subsidiary conditions[26]). On the other hand, we know from the discussion of this section that an infinite number of SU(6) multiplets suitably chosen is sufficient to guarantee completeness. The natural question to ask is how the higher multiplets are weighted in the completeness sum if this is taken between two channels, and whether there is any meaning in a statement that already few SU(6) multiplets are sufficient to conserve probability in a "good" approximation. The answer we will give makes sense certainly only in so far as we consider the theories of Section 4 to be reasonable.

Generalizing the results obtained for SL(2,C) (see Section 4.2.6), which could be explained in simple physical terms of centrifugal barriers and increase of number of open channels, we suggest:

If a channel is open to several SU(6) multiplets of one infinite representation of SL(6,C), is is mainly coupled to the lowest multiplet! We note, however, that the meaning of "lowest" is unclear in this context. It could mean lowest in the sense of "highest weight" but also in the sense of generations which have been introduced in Section 4.2.3. Both notions coincide for the group SL(2,C). If this suggestion turns out to be correct in any-well-defined version, it implies that in the completeness sum only a few "low" multiplets must be taken into account. This would not only justify the theories of Section 3 physically, but also bear some resemblance to the notion "saturation" used in the current algebra approach.

Another implication of fundamental importance would be that the higher rungs will be unobservable in the near future.

The unitarity problem was the crucial defect of the models treated in Section 3. The critical problem involved in the models of Section 4 is connected with the substitution rule and crossing symmetry.

Let us consider a process described by a finite number of invariant functions D_i. The symmetry would yield predictions for the Mandelstam amplitudes A_i invariant with respect to the Poincaré group and unitary symmetry SU(3)

$$A_i = \sum_j K_{ij} D_j , \quad i = 1, 2, \ldots, n , \quad j = 1, 2, \ldots m .$$

The structure of the functions D_j is undetermined by the symmetry. In a local field theory of global interacting fields one could perhaps make them satisfy the substitution rule, but we may neglect this. Eliminating them we obtain n-m identities

$$\sum_i \varkappa_{l i} A_i = 0 , \quad l = 1, 2, \ldots, n - m .$$

In the simplest case of only one D-function we obtain

$$A_i K_j - K_i A_j = 0$$

for any pair i, j. Since the K-functions are analytic functions prescribed by the theory of representations for the group G and $P \otimes$ SU(3), these relations enable us to study

[26]) The idea that the 72 (respectively 143) momenta are to be interpreted as "internal" and generate higher SU(6) multiplets if added to a basic multiplet, allows us to connect the models with finite and infinite multiplets in an intuitive manner. This idea is perhaps also mathematically formalizable.

analytic properties of the functions A; in particular, whether they satisfy postulates like crossing symmetry. As has been inspected from definite examples (say those of Ref. [349]), these identities for the amplitudes A inhibit them from showing crossing symmetry and obeying the substitution rule, in contradiction to the postulates of familiar S-matrix theories.

Such a difficulty could have been suspected from a study of the little groups. For the inhomogeneous Lorentz group these little groups are

$$\begin{array}{ccc} SU(2) & SU(1,1) & SU(2) \\ \text{for } p^2 = m^2 > 0\,,\ p^0 > 0\,, & \text{for } p^2 = m^2 < 0\,, & \text{for } p^2 = m^2 > 0\,,\ p^0 < 0\,. \end{array}$$

For the same situations we would expect particle classification in a relativistic SU(6) model of the kind

$$SU(6) \qquad SU(3,3) \qquad SU(6)\,.$$

Indeed, the models with subsidiary conditions do give these little groups if the orbits are chosen appropriately. In the models with infinite multiplets we have instead

$$SU(2) \otimes SL(6,C) \qquad SU(1,1) \otimes SL(6,C) \qquad SU(2) \otimes SL(6,C)\,.$$

The factors SU(2) and SU(1,1) can be dropped since we have limited our investigations to representations with orbital spin zero. Within the representation of SL(6,C) we ascribe particles to boosted SU(6) multiplets. These boosters can be continued analytically from positive energy to negative energy, but they remain inside SL(6,C) and are therefore represented by unitary operators. This implies that the algebra which transforms the states of a fixed SU(6) multiplet into themselves is always isomorphic to an SU(6) algebra. The algebra classifying the SU(6) multiplets is therefore SU(6) in all three cases.

This difference of analytic behaviour in the mass between the description of particles in terms of spinorial representations and unitary representations of the non-compact group reflects itself in the analytic properties of the kinematic functions.

Further references: [206, 372].

PART II

APPLICATIONS

5. Applications of the static, collinear and coplanar subgroup symmetries

5.1 *General remarks*

5.1.1 Introduction

In the following sections we want to compare the models described in Part I with experiment. We shall be mainly concerned with the subgroup chains described in Section 2.3, namely the chain $S[U(6) \otimes U(6)]$ for particles at rest, $SU(6)_W$ for collinear processes, and $S[U(3) \otimes U(3)]$ for coplanar processes and the chain SU(6) (rest) $\supset$ $S[U(3) \otimes U(3)]$ (collinear) $\supset$ SU(3) (coplanar). The second chain consists of subgroups of the first chain. As was pointed out in Section 3.2, the predictions of these chains are the same as those obtained from the inhomogeneous groups SU(6,6) and SL(6,C) with supplementary conditions[27]). One should also mention that SU(6,6) or SL(6,C) symmetries broken with kinetic spurions yield the same results.

All these models can of course have only approximate validity. As is already the case for the SU(3) symmetry, mass splitting may play an important role. The underlying hope is that a transition matrix or a form factor may be divided into two factors, a dynamical one (coupling constants, etc.) where the symmetry holds, and a kinematical (phase space, etc.) where the true masses are used. Such a division is of course ambiguous, and it is an open problem to find a consistent prescription which agrees with experiment. For example, the symmetry relates the decays $V \to VP$ and $V \to PP$, where V is a spin 1^- and P a spin 0^- particle. But the coupling constants have different dimensions. In order to make meaningful comparisons, one should multiply one or both coupling constants by factors depending on the mass. Should this be some mean mass of the multiplet, or the mass of the decaying particle? Group theory alone cannot give an answer to this question. Because we wanted to get numbers which the experimentalist could immediately use, we have in many cases chosen a definite prescription. Due to the arbitrariness just mentioned, other choices may turn out to be better. However, accumulation of experimental results to be fitted by the model will again reduce this freedom. It may even turn out that a successful prescription found empirically could give some clues to the dynamics. An example is the Cabibbo theory of weak interactions.

In this theory, the ratio of comparable leptonic decays with $\Delta S = 0$ and $\Delta S = 1$ is given by the Cabibbo angle θ and SU(3). For example, $\pi \to \mu\,\nu$ and $K \to \mu\,\nu$ is de scribed by the interaction

$$\cos\theta\, g_\pi\, \overline{\psi}\, \gamma_\mu\, \gamma_5\, \psi\, \partial^\mu\, \pi \quad \text{and} \quad \sin\theta\, g_K\, \overline{\psi}\, \gamma_\mu\, \gamma_5\, \psi\, \partial^\mu\, K\,.$$

If SU(3) relates g_K to g_π, one gets for $\mathrm{tg}\,\theta$ the value obtained from other processes (such as β-decay, etc.). If, however, SU(3) applies to the dimensionless coupling constants g_B/m_B[28]), one would get a quite different value for $\mathrm{tg}\,\theta$, destroying the universality of the theory.

Another ambiguity due to the mass differences comes from the fact that one is obliged in certain cases to compare processes at a different energy or a different momentum transfer. Extrapolations are needed in these cases and only the dynamics can give information (see Ref. [277]).

We should, however, mention that there exist a certain number of examples where the kinematics is such that many ambiguities discussed previously disappear. This is the case for the magnetic form factors of the nucleons, which is one of the main successes of the models. The same is true for certain radiative decays $V \to P\gamma$ and certain baryon-meson scatterings in the forward and backward directions, for which precise experiments are still lacking. These should be considered as test cases.

Considering now what we called the "dynamical factor", we remark that there are a priori reasons to believe that the symmetry is not equally good for all processes. If

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[27]) The models based on the inhomogeneous groups may give some additional restrictions such as the value of the form factor at some particular points. See discussion in Section 3.2.

[28]) where B is the boson π or K.

a scattering is dominated by peripheral mechanisms, then the matrix elements are sensitive to the mass of the exchanged particle. Mass differences play an even more important role in the peripheral model with absorption.

So far, the symmetry was applied only to real particles. However, especially for vertex functions, one or more particles may be off the mass-shell. In these cases we assume that the relations obtained from the symmetry for real processes can be analytically continued to unphysical situations.

It was already mentioned in Section 3.2 that unitarity and crossing symmetry give additional restrictions which, together with the symmetry, may be too severe. However, nobody (including S. Coleman) has worked out these restrictions in a realistic case[29]).

In spite of all objections, we present the predictions of the models as they stand. We hope that in many cases they will provide at least a useful 0^{th} approximation.

We have tried to give the results in a form immediately useful to the experimentalist. This was not always possible, due to the great number of processes. The relevant literature quoted after each section should fill these gaps.

Although in the following section we analyse only the predictions of the two subgroup chains, we give also the literature for the static SU(6) model, as well as for the models based on SU(6,6) without "irregular" couplings. The latter model gives for collinear processes the same predictions as $SU(6)_W$, provided all particles belong to irreducible representations of $SU(6)_W$, the reason being that irregular couplings are invariants of $SU(6)_W$ under collinear conditions; their value depends then only on the irreducible representations considered. On the other hand, static SU(6) is non-relativistic and is based on strict conservation of spin, in contradistinction with the models we study here.

In the references quoted after each section, SU(6,6) includes $SU(6)_W$ and coplanar $S[U(3) \otimes U(3)]$, and SL(6,C) includes collinear $S[U(3) \otimes U(3)]$.

5.1.2 Notations

a) The chain $SU(6) \supset S[U(3) \otimes U(3)] \supset SU(3)$

In this model, the group SU(6) classifies particles at *rest*. The infinitesimal transformations of the fundamental, sixdimensional representation act on the quark wave functions in the following way:

$$\delta q_A \equiv \delta q_{a\alpha} = \frac{1}{2}\, i\, (\lambda_i)_a^b\, (\sigma_\mu)_\alpha^\beta\, q_{b\beta}\, \delta\varepsilon_{i\mu} \tag{1}$$

$$A = 1, \ldots 6; \quad a = 1, \ldots 3; \quad \alpha = 1, 2; \quad i = 0, \ldots 8; \quad \mu = 0, \ldots 3$$

λ_i are the eight Gell-Mann 3×3 matrices together with the unit matrix, σ_μ the three Pauli 2×2 matrices together with the unit matrix, $\delta\varepsilon_{i\mu}$ the infinitesimal parameters. The condition of unimodularity is expressed by $\delta\varepsilon_{00} = 0$.

The decomposition $SU(6) \supset SU(3) \otimes SU(2)$ reads

$$6 = (3, 2)\,.$$

[29]) The objections of Refs. [49] and [60] do not apply to the models discussed here. See discussion in Section 3.2.

The conjugate representation is given by:

$$\delta q^A \equiv \delta q^{a\alpha} = -\frac{1}{2} i (\lambda_i)^a_b (\sigma_\mu)^\alpha_\beta q^{b\beta} \delta\varepsilon_{i\mu}.$$

Under *charge conjugation* C we want to leave the ordinary spin unchanged but change an SU(3) representation into its conjugate. So we define

$$(\bar{q}_A)^c \equiv (\bar{q}_{a\alpha})^c = q^a_\alpha = \varepsilon_{\alpha\beta} q^{a\beta} \qquad (\bar{q}^A)^c \equiv (\bar{q}^{a\alpha})^c = q^\alpha_a = \varepsilon^{\alpha\beta} q_{a\beta} \tag{2}$$

where $\bar{q}$ is the charge conjugate wave function.

We recall that $\varepsilon_{\alpha\beta} = \varepsilon^{\alpha\beta}$ is the invariant metric tensor of SU(2).

The negative parity mesons are described by tensors of the representation 35 which transform like $\bar{q}^{a\alpha} q_{b\beta}$

$$M^A_B = \frac{1}{\sqrt{2}} P^{(8)a}_{\ \ b} \delta^\alpha_\beta + V^{(9)a}_{\ \ b} S^\alpha_\beta$$
$$V^{(9)a}_{\ \ b} = V^{(8)a}_{\ \ b} + \frac{1}{\sqrt{3}} V^{(1)} \delta^a_b$$
$$P^{(8)a}_{\ \ a} = V^{(8)a}_{\ \ a} = S^\alpha_\alpha = 0 \tag{3}$$

P stands for pseudoscalar and V for vector.
The SU(6) $\supset$ SU(3) $\otimes$ SU(2) decomposition is therefore:

$$35 = (8, 1) \oplus (8, 3) \oplus (1, 3)$$

Under *charge conjugation* we get, according to formula (2):

$$(\bar{M}^{a\alpha}_{b\beta})^c = \varepsilon^{\alpha\gamma} \varepsilon_{\beta\delta} M^{\beta\delta}_{\alpha\gamma}. \tag{4}$$

This agrees with the positive C-parity of P and the negative C-parity of V. Furthermore, here we can put $\bar{M} = M$, there being no extra quantum numbers besides I_3 and Y distinguishing mesons and antimesons.

If an interaction is invariant under SU(6)$_\sigma$ and C, it is also invariant under C' which is the product of C and the rotation $\exp[i(\sigma_y/2)\pi]$ (acting on a quark)[30]. This rotation changes the sign of the z-component of the quark spin S_z, and is numerically equal to the matrix $\varepsilon^{\alpha\beta}$. We have thus:

$$(\bar{q}^{a\alpha})^{c'} = q_{a\alpha} \qquad (M^{a\alpha}_{b\beta})^{c'} = M^{b\beta}_{a\alpha} \tag{2 bis}$$

$P^{(8)}$ and $V^{(8)}$ can also be written in matrix form:

$$P^{(8)} = \begin{pmatrix} \frac{1}{\sqrt{6}} \eta^{(8)} + \frac{1}{\sqrt{2}} \pi^0 & \pi^+ & K^+ \\ \pi^- & \frac{1}{\sqrt{6}} \eta^{(8)} - \frac{1}{\sqrt{2}} \pi^0 & K^0 \\ K^- & K^0 & -\frac{2}{\sqrt{6}} \eta^{(8)} \end{pmatrix} \tag{5}$$

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[30]) We thank Dr. J. S. BELL for discussions on this point.

$$V^{(8)} = \begin{pmatrix} \frac{1}{\sqrt{6}}\,\varphi^{(8)} + \frac{1}{\sqrt{2}}\,\varrho^0 & \varrho^+ & K^{*+} \\ \varrho^- & \frac{1}{\sqrt{6}}\,\varphi^{(8)} - \frac{1}{\sqrt{2}}\,\varrho^0 & K^{*0} \\ K^{*-} & K^{*0} & -\frac{2}{\sqrt{6}}\,\varphi^{(8)} \end{pmatrix} \tag{6}$$

$\eta^{(8)}$ is a linear combination of the η (549 MeV) and some other $I = 0$ pseudoscalar meson. The most likely candidate is $\eta' = x_0$ (958 MeV) although its quantum numbers are not definitely established. If $\eta^{(1)}$ is an SU(3) singlet pseudoscalar meson, one can write:

$$\eta = \eta^{(8)} \cos\alpha + \eta^{(1)} \sin\alpha \qquad \eta' = -\eta^{(8)} \sin\alpha + \eta^{(1)} \cos\alpha\,. \tag{7}$$

Using the Gell-Mann/Okubo mass formula and the actual masses of η and η', one gets:

$$\cos\alpha = 0.98 \qquad \sin\alpha = \pm\,0.18 \tag{8}$$

so that the mixing angle is quite small. It would be even smaller if η' turns out to be a more massive resonance.

The vector meson $\varphi^{(8)}$ is a linear combination of ω (783 MeV) and φ (1019 MeV). Writing again

$$\varphi = \varphi^{(8)} \cos\lambda + \varphi^{(1)} \sin\lambda \qquad \omega = -\,\varphi^{(8)} \sin\lambda + \varphi^{(1)} \cos\lambda \tag{9}$$

one gets, using the masses of the "nonet",

$$\cos\lambda = 0.766 \qquad \sin\lambda = \pm 0.643\,. \tag{10}$$

Such a mixing angle allows, as we shall see, a small coupling for $\varphi \to \varrho\,\pi$ and $\varphi \to \varrho\,\gamma$ (see Sections 5.3 and 5.5). The experimental branching ratio $\varphi \to 3\pi/\varphi \to K\bar{K}$ is much larger than suggested by earlier experiments. But since the phase space is much more favourable for $\varphi \to \varrho\,\pi$ than for $\varphi \to K\bar{K}$, the coupling constant for $\varphi \to \varrho\,\pi$ is still small, but not zero. Nothing is known about $\varphi \to \varrho\,\gamma$.

In a simple quark model, the mixing angle would be, instead of formula (10):

$$\cos\lambda = \frac{\sqrt{2}}{\sqrt{3}} \qquad \sin\lambda = -\frac{1}{\sqrt{3}} = -0.577\,. \tag{11}$$

In many models, including those discussed here, formula (11) would forbid $\varphi \to \varrho\,\pi$ and $\varphi \to \varrho\,\gamma$ completely.

We shall use formula (10), but take only the negative value for $\sin\lambda$.

It is an open question to understand why α is so much smaller than λ. If particles are classified according to SU(6), where $P^{(8)}$ and $V^{(9)}$ belong to 35, and $P^{(1)}$ is a SU(6) scalar, this is not so surprising. But if the rest group is S[U(6) $\otimes$ U(6)], the negative parity mesons belong to 36, including $P^{(1)}$, as we shall see below. Also in the quark model, it is difficult to understand why $\alpha \ll \lambda$.

The $^1/_2{}^+$ octet of "stable" baryons and $^3/_2{}^+$ decuplet resonances are assigned to the representation 56, which has the SU(3) $\otimes$ SU(2) decomposition

$$56 = (10,\,4) \oplus (8,\,2) \tag{12}$$

and is described by the tensor:

$$B_{(ABC)} = D_{(abc)}\, S_{(\alpha\beta\gamma)} + \frac{1}{3\sqrt{2}} \{0_a^d\, \varepsilon_{dbc}\, S_\alpha\, \varepsilon_{\beta\gamma} + 0_b^d\, \varepsilon_{dca}\, S_\beta\, \varepsilon_{\gamma\alpha} + 0_c^d\, \varepsilon_{dab}\, S_\gamma\, \varepsilon_{\alpha\beta}\} \tag{13}$$

where D stands for decuplet and 0 for octet, and $S_{(\alpha\beta\gamma)}$ is a spin $^3/_2$ tensor (see Section 1.3.1 for explicit expressions).

We now consider the *collinear* group S[U(3) ⊗ U(3)] which applies to particles all moving in the same space direction. Its generators are:

$$\frac{1}{2}\, \lambda_i\, \frac{(1 \pm \sigma_z)}{2}\, . \tag{14}$$

This is the subgroup of SU(6) which commutes with Lorentz transformations along the z-axis. Choosing the z-direction as quantization axis, the two U_3 groups transform the $S_z = \pm\, ^1/_2$ quark states separately. We introduce the notation:

$$q_{a1} \equiv q_a \in (3, 1) \quad (S_z = +1/2) \qquad q_{a2} \equiv q_{\bar{a}} \in (1, 3) \quad (S_z = -1/2)\, . \tag{15}$$

For the contravariant representation, S_z change sign

$$q^{a1} \equiv q^a \in (\bar{3}, 1) \quad (S_z = -1/2) \qquad q^{a2} \equiv q^{\bar{a}} \in (1, \bar{3}) \quad (S_z = +1/2) \tag{16}$$

Under the *parity* operation, the momentum of all particles changes sign, but since the momenta do not appear explicitly in this formalism, we define $P' = R_x(\pi) P$, where $R_x(\pi)$ is a rotation of π around the x-axis. P' leaves the momenta (which are along the z-axis) unchanged and changes the sign of S_z. Hence

$$(q_a)^{P'} = q_{\bar{a}} \quad \text{or} \quad (3, 1) \overset{P'}{\longleftrightarrow} (1, 3) \tag{17}$$

charge conjugation affects only the SU(3) properties, not S_z. Hence

$$(\bar{q}_a)^c = q^{\bar{a}} \quad \text{or} \quad (3, 1) \overset{C}{\longleftrightarrow} (1, \bar{3})\, . \tag{18}$$

The mesonic representation 35 of SU(6) splits under S[U(3) ⊗ U(3)] in the following way:

$$\begin{aligned}
M_B^A\,|_{\alpha=1,\beta=1} &\equiv M_b^a + \frac{1}{\sqrt{6}}\, \delta_b^a\, M = \frac{1}{\sqrt{2}}\, P^{(8)}{}_b^a + V^{(8,0)}{}_b^a\, S_1^1 + \frac{1}{\sqrt{3}}\, V^{(1,0)}\, \delta_b^a\, S_1^1 \\
M_B^A\,|_{\alpha=2,\beta=2} &\equiv M_{\bar{b}}^{\bar{a}} + \frac{1}{\sqrt{6}}\, \delta_{\bar{b}}^{\bar{a}}\, M = \frac{1}{\sqrt{2}}\, P^{(8)}{}_b^a + V^{(8,0)}{}_b^a\, S_2^2 + \frac{1}{\sqrt{3}}\, V^{(1,0)}\, \delta_b^a\, S_2^2 \\
M_B^A\,|_{\alpha=2,\beta=1} &\equiv M_b^{\bar{a}} = V^{(8+)}{}_b^a\, S_1^2 + \frac{1}{\sqrt{3}}\, \delta_b^a\, V^{(1+)} S_1^2 \\
M_B^A\,|_{\alpha=1,\beta=2} &\equiv M_{\bar{b}}^a = V^{(8-)}{}_b^a\, S_2^1 + \frac{1}{\sqrt{3}}\, \delta_b^a\, V^{(1-)}\, S_2^1
\end{aligned} \tag{19}$$

$$S_1^1 = -\, S_2^2$$

or

$$35 = (8, 1) \oplus (1, 8) \oplus (1, 1) \oplus (3, \bar{3}) \oplus (\bar{3}, 3) \tag{20}$$

and $V^{(8+)}$ is, for example, a vector octet with $S_z = +\,1$.

Since the mesons have negative intrinsic parity, one has

$$(M^a_b)^P = -M^{\bar a}_{\bar b} \qquad (M^{\bar a}_{\bar b})^P = -M^a_b. \tag{21}$$

Because the $I = 0$, $Y = 0$ members of $P^{(8)}$, respectively $V^{(8)}$ are even, respectively odd under C, one gets:

$$(M^a_b)^C = M^{\bar b}_{\bar a} \qquad (M^a_{\bar b})^C = -M^b_{\bar a} \qquad (M^{\bar a}_b)^C = -M^{\bar b}_a. \tag{22}$$

The baryonic representation 56 splits in the following way:

$$\begin{aligned}
B_{ABC}\,|_{\alpha=1,\beta=1,\gamma=1} &\equiv B_{abc} = D_{abc}\, S_{111} \\
B_{ABC}\,|_{\alpha=2,\beta=2,\gamma=2} &\equiv B_{\bar a\bar b\bar c} = D_{abc}\, S_{222} \\
B_{ABC}\,|_{\alpha=1,\beta=1,\gamma=2} &\equiv B_{ab\bar c} = \sqrt{3}\, D_{abc}\, S_{122} + \frac{1}{\sqrt{6}}\,(0^d_a\, \varepsilon_{dbc} + 0^d_b\, \varepsilon_{dab})\, S_1 \\
B_{ABC}\,|_{\alpha=1,\beta=2,\gamma=2} &\equiv B_{a\bar b\bar c} = \sqrt{3}\, D_{abc}\, S_{122} + \frac{1}{\sqrt{6}}\,(0^d_b\, \varepsilon_{dac} + 0^d_c\, \varepsilon_{dab})\, S_2
\end{aligned} \tag{23}$$

or

$$56 = (10, 1) \oplus (1, 10) \oplus (6, 3) \oplus (3, 6)\,. \tag{24}$$

b) The chain S[U(6) ⊗ U(6)] ⊃ $SU(6)_W$ ⊃ S[U(3) ⊗ U(3)]

Particles at *rest* are classified according to S[U(6) ⊗ U(6)]. The first U(6) group applies only to quarks, the second only to antiquarks (see Section 2.3). Hence, for systems built of quarks only, one gets the same classification as with $SU(6)_\sigma$. But mesons are $q\bar q$ systems, which have negative parity. Hence, the 0^- and 1^- mesons belong to the representation $(6,\bar 6)$, and we have an additional pseudoscalar meson, compared with $SU(6)_\sigma$[31]).

Particles moving along the z-axis are classified according to the collinear group $SU(6)_W$. It is the subgroup of S[U(6) ⊗ U(6)] which commutes with Lorentz transformations along the z-axis. Quarks transform the same way as under $SU(6)_\sigma$ (see Eq. (1)].

But antiquarks behave differently under the subgroup $SU(2)_W$, as was shown in Section 2.3:

$$\begin{aligned}
\delta\bar q_{a\alpha} &= \frac{1}{2}\, i(\lambda_i)^b_a\, (\sigma_\mu)^\beta_\alpha\, \bar q_{b\beta}\, \delta\varepsilon_{i\mu} \quad \text{foi} \quad \mu = 0, 3 \\
\delta\bar q_{a\alpha} &= -\frac{1}{2}\, i(\lambda_i)^b_a\, (\sigma_\mu)^\beta_\alpha\, \bar q_{b\beta}\, \delta\varepsilon_{i\mu} \quad \text{for} \quad \mu = 1, 2
\end{aligned} \tag{25}$$

where $\bar q_{a\alpha}$ is defined by Eq. (2).

This difference arises from the fact that we define antiquarks to transform in the same way as quarks under $SU(2)_\sigma$.

This is, however, not very convenient for calculations where invariance under $SU(6)_W$ is assumed. Since $S_z = W_z$ (for quarks and antiquarks) is conserved in collinear reactions, we can define a new antiquark wave function:

$$\bar q'_{a\alpha} = (\sigma_z)^\beta_\alpha\, \bar q_{a\beta} \tag{26}$$

---

31) For positive parity mesons one could use a quark and a pseudoantiquark, thus getting the representation (35,1).

Under $SU(6)_W$, the transformation properties are now:

$$\delta \bar{q}'_{a\alpha} = \frac{1}{2} i(\lambda_i)^b_a (\sigma_\mu)^\beta_\alpha \bar{q}'_{b\beta} \, \delta\varepsilon_{i\mu} \quad \mu = 0, \dots 3 . \tag{27}$$

The negative parity mesons are described by tensors of the representation 35 of $SU(6)_W$, transforming like $\bar{q}'^{a\alpha} q_{b\beta}$. But since $\bar{q}'$, has different transformation properties under $SU(6)_\sigma$, we now have:

$$\begin{aligned}
M^A_B &= \frac{1}{\sqrt{2}} V^{(80)a}_{\ \ b} \delta^\alpha_\beta + M^{(9)a}_{\ \ b} w^\alpha_\beta \\
M^{(9)a}_{\ \ b} &= M^{(8)a}_{\ \ b} + \frac{1}{\sqrt{3}} M^{(1)} \delta^a_b \\
M^{(9)a}_{\ \ b} w^1_1 &= - M^{(9)a}_{\ \ b} w^2_2 = \frac{1}{\sqrt{2}} P^{(9)a}_{\ \ b} \\
M^{(9)a}_{\ \ b} w^2_1 &= V^{(9+)a}_{\ \ b} \quad M^{(9)a}_{\ \ b} w^1_2 = V^{(9-)a}_{\ \ b}
\end{aligned} \tag{28}$$

where 0, +, − stand for $S_z = 0, +1, -1$.

If an interaction is invariant under $SU(6)_W$, we can again define a new operation C' such that [see (2bis)]

$$(\bar{q}'_{a\alpha})^{C'} = q^{a\alpha} . \tag{29}$$

We now add a few remarks on *parity* P. Again we consider the operation $P' = R_x(\pi) P$, defined above. For both $SU(6)_\sigma$ and $SU(6)_W$, we define

$$\begin{aligned}
(q_{a1})^{P'} &= q_{a2} \\
(q_{a2})^{P'} &= q_{a1}
\end{aligned} \tag{30}$$

apart from an arbitrary phase.

Since a quark-antiquark system has negative intrinsic parity, one has

$$\begin{aligned}
(\bar{q}_{a1})^{P'} &= - \bar{q}_{a2} \\
(\bar{q}_{a2})^{P'} &= - \bar{q}_{a1} .
\end{aligned} \tag{31}$$

In $SU(6)\sigma$, negative parity mesons transform like $q_{a\alpha} \bar{q}^{b\beta}$ so that

$$\begin{aligned}
(M^{b1}_{a1})^{P'} &= - M^{b2}_{a2} \\
(M^{b2}_{a1})^{P'} &= - M^{b1}_{a2}
\end{aligned} \tag{32}$$

etc.

Hence, a coupling of three negative parity mesons

$$M_{1\,B}^{\ A} M_{2\,C}^{\ B} M_{3\,A}^{\ C} \tag{33}$$

is forbidden by $SU(6)_\sigma$ and parity.

However, for $SU(6)_W$, negative parity mesons transform like $q_{a\alpha} \bar{q}'^{b\beta}$, where $\bar{q}'$ was defined in Eq. (26). We now have

$$\begin{aligned}
(\bar{q}'_{a1})^{P'} &= q_{a2} \\
(\bar{q}'_{a2})^{P'} &= q_{a1}
\end{aligned} \tag{34}$$

and correspondingly

$$(M^{b\,1}_{a\,1})^{P'} = + M^{b\,2}_{a\,2}\,. \tag{35}$$

The three-meson coupling Eq. (33) is now allowed, the indices of course referring to $SU(6)_W$.

Particles moving in a plane are classified according to the *coplanar* group S[U(3) ⊗ U(3)]. The first U(3) group refers to $S_z = +1$, the recond to $S_z = -1$, but the quantization axis is now orthogonal to the plane. In the following, we shall only use the fact that the group yields conservation of S_z, for processes where no antiquarks are involued, so we shall not analyse the formalism any further.

5.2 *BBM vertex*

We consider the baryons belonging to the representation 56 of $SU(6)_\sigma$ and the mesons of 35 of $SU(6)_\sigma$, respectively 36 of S[U(6) ⊗ U(6)].

5.2.1 Collinear S[U(3) ⊗ U(3)]

The different S_z components belong to different representations and thus give rise to different invariants, except that parity relates S_z and $-S_z$.

There are five invariants

$$I = a_1\, \bar{B}^{a\,b\,c}\, B_{a\,b\,d}\, M^d_c + a_2\, \bar{B}^{a\,b\,c}\, B_{a\,b\,\bar{e}}\, M^{\bar{e}}_c + a_3\, \bar{B}^{a\,b\,\bar{c}}\, B_{a\,b\,\bar{e}}\, M^{\bar{e}}_{\bar{c}} + a_4\, \bar{B}^{a\,b\,c}\, B_{a\,d\,\bar{c}}\, M^d_{\bar{b}} + a_5\, \bar{B}^{a\,\bar{b}\,\bar{c}}\, B_{a\,\bar{b}\,d}\, M^d_{\bar{c}} + (P) \tag{1}$$

where (P) are the terms obtained by parity transformation.

Using Eqs. (19) and (23) of Section 5.1.2 it is seen that a_3 and a_4 contribute to the vertex $\bar{O}OP$, and hence no new relations beyond SU(3) result for this vertex. The predictions which one gets for the other SU(3) multiplets are difficult to test since they hold only for definite helicity states. They have been quoted in Section 3.2.3.

5.2.2 $SU(6)_W$

Considering only the representation 35 of $SU(6)_W$, we have one invariant:

$$I = \bar{B}^{A\,B\,C}\, B_{A\,B\,D}\, M^D_C\,. \tag{2}$$

Using Eqs. (13) and (28) of Section 5.1.2 one finds:

$$\begin{aligned}
I = {} & \bar{D}^{a\,b\,c}\, D_{a\,b\,d} \left(\frac{1}{\sqrt{2}}\, V^{(8,0)d}_{\ \ c}\, \delta^\delta_\gamma + M^{(9)d}_{\ \ c}\, w^\delta_\gamma \right) \bar{S}^{\alpha\beta\gamma}\, S_{\alpha\beta\delta} \\
& + \frac{2}{3\sqrt{2}}\, \bar{0}^a_e\, \varepsilon^{e\,b\,c}\, D_{a\,b\,d}\, M^{(9)\,d}_{\ \ c}\, w^\delta_\gamma\, \bar{S}^\alpha\, \varepsilon^{\beta\gamma}\, S_{\alpha\beta\delta} \\
& + \frac{2}{3\sqrt{2}}\, \bar{D}^{a\,b\,c}\, 0^f_a\, \varepsilon_{f\,b\,d}\, M^{(9)\,d}_{\ \ c}\, w^\delta_\gamma\, \bar{S}^{\alpha\beta\gamma}\, \varepsilon_\alpha\, S_{\beta\delta} \\
& + \frac{1}{3\sqrt{2}}\, \mathrm{tr}\, (\bar{0}\, V^{(8,0)}\, 0 - \bar{0}0V^{(8,0)})\, \bar{S}^\alpha\, S_\alpha \\
& + \frac{1}{9}\, \mathrm{tr}\, [3\, (\bar{0}M^{(9)}0 + \bar{0}0M^{(9)}) + 2\, (\bar{0}M^{(9)}0 - \bar{0}0M^{(9)})]
\end{aligned}$$

$$\mathrm{tr}\, \bar{0}V0 \equiv \bar{0}^a_e\, V^d_a\, 0^e_d\,.$$

The last term contributes to the vertex $\bar{O}OP$ and gives $D/F = 3/2$, where D, respectively F are the symmetric, respectively antisymmetric couplings of two SU(3) octets with a third octet.

For the decay $D \to O + P$, one gets the same results as with SU(3), which are known to be not very accurate, so that SU(3) symmetry breaking is not negligible.

One also gets a relation between the $\bar{D}OP$ and $\bar{O}OP$ coupling constants. But since they have different dimensions, they are related by an unknown function of the masses. Various recipes have been proposed (see Refs. [110, 186]).

Literature: SU(6)σ: [80, 186]; SL(6,C): [150, 209, 334, 339, 384]; SU(6,6): [40, 110, 120, 138, 243, 359].

Literature on the decay of higher baryon resonances: [106, 200, 201].

5.3 *MMM vertex*

We consider the vertex of three negative parity mesons belonging to the representation 35 of SU(6)$_\sigma$ or $(6,\overline{6})$ of S[U(6) ⊗ U(6)].

5.3.1 Collinear S[U(3) ⊗ U(3)]

Using P and C invariance [Section 5.1.2, Eqs. (21) and (22)], one gets two invariants:

$$I_1 = a_1 \left\{ M_{1\,a}^{\;b}\, M_{2\,b}^{\;c}\, M_{3\,c}^{\;a} - M_{1\,\bar a}^{\;\bar b}\, M_{2\,\bar b}^{\;\bar c}\, M_{3\,\bar c}^{\;\bar a} + M_{1\,\bar a}^{\;\bar b}\, M_{3\,\bar b}^{\;c}\, M_{2\,\bar c}^{\;\bar a} - M_{1\,a}^{\;b}\, M_{3\,b}^{\;c}\, M_{2\,c}^{\;a} \right\}$$

$$I_2 = a_2 \left\{ M_{1\,a}^{\;b}\, M_{2\,b}^{\;\bar c}\, M_{3\,\bar c}^{\;a} - M_{1\,\bar a}^{\;\bar b}\, M_{2\,\bar b}^{\;c}\, M_{3\,c}^{\;\bar a} + M_{1\,\bar a}^{\;\bar b}\, M_{3\,\bar b}^{\;c}\, M_{2\,c}^{\;\bar a} - M_{3\,a}^{\;b}\, M_{3\,b}^{\;\bar c}\, M_{2\,\bar c}^{\;a} \right\}.$$

With the expressions [Section 5.1.2, Eq. (19)] one finds:

$$I_1 = a_1 \frac{1}{\sqrt{2}} \{ \sum_P \mathrm{tr}\,[P_1^{(8)}\, P_2^{(8)}]\, V_3^{(8,0)} + \mathrm{tr}\,[V_1^{(8,0)}\, V_2^{(8,0)}]\, V_3^{(8,0)} \} \tag{1}$$

where P is a cyclic permutation of 1, 2, 3,

$$I_2 = a_2 \frac{1}{\sqrt{2}} \{ \mathrm{tr}\, P_1^{(8)}\, (V_2^{(9+)}, V_3^{(9-)}) - \mathrm{tr}\, P_1^{(8)}\, (V_2^{(9-)}, V_3^{(9+)}) + \mathrm{tr}\, V_1^{(8,0)}\, [V_2^{(8+)}, V_3^{(8-)}] + \mathrm{tr}\, V_1^{(8,0)}\, [V_2^{(8-)}, V_3^{(8+)}] \}. \tag{2}$$

Notice that *tr* $P^{(8)}\, V^{(1+)}\, V^{(1-)} = 0$ and also that the three-meson couplings involving $P^{(1)}$ and $V^{(1,0)}$ are unrelated to the preceding ones. Everywhere $[A, B] = AB - BA$, $(A, B) = AB + BA$.

5.3.2 SU(6)$_W$

In addition to a trivial coupling of $V^{(1,0)}$ which does not belong to the representation 35 of SU(6)$_W$, there are two invariants:

$$I = b_1\, \mathrm{tr}\, M_1\, M_2\, M_3 + b_2\, \mathrm{tr}\, M_1\, M_3\, M_2$$

where M is a traceless tensor belonging to the representation 35 of SU(6)$_W$. One notices that each term conserves parity, provided that the intrinsic parity of the mesons is negative [see Section 5.1.2, Eq. (35)]. For *positive parity*, such a coupling would be forbidden. This may have interesting consequences for bootstrap calculations.

Charge conjugation invariance [Section 5.1.2, Eq. (29)] implies $b_1 = -b_2$. Remembering that pseudoscalar mesons belong to a W-spin triplet and the zero helicity states of vector mesons to a W-spin singlet, one finds, using Eq. (28) of Section 5.1.2

$$\begin{aligned} I &= b_1 \operatorname{tr} [M_1, M_2]\, M_3 \\ &= \sum_P \frac{b_1}{\sqrt{2}} \operatorname{tr} \Big\{ (V_1^{(9+)}, V_1^{(9-)})\, P_3^{(9)} - (V_1^{(9-)}, V_2^{(9+)})\, P_3^{(9)} \\ &+ \frac{1}{3} [V_1^{(8,0)}, V_2^{(8,0)}]\, V_3^{(8,0)} + [P_1^{(8)}\, P_2^{(8)}]\, V_3^{(8,0)} \\ &+ [V_1^{(8+)}, V_2^{(8-)}]\, V_3^{(8,0)} + [V_1^{(8-)}\, V_2^{(8+)}]\, V_3^{(8,0)} \Big\} . \end{aligned} \tag{3}$$

Notice that the coupling of the 0^- SU(3) singlet is now related to the others, using $P^{(9)} = P^{(8)} + 1/\sqrt{3}\, P^{(1)}$.

Comparing I with I_1 and I_2 of the preceding paragraph, one sees that $SU(6)_W$ invariance gives the additional relation

$$a_1 = a_2 .$$

5.3.3 Applications

i) From Eqs. (1), (2) and (3), one can get relations between coupling constants. Since they have in general different dimensions, one needs, as already said, a certain prescription. Following Ref. [355], one finds with $SU(6)_W$:

$$\frac{g_{\varrho\omega\pi}}{g_{\varrho\pi\pi}} = \frac{2}{\mu} \tag{4}$$

where μ is the mass of a vector meson.

ii) Using the "ideal" mixing angle [Section 5.1.2, Eq. (11)], $S[U(3) \otimes U(3)]$ and $SU(6)_W$ forbid the decay $\varphi \to \varrho\pi$. With the mixing angle [Section 5.1.2, Eq. (10)], one gets a small decay rate which, by $SU(6)_W$, is related to $\varrho \to \pi\pi$.

iii) An experimental test is provided by the decay of one meson into two mesons. But, due to phase space, the only decays of interest are, in addition to $\varphi \to \varrho\pi$, the three decays

$$\varphi \to K\bar{K}, \quad K^* \to K\pi, \quad \varrho \to \pi\pi .$$

Because of C invariance, the decay $\varphi \to K\bar{K}$ proceeds only through the SU(3) octet part of φ. Hence, these decays are already related by SU(3) alone. Using an interaction $g\, tr(V^\mu\, P \overleftrightarrow{\partial}_\mu P)$, one finds a phase-space factor P^3/m_v^2, where m is the mass of the decaying meson and P the centre-of-mass momentum. The results are shown in the Table. For the experimental values, see Ref. [325.]

Transition	g^2	P^3/m_v^2	Γ_{th}(MeV)	Γ_{exp}(MeV)
$\varrho\,\pi\,\pi$	2	77.8	172	125 ± 15
$K^*\,K\,\pi$	$^3/_2$	30.1	Input	50.0 ± 1.5
$\varphi\,K^+\,K^-$	0.88	1.92	1.88	1.26 ± 0.5
$\varphi\,K_1^0\,K_2^0$	0.88	1.16	1.13	1.26 ± 0.5

Note added in proof: the values of the last column keep changing.

Literature: $SU(6)_\sigma$: [172, 242]; SL(6,C): [336]; SU(6,6): [33, 182, 217, 270, 355]. Literature on the decay of higher meson resonances: [74, 88, 107, 109, 200, 215].

5.4 $B B \gamma$ *vertex*

The main successes of SU(3) have been obtained in situations where *two* hadrons interact with the electromagnetic or the weak currents. This may be due to the fact that the low number of baryons minimizes the symmetry-breaking effects and/or that they are coupled to a conserved or almost conserved current. It is therefore important to see if $SU(6)_\sigma$ and its relativistic generalizations are equally successful in the same situations. This is indeed the case for the $BB\gamma$ vertex, treated in this section. The $MM\gamma$ vertex will be studied in Section 5.5. For β-decay one gets the ratio $G_A/G_V = -5/3$.

The first success was the prediction of the ratio of the total magnetic moments of the proton and the neutron, from $SU(6)_\sigma$ (Ref. [46])

$$\frac{\mu(p)}{\mu(n)} = -\frac{3}{2}. \tag{1}$$

This result remains valid in $SU(6)_W$ (Ref. [38]) and collinear $S[U(3) \otimes U(3)]$ (Refs. [339, 388]), and can even be generalized to the Sachs magnetic form factors:

$$\frac{G_M^{(P)}(q^2)}{G_M^{(n)}(q^2)} = -\frac{3}{2}. \tag{2}$$

This relation holds for any q^2 and is in remarkable agreement with experiment.

In order to get Eq. (2), one needs two hypotheses:

i) The baryons belong to the representation 56 of $SU(6)_\sigma$ or $S[U(6) \otimes U(6)]$.

ii) The magnetic form factor transforms like a component of the representation 35.

If the baryons belong instead to the representation 20, one gets $-1/2$ for the same ratio, and if they belong to 70, no relation follows, since there are then two reduced matrix elements.

The second hypothesis is the simplest that one can make, and is certainly true for quarks. We assume its validity for all representations.

The matrix elements of the quark current are:

$$\begin{aligned}
&\langle p_1 \mid \overline{\psi}\, \gamma_\mu\, Q\, \psi \mid p_2 \rangle \\
&= \frac{1}{2M}\left(1 + \frac{q^2}{4M^2}\right)^{-1} \{G_E(q^2)\, \overline{u}\, K_\mu\, Q\, u - G_M(q^2)\, \overline{u}\, i\, r_\mu\, Q\, u\} \\
&K = p_1 + p_2 \quad q = p_1 - p_2 \quad r_\mu = \varepsilon_{\mu\nu\varrho\sigma}\, K^\nu\, q^\varrho\, \gamma^\sigma\, \gamma_5
\end{aligned} \tag{3}$$

G_E and G_M are the Sachs form factors and Q the SU(3) charge matrix

$$Q = \frac{1}{3}\begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix}. \tag{4}$$

In the Breit system, where $\boldsymbol{p}_1 = -\boldsymbol{p}_2 = {}^1/_2\,\boldsymbol{q}$, we can apply the collinear groups $S[U(3) \otimes U(3)]$ and $SU(6)_W$. Taking the z-axis along $\boldsymbol{p}$, the only non-zero matrix elements are:

$$\bar{u}\, K_0\, Q\, u = \frac{4\,E^2}{E+M}\, \chi^+\, Q\, \chi \tag{5}$$

where χ is the spin ${}^1/_2$ quark wave function, and

$$\begin{aligned} u\, r_1\, Q\, u &= \frac{4\,E^2}{E+M}\, 2\,p\, \chi^+ \sigma_2\, Q\, \chi \\ \bar{u}\, r_2\, Q\, u &= -\frac{4\,E^2}{E+M}\, 2\,p\, \chi^+ \sigma_1\, Q\, \chi\,. \end{aligned} \tag{6}$$

5.4.1 Collinear S[U(3) ⊗ U(3)]

We assume that the transformation properties of the vertex abstracted from Eqs. (5) and (6) also hold for the $B\,B\,\gamma$ vertex. This means that the various S_z components of the virtual photon behave like:

$$\begin{aligned} S_z &= 0 \sim (8,1) \oplus (1,8) \sim Q^a_b + Q^{\bar{a}}_{\bar{b}} \\ S_z &= 1 \sim (3,\bar{3}) \qquad\quad\;\; \sim Q^{\bar{a}}_b \\ S_z &= -1 \sim (\bar{3},3) \qquad\quad \sim Q^a_{\bar{b}} \end{aligned} \tag{7}$$

and we have given both the representation and the tensor.

For the transition $\langle S_z = {}^1/_2 \mid j \mid S_z = {}^1/_2\rangle$ one has two invariants:

$$\begin{aligned} I_1 &= \bar{B}^{a\,b\,\bar{c}}\, B_{a\,d\,\bar{c}}\, Q^d_b \\ I_2 &= \bar{B}^{a\,b\,\bar{c}}\, B_{a\,b\,\bar{e}}\, Q^{\bar{e}}_{\bar{c}}\,. \end{aligned} \tag{8}$$

Using Eq. (23) of Section 5.1.2, one gets:

$$\begin{aligned} I_1 &= 3\,\bar{D}^{a\,b\,c}\, D_{a\,c\,d}\, Q^d_b\, \bar{S}^{112}\, S_{112} + \frac{1}{6}\,(4\,\mathrm{tr}\,\bar{0} Q 0 - \mathrm{tr}\,\bar{0} 0 Q)\, \bar{S}'\, S\,, \\ I_2 &= 3\,\bar{D}^{a\,b\,c}\, D_{a\,c\,d}\, Q^d_c\, \bar{S}^{112}\, S_{112} - \frac{1}{6}\,(2\,\mathrm{tr}\,\bar{0} Q 0 + 4\,\mathrm{tr}\,\bar{0} 0 Q)\, \bar{S}'\, S\,. \end{aligned}$$

Hence, for the electric form factor of the baryons, one gets nothing new compared to SU(3).

For the transition $\langle S_z = {}^1/_2 \mid j \mid S_z = -{}^1/_2\rangle$ one gets only one invariant:

$$\begin{aligned} I_3 = \bar{B}^{a\,b\,\bar{c}}\, B_{a\,\bar{c}\,\bar{e}}\, Q^{\bar{e}}_b &= 3\,\bar{D}^{a\,b\,c}\, D_{a\,c\,d}\, Q^d_b\, \bar{S}^{112}\, S_{122} \\ &+ \frac{1}{\sqrt{2}}\, \bar{D}^{a\,b\,c}\, \varepsilon_{a\,d\,e}\, 0^e_c\, Q^d_b\, \bar{S}^{112}\, S_2 + \frac{1}{\sqrt{2}}\, \bar{0}^a_e\, \varepsilon^{b\,c\,e}\, D_{a\,c\,d}\, Q^d_b\, \bar{S}^1\, S_{122} \\ &+ \frac{1}{6}\,\mathrm{tr}\,[3\,\bar{0}\,(Q\,0 + 0\,Q) + 2\,\bar{0}\,(Q\,0 - 0\,Q)\, \bar{S}^1\, S_2\,. \end{aligned} \tag{9}$$

From this follows the result of Eq. (2).

The transition $\langle S_z = {}^3/_2 \mid j \mid S_z = {}^1/_2\rangle$ is given by a different invariant

$$I_4 = \overline{B}^{abc} B_{ab\bar{e}} Q^{\bar{e}}_c = \overline{D}^{abc} \overline{S}^{111} \left(\sqrt{3}\, D_{abd} S_{112} + \frac{2}{\sqrt{6}}\, 0^e_a \varepsilon_{ebd} S_1\right) Q^d_c$$

so that the predictions for the transitions $\overline{D}\, O\, \gamma$ and $\overline{D}\, D\, \gamma$ are rather weak.

5.4.2 $SU(6)_W$

Writing for the tensor representing the current:

$$J^A_B = Q^a_b S^\alpha_\beta \qquad S^\alpha_\alpha \neq 0$$

one gets the unique invariant

$$\begin{aligned} I = {} & 3\, \overline{B}^{ABC} J^D_C B_{ABD} = 3\, \overline{D}^{abc} D_{abd} Q^d_c \overline{S}^{\alpha\beta\gamma} S_{\alpha\beta\delta} S^\delta_\gamma \\ & + \sqrt{2}\, D^{abc} Q^d_c \varepsilon_{bdf} 0^f_a S^{\alpha\beta\gamma} S^\delta_\gamma \varepsilon_{\beta\delta} S_\alpha \\ & + \sqrt{2}\, \varepsilon^{bcf} \overline{0}^a_f Q^d_c D_{abd} \overline{S}^\alpha \varepsilon^{\beta\gamma} S^\delta_\gamma S_{\alpha\beta\delta} \end{aligned} \qquad (10)$$

$$+ \operatorname{tr} \overline{0}\,(Q0 - 0Q)\, \overline{S}^\alpha S_\alpha + \operatorname{tr}\left[\overline{0}\,(Q0 + 0Q) + \frac{2}{3}\, \overline{0}(Q0 - 0Q)\right] \overline{S}^\alpha S_\beta \left(S^\beta_\alpha - \frac{1}{2}\, \delta^\beta_\alpha S^\gamma_\gamma\right).$$

From this one gets the following predictions:

i) Putting $S^\alpha_\beta = \delta^\alpha_\beta$, one sees that $G_E(q^2)$ is proportional to the charge.

ii) Putting $S^\alpha_\beta = (\sigma_3)^\alpha_\beta$ one finds

$$\begin{aligned} & \langle N^{*+} S_z = \tfrac{1}{2} \mid j \mid N^{*+} S_z = \tfrac{1}{2} \rangle = \langle p\, S_z = \tfrac{1}{2} \mid j \mid p\, S_z = \tfrac{1}{2} \rangle \\ & \quad = -\frac{3}{2} \langle n\, S_z = \tfrac{1}{2} \mid j \mid S_z = \tfrac{1}{2} \rangle = \frac{3}{2\sqrt{2}} \langle N^{*+} S_z = \tfrac{1}{2} \mid j \mid p\, S_z = \tfrac{1}{2} \rangle \end{aligned} \qquad (11)$$

where j refers to the spin 1 part only. From this, relation (2) follows again. One finds also from Eq. (10) that the $N^* N \gamma$ transition is pure $M(1)$ (see Ref. [202]). Of course, one has to add to Eq. (11) all relations following from SU(3) alone.

One gets more information if one assumes that the form factor is dominated by a vector meson pole (see Refs. [39] and [143]).
However, one gets a different result if one makes the pole approximation in the F functions (Dirac and Pauli form factors), or in the G functions (Sachs form factors). In the first case one gets, for the *proton*:

$$\frac{F_1(q^2)}{F_2(q^2)} = \mu\, ; \qquad \frac{G_M(q^2)}{G_E(q^2)} = \frac{1}{2\,M} \, \frac{1 + 2\, M/\mu}{1 + q^2/2\, M\mu}\, ; \qquad \mu_p = 1 + \frac{2\,M}{\mu} \qquad (12)$$

and in the second case

$$\frac{G_M(q^2)}{G_E(q^2)} = \frac{1}{\mu}\, ; \qquad \mu_p = \frac{2\,M}{\mu} \qquad (13)$$

where M is the mass of the baryon, μ the mass of the meson and μ_p the total magnetic moment of the proton. Equation (13) agrees rather well with experiment.

Literature: SU(3): [95]; $SU(6)_\sigma$ or $SU(4)_\sigma$: [6, 24, 46, 58, 170, 199, 228, 236, 254, 352]; SL(6,C) or SL(4,C): [118, 154, 209, 288, 334, 339, 385, 388]; SU(6,6) or SU(4,4): [3, 38, 39, 55, 92, 111, 123, 143, 266, 295, 296, 305, 313].
Literature on radiative hyperon decays with creation of a Dalitz pair: [124]; Literature on the electroproduction of resonances: [173]; Literature on the electromagnetic properties of the 70-plet: [87]; see also the literature on the $B\ B\ M$-vertex.

Photoproduction process $B\ \gamma \to B\ M$

The main experimental information comes from the resonance region, where the problem is reduced to the determination of the coupling constant $B^*\ B\gamma$. Below the B^* threshold, symmetry breaking plays probably an important role.

Literature on $B\gamma \to B\,M$.

$SU(6)_\sigma$: [116, 225, 312, 380]; SL(6,C): [71]; SU(6,6): [76, 113, 225, 249, 250]; Quark model: [102].

5.5 $M\ M\ \gamma$ *vertex*

We consider the interaction of two negative parity mesons belonging to the representation 35 of $SU(6)_\sigma$ or 36 of $S[U(6) \otimes U(6)]$ with the photon.

Since the predictions of the collinear groups for the $B\ B\gamma$ vertex have so far been successful, it is important to have another test where only two hadrons interact. There should soon be enough experimental information available concerning the radiative decays of vector mesons. In at least three cases, namely $\varrho \to \pi\gamma$, $\omega \to \pi\gamma$, and $\varphi \to \eta\gamma$, the kinematical factors are similar, so that the complications discussed in the Introduction should not arise. There is, of course, still the essential difficulty that in the symmetry limit with degenerate masses the photon has zero energy, so that an extrapolation to the real masses is needed.

5.5.1 Collinear $S[U(3) \otimes U(3)]$

We assume the hypothesis of Section 5.4, Eq. (7). Using P and C invariance, one gets three invariants:

$$I_1 = G_E^{(1)} \left\{ Q^a_b \left[{M_1}^b_c\, {M_2}^c_a - {M_2}^b_c\, {M_1}^c_a \right] + Q^{\bar a}_{\bar b} \left[{M_1}^{\bar b}_{\bar c}\, {M_2}^{\bar c}_{\bar a} - {M_2}^{\bar b}_{\bar c}\, {M_1}^{\bar c}_{\bar c} \right] \right\}$$

$$I_2 = G_E^{(2)} \left\{ Q^a_b \left[{M_1}^b_{\bar c}\, {M_2}^{\bar c}_a - {M_2}^b_{\bar c}\, {M_1}^{\bar c}_a \right] + Q^{\bar a}_{\bar b} \left[{M_1}^{\bar b}_c\, {M_2}^c_{\bar a} - {M_2}^{\bar b}_c\, {M_1}^c_{\bar a} \right] \right\}$$

$$I_3 = G_M \left\{ Q^{\bar a}_b \left[{M_1}^b_{\bar c}\, {M_2}^{\bar c}_{\bar a} + {M_2}^b_c\, {M_1}^c_{\bar a} \right] - Q^a_{\bar b} \left[{M_1}^{\bar b}_c\, {M_2}^c_a + {M_2}^{\bar b}_{\bar c}\, {M_1}^{\bar c}_a \right] \right\}$$

Using the numerical convention for the matrix elements

$$Q^a_b = Q^{\bar a}_{\bar b} = \frac{1}{\sqrt{2}}\, Q^a_{\bar b} = \frac{1}{\sqrt{2}}\, Q^{\bar a}_b \,.$$

one finds

$$I_1 = G_E^{(1)} \left\{ \mathrm{tr}\, Q\, [P_1^{(8)}, P_2^{(8)}] + \mathrm{tr}\, Q\, [V_1^{(8,0)}, V_2^{(8,0)}] \right\}$$

$$I_2 = G_E^{(2)} \left\{ \mathrm{tr}\, Q\, [V_1^{(8+)}, V_2^{(8-)}] + \mathrm{tr}\, Q\, [V_1^{(8-)}, V_2^{(8+)}] \right\}$$

$$I_3 = G_M \left\{ \mathrm{tr}\, Q^{(+)}\, (V_1^{(9-)}, P_2^{(8)}) - \mathrm{tr}\, Q^{(-)}\, (V_1^{(9+)}, P_2^{(8)}) - \mathrm{tr}\, Q^{(+)}\, [V_1^{(8-)}, V_2^{(8,0)}] + \mathrm{tr}\, Q^{(-)}\, [V_1^{(8+)}, V_2^{(8,0)}] \right\}$$

$[A, B]$ stands for the commutator, (A, B) for the anticommutator.

5.5.2 $SU(6)_W$

For each form factor there is one invariant

$$I = \mathrm{tr}\, Q\,[M_1, M_2]\,.$$

Extracting from this the (8,1) part of Q for the charge form factor and the (8,3) part for the magnetic form factor, one finds one additional relation:

$$G_E^{(1)} = G_E^{(2)}$$

and in I_3 one should replace $P^{(8)}$ by $P^{(9)} = P^{(8)} + (1/\sqrt{3})\, P^{(1)}$.

5.5.3 Decay $M_1 \to M_2 \gamma$

This is the most interesting case for comparison with experiment, since many decays are energetically possible, and experimental information will soon be available. Recall that for a real photon only magnetic transitions are allowed. Hence, there is only one form factor $G_M(q_2)$. In addition, $SU(6)_W$ relates $P^{(1)}$ to $P^{(8)}$. Without further knowledge of $G_M(q^2)$, we assume it to vary slowly with q^2 using, however, a phase-space factor calculated with the physical masses. For the decays $V \to P + \gamma$ or $P \to V + \gamma$ we start with the interaction

$$\varepsilon^{\alpha\beta\gamma\delta}\, \partial_\alpha A_\beta\, \partial_\gamma V_\delta\, P\,.$$

In momentum space, with the momentum $\boldsymbol{q}$ of the photon along say the third axis, one typical term will be

$$G_M(q^2)\, q_3\, M_V\, \varepsilon^{3102}\, A_1\, V_2\, P\,.$$

Assuming the symmetry relations for $G_M(q^2)$, one gets a "phase-space factor" proportional to q^3.

Considering the decay of a vector nonet into a pseudoscalar nonet and a photon, SU(3) alone (with C invariance) describes these decays in terms of three coupling constants (Ref. [180])

$$g_{88},\ g_{81},\ g_{18}\,.$$

From the foregoing expressions, one finds for

a) $S[U(3) \oplus U(3)]$ collinear: $g_{18} = \sqrt{2}\, g_{88}$

b) $SU(6)_W$: $g_{18} = g_{81} = \sqrt{2}\, g_{88}$.

In the following Table, the predictions of SU(3), $S[U(3) \otimes U(3)]$, and $SU(6)_W$ are given.

Remarks

Column 6: The values $\sin\lambda = -0.643$ and $\sin\alpha = \pm 0.183$ were used. This choice minimizes $\varphi \to \pi^0\gamma$ [see Section 5.1.2, Eqs. (8) and (10)].

Column 7: To give a rough idea for the order of magnitude, the input $\Gamma(\omega\pi^0\gamma) = 1.2$ MeV was used, although this is only an *upper limit* with large errors (10%). The five first decay widths are then calculated using $U(3) \otimes U(3)$, the others using $SU(6)_W$, and multiplying with q^3.

Column 8: The values for Γ are taken from ROSENFELD et al., UCRL-8030 (Rev. 1. 10. 1965). The width of $X_0 = \eta'$ is unknown, but the values of the Table can be used to give a lower limit to it. With

$$\frac{\Gamma(X_0 \to \varrho\gamma)}{\Gamma(X_0 \to \text{all})} < \frac{1}{4}$$

one gets

$$\Gamma_{X_0} > 0.24 \text{ MeV} \quad \text{or} \quad \Gamma_{X_0} > 0.40 \text{ MeV}$$

according to the two possible solutions. This is a prediction of $SU(6)_W$ only.

As is apparent from the Table, three decays are particularly well suited for comparison with experiment, because q^3 has nearly the same value.

From collinear $S[U(3) \otimes U(3)]$ one gets:

$$\frac{\Gamma(\varrho\pi\gamma)}{\Gamma(\omega\pi\gamma)} = 0.104 \ .$$

In addition, $SU(6)_W$ predicts:

$$\frac{\Gamma(\varphi\eta\gamma)}{\Gamma(\omega\pi\gamma)} = 0.203 \quad \text{or} \quad 0.33$$

Table

Predictions of SU(3), $S[U(3) \otimes U(3)]$ and $SU(6)_W$ for the decays $M_1 \to M_2\,\gamma$

Transition	g_{88}	g_{18}	g_{81}	$q^3 \times 10^{-6}$	$\lvert g \rvert^2$	Γ_{if} (MeV)	Γ_{if}/Γ
$\varrho^+\,\pi^-\,\gamma$	1			50.66	1	0.12	10^{-3}
$\varrho^0\,\pi^0\,\gamma$	1			50.90	1	0.12	10^{-3}
$K^{*+}\,K^-\,\gamma$	1			29.37	1	0.07	1.5×10^{-3}
$K^{*0}\,\bar{K}^0\,\gamma$	-2			28.73	4	0.28	6.0×10^{-3}
					From $S[U(3) \otimes U(3)]$ $\sqrt{2}\,g_{88} = g_{18}$		
$\varphi\,\pi^0\,\gamma$	$\sqrt{3}\cos\lambda$	$\sqrt{3}\sin\lambda$		125.75	0.06	0.02	1.5×10^{-2}
$\omega\,\pi^0\,\gamma$	$-\sqrt{3}\sin\lambda$	$\sqrt{3}\cos\lambda$		54.88	8.94	1.2 =	10^{-1}
					From $SU(6)_W$ $\sqrt{2}\,g_{88} = g_{18} = g_{81}$	Input	
$\varrho^0\,\eta\,\gamma$	$\sqrt{3}\cos\alpha$		$\sqrt{3}\sin\alpha$	6.40	4.63	0.073	6×10^{-4}
					1.57	0.025	2×10^{-4}
$\varphi\,\eta\,\gamma$	$-\cos\lambda\,\cos\alpha$	$\sin\lambda\,\cos\alpha$	$\cos\lambda\,\sin\alpha$	47.44	2.10	0.24	7.3×10^{-2}
					3.40	0.40	12.1×10^{-2}
$\omega\,\eta\,\gamma$	$\sin\lambda\,\cos\alpha$	$\cos\lambda\,\cos\alpha$	$-\sin\lambda\,\sin\alpha$	7.93	0.36	0.007	6×10^{-4}
					0.07	0.002	1.7×10^{-4}
$X^0\,\varrho^0\,\gamma$	$-\sqrt{3}\sin\alpha$		$\sqrt{3}\cos\alpha$	5.33	4.37	0.057	
					7.43	0.096	
$\varphi\,X^0\,\gamma$	$\cos\lambda\,\sin\alpha$	$-\sin\lambda\,\sin\alpha$	$\cos\lambda\,\cos\alpha$	0.20	1.88	0.0009	3×10^{-4}
					0.57	0.0003	10^{-4}
$X^0\,\omega\,\gamma$	$-\sin\lambda\,\sin\alpha$	$-\cos\lambda\,\sin\alpha$	$-\sin\lambda\,\cos\alpha$	4.10	0.66	0.007	
					0.95	0.009	

where the two solutions refer to the two possible signs of $\sin\alpha$. From $S[U(3) \otimes U(3)]$ one gets the relation for amplitudes, corrected for phase space:

$$0.284 \mid \omega \, \eta \, \gamma \mid = 0.093 \mid \varphi \, \eta \, \gamma \mid + 0.06 \mid \omega \, \pi \, \gamma \mid .$$

Only upper limits are known for the experimental quantities involved.

Finally we remark that BECCHI and MORPURGO (Ref. [43]) calculate the absolute rate of $\omega \to \pi \gamma$, using a quark model and a quark magnetic moment deduced from the proton magnetic moment. They find

$$\Gamma(\omega \, \pi \, \gamma) = 1.17 \text{ MeV}$$

in agreement with experiment. They also get the other results of the Table, although the physical assumptions of their quark model are different from $SU(6)_W$.

Literature: SU(3): [180]; $SU(6)_\sigma$: [22, 23, 30, 205, 276, 311, 369, 370]; SL(6,C): [205, 336, 398, 399].

Literature: SU(3): [180]; $SU(6)_\sigma$: [22, 23, 30, 205, 276, 311, 369, 370]; SL(6,C): [205, 336, 398, 399]; SU(6,6): [63]; Quark model: [21, 43, 100, 379].
Reference on the meson decay with creation of a Dalitz pair: [124]. See also the references on the $M\,M\,M$ vertex.

5.6 *B B B B four-point function*

To our knowledge, all the works which treat the problem of baryon-baryon scattering with B belonging to the representation 56 of SU(6), in the framework of a U(6,6) theory, omit irregular couplings. Hence, one can extract from these works the predictions of collinear $SU(6)_W$, but not of *coplanar* $S[U(3) \otimes U(3)]$. The reason is again that the 56 baryon states belong to an irreducible representation of $SU(6)_W$, and therefore the irregular couplings give no new invariants. On the other hand, they belong to a reducible representation of $S[U(3) \otimes U(3)]$. For example, the spin $^1/_2$ octet belongs to $(6,3) \oplus (3,6)$, and the irregular couplings may take different values for (6,3) and (3,6), and hence are important.

However, the most important prediction of *coplanar* $S[U(3) \otimes U(3)]$ can be obtained without any work: the component of the spin along an axis orthogonal to the scattering plane is conserved. For spin $^1/_2$-spin $^1/_2$ scattering, this means that there are no $\Delta l = 2$ transitions.

For *proton-proton scattering*, the most general form of the scattering matrix in spin space is, in standard notations (Ref. [281]):

$$M(\boldsymbol{k}, \boldsymbol{k}') = B\,P_S + C\,(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\,\boldsymbol{n} + N(\boldsymbol{\sigma}_1 \boldsymbol{n})\,(\boldsymbol{\sigma}_2 \boldsymbol{n})\,P_T$$
$$+ \frac{1}{2}\,G\,[(\boldsymbol{\sigma}_1 \boldsymbol{K})\,(\boldsymbol{\sigma}_2 \boldsymbol{K}) + (\boldsymbol{\sigma}_1 \boldsymbol{P})\,(\boldsymbol{\sigma}_2 \boldsymbol{P})]\,P_T + \frac{1}{2}\,H\,[(\boldsymbol{\sigma}_1 \boldsymbol{K})\,(\boldsymbol{\sigma}_2 \boldsymbol{K}) - (\boldsymbol{\sigma}_1 \boldsymbol{P})\,(\boldsymbol{\sigma}_2 \boldsymbol{P})]\,P_T$$

where $\boldsymbol{n}$, $\boldsymbol{K}$, $\boldsymbol{P}$ are unit vectors along $\boldsymbol{k} \times \boldsymbol{k}'$, $\boldsymbol{k} + \boldsymbol{k}'$ and $\boldsymbol{k} - \boldsymbol{k}'$, and P_S and P_T are singlet and triplet projection operators.

The same form holds for particles belonging to a same I-spin, U-spin or V-spin multiplet (Ref. [332]) but otherwise could be more general.

The term multiplying H is responsible for $\Delta l = 2$ transitions. Hence, coplanar S[U(3) ⊗ U(3) entails

$$H = 0 .$$

In terms of triple scattering parameters, this means (Ref. [146]):

$$R = A' = 0$$

$$R' + A = 0$$

$$| C_{Kp} | \leqslant \cos\alpha_L$$

α_L is the angle between the two final baryons in the lab. frame.

For proton-proton scattering, the Pauli principle leads to the additional relations at 90° c.m.:

$$A(90°) - R(90°) \cot\theta_L = 0$$

$$2\,[1 - D(90°)] - [1 - C_{NN}(90°)] = 0$$

$$D(90°) \geqslant 0$$

θ_L is the laboratory scattering angle.

Comparison of these predictions with experiment has been done in Refs. [146] and [234]. The agreement is very poor.

For *collinear* processes, the most general scattering amplitude is obtained by setting

$$C = 0 \quad N = \frac{1}{2}\,(G - H)$$

so that $SU(6)_W$ entails in addition

$$N = \frac{1}{2}\,G\,.$$

In view of the bad prediction of the coplanar group, it is not worth while pursuing the analysis of $SU(6)_W$.

On the other hand, the hierarchy of groups

$$SU(6)_\sigma \supset \text{collinear } S[U(3) \otimes U(3)] \supset \text{coplanar } SU(3)$$

gives no additional restrictions for coplanar processes. For collinear processes, one gets predictions for reactions involving *polarized* particles. There is not yet enough experimental material to discuss these predictions in true detail.

Literature: SU(3): [332]; $SU(6)_\sigma$ at threshold: [36, 59, 91, 377]; SL(6,C): 267]; SU(6,6): [7, 72, 146, 234, 252]. Literature on baryon-antibaryon scattering: SU(6,6): [7, 72, 125, 265].

5.7 *B B M M four-point function*

There are doubts if higher symmetries like SU(3) or SU(6) should be applied at all to processes where four hadrons are involved. For example, in the case of baryon-meson scattering, one striking empirical fact is the existence, in many processes, of forward and/or backward peaks, which can be explained by the dominance of peripheral graphs. These are characterized by the exchange of a meson or a baryon, and give a pole contribution which is most important if the mass of the exchanged particle is small. Especially in the case of pseudoscalar meson exchange, the large mass difference between the pion and the kaon will introduce large symmetry-breaking effects. These are, however, smaller if a baryon is exchanged. Other, inelastic effects, also depend strongly on mass differences.

Another serious difficulty is the dependence of the kinematical factors on mass differences. Some reactions are endothermic, such as $p\pi^- \rightarrow \Sigma^- K^+$, others are exothermic, like $p K^- \rightarrow \Sigma^- \pi^+$. So one does not know exactly at what energy one should compare these processes. A prescription has been given in Ref. [277].

There are, however, reactions where both of these difficulties are minimal. For example, both collinear groups predict for the cross-sections in the forward and backward direction

$$\sigma\,(p\,K^- \rightarrow \Xi\,K^+) = 4\,\sigma\,(p\,K^- \rightarrow \Xi^\circ\,K^\circ)\,.$$

The kinematical factors are here almost the same. In the backward direction, these processes seem to be dominated by baryon exchange, for example Λ and Σ, whose mass difference is relatively small. In fact, the above prediction is consistent with present experiments (Ref. [28]).

A new, practical difficulty appears for collinear processes. Since the differential cross-sections are only known for a small number of angles, it is difficult to extrapolate them to 0° or 180°. Only improved statistics and better angular resolution will solve this problem.

One may argue that mass differences would play a smaller role for reactions at high energy and high momentum transfer. However, the cross-sections for two-body processes are very small under such conditions, so that experimental verification becomes extremely difficult. Furthermore, the collinear groups would then not apply, since they involve zero momentum transfer in one or the other channel. But one of the successes of the models discussed here are the Johnson-Treiman (Ref. [226]) relations, which are derived using the collinear groups. (They can, however, also be derived using other models.)

Eventually, only experiment will decide if the theory is useful. Hence, in the following we give the main predictions, first for baryon (56) meson (35) scattering, then for the corresponding annihilation channel.

5.7.1 Baryon (56) meson (35) scattering

a) Collinear S[U(3) ⊗ U(3)]

The number of invariants is very large and it would be useless to give them all. An extensive list of predictions is contained in Refs. [267] and [331].

Let us first remember that the zero helicity states of vector mesons belong to a different representation than the ± 1 helicity states. The same is true for the $\pm {}^1/_2$ and $\pm {}^3/_2$ states of spin ${}^3/_2$ baryons. Hence, no simple predictions concerning polarization arise.

The most interesting predictions are of two kinds:

i) Forward and backward scattering

$$\sigma\,(K^- p \rightarrow K^+ \Xi^-) = 4\,\sigma\,(K^- p \rightarrow K^\circ\,\Xi^\circ) \tag{1a}$$

$$\sigma\,(K^- p \rightarrow K^0\,\Xi^\circ) = \sigma\,(\pi^- p \rightarrow K^+ \Sigma^-) \tag{1b}$$

$$\sigma\,(K^- p \rightarrow K^\circ\,\Xi^\circ) = \sigma\,(K^- p \rightarrow \pi^+ \Sigma^-)\,. \tag{1c}$$

As stated above, Eq. (1a) gives the least ambiguous test of the model. Within (large) experimental errors, it is satisfied in the backward direction (Ref. [28]. In the forward direction, there is no reliable experimental value for $\sigma\,(K^- p \to K^0 \Xi^0)$, due to the difficulty of identifying the process.

Equation (1c) follows from SU(3) alone (Refs. [147, 258]). Equations (1b) and (1c) disagree with experiment, maybe for the reasons outlined at the beginning.

ii) Total cross-sections

Using the optical theorem, the symmetry gives a linear combination of the Johnson-Treiman relations (Ref. [226]).

$$(K^+ p) + (\pi^- p) + (\overline{K^\circ}\, p) = (K^- p) + (\pi^+ p) + (K^\circ p)$$

where (Mp) means $\sigma_{\text{total}}\,(Mp)$.

b) Collinear $SU(6)_W$

There are four invariants:

$$\begin{aligned} I_1 &= a_1\, \overline{B}^{ABC}\, B_{ABC}\, \overline{M}^D_E\, M^E_D \\ I_2 &= a_2\, \overline{B}^{ABC}\, B_{ABD}\, \overline{M}^D_E\, M^E_C \\ I_3 &= a_3\, \overline{B}^{ABC}\, B_{ABD}\, M^D_E\, \overline{M}^E_C \\ I_4 &= a_4\, \overline{B}^{ABC}\, B_{ADE}\, \overline{M}^D_B\, M^E_C\,. \end{aligned}$$

Contrary to the assertion of Ref. [65], $SU(6)_W$ gives exactly the same number of invariants as a U(6,6) theory with irregular couplings.

An extensive list of predictions is contained in Ref. [75]. We give a few examples:

i) Forward and backward scattering

In addition to the $S[U(3) \otimes U(3)]$ predictions one gets, for example,

$$\sigma\,(K^+ p \to K^\circ N^{*++}) = \sigma\,(K^\circ p \to K^+ N^{*\circ}) = \sigma\,(K^- p \to K^- N^{*+}) = \sigma\,(\overline{K^\circ}\, p \to K^- N^{*++}) = 0\,.$$

This follows from an SU(2) subgroup of $SU(6)_W$ (Ref. [264])

$$\sigma\,(K^+ p \to K^{*+} p) = \frac{2}{3}\,\sigma\,(K^+ p \to K^{*\circ} N^{*++}) = \frac{16}{3}\,\sigma\,(K^\circ p \to K^+ n)\,.$$

All reactions $B + M \to B' + M'$, such that the reaction in the crossed channel $B + \overline{B}' \to M' + \overline{M}$ goes only through the representation 405, are given by *one* amplitude. This is the case for $B\overline{B}'$ systems with $Y = 2$ and/or $I = 3/2$, $I = 2$, $W = 2$. All reactions $P + B \to P + B^*$ are in this category (Ref. [75]). So, for example,

$$\frac{1}{12}\,\sigma\,(K^- p \to K^+ \Xi^-) = \frac{1}{48}\,\sigma\,(K^- p \to K^{*+} \Xi^-) = \frac{1}{24}\,\sigma\,(K^- p \to \pi^- K^{*+})$$
$$= \frac{1}{96}\,\sigma\,(K^- p \to \pi\, K^{*-}) = \frac{1}{288}\,\sigma\,(\pi^- p \to \pi^+ N^{*-}) = \frac{1}{24}\,\sigma\,(\pi^+ p \to \pi^+ N^{*+}) = \text{etc.}$$

As pointed out in Ref. [223], many of these predictions are in violent disagreement with experiment. On the other hand, one important agreement has been found in Ref. [299]: $SU(6)_W$ gives a relation between the $I = 1/2$ and $I = 3/2$ amplitudes of $\pi N \to \pi N^*$, which agrees with the determined scattering lengths. This example is relevant in so far as the kinematical factors do not break the symmetry. However, the prediction $\sigma (K^- p \to \pi^- Y^{*+}) = 4 \sigma (K^- p \to \pi^+ Y^{*-})$ again disagrees with empirical data.

ii) Total cross-sections

Both Johnson-Treiman relations follow from $SU(6)_W$, namely

$$(K^- p) + 2(\pi^+ p) = (K^+ p) + 2(\pi^- p) \qquad \text{(JT1)}$$

$$(K^- p) + 2(K^0 p) = (K^+ p) + 2(\bar{K}^0 p) . \qquad \text{(JT2)}$$

The relation obtained from collinear $S[U(3) \otimes U(3)]$ is the sum of these two relations, which we call $(JT\Sigma)$.

In the following Table, we give the experimental results for the left-hand side of the equations, and of the difference Δ between the two sides (Ref. [156]).

Table

	$JT\,1$		$JT\,2$		$JT\,\Sigma$	
Momentum (GeV/c)	$(K^- p) + 2(\pi^+ p)$ (mb)	Δ (mb)	$(K^- p) + 2(K^0 P)$ (mb)	Δ (mb)	$(K^- p) + (\pi^+ p) + (K^0 p)$ (mb)	Δ (mb)
6	76.4 ± 0.7	+2.4 ± 1.4	59.0 ± 1.2	−1.8 ± 2.1	67.7 ± 1.0	+0.3 ± 1.8
8	73.8 ± 0.6	+1.5 ± 1.3	58.8 ± 1.0	+2.1 ± 1.9	66.3 ± 0.8	+1.8 ± 1.6
10	72.1 ± 0.6	+1.8 ± 1.3	57.5 ± 1.0	−1.0 ± 1.9	64.8 ± 0.8	+0.4 ± 1.6
12	70.0 ± 0.6	+0.9 ± 1.3	56.8 ± 1.0	−0.9 ± 1.9	63.4 ± 0.8	0 ± 1.6
14	69.3 ± 0.6	+1.1 ± 1.3	56.5 ± 1.0	−1.1 ± 1.9	62.9 ± 0.8	0 ± 1.6
16	68.1 ± 0.8	+0.9 ± 1.5	56.1 ± 1.2	−1.5 ± 2.5	62.1 ± 1.0	−0.3 ± 2.0
18	68.0 ± 1.2	+0.9 ± 1.9	56.2 ± 1.6	−1.5 ± 3.9	62.1 ± 1.4	−0.3 ± 2.9

The agreement of the $S[U(3) \otimes U(3)]$ prediction is systematically better than the additional $SU(6)_W$ prediction.

There exist other models which predict the Johnson-Treiman relations, for example, the quark model (Ref. [262]), or the octet dominance model (Ref. [36]), so that they are not a decisive test of any of these models.

iii) Polarizations

$SU(6)_W$ [but not $U(3) \otimes U(3)$] predicts for forward or backward production zero polarization of the resonances (meaning that the density matrix elements ϱ_{mm} are equal for different m values), for example, in the following cases:

$$K^+ p \to K^{+*} p \,, \quad K^- p \to K^{-*} p$$

$$\pi^+ p \to \varrho^+ p \,, \quad \pi^- p \to \varrho^- p$$

$$K^+ p \to K^{*+} N^{*+} \,, \quad \text{etc.}$$

In the first or last case, for example, this disagrees with experiment (Ref. [223]).

5.7.2 Baryon-antibaryon annihilation into two mesons

This is, of course, just the crossed channel of baryon-meson scattering, so one gets the same number of amplitudes and the same number of predictions. However, most of the latter are not very interesting, because they involve processes which are very difficult to realize in the laboratory, such as $\bar{\Xi}^- p \to K^+ K^+$, etc.

One important restriction arises from the fact that $p\bar{p}$ annihilation at rest into two mesons proceeds mainly through an S-state. In the following, we shall assume this to be always the case. Parity conservation implies that the mesons are in an odd angular momentum state. Using generalized Bose statistics for the mesons, one finds that only antisymmetric products of representations of the collinear groups contribute. For $SU(6)_W$, there is only one amplitude, namely $(35)_a$.

This case has been analysed in Ref. [367] for both $SU(6)_W$ and collinear $S(U(3) \otimes U(3))$. Comparison of $SU(6)_W$ predictions with experiment can be found in Ref. [57].

a) Collinear $S(U(3) \otimes U(3))$

$$(\bar{p}\, p \to \pi\, \varphi) = (\bar{p}\, p \to \eta\, \varphi) = (p\, \bar{p} \to \varrho\, \varphi) = (p\, \bar{p} \to \omega\, \varphi) = 0\,.$$

This was also obtained in Ref. [14], using static SU(6) with a spurion.

There are four other relations involving polarized particles.

$$\langle K^+ K^- \mid T \mid p^\uparrow \bar{p}^\downarrow\rangle = \langle K^{*+\rightarrow}, K^{*-\rightarrow} \mid T \mid p^\uparrow \bar{p}^\downarrow\rangle$$
$$\langle K^0 \overline{K^0} \mid T \mid p^\uparrow \bar{p}^\downarrow\rangle = \langle K^{*0\rightarrow}, \overline{K^{*0}}^{\rightarrow} \mid T \mid p^\uparrow \bar{p}^\downarrow\rangle$$
$$\langle K^+ K^{*-\uparrow} \mid T \mid p^\uparrow \bar{p}^\uparrow\rangle = \langle K^{*+}, K^{*-\uparrow} \mid T \mid p^\uparrow \bar{p}^\uparrow\rangle$$
$$\langle K^0 \overline{K^{*0}}^{\uparrow} \mid T \mid p^\uparrow \bar{p}^\uparrow\rangle = \langle K^{*0\rightarrow}, \overline{K^{*0}}^{\uparrow} \mid T \mid p^\uparrow \bar{p}^\uparrow\rangle$$

where $\overset{*}{K}{}^\uparrow$, $\overset{*}{K}{}^\rightarrow$, $\overset{*}{K}{}^\downarrow$ are the $S_z = 1, 0, -1$ states.

b) $SU(6)_W$

We give the results of Ref. [57]. It is assumed that the $\bar{p}n$ annihilation at rest also goes through the S-state.

Table

Rates for two-meson annihilations of antiprotons at rest on protons and neutrons. $SU(6)_W$ values. The rates reported are normalized to $^3S_1\, p\bar{p} \to \bar{K}^{*0} K^0$ taken equal to one.

Channel	Rate	Channel	Rate
$^3S_1\, p\bar{p} \to \bar{K}^{*0}\, K^0$	1	$^1S_0\, p\bar{p} \to \bar{K}^{*0}\, K^0$	0.5
$\to K^{*-}\, K^+$	16	$\to K^{*-}\, K^+$	8
$\to \varrho^{0\pm}\, \pi^{0\mp}$	9+9+9	$\to \varrho^{\pm}\, \pi^{\mp}$	12.5+12.5
$\to \omega^0\, \pi^0$*)	25	$^1S_0\, np \to K^{*0}\, K^-$	12.5
$\to \varphi\, \pi^0$*)	0	$\to K^{*-}\, K^0$	12.5
$^3S_1\, n\bar{p} \to K^{*-}\, K^0$	25	$\to \varrho^0\, \pi^-$	25
$\to K^{*0}\, K^-$	25	$^3S_1\, p\bar{p} \to \pi^-\, \pi^+$	4.5
$\to \omega^0\, \pi^-$*)	50	$\to K^-\, K^+$	18
$\to \varphi\, \pi^-$*)	0	$\to \bar{K}^0\, K^0$	4.5
		$^3S_1\, n\bar{p} \to \pi^-\, \pi^0$	9
		$\to K^-\, K^0$	4.5

*) We have assumed:

$$\omega^0 = \frac{1}{\sqrt{3}} \varphi_8 + \sqrt{\frac{2}{3}} \varphi_1 \qquad \varphi = -\sqrt{\frac{2}{3}} \varphi_8 + \sqrt{\frac{1}{3}} \varphi_1 .$$

The following predictions involve final states with nearly the same mass:

1)
$$\frac{p\,\bar{p} \to K^+ K^-}{p\,\bar{p} \to K^0 \bar{K}^0} = 4$$

(see also Refs. [14, 202, 218]).
The experimental ratio is 2.0 ± 0.26, Ref. [57].

2)
$$\frac{({}^3S_1\, p\,\bar{p} \to \bar{K}^{*0} K^0) + ({}^3S_1\, p\,\bar{p} \to K^{*0} \bar{K}^0)}{({}^3S_1\, p\,\bar{p} \to K^{*-} K^+) + ({}^3S_1\, p\,\bar{p} \to K^{*+} K^-)} = \frac{1}{16} .$$

A lower limit for the experimental ratio is 0.87 ± 0.23.

3)
$$\frac{{}^3S_1\, p\,\bar{p} \to \bar{K}^{0*} K^0}{{}^1S_0\, p\,\bar{p} \to \bar{K}^{0*} K^0} = \frac{1}{8}$$

4)
$$\frac{{}^3S_1\, p\,\bar{p} \to \varrho\,\pi}{{}^1S_0\, p\,\bar{p} \to \varrho\,\pi} = \frac{27}{25}$$

5) Assuming an $\omega - \varphi$ mixing angle of 40°, one gets

$$\frac{{}^3S_1\, n\,\bar{p} \to \varphi\,\pi^-}{{}^3S_1\, n\,\bar{p} \to \omega\,\pi^-} = 0.007$$

which disagrees with the experimental lower limit.

Literature on meson-baryon scattering: SU(3): [147, 258, 277]; $SU(6)_\sigma$: [35, 101, 226, 314]; SL(6,C): [267, 331, 337]; SU(6,6): [26, 60, 65, 75, 85, 90, 98, 115, 197, 271, 299, 318, 402]; other models: [35, 262, 363]; comparison with experiment: [179, 223].

Literature on baryon-antibaryon annihilation: $SU(6)_\sigma$: [14, 29, 127, 198, 241, 244]; SL(6,C): [71, 367, 400]; SU(6,6): [57, 82, 108, 192, 195, 202, 218, 219, 223, 251, 255, 267, 397].

5.8 *Non-leptonic decays of hyperons and omega*

In this paragraph we deal with the weak decays

hyperon or $\Omega^- \to$ nucleon or hyperon + pseudoscalar meson.

For the general theory see Ref. [54].

The decay amplitudes consist of two parts which conserve and violate parity, respectively. They are handled separately. The SU(6) treatment of these decays is a generalization of the SU(3) treatment in so far as the characteristic selection rule $\Delta I = 1/2$ is incorporated in the ansatz by means of a spurion which belongs to the adjoint representation:

$$\mathrm{SU}(3) : S^a_b = (\delta^a_3\, \delta^2_b + \delta^a_2\, \delta^3_b) \text{ belonging to the octet,}$$
$$\mathrm{SU}(6) : S^A_B = (\delta^a_3\, \delta^2_b + \delta^a_2\, \delta^3_b)\, \delta^\alpha_\beta \text{ from the 35-plet.}$$

For the discussion of the (+)-sign in these spurions we refer to Ref. [54]. The spurion is assumed to bear the eigenparity plus or minus corresponding to parity conservation or violation. In S[U(6) ⊗ U(6)] the ansatz for the spurion can be made analogously. The representations to be used are the products of quark and pseudoquark with pseudoantiquark and antiquark, respectively (parity plus) or of quark with antiquark (parity minus). They belong to one 143-plet of SU(6,6).

We shall use the collinear groups to describe the P- and D-waves; S-waves can be handled with the static groups. We know that the collinear group approach is equivalent to a certain technique which uses "kinetic spurions" (see Part I, Section 3.2 for the discussion of this technique). These kinetic spurions are SU(3) singlets. The appearance of two types of spurions, kinetic and weak ones, leads us to suggest a different method of treating the non-leptonic weak decays which is no longer equivalent to the collinear group technique: we ascribe the transformation properties of the weak spurion to the kinetic ones and take the latter from SU(3) octets [weak spurion (8,1) ⊂ 35]×[kinetic spurion (1,3) ⊂ 35] → [weak kinetic spurion (8,3) ⊂ 35]. We do not go into the details of such a "combined spurion" approach, but refer to the original publications, Refs. [25, 135, 220, 235, 327]. The results obtained are in general more restrictive than those of the collinear groups. Only in the case of the SU(6,6) group is this approach accidentally equivalent to the $SU(6)_W$ technique.

The parity violating amplitude can be handled with the static models as long as only S-waves are concerned. The Ω^- decay proceeds, however, possibly partly through D-waves. Therefore, we cannot a priori neglect them.

We note the relations implied by the $\Delta I = {}^1/_2$ rule

$$\sqrt{2}\ \langle \Sigma^+ \to p\,\pi^0\rangle = \langle \Sigma^+ \to n\,\pi^+\rangle - \langle \Sigma^- \to n\,\pi^-\rangle \tag{1}$$

$$\langle \Lambda \to p\,\pi^-\rangle = -\sqrt{2}\ \langle \Lambda \to n\,\pi^0\rangle \tag{2}$$

$$\langle \Xi^- \to \Lambda\,\pi^-\rangle = \sqrt{2}\ \langle \Xi^0 \to \Lambda\,\pi^0\rangle \tag{3}$$

$$\langle \Omega^- \to \Xi^0\,\pi^-\rangle = -\sqrt{2}\ \langle \Omega^- \to \Xi^-\,\pi^0\rangle\,. \tag{4}$$

5.8.1 Collinear S[U(3) ⊗ U(3)]

We follow along the lines of Ref. [16]. The technique of constructing invariants is quite similar to that used for the baryon-meson scattering process, since the weak spurion belongs either to the scalar or to the pseudoscalar octet contained in the 35-plet, and is correspondingly attributed to the representations of the subgroup S[U(3) ⊗ U(3)].

It is well known that in addition to the unitary symmetry certain discrete symmetries are necessary to yield restrictions on the non-leptonic decays in the SU(3) scheme. The situation here is similar. It is simplest to generalize the concept of "CP invariance", see Ref. [171] to the SU(6) approach, since the dynamical assumptions involved do not depend intrinsically on the group structure. We therefore give the results with or without additional CP invariance. We find *without* CP invariance for the parity conserving amplitudes

$$\langle \Omega^- \to \Lambda\,K^-\rangle_p = \frac{\sqrt{3}}{2}\left\{\langle \Lambda \to p\,\pi^-\rangle_p - 2\,\langle \Xi^- \to \Lambda\,\pi^-\rangle_p - \sqrt{3}\ \langle \Sigma^+ \to p\,\pi^0\rangle_p\right\} \tag{1}$$

and for the parity violating amplitudes

$$\langle\Omega^- \to \Lambda K^-\rangle_D = \frac{\sqrt{15}}{2}\left\{\langle\Lambda \to p\,\pi^-\rangle_S - 2\,\langle\Xi^- \to \Lambda\,\pi^-\rangle_S - \sqrt{3}\,\langle\Sigma^+ \to p\,\pi^0\rangle_S\right\}. \qquad (2)$$

If CP invariance *is* imposed, we obtain three additional relations for the parity violating amplitudes

$$\langle\Sigma^+ \to n\,\pi^+\rangle_S = 0 \qquad (3)$$

$$\langle\Omega^- \to \Lambda K^-\rangle_D = 0 \qquad (4)$$

$$2\,\langle\Omega^- \to \Xi^0\,\pi^-\rangle_D = -\sqrt{10}\,\langle\Lambda \to p\,\pi^-\rangle_S + \sqrt{15}\,\langle\Sigma^- \to n\,\pi^-\rangle_S\,. \qquad (5)$$

Static $SU(6)_\sigma$ gives the same results for the parity violating amplitudes, only all D-waves must be put equal zero (see Refs. [13, 17, 376]).

If we take Eqs. (2) and (4) together we obtain the first component of the triangle relation of SUGAWARA and LEE, Refs. [253] and [374]:

$$\langle\Lambda \to p\,\pi^-\rangle_S - 2\,\langle\Xi^- \to \Lambda\,\pi^-\rangle_S - \sqrt{3}\,\langle\Sigma^+ \to p\,\pi^0\rangle_S = 0\,.$$

The second component can be obtained if we assume ad hoc that in Eq. (1)

$$\langle\Omega^- \to \Lambda K^-\rangle_p = 0\,.$$

In the unitary symmetry scheme both triangle relations can be obtained with the assumption of R-invariance.

Modifications of the weak spurion, for example, the generalization due to the current-current hypothesis, are discussed in Refs. [13, 273, 378]. Some dynamic assumptions based on dispersion theory have been introduced in Ref. [67].

5.8.2 $SU(6)_W$

We rely on the results of Refs. [164, 214, 237, 293]. Similar results have been found in Refs. [158, 239]. The weak spurion is taken from Ref. [143]. We find *without CP* invariance for the parity conserving amplitudes

$$\langle\Omega^- \to \Lambda K^-\rangle_p = \frac{\sqrt{3}}{2}\left\{\langle\Lambda \to p\,\pi^-\rangle_p - 2\,\langle\Xi^- \to \Lambda\,\pi^-\rangle_p - \sqrt{3}\,\langle\Sigma^+ \to p\,\pi^0\rangle_p\right\} \qquad (1)$$

$$\langle\Omega^- \to \Lambda K^-\rangle_p = -\frac{3\sqrt{2}}{5}\,\langle\Sigma^+ \to n\,\pi^+\rangle_p \qquad (2)$$

$$\sqrt{2}\,\langle\Sigma^+ \to n\,\pi^+\rangle_p - 3\,\langle\Sigma^+ \to p\,\pi^0\rangle_p + \frac{1}{\sqrt{3}}\,\langle\Lambda \to p\,\pi^-\rangle_p = 0 \qquad (3)$$

$$\langle\Omega^- \to \Xi^-\,\pi^0\rangle_p + \frac{\sqrt{6}}{5}\,\langle\Sigma^+ \to n\,\pi^+\rangle_p - \sqrt{6}\,\langle\Sigma^- \to n\,\pi^-\rangle_p = 0\,. \qquad (4)$$

A more useful combination of Eqs. (1), (2) and (3) is

$$\langle\Sigma^+ \to p\,\pi^0\rangle_p - \frac{1}{\sqrt{3}}\,\langle\Xi^- \to \Lambda\,\pi^-\rangle_p - \frac{3\sqrt{2}}{10}\,\langle\Sigma^+ \to n\,\pi^+\rangle_p = 0\,. \qquad (1')$$

For the parity violating amplitudes we obtain

$$\langle \Omega^- \to \Lambda\, K^- \rangle_D = 0 \tag{5}$$

$$\langle \Lambda \to p\, \pi^- \rangle_S - 2\, \langle \Xi^- \to \Lambda\, \pi^- \rangle_S - \sqrt{3}\, \langle \Sigma^+ \to p\, \pi^0 \rangle_S = 0 \tag{6}$$

$$\frac{1}{\sqrt{3}}\, \langle \Lambda \to p\, \pi^- \rangle_S + \sqrt{2}\, \langle \Sigma^+ \to n\, \pi^+ \rangle + \langle \Sigma^+ \to p\, \pi^0 \rangle_S = 0\,. \tag{7}$$

If CP invariance is superimposed we get

$$\langle \Sigma^+ \to n\, \pi^+ \rangle_S = 0 \tag{8}$$

$$\langle \Omega^- \to \Xi^-\, \pi^0 \rangle_D = 0 \tag{9}$$

We note a useful combination of Eqs. (8) and (7)

$$\langle \Lambda \to p\, \pi^- \rangle_S + \sqrt{3}\, \langle \Sigma^+ \to p\, \pi^0 \rangle_S = 0\,. \tag{8'}$$

We recognize that the result

$$\langle \Sigma^+ \to n\, \pi^+ \rangle_S = 0$$

appears again only as a consequence of CP invariance. A remark made in Ref. [158] is perhaps interesting in this context: if the formalism of spinor invariants is used, the magnitude of the amplitude $\langle \Sigma^+ \to n\, \pi^+ \rangle_s$ turns out to be of the order of the meson-baryon mass ratio.

5.8.3 Comparison with experimental data

We use the data on hyperons as compiled in Ref. [362], data about the Ω^- are not yet available (following Table).

Only the prediction

$$\langle \Sigma^+ \to n\, \pi^+ \rangle_S = 0 \tag{1}$$

and the Sugawara-Lee relation

$$\langle \Lambda \to p\, \pi^- \rangle_S - 2\, \langle \Xi^- \to \Lambda\, \pi^- \rangle_S - \sqrt{3}\, \langle \Sigma^+ \to p\, \pi^0 \rangle_S = 0 \tag{2}$$

Table

Units are 10^5 $(1/\mathrm{MeV\ sec})^{1/2}$. A and B are the familiar covariant amplitudes corresponding to S- and P-waves. The phase of $A(\Lambda^0_-)$ is chosen positive. All other phases are chosen to fit the Sugawara-Lee relation for the A-amplitudes. We took the solution with $A(\Sigma^+_+)$ small (see Ref. [362]). The ambiguity of the value of the amplitude $\langle \Sigma^+ \to p\, \pi^0 \rangle$ is due to the fact that only the asymmetry parameter α has been measured.

	$\langle \Lambda \to p\, \pi^- \rangle$	$\langle \Xi^- \to \Lambda\, \pi^- \rangle$	$\langle \Sigma^- \to n\, \pi^- \rangle$	$\langle \Sigma^+ \to n\, \pi^+ \rangle$	$\langle \Sigma^+ \to p\, \pi^0 \rangle$
A	$+0.132 \pm 0.007$	$+0.169 \pm 0.004$	$+0.158 \pm 0.002$	-0.004 ± 0.007	-0.079 ± 0.020 a)
					-0.144 ± 0.011 b)
B	$+0.858 \pm 0.119$	-0.697 ± 0.125	-0.127 ± 0.168	$+1.632 \pm 0.042$	$+1.443 \pm 0.114$ a)
					$+0.785 \pm 0.200$ b)

can be tested from the set of $S[U(3) \otimes U(3)]$ results. At least one of the solutions which come out from the experiments fits the relations reasonable well. The single new S-wave result from $SU(6)_W$, Eq. (8′), is also in good agreement if solution a) (see the text belonging to the Table for the amplitude $\langle \Sigma^+ \to p\,\pi^0 \rangle_s$) is chosen.

The P-wave predictions from $SU(6)_W$ are, however, systematically bad as has been pointed out by several authors (see Ref. [158]). A typical case is the second component of the Sugawara-Lee relation which appears as

$$\langle \Lambda \to p\,\pi^- \rangle_p - 2\,\langle \Xi^- \to \Lambda\,\pi^- \rangle_p - \sqrt{3}\,\langle \Sigma^+ \to p\,\pi^0 \rangle_p = -\frac{2}{5}\sqrt{6}\,\langle \Sigma^+ \to n\,\pi^+ \rangle_p\,. \tag{3}$$

If a solution is chosen which makes $\langle \Sigma^+ \to n\,\pi^+ \rangle_s$ small (as was done in the Table) the amplitude $\langle \Sigma^+ \to n\,\pi^+ \rangle_p$ is necessarily big. The $SU(6)_W$ predictions are therefore not consistent.

Modifications of the weak spurion which are aimed at overcoming this difficulty have been studied in Refs. [158] and [165].

Literature on non-leptonic hyperon decays: $SU(6)_\sigma$: [2, 13, 17, 25, 67, 70, 220, 221, 235, 273, 327, 376, 378]; SL(6,C): [12, 16, 71, 328]; SU(6,6): [158, 164, 165, 214, 237, 239, 393, 303].

Literature on weak interactions in general and the structure of hadronic weak currents: $SU(6)_\sigma$: [18, 47, 70, 163, 282]; SL(6,C): [212, 335, 368]; SU(6,6): [3, 212, 237, 295, 396, 401].

Literature on neutrino-induced processes: $SU(6)_\sigma$: [9, 10, 304]; SL(6,C): [11]; SU(6,6): [11, 19, 238].

5.9 *Conclusions*

There is not yet sufficient experimental material to be able to arrive at a definitive conclusion about the usefulness of the concept of the chains of subgroups which are relativistic generalizations of SU(6).

Both collinear groups $S[U(3) \otimes U(3)]$ and $SU(6)_W$ give correct and important predictions for the $BB\gamma$ vertex. For the $MM\gamma$ vertex, experiments will soon be available.

For the BBM and MMM vertices, the accuracy of the collinear groups is limited by the accuracy of SU(3). The same is true for scattering processes involving four hadrons. For the total cross-section BM, $SU(6)_W$ gives the two Johnson-Treiman relations which agree fairly well with experiment, and collinear $S[U(3) \otimes U(3)]$ gives the linear combination of these relations which is in best agreement. There are a certain number of predictions of collinear $SU(6)_W$ for the processes $BM \to BM$ and $B\bar{B} \to MM$, and of the *coplanar* subgroup $S[U(3) \otimes U(3)]$ for proton-proton scattering which disagree rather violently with the empirical facts. Further tests of the collinear group $S[U(3) \otimes U(3)]$ probably require the measurement of polarizations.

6. Appendix

Table 1

The decomposition of representations of SU(6) into irreducible representations of $SU(3) \otimes SU(2)$

Indices

$A, B, \ldots$ run from 1 to 6, $a, b, \ldots$ from 1 to 3, $\alpha, \beta, \ldots$ from 1 to 2.

The following relations are always assumed:

$$t^A_A = r^a_a = s^\alpha_\alpha = s^{\alpha\beta}_{\alpha\gamma} = r^{ab}_{ac} = t^{AB}_{AC} = 0 .$$

Indices in round brackets are symmetrized, and those in square brackets are antisymmetrized.

The normalization factors have been chosen in the usual way, namely the norm of an SU(6) tensor is given by the sum of products of norms of SU(3), SU(2) tensors:

Example:

$$| 35 |^2 = | (8,3) |^2 + | (8,1) |^2 + | (1,3) |^2$$

$$t^B_A \, t^A_B = r^a_b \, r^b_a \, s^\alpha_\beta \, s^\beta_\alpha + r'^a_b \, r'^b_a + s'^\alpha_\beta \, s'^\beta_\alpha .$$

$t_A =$ 6

$r_a \, s_\alpha$ (3,2)

$t^A =$ $\overline{6}$

$r^a \, s^\alpha$ $(\overline{3},2)$

$t^A_B =$ 35

$r^a_b \, s^\alpha_\beta$ (8,3)

$+ \frac{1}{\sqrt{2}} r^a_b \, \delta^\alpha_\beta$ (8,1)

$+ \frac{1}{\sqrt{3}} \delta^a_b \, s^\alpha_\beta$ (1,3)

$t_{[ABC]} =$ 20

$\frac{1}{\sqrt{6}} \varepsilon_{abc} \, (s_{\alpha\beta\gamma})$ (1,4)

$+ \frac{1}{3\sqrt{6}} \left[r_{[ab]c} \, s_{(\alpha\beta)\gamma} + r_{[bc]a} \, s_{(\beta\gamma)\alpha} + r_{[ca]b} \, s_{(\gamma\alpha)\beta}\right]$ (8,2)

$t_{(ABC)} =$ 56

$r_{(abc)} \, s_{(\alpha\beta\gamma)}$ (10,4)

$+ \frac{1}{3\sqrt{2}} \left[r_{[ab]c} \, s_{[\alpha\beta]\gamma} + r_{[bc]a} \, s_{[\beta\gamma]\alpha} + r_{[ca]b} \, s_{[\gamma\alpha]\beta}\right]$ (8,2)

$t_{[AB]C} = - t_{[BC]A} - t_{[CA]B} =$ 70

$\frac{1}{\sqrt{2}} r_{[ab]c} \, s_{(\alpha\beta\gamma)}$ (8,4)

$+ \frac{1}{6} \varepsilon_{abc} \, s_{(\alpha\beta)\gamma}$ (1,2)

$+ \frac{1}{3\sqrt{6}} \left[2 \, r_{[ab]c} \, s_{(\alpha\beta)\gamma} - r_{[bc]a} \, s_{(\beta\gamma)\alpha} - r_{[ca]b} \, s_{(\gamma\alpha)\beta}\right]$ (8,2)

$+ \frac{1}{\sqrt{2}} r_{(abc)} \, s_{[\alpha\beta]\gamma}$ (10,2)

$$t^{[CD]}_{[AB]} = \qquad 189$$

$$\frac{1}{12\sqrt{16}}\left[3\,\delta^{(cd)}_{(ab)}\,\delta^{[\gamma\delta]}_{[\alpha\beta]} - 2\,\delta^{[cd]}_{[ab]}\,\delta^{(\gamma\delta)}_{(\alpha\beta)}\right] \qquad (1,1)$$

$$+\frac{1}{8\sqrt{30}}\left[5\,(\delta r)^{[cd]}_{[ab]}\,\delta^{(\gamma\delta)}_{(\alpha\beta)} - 3\,(\delta r)^{(cd)}_{(ab)}\,\delta^{[\gamma\delta]}_{[\alpha\beta]}\right] \qquad (8,1)$$

$$+\frac{1}{2}\,r^{(cd)}_{(ab)}\,\delta^{[\gamma\delta]}_{[\alpha\beta]} \qquad (27,1)$$

$$+\frac{1}{8\sqrt{3}}\left[(\delta r)^{(cd)}_{[ab]}\,(\delta s)^{[\gamma\delta]}_{(\alpha\beta)} - (\delta r)^{[cd]}_{(ab)}\,(\delta s)^{(\gamma\delta)}_{[\alpha\beta]}\right] \qquad (8,3)$$

$$+\frac{1}{16\sqrt{3}}\left[(\delta r)^{(cd)}_{[ab]}\,(\delta s)^{[\gamma\delta]}_{(\alpha\beta)} + (\delta r)^{[cd]}_{(ab)}\,(\delta s)^{(\gamma\delta)}_{[\alpha\beta]} - 3\,(\delta r)^{[cd]}_{[ab]}\,(\delta s)^{(\gamma\delta)}_{(\alpha\beta)}\right] \qquad (8,3)'$$

$$+\frac{1}{4}\,\varepsilon^{cde}\,r_{(abe)}\,(\delta s)^{(\gamma\delta)}_{[\alpha\beta]} \qquad (10,3)$$

$$+\frac{1}{4}\,\varepsilon_{abe}\,r^{(cde)}\,(\delta s)^{[\gamma\delta]}_{(\alpha\beta)} \qquad (\overline{10},3)$$

$$+\frac{1}{2\sqrt{3}}\,\delta^{[cd]}_{[ab]}\,s^{(\gamma\delta)}_{(\alpha\beta)} \qquad (1,5)$$

$$+\frac{1}{2}\,(\delta r)^{[cd]}_{[ab]}\,s^{(\gamma\delta)}_{(\alpha\beta)} \qquad (8,5)$$

$$t^{[CD]}_{(AB)} = \qquad 280$$

$$\frac{1}{24}\left[(\delta r)^{[cd]}_{(ab)}\,\delta^{(\gamma\delta)}_{(\alpha\beta)} - 3\,(\delta r)^{(cd)}_{[ab]}\,\delta^{[\gamma\delta]}_{(\alpha\beta)}\right] \qquad (8,1)$$

$$+\frac{1}{2\sqrt{6}}\,\varepsilon^{cde}\,r_{(abe)}\,\delta^{(\gamma\delta)}_{(\alpha\beta)} \qquad (10,1)$$

$$+\frac{1}{2\sqrt{6}}\,\varepsilon_{abe}\,r^{(cde)}\,\delta^{[\gamma\delta]}_{[\alpha\beta]} \qquad (\overline{10},1)$$

$$+\frac{1}{8\sqrt{5}}\left[\delta^{(cd)}_{(ab)}\,(\delta s)^{[\gamma\delta]}_{(\alpha\beta)} - 2\,\delta^{[cd]}_{[ab]}\,(\delta s)^{(\gamma\delta)}_{[\alpha\beta]}\right] \qquad (1,3)$$

$$+\frac{1}{8\sqrt{15}}\left[(\delta r)^{(cd)}_{(ab)}\,(\delta s)^{[\gamma\delta]}_{(\alpha\beta)} - 5\,(\delta r)^{[cd]}_{[ab]}\,(\delta s)^{(\gamma\delta)}_{[\alpha\beta]}\right] \qquad (8,3)$$

$$+\frac{1}{8\sqrt{6}}\left[(\delta r)^{(cd)}_{(ab)}\,(\delta s)^{[\gamma\delta]}_{(\alpha\beta)} + (\delta r)^{[cd]}_{[ab]}\,(\delta s)^{(\gamma\delta)}_{[\alpha\beta]} - (\delta r)^{[cd]}_{(ab)}\,(\delta s)^{(\gamma\delta)}_{(\alpha\beta)}\right] \qquad (8,3)'$$

$$+\frac{1}{4\sqrt{2}}\,\varepsilon^{cde}\,r_{(abe)}\,(\delta s)^{(\gamma\delta)}_{(\alpha\beta)} \qquad (10,3)$$

$$+\frac{1}{2\sqrt{2}}\,r^{(cd)}_{(ab)}\,(\delta s)^{[\gamma\delta]}_{(\alpha\beta)} \qquad (27,3)$$

$$+\frac{1}{2\sqrt{3}}\,(\delta r)^{[cd]}_{(ab)}\,s^{(\gamma\delta)}_{(\alpha\beta)} \qquad (8,5)$$

$$+\frac{1}{\sqrt{2}}\,\varepsilon^{cde}\,r_{abe}\,s^{(\gamma\delta)}_{(\alpha\beta)} \qquad (10,5)$$

$t^{(CD)}_{(AB)} =$ 405

$$\frac{1}{12\sqrt{14}} \left[\delta^{(cd)}_{(ab)}\, \delta^{(\gamma\delta)}_{(\alpha\beta)} - 6\, \delta^{[cd]}_{[ab]}\, \delta^{[\gamma\delta]}_{[\alpha\beta]}\right] \qquad (1,1)$$

$$+ \frac{1}{16\sqrt{15}} \left[(\delta r)^{(cd)}_{(ab)}\, \delta^{(\gamma\delta)}_{(\alpha\beta)} - 15\, (\delta r)^{[cd]}_{[ab]}\, \delta^{[\gamma\delta]}_{[\alpha\beta]}\right] \qquad (8,1)$$

$$+ \frac{1}{2\sqrt{3}}\, s^{(cd)}_{(ab)}\, \delta^{(\gamma\delta)}_{(\alpha\beta)} \qquad (27,1)$$

$$+ \frac{1}{8\sqrt{3}} \left[(\delta r)^{(cd)}_{[ab]}\, (\delta s)^{(\gamma\delta)}_{[\alpha\beta]} - (\delta r)^{[cd]}_{(ab)}\, (\delta s)^{[\gamma\delta]}_{(\alpha\beta)}\right] \qquad (8,3)$$

$$+ \frac{1}{16\sqrt{30}} \left[5\, (\delta r)^{(cd)}_{[ab]}\, (\delta s)^{(\gamma\delta)}_{[\alpha\beta]} + 5\, (\delta r)^{[cd]}_{(ab)}\, (\delta s)^{[\gamma\delta]}_{(\alpha\beta)} - 3\, (\delta r)^{(cd)}_{(ab)}\, (\delta s)^{(\gamma\delta)}_{(\alpha\beta)}\right] \qquad (8,3)'$$

$$+ \frac{1}{4}\, \varepsilon^{cde}\, r_{(abe)}\, (\delta s)^{[\gamma\delta]}_{(\alpha\beta)} \qquad (10,3)$$

$$+ \frac{1}{4}\, \varepsilon_{abe}\, r^{(cde)}\, (\delta s)^{(\gamma\delta)}_{[\alpha\beta]} \qquad (\overline{10},3)$$

$$+ \frac{1}{4}\, r^{(cd)}_{(ab)}\, (\delta s)^{(\gamma\delta)}_{(\alpha\beta)} \qquad (27,3)$$

$$+ \frac{1}{2\sqrt{6}}\, \delta^{(cd)}_{(ab)}\, s^{(\gamma\delta)}_{(\alpha\beta)} \qquad (1,5)$$

$$+ \frac{1}{2\sqrt{5}}\, (\delta r)^{(cd)}_{(ab)}\, s^{(\gamma\delta)}_{(\alpha\beta)} \qquad (8,5)$$

$$+ r^{(cd)}_{(ab)}\, s^{(\gamma\delta)}_{(\alpha\beta)} \qquad (27,5)$$

The following tensors have been made use of:

$$r_{[ab]c} = \varepsilon_{abd}\, r^d_c\,,$$

$$s_{(\alpha\beta)\gamma} = \varepsilon_{\alpha\gamma}\, s_\beta + \varepsilon_{\beta\gamma}\, s_\alpha\,,$$

$$s_{[\alpha\beta]\gamma} = \varepsilon_{\alpha\beta}\, s_\gamma\,,$$

and

$$\delta^{(cd)}_{(ab)} = \delta^c_a\, \delta^d_b + \delta^d_a\, \delta^c_b\,,$$

$$\delta^{[cd]}_{[ab]} = \delta^c_a\, \delta^d_b - \delta^d_a\, \delta^c_b\,,$$

$$\delta^{(\gamma\delta)}_{(\alpha\beta)} = \delta^\gamma_\alpha\, \delta^\delta_\beta + \delta^\delta_\alpha\, \delta^\gamma_\beta\,,$$

$$\delta^{[\gamma\delta]}_{[\alpha\beta]} = \delta^\gamma_\alpha\, \delta^\delta_\beta - \delta^\delta_\alpha\, \delta^\gamma_\beta\,,$$

$$(\delta r)^{(cd)}_{(ab)} = \delta^c_a\, r^d_b + \delta^d_a\, r^c_b + \delta^c_b\, r^d_a + \delta^d_b\, r^c_a\,,$$

$$(\delta r)^{[cd]}_{(ab)} = \delta^c_a\, r^d_b - \delta^d_a\, r^c_b + \delta^c_b\, r^d_a - \delta^d_b\, r^c_a\,,$$

$$(\delta r)^{(cd)}_{[ab]} = \delta^c_a\, r^d_b + \delta^d_a\, r^c_b - \delta^c_b\, r^d_a - \delta^d_b\, r^c_a\,,$$

$$(\delta r)^{[cd]}_{[ab]} = \delta^c_a\, r^d_b - \delta^d_a\, r^c_b - \delta^c_b\, r^d_a + \delta^d_b\, r^c_a\,.$$

The tensors (δs) are constructed analogously.

Table 2
Multiplication of representations of SU(6)

	6	21	15
$6\ (1)$	$21\ (2) \oplus 15\ (1^2)$		
$\overline{6}\ (1^5)$	$35\ (21^4) \oplus 1$		
$21\ (2)$	$56\ (3) \oplus 70\ (21)$	$126\ (4) \oplus 210\ (31) \oplus 105\ (2^2)$	
$\overline{21}\ (2^5)$	$\overline{6}\ (1^5) \oplus \overline{120}\ (32^4)$	$1 \oplus 35\ (21^4) \oplus 405\ (42^4)$	
$15\ (1^2)$	$20\ (1^3) \oplus 70\ (21)$	$105'\ (21^2) \oplus 210\ (31)$	$105'\ (21^2) \oplus 105\ (2^2) \oplus \overline{15}\ (1^4)$
$\overline{15}\ (1^4)$	$84\ (21^3) \oplus \overline{6}\ (1^5)$	$280\ (31^3) \oplus 35\ (21^4)$	$189\ (2^21^2) \oplus 35\ (21^4) \oplus 1$
$126\ (4)$	$252\ (5) \oplus 504\ (41)$	$462\ (6) \oplus 1050\ (51) \oplus 1134\ (42)$	$1050\ (51) \oplus 840\ (41^2)$
$\overline{126}\ (4^5)$	$\overline{56}\ (3^5) \oplus \overline{700}\ (54^4)$	$2310\ (64^4) \oplus \overline{315}\ (43^4) \oplus \overline{21}\ (2^5)$	$\overline{1575}\ (5^24^3) \oplus \overline{315}\ (43^4)$
$210\ (31)$	$504\ (41) \oplus 420\ (32) \oplus 336\ (31^2)$	$1050\ (51) \oplus 1134\ (42) \oplus 840\ (41^2) \oplus 490\ (3^2) \oplus 896\ (321)$	$1134\ (42) \oplus 840\ (41^2) \oplus 896\ (321) \oplus 280\ (31^3)$
$\overline{210}\ (3^42)$	$\overline{56}\ (3^5) \oplus \overline{70}\ (2^41) \oplus \overline{1134}\ (43^22)$	$\overline{3675}\ (53^32) \oplus \overline{384}\ (32^31) \oplus \overline{315}\ (43^4) \oplus \overline{21}\ (2^5) \oplus \overline{15}\ (1^4)$	$\overline{2430}\ (4^23^22) \oplus \overline{384}\ (32^31) \oplus \overline{315}\ (43^4) \oplus \overline{21}\ (2^5)$

Table 2 (Continuation)

	6	21	15
105 (2^2)	$420\ (32) \oplus 210'\ (2^21)$	$1134\ (42) \oplus 896\ (321) \oplus 175\ (2^3)$	$490\ (3^2) \oplus 896\ (321) \oplus 189\ (2^21^2)$
$\overline{105}$ (2^4)	$\overline{560}\ (32^3) \oplus \overline{70}\ (2^41)$	$\overline{1800}\ (42^3) \oplus \overline{384}\ (32^31) \oplus \overline{21}\ (2^5)$	$\overline{1176}\ (3^22^2) \oplus \overline{384}\ (32^31) \oplus \overline{15}\ (1^4)$
$105'$ (21^2)	$336\ (31^2) \oplus 210'\ (2^21) \oplus 84\ (21^3)$	$840\ (41^2) \oplus 896\ (321) \oplus 280\ (31^3) \oplus 189\ (2^21^2)$	$896\ (321) \oplus 175\ (2^3) \oplus 280\ (31^3) \oplus 189\ (2^21^2) \oplus 35\ (21^4)$
$\overline{105'}$ (2^31^2)	$540\ (32^21^2) \oplus \overline{70}\ (2^41) \oplus 20\ (1^3)$	$1701\ (42^21^2) \oplus \overline{384}\ (32^31) \oplus 105'\ (21^2) \oplus \overline{15}\ (1^4)$	$\overline{1050''}\ (3^221^2) \oplus \overline{384}\ (32^31) \oplus 105'\ (21^2) \oplus \overline{21}\ (2^5) \oplus \overline{15}\ (1^4)$
252 (5)	$462\ (6) \oplus 1050\ (51)$	$792\ (7) \oplus 1980\ (61) \oplus 2520\ (52)$	$1980\ (61) \oplus 1800\ (51^2)$
$\overline{252}$ $5^5)$	$\overline{1386}\ (65^4) \oplus \overline{126}\ (4^5)$	$\overline{56}\ (3^5) \oplus \overline{700}\ (54^4) \oplus \overline{4536}\ (75^4)$	$\overline{3080}\ (6^25^3) \oplus \overline{700}\ (54^4)$
20 (1^3)	$105'\ (21^2) \oplus \overline{15}\ (1^4)$	$336\ (31^2) \oplus 84\ (21^3)$	$210\ (2^21) \oplus 84\ (21^3) \oplus \overline{6}\ (1^5)$
56 (3)	$210\ (31) \oplus 126\ (4)$	$252\ (5) \oplus 504\ (41) \oplus 420\ (32)$	$504\ (41) \oplus 336\ (31^2)$
$\overline{56}$ (3^5)	$\overline{315}\ (43^4) \oplus \overline{21}\ (2^5)$	$\overline{1050}\ (53^4) \oplus \overline{120}\ (32^4) \oplus \overline{6}\ (1^5)$	$\overline{720}\ (4^23^3) \oplus \overline{120}\ (32^4)$
70 (21)	$210\ (31) \oplus 105\ (2^2) \oplus 105'\ (21^2)$	$504\ (41) \oplus 420\ (32) \oplus 336\ (31^2) \oplus 210'\ (2^21)$	$84\ (21^3) \oplus 420\ (32) \oplus 336\ (31^2) \oplus 210'\ (2^21)$
$\overline{70}$ (2^41)	$\overline{384}\ (32^31) \oplus \overline{21}\ (2^5) \oplus \overline{15}\ (1^4)$	$\overline{1260}\ (42^31) \oplus \overline{120}\ (32^4) \oplus 84\ (21^3) \oplus \overline{6}\ (1^5)$	$840\ (3^22^21) \oplus \overline{120}\ (32^4) \oplus 84\ (21^3) \oplus \overline{6}\ (1^5)$

Table 3

Multiplication of representations of SU(6)

	35	189	280	$\overline{280}$	405	20	56	$\overline{56}$	70	$\overline{70}$	700	$\overline{700}$
35	×											
189	×	×										
280	×	×	×	×								
405	×	×	×		×							
20	×	×	×		×	×						
56	×	×	×	×	×	×	×	×				
70	×	×	×	×	×	×	×	×	×	×		
700	×	×	×	×	×	×	×	×	×	×	×	×

$35\ (21^4) \times 35\ (21^4) = 405\ (42^4) \oplus 280\ (31^3) \oplus \overline{280}\ (3^22^3) \oplus 189\ (2^21^2) \oplus 2(35)\ (21^4) \oplus 1\ (0)$

$189\ (2^21^2) \times 35\ (21^4) = 3675\ (432^21) \oplus 896\ (321) \oplus \overline{896}\ (3^321) \oplus 280\ (31^3) \oplus \overline{280}\ (3^22^3) \oplus 2(189)\ (2^21^2) \oplus 175\ (2^3) \oplus 35\ (21^4).$

$189\ (2^21^2) \times 189\ (2^21^2) = 1\ (0) \oplus 2(35)\ (21^4) \oplus 405\ (42^4) \oplus 280\ (31^3) \oplus \overline{280}\ (3^22^3) \oplus 3(189)\ (2^21^2) \oplus (175)\ (2^3) \oplus 2(896)\ (321) \oplus 2(\overline{896})\ (3^321) \oplus 490\ (3^2) \oplus \overline{490}\ (3^4) \oplus 3969\ (43^21^2) \oplus 5670\ (43^22) \oplus \overline{5670}\ (4^221^2) \oplus 2(3675)\ (432^21) \oplus 6720\ (4^22^2).$

$280\ (31^3) \times 35\ (21^4) = 3200\ (52^31) \oplus 405\ (42^4) \oplus 3675\ (432^21) \oplus 840\ (41^2) \oplus 2(280)\ (31^3) \oplus 896\ (321) \oplus 189\ (2^21^2) \oplus 35\ (21^4).$

$280\ (31^3) \times 189\ (2^21^2) = 14175\ (532^2) \oplus 12250\ (5321^2) \oplus 3200\ (52^31) \oplus 1134\ (42) \oplus 5670\ (43^22) \oplus 840\ (41^2) \oplus 3969\ (43^21^2) \oplus 2(3675)\ (432^21) \oplus 405\ (42^4) \oplus 2(896)\ (321) \oplus \overline{896}\ (3^321) \oplus 2(280)\ (31^3) \oplus \overline{280}\ (3^22^3) + 175\ (2^3) \oplus 189\ (2^21^2) \oplus 35\ (21^4).$

$280\ (31^3) \times 280\ (31^3) = 9625\ (62^3) \oplus 8910\ (62^21^2) \oplus 1050\ (51) \oplus 14175\ (532^2) \oplus 12250\ (5321^2) \oplus 2(3200)\ (52^31) \oplus 1134\ (42) \oplus 2(840)\ (41^2) \oplus 6720\ (4^22^2) \oplus 5670\ (4^221^2) \oplus 2(3675)\ (432^21) \oplus 405\ (42^4) \oplus 490\ (3^2) \oplus 2(896)\ (321) \oplus 280\ (31^3) \oplus \overline{280}\ (3^22^3) \oplus 189\ (2^21^2).$

$\overline{280}\ (3^22^3) \times 280\ (31^3) = 1\ (0) \oplus 29700\ (643^22) \oplus 12250\ (5321^2) \oplus \overline{12250}\ (54^232) \oplus 3200\ (52^31) \oplus \overline{3200}\ (543^3) \oplus 896\ (321) \oplus \overline{896}\ (3^321) \oplus 280\ (31^3) \oplus \overline{280}\ (3^32^3) \oplus 2695\ (63^4) \oplus 3969\ (43^21^2) \oplus 2(405)\ (42^4) \oplus 2(3675)\ (432^21) \oplus 175\ (2^3) \oplus 2(189)\ (2^21^2) \oplus 2(35)\ (21^4).$

$405\ (42^4) \times 35\ (21^4) = 2695\ (63^4) + 3200\ (52^31) \oplus \overline{3200}\ (543^3) \oplus 3675\ (432^21) \oplus 2(405)\ (42^4) \oplus 280\ (31^3) \oplus \overline{280}\ (3^22^3) \oplus 35\ (21^4).$

$$405\,(42^4) \times 189\,(2^21^2) = 29700\,(643^22) \oplus 12250\,(5321^2) \oplus \overline{12250}\,(54^232) \oplus 3200\,(52^31) \oplus \overline{3200}\,(543^3) \oplus 840\,(41^2) \oplus \overline{840}\,(4^33^2) \oplus 896\,(321) \oplus \overline{896}\,(3^321) \oplus 280\,(31^3) \oplus \overline{280}\,(3^22^3) \oplus 2(3675)\,(432^21) \oplus 3969\,(43^21^2) \oplus 405\,(42^4) \oplus 189\,(2^21^2).$$

$$405\,(42^4) \times 280\,(31^3) = 19845\,(73^32) \oplus 8910\,(62^21^2) \oplus 2695\,(63^4) \oplus 29700\,(643^22) \oplus 2(3200)\,(52^31) \oplus 12250\,(5321^2) \oplus \overline{3200}\,(543^3) \oplus 840\,(41^2) \oplus 2(3675)\,(432^21) \oplus 405\,(42^4) \oplus 2(280)\,(31^3) \oplus 896\,(321) \oplus 14175\,(5^23^22) \oplus 5670\,(4^221^2) \oplus \overline{280}\,(3^22^3) \oplus 189\,(2^21^2) \oplus 35\,(21^4).$$

$$405\,(42^4) \times 405\,(42^4) = 12740\,(84^4) \oplus 19845\,(73^32) \oplus \overline{19845}\,(754^3) \oplus 2(2695)\,(63^4) \oplus 9625\,(62^3) \oplus \overline{9625}\,(6^24^3) \oplus 2(3200)\,(52^31) \oplus 2(\overline{3200})\,(543^3) \oplus 14175\,(532^2) \oplus \overline{14175}\,(5^23^22) \oplus 2(3675)\,(432^21) \oplus 3(405)\,(42^4) \oplus 6720\,(4^22^2) \oplus 280\,(31^3) \oplus \overline{280}\,(3^22^3) \oplus 2(35)\,(21^4) \oplus 189\,(2^21^2) \oplus 1\,(0).$$

$$20\,(1^3) \times 35\,(21^4) = 540\,(32^21^2) \oplus 70\,(21) \oplus \overline{70}\,(2^41) \oplus 20\,(1^3).$$

$$20\,(1^3) \times 189\,(2^21^2) = 1960\,(3^221) \oplus \overline{560}\,(32^3) \oplus 560\,(3^21^3) \oplus 540\,(32^21^2) \oplus 70\,(21) \oplus \overline{70}\,(2^41) \oplus 20\,(1^3).$$

$$20\,(1^3) \times 280\,(31^3) = 3240\,(42^21) \oplus 1134\,(421^3) \oplus \overline{560}\,(32^3) \oplus 540\,(32^21^2) \oplus 70\,(21) \oplus 56\,(3).$$

$$20\,(1^3) \times 405\,(42^4) = 5292\,(53^22^2) \oplus 1134\,(421^3) \oplus \overline{1134}\,(43^32) \oplus 540\,(32^21^2).$$

$$20\,(1^3) \times 20\,(1^3) = 1\,(0) \oplus 35\,(21^4) \oplus 175\,(2^3) \oplus 189\,(2^21^2).$$

$$56\,(3) \times 35\,(21^4) = 1134\,(421^3) \oplus 700\,(51^4) \oplus 70\,(21) \oplus 56\,(3).$$

$$56\,(3) \times 189\,(2^21^2) = 5670\,(521^2) \oplus 3240\,(42^21) \oplus 1134\,(421^3) \oplus 540\,(32^21^2).$$

$$56\,(3) \times 280\,(31^3) = 3080\,(61^3) \oplus 5670\,(521^2) \oplus 700\,(51^4) \oplus 1134\,(421^3) \oplus 560\,(3^21^3) \oplus 4536'\,(431^2).$$

$$56\,(3) \times \overline{280}\,(3^22^3) = 8624\,(632^3) \oplus 5292\,(53^22^2) \oplus 1134\,(421^3) \oplus 540\,(32^21^2) \oplus 70\,(21) \oplus 20\,(1^3).$$

$$56\,(3) \times 405\,(42^4) = 4536\,(72^4) \oplus 8624\,(632^3) \oplus 700\,(51^4) \oplus 7000\,(542^3) \oplus 1134\,(421^3) \oplus 560\,(3^21^3) \oplus 70\,(21) \oplus 56\,(3).$$

$$56\,(3) \times 20\,(1^3) = 840\,(41^2) \oplus 280\,(31^3).$$

$$56\,(3) \times 56\,(3) = 1050\,(51) \oplus 1134\,(42) \oplus 490\,(3^2) \oplus 462\,(6).$$

$$\overline{56}\,(3^5) \times 56\,(3) = 2695\,(63^4) \oplus 405\,(42^4) \oplus 35\,(21^4) + 1\,(0).$$

70 (21),	×	35 (21^4)	=	1134 (421^3),	⊕ 56 (3),	⊕ 560 $(3^2 1^3)$,	⊕ 2(70) (21),	⊕ 540 $(32^2 1^2)$,	⊕ 20 (1^3).
70 (21),	×	70 (21)	=	1134 (42),	⊕ 840 (41^2),	⊕ 490 (3^2),	⊕ 2(896) (321),	⊕ (175) (2^3),	⊕ 280 (31^3),
				⊕ 189 $(2^2 1^2)$.					
70 (21),	×	20 (1^3)	=	896 (321),	⊕ 280 (31^3),	⊕ 189 $(2^2 1^2)$,	⊕ 35 (21^4).		
70 (21),	×	56 (3)	=	1050 (51),	⊕ 1134 (42),	⊕ 840 (41^2),	⊕ 896 (321).		
70 (21),	×	$\overline{56}$ (3^5)	=	$\overline{3200}$ (543^3),	⊕ 405 (42^4),	⊕ $\overline{280}$ $(3^2 2^3)$,	⊕ 35 (21^4).		
70 (21),	×	280 (31^3)	=	5670 (521^2),	⊕ 700 (51^4),	⊕ 1960 $(3^2 21)$,	⊕ 3240 $(42^2 1)$,	⊕ 560 $(3^2 1^3)$,	⊕ 2(1134) (421^3),
				⊕ 4536′ (431^2),	⊕ 540 $(32^2 1^2)$,	⊕ 70 (21),	⊕ 56 (3).		
70 (21),	×	189 $(2^2 1^2)$	=	4536′ (431^2),	⊕ 3240 $(42^2 1)$,	⊕ 1134 (421^3),	+ 560 $(3^2 1^3)$,	⊕ $\overline{560}$ (32^3),	⊕ 2(540) $(32^2 1^2)$,
				⊕ 1960 $(3^2 21)$,	⊕ 70 (21),	⊕ $\overline{70}$ $(2^4 1)$,	⊕ 20 (1^3).		
70 (21),	×	$\overline{280}$ $(3^2 2^3)$	=	7000 (542^3),	⊕ 5292 $(53^2 2^2)$,	⊕ 1134 (421^3),	⊕ $\overline{1134}$ $(43^3 2)$,	⊕ 3240 $(4^2 32^2)$,	⊕ 2(540) $(32^2 1^2)$,
				⊕ 560 $(3^2 1^3)$,	⊕ 70 (21),	⊕ $\overline{70}$ $(2^4 1)$,	⊕ 20 (1^3).		
70 (21),	×	405 (42^4)	=	8624 (632^3),	⊕ 700 (51^4),	⊕ 5292 $(53^2 2^2)$,	⊕ 3240 $(4^2 32^2)$,	⊕ 2(1134) (421^3),	⊕ 7000 (542^3),
				⊕ 560 $(3^2 1^3)$,	⊕ 540 $(32^2 1^2)$,	⊕ 70 (21),	⊕ 56 (3).		
$\overline{70}$ $(2^4 1)$,	×	70 (21)	=	1 (0),	⊕ 2(35) (21^4),	⊕ 189 $(2^2 1^2)$,	⊕ 280 (31^3),	⊕ $\overline{280}$ $(3^2 2^3)$,	⊕ 405 (42^4),
				⊕ 3675 $(432^2 1)$.					
700 (51^4),	×	35 (21^4)	=	4536 (72^4),	⊕ 8624 (632^3),	⊕ 3080 (61^3),	⊕ 5670 (521^2),	⊕ 2(700) (51^4),	⊕ 1134 (421^3),
				⊕ 56 (3).					
700 (51^4),	×	189 $(2^2 1^2)$	=	49000 $(732^2 1)$,	⊕ 11550 (621),	⊕ 3080 (61^3),	⊕ 32340 $(63^2 21)$,	⊕ 6000 (52^2),	⊕ 2(5670) (521^2),
				⊕ 8624 (632^3),	⊕ 700 (51^4),	⊕ 5292 (532^2),	⊕ 3240 $(42^2 1)$,	⊕ 1134 (421^3).	
700 (51^4),	×	280 (31^3)	=	24024 $(82^3 1)$,	⊕ 6160 (71^2),	⊕ 49000 $(732^2 1)$,	⊕ 11550 (621),	⊕ 4536 (72^4),	⊕ 2(3080) (61^3),
				⊕ 49896 $(642^2 1)$,	⊕ 11340 (531),	⊕ 8624 (632^3),	⊕ 2(5670) (521^2),	⊕ 700 (51^4),	⊕ 7000 (542^3),
				⊕ 4536′ (431^2),	⊕ 1134 (421^3).				
700 (51^4),	×	$\overline{280}$ $(3^2 2^3)$	=	43680 (843^3),	⊕ 49000 $(732^2 1)$,	⊕ 4536 (72^4),	⊕ 31360 $(74^2 3^2)$,	⊕ 32340 $(63^2 21)$,	⊕ 2(8624) (632^3).
				⊕ 5670 (521^2),	⊕ 700 (51^4),	⊕ 5292 $(53^2 2^2)$,	⊕ 3240 $(42^2 1)$,	⊕ 2(1134) (421^3),	⊕ 56 (3).
				⊕ 540 $(32^2 1^2)$,	⊕ 70 (21).				

700 (51^4),	$\times$	405 (42^4)	$=$	20580 (93^4),	$\oplus$ 24024 (82^31),	$\oplus$ 43680 (843^3),	$\oplus$ 49000 (732^21),	$\oplus$ $2(4536)$ (72^4),	$\oplus$ 3080 (61^3),
				$\oplus$ 45360 (753^3),	$\oplus$ 49896 (642^21),	$\oplus$ $2(8624)$ (632^3),	$\oplus$ $2(700)$ (51^4),	$\oplus$ 5670 (521^2),	$\oplus$ 7000 (542^3),
				$\oplus$ $4536'$ (431^2),	$\oplus$ $2(1134)$ (421^3),	$\oplus$ 560 (3^21^3),	$\oplus$ 70 (21),	$\oplus$ 56 (3).	
700 (51^4),	$\times$	20 (1^3)	$=$	8910 (62^21^2),	$\oplus$ 1050 (51),	$\oplus$ 3200 (52^31),	$\oplus$ 840 (41^2),		
700 (51^4),	$\times$	56 (3)	$=$	4290 (81^4),	$\oplus$ 10368 (721^3),	$\oplus$ 462 (6),	$\oplus$ 12936 (631^3),	$\oplus$ 1050 (51),	$\oplus$ $1134'$ (42),
				$\oplus$ 8960 (541^3).					
700 (51^4),	$\times$	$\overline{56}$ (3^5)	$=$	12740 (84^4),	$\oplus$ 19845 (73^32),	$\oplus$ 3200 (52^31),	$\oplus$ 2695 (63^4),	$\oplus$ 405 (42^4),	$\oplus$ 280 (31^3),
				$\oplus$ 35 (21^4).					
700 (51^4),	$\times$	70 (21)	$=$	10368 (721^3),	$\oplus$ 462 (6),	$\oplus$ 12250 (5321^2),	$\oplus$ 1134 (42),	$\oplus$ 12936 (631^3),	$\oplus$ $2(1050)$ (51),
				$\oplus$ 840 (41^2),	$\oplus$ 8910 (62^21^2).				
700 (51^4),	$\times$	$\overline{70}$ (2^41)	$=$	19845 (73^32),	$\oplus$ 9625 (62^3),	$\oplus$ 8910 (62^21^2),	$\oplus$ 2695 (63^4),	$\oplus$ $2(3200)$ (52^31),	$\oplus$ 840 (41^2),
				$\oplus$ 405 (42^4),	$\oplus$ 280 (31^3).				
700 (51^4),	$\times$	700 (51^4)	$=$	22275 $(10,2^4)$	$\oplus$ 15015 (91^3),	$\oplus$ 56595 (932^3),	$\oplus$ 36855 (821^2),	$\oplus$ $2(4290)$ (81^4),	$\oplus$ 78000 (842^3),
				$\oplus$ 49896 (731^2),	$\oplus$ $2(10368)$ (721^3),	$\oplus$ 68040 (752^3),	$\oplus$ 462 (6),	$\oplus$ 43120 (641^2),	$\oplus$ $2(12936)$ (631^3),
				$\oplus$ 26950 (6^22^3),	$\oplus$ 1050 (51),	$\oplus$ 17010 (5^21^2),	$\oplus$ $2(8960)$ (541^3),	$\oplus$ 1134 (42),	$\oplus$ 490 (3^2).
$\overline{700}$ (54^4),	$\times$	700 (51^4)	$=$	47628 $(10,5^4)$,	$\oplus$ 87808 (965^3),	$\oplus$ $\overline{87808}$ (94^33),	$\oplus$ 154791 (854^23),	$\oplus$ $2(12740)$ (84^4),	$\oplus$ 19845 (73^32),
				$\oplus$ $\overline{19845}$ (754^3),	$\oplus$ 29700 (643^22),	$\oplus$ $2(2695)$ (63^4),	$\oplus$ 3200 (52^31),	$\oplus$ $\overline{3200}$ (543^3),	$\oplus$ 3675 (432^21),
				$\oplus$ $2(405)$ (42^4),	$\oplus$ 280 (31^3),	$\oplus$ $\overline{280}$ (3^22^3),	$\oplus$ 189 (2^21^2),	$\oplus$ $2(35)$ (21^4),	$\oplus$ 1 (0).

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