Zeitschrift: Helvetica Physica Acta

Band: 39 (1966)

Heft: 5

Artikel: Representation of group generators by Boson or Fermion operators

application to spin perturbation theory

Autor: Enz, Charles P.

DOI: https://doi.org/10.5169/seals-113699

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 26.11.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Representation of Group Generators by Boson or Fermion Operators Application to Spin Perturbation Theory

by Charles P. Enz

Institut de Physique Théorique, Université de Genève

(27. V. 66)

A representation of spin by fermion operators was known for a long time for S = 1/2 [1]¹), whereas the generalization to S > 1/2 is more recent [2]. The interest of such a representation is to avoid the complications of a generalized Wick's theorem valid for spin [3]. Here we remark first that such representations have a much larger generality, valid for fermions as well as for bosons [4]. We then discuss the elements of perturbation theory in the representation of spin by fermion operators.

1. Let G_i (i = 1, ..., r) be the generators of a simple Lie group numbered such that the first l commute with each other. A basis of an irreducible representation D^N may then be chosen such that

$$G_i \mid v \rangle = m_i(v) \mid v \rangle; \ i = 1, \dots l; \ v = 0, 1, \dots N - 1.$$
 (1)

One calls r the order and l the rank of the group, N the dimension and $m_i(r)$ the weights of the representation [5].

Let a_{ν} and a_{ν}^{*} be boson or fermion operators such that

$$[a_{\nu} \, a_{\nu'}^*]_{\pm} = \delta_{\nu \, \nu'}. \tag{2}$$

It is then easy to prove that the expressions $\sum_{\nu\nu'} a_{\nu}^* g_i^{\nu\nu'} a_{\nu'}$ $(i=1,\ldots r)$ satisfy the commutation rules of the G_i , $[G_i G_j]_- = \sum_k \gamma_{ij}^k G_k$, if the matrices g_i do so.

If | > is the vacuum state of these fictitious particles such that

$$a_{\nu} \mid \rangle = 0 \tag{3}$$

then the identification

$$\mid \nu \rangle = a_{\nu}^{*} \mid \rangle \tag{4}$$

$$G_i = \sum_{v,v'} a_v^* g_i^{vv'} a_{v'} \tag{5}$$

reproduces all the properties of the representation (1), of which the matrices g_i are a realization.

The inconvenience of this representation by *fictitious* particles ν is that neither the vacuum $| \rangle$ nor the states with more than one particle, $a_{\nu_i}^* \dots a_{\nu_n}^* | \rangle$, n > 1, have

¹⁾ Numbers in brackets refer to References, page 465.

physical meaning. For fermions $n \leq N$ and hence the dimension of the Hilbert space of all particles is 2^N .

The case of spin is obtained with the group SU(2) for which r=3, l=1, and $G_1=S_3$, $G_2=S_+$, $G_3=S_-$, N=2 S+1, m(v)=S-v.

For spins localized on different atoms labeled by n one has [2]

$$S_n = \sum_{vv'} a_{vn}^* \, s^{vv'} \, a_{v'n} \,. \tag{6}$$

It is interesting to note that in elementary particle theory, localized G_i have an interpretation as generalized charge densities and (5) expresses their bilinear form in the fields of the particles ν which, in this case, are *physical*.

2. Consider now perturbation theory for the coupling of one single spin, Equation (6) with indices n omitted. The unperturbed hamiltonian is, for an external magnetic field \mathfrak{H} ,

$$H_0 = -\gamma S_z = -\gamma \sum_{\nu=0}^{2S} (S - \nu) a_{\nu}^* a_{\nu}; \ \gamma = 2 \mu_B \, \mathfrak{H} > 0.$$
 (7)

 $\mid 0 \rangle = a_0^* \mid \rangle$ is the ground state. The expressions to be calculated are of the form $\langle \prod_{\nu=0}^{2S} \tau_{\nu} \rangle$. Here τ_{ν} is a product ordered according to imaginary times of operators $a_{\nu}^*(-i\tau)$, $a_{\nu}(-i\tau)$ where $O(t) = e^{iH_0t} O(e^{-iH_0t})$. $\langle \cdot \rangle$ is an unperturbed canonical average taken over the physical states $|\nu\rangle$, i.e.

$$\langle O \rangle \equiv \sum_{\nu=0}^{2S} \langle \nu \mid e^{-\beta H_0} O \mid \nu \rangle / \sum_{\nu=0}^{2S} \langle \nu \mid e^{-\beta H_0} \mid \nu \rangle. \tag{8}$$

One calculates

$$\langle \nu \mid e^{-\beta H_0} \prod_{\nu'} \tau_{\nu'} \mid \nu \rangle = e^{\beta \gamma (S-\nu)} \langle \nu \mid \tau_{\nu} \mid \nu \rangle \prod_{\nu' \neq \nu} \langle \mid \tau_{\nu'} \mid \rangle. \tag{9}$$

Here $\langle \mid \tau_{\nu} \mid \rangle$ is obtained by applying the usual diagram technique for zero temperature and is expressed in terms of free propagators which, however, do not have the usual form because of the imaginary time order. They are

$$G_{\nu}(\tau) \equiv \langle \mid T_{\tau} \left(c_{\nu}(-i \tau) \ a_{\nu}^{*}(0) \right) \mid \rangle$$

$$= \frac{1}{2\pi} \int_{-i\gamma (S-\nu+0)-\infty}^{-i\gamma (S-\nu+0)+\infty} d\omega \ \frac{e^{i \omega \tau}}{i \omega - \gamma (S-\nu)} = \begin{cases} e^{\tau \gamma (S-\nu)}; \tau > 0 \\ 0 ; \tau < 0 \end{cases}.$$

$$(10)$$

The expression $\langle v \mid \tau_{\nu} \mid \nu \rangle$ is not directly accessible to the usual technique because the normal products obtained by applying the ordinary Wick's theorem to τ_{ν} , do not all vanish when taken between the states $\mid \nu \rangle$. To handle it, we introduce the unperturbed canonical average taken over the Hilbert space of the fictitious particles. For fermions this is

$$\ll O \gg \equiv \sum_{\{n_{\nu}=0,1\}} \langle | \prod_{\nu} (a_{\nu})^{n_{\nu}} e^{-\beta H_{0}} O \prod_{\nu} (a_{\nu}^{*})^{n_{\nu}} | \rangle
\times \{ \sum_{\{n_{\nu}=0,1\}} \langle | \prod_{\nu} (a_{\nu})^{n_{\nu}} e^{-\beta H_{0}} \prod_{\nu} (a_{\nu}^{*})^{n_{\nu}} | \rangle \}^{-1} .$$
(11)

One then finds that $\langle v \mid \tau_v \mid v \rangle$ can be expressed in terms of $\langle \tau_v \mid v \rangle$ and $\langle \tau_v \mid v \rangle$,

$$e^{\beta \gamma (S-\nu)} \langle \nu \mid \tau_{\nu} \mid \nu \rangle = (1 + e^{\beta \gamma (S-\nu)}) \ll \tau_{\nu} \gg - \langle \mid \tau_{\nu} \mid \rangle. \tag{12}$$

 $\ll \tau_{\nu} \gg$ can be calculated by the usual diagram technique for *finite temperature* [6] and is expressed in terms of free propagators,

$$g_{\nu}(\tau) \equiv \ll T_{\tau} \left(a_{\nu}(-i \tau) a_{\nu}^{*}(0) \right) \gg = \frac{1}{\beta} \sum_{r=-\infty}^{+\infty} \frac{e^{i \omega_{r} \tau}}{-i \omega_{r} - \gamma (S - \nu)}; \ \omega_{r} = \frac{\pi}{\beta} (2 r + 1).$$
 (13)

The general result now is

$$\langle \prod_{\nu} \tau_{\nu} \rangle = e^{-\beta \gamma S} \frac{1 - e^{-\beta \gamma}}{1 - e^{-\beta \gamma (2S + 1)}} \left\{ \sum_{\nu=0}^{2S} \left(1 + e^{\beta \gamma (S - \nu)} \right) \times \ll \tau_{\nu} \gg \prod_{\nu' + \nu} \langle \mid \tau_{\nu'} \mid \rangle - (2S + 1) \langle \mid \prod_{\nu} \tau_{\nu} \mid \rangle \right\}. \tag{14}$$

Since $\prod_{\nu} \tau_{\nu}$ usually is a product of operators (6) one has $\langle |\prod_{\nu} \tau_{\nu}| \rangle = 0$. For low temperatures, $\beta \gamma \gg 1$, Equation (14) simplifies considerably:

$$\langle \prod_{\nu} \tau_{\nu} \rangle \cong \ll \tau_{0} \gg \prod_{\nu \neq 0} \langle \mid \tau_{\nu} \mid \rangle.$$
 (15)

The procedure described here is to be compared with that of Abrikosov's second paper [7] while in his first paper [2], Abrikosov uses an artificial hamiltonian which is obtained from (7) by the substitution $\gamma(S-\nu) \to \lambda$. In this case, Equation (14) goes over into

$$\langle \prod_{\nu} \tau_{\nu} \rangle = \frac{e^{\beta \lambda} + 1}{2 S + 1} \sum_{\nu} \langle \tau_{\nu} \rangle \prod_{\nu' + \nu} \langle | \tau_{\nu'} | \rangle. \tag{16}$$

I gratefully acknowledge discussions with Dr. B. Giovannini and Dr. L. Horwitz.

References

[1] P. W. Anderson, in Solid State Physics, edited by F. Seitz and D. Turnbull, vol. 14, p. 112 (1963). A representation for S=1/2 by boson operators has been used by Schwinger in an unpublished paper of 1952 to descripe arbitrary angular momentum. See J. Schwinger, in Quantum Theory of Angular Momentum, edited by L. C. Biedenharn and H. van Dam (Academic Press, 1965), p. 229. For our purpose, Schwinger's representation has the disadvantage that the physical states for spin S>1/2 are generated from the vacuum by a product of 2 S creation operators instead of only one, i. e. in our notation

$$|v\rangle = [(2 S - v)! v!]^{-1/2} (a_0^*)^{2S - v} (a_1^*)^v]\rangle$$

instead of Equation (4).

- [2] A. A. Abrikosov, Physics 2, 5 (1965). It has been proposed independently by B. Giovannini, see C. P. Enz and B. Giovannini, communications to the Bern meeting of the Swiss Physical Soc. (April 1966), Helv. phys. Acta 39, 224 (1966).
- [3] B. GIOVANNINI, Scientific Papers of the College of General Education, University of Tokyo 15, 49 (1965).
- [4] The representation of Lie algebras by boson or fermion operators has been used extensively by H. L. LIPKIN, in 'Lie Groups for Pedestrians' (North Holland, 1965).
- [5] R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee, Rev. Mod. Phys. 34, 1 (1962).
- [6] C. P. Enz, Helv. phys. Acta 38, 150 (1965). The derivation by taking the limit of infinite volume, as given, e.g., by A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, Quantum Field Theory in Statistical Physics (Prentice-Hall, 1963), is obviously not valid here.
- [7] A. A. Abrikosov, Physics 2, 61 (1966).