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Representation of Group Generators by Boson or Fermion Operators Application to Spin Perturbation Theory

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A representation of spin by fermion operators was known for a long time for $S = 1/2$ [1] [1]), whereas the generalization to $S > 1/2$ is more recent [2]. The interest of such a representation is to avoid the complications of a generalized Wick's theorem valid for spin [3]. Here we remark first that such representations have a much larger generality, valid for fermions as well as for bosons [4]. We then discuss the elements of perturbation theory in the representation of spin by fermion operators.

1. Let G_i $(i = 1, \dots r)$ be the generators of a simple Lie group numbered such that the first l commute with each other. A basis of an irreducible representation D^N may then be chosen such that

$$G_i \, |\nu\rangle = m_i(\nu) \, |\nu\rangle; \; i = 1, \dots l; \; \nu = 0, 1, \dots N - 1 \,. \tag{1}$$

One calls r the order and l the rank of the group, N the dimension and $m_i(\nu)$ the weights of the representation [5].

Let a_ν and a_ν^* be boson or fermion operators such that

$$[a_\nu \, a_{\nu'}^*]_\mp = \delta_{\nu \nu'} \,. \tag{2}$$

It is then easy to prove that the expressions $\sum_{\nu \nu'} a_\nu^* \, g_i^{\nu \nu'} \, a_{\nu'}$ $(i = 1, \dots r)$ satisfy the commutation rules of the G_i, $[G_i \, G_j]_- = \sum_k \gamma_{ij}^k \, G_k$, if the matrices g_i do so.

If $| \, \rangle$ is the vacuum state of these fictitious particles such that

$$a_\nu \, |\rangle = 0 \tag{3}$$

then the identification

$$|\, \nu\rangle = a_\nu^* \, |\rangle \tag{4}$$

$$G_i = \sum_{\nu \nu'} a_\nu^* \, g_i^{\nu \nu'} \, a_{\nu'} \tag{5}$$

reproduces all the properties of the representation (1), of which the matrices g_i are a realization.

The inconvenience of this representation by *fictitious* particles ν is that neither the vacuum $| \, \rangle$ nor the states with more than one particle, $a_{\nu_i}^* \dots a_{\nu_n}^* \, | \, \rangle$, $n > 1$, have

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[1]) Numbers in brackets refer to References, page 465.

physical meaning. For fermions $n \leq N$ and hence the dimension of the Hilbert space of all particles is 2^N.

The case of spin is obtained with the group $SU(2)$ for which $r = 3$, $l = 1$, and $G_1 = S_3$, $G_2 = S_+$, $G_3 = S_-$, $N = 2\,S + 1$, $m(\nu) = S - \nu$.

For spins localized on different atoms labeled by n one has [2]

$$\boldsymbol{S}_n = \sum_{\nu\nu'} a^*_{\nu n}\, \boldsymbol{s}^{\nu\nu'}\, a_{\nu' n}\,. \tag{6}$$

It is interesting to note that in elementary particle theory, localized G_i have an interpretation as generalized charge densities and (5) expresses their bilinear form in the fields of the particles ν which, in this case, are *physical*.

2. Consider now perturbation theory for the coupling of one single spin, Equation (6) with indices n omitted. The unperturbed hamiltonian is, for an external magnetic field $\mathfrak{H}$,

$$H_0 = -\gamma\, S_z = -\gamma \sum_{\nu=0}^{2S} (S-\nu)\, a^*_\nu\, a_\nu;\ \gamma = 2\,\mu_B\, \mathfrak{H} > 0\,. \tag{7}$$

$|\,0\,\rangle = a^*_0\,|\,\rangle$ is the ground state. The expressions to be calculated are of the form $\langle \prod_{\nu=0}^{2S} \tau_\nu\rangle$. Here τ_ν is a product ordered according to imaginary times of operators $a^*_\nu(-\,i\,\tau)$, $a_\nu(-\,i\,\tau)$ where $O(t) = e^{iH_0 t}\, O\, e^{-iH_0 t}$. $\langle\ \rangle$ is an unperturbed canonical average taken over the physical states $|\,\nu\,\rangle$, i.e.

$$\langle O\rangle \equiv \sum_{\nu=0}^{2S} \langle\nu|\, e^{-\beta H_0}\, O\,|\,\nu\rangle \Big/ \sum_{\nu=0}^{2S} \langle\nu\,|e^{-\beta H_0}|\,\nu\rangle\,. \tag{8}$$

One calculates

$$\langle\nu\,|\, e^{-\beta H_0} \prod_{\nu'} \tau_{\nu'}\,|\,\nu\rangle = e^{\beta\gamma(S-\nu)}\, \langle\nu\,|\,\tau_\nu\,|\,\nu\rangle \prod_{\nu'\neq\nu} \langle\,|\,\tau_{\nu'}\,|\,\rangle\,. \tag{9}$$

Here $\langle\,|\,\tau_\nu\,|\,\rangle$ is obtained by applying the usual diagram technique for *zero temperature* and is expressed in terms of free propagators which, however, do not have the usual form because of the imaginary time order. They are

$$G_\nu(\tau) \equiv \langle|\, T_\tau\,\big(c_\nu(-\,i\,\tau)\, a^*_\nu(0)\big)\,|\rangle$$
$$= \frac{1}{2\,\pi} \int_{-i\gamma(S-\nu+0)-\infty}^{-i\gamma(S-\nu+0)+\infty} d\omega\, \frac{e^{i\omega\tau}}{i\,\omega-\gamma\,(S-\nu)} = \begin{cases} e^{\tau\gamma(S-\nu)}; & \tau > 0 \\ 0 & ;\ \tau < 0 \end{cases}. \tag{10}$$

The expression $\langle\nu\,|\,\tau_\nu\,|\,\nu\rangle$ is not directly accessible to the usual technique because the normal products obtained by applying the ordinary Wick's theorem to τ_ν, do not all vanish when taken between the states $|\,\nu\,\rangle$. To handle it, we introduce the unperturbed canonical average taken over the Hilbert space of the fictitious particles. For fermions this is

$$\langle\!\langle O\rangle\!\rangle \equiv \sum_{\{n_\nu=0,1\}} \langle|\, \prod_\nu (a_\nu)^{n_\nu}\, e^{-\beta H_0}\, O \prod_\nu (a^*_\nu)^{n_\nu}\,|\rangle$$
$$\times \Big\{ \sum_{\{n_\nu=0,1\}} \langle|\, \prod_\nu (a_\nu)^{n_\nu}\, e^{-\beta H_0} \prod_\nu (a^*_\nu)^{n_\nu}\,|\rangle\Big\}^{-1}\,. \tag{11}$$

One then finds that $\langle \nu \mid \tau_\nu \mid \nu \rangle$ can be expressed in terms of $\langle \mid \tau_\nu \mid \rangle$ and $\ll \tau_\nu \gg$,

$$e^{\beta \gamma (S-\nu)} \langle \nu \mid \tau_\nu \mid \nu \rangle = (1 + e^{\beta \gamma (S-\nu)}) \ll \tau_\nu \gg - \langle \mid \tau_\nu \mid \rangle . \qquad (12)$$

$\ll \tau_\nu \gg$ can be calculated by the usual diagram technique for *finite temperature* [6] and is expressed in terms of free propagators,

$$g_\nu(\tau) \equiv \ll T_\tau \left(a_\nu(-i\tau)\, a_\nu^*(0)\right) \gg = \frac{1}{\beta} \sum_{r=-\infty}^{+\infty} \frac{e^{i\omega_r \tau}}{i\,\omega_r - \gamma\,(S-\nu)} ; \quad \omega_r = \frac{\pi}{\beta}\,(2r+1) . \qquad (13)$$

The general result now is

$$\langle \prod_\nu \tau_\nu \rangle = e^{-\beta\gamma S} \frac{1-e^{-\beta\gamma}}{1-e^{-\beta\gamma(2S+1)}} \left\{ \sum_{\nu=0}^{2S} (1 + e^{\beta\gamma(S-\nu)}) \right.$$
$$\left. \times \ll \tau_\nu \gg \prod_{\nu' \neq \nu} \langle \mid \tau_{\nu'} \mid \rangle - (2S+1) \langle \mid \prod_\nu \tau_\nu \mid \rangle \right\} . \qquad (14)$$

Since $\prod_\nu \tau_\nu$ usually is a product of operators (6) one has $\langle \mid \prod_\nu \tau_\nu \mid \rangle = 0$. For low temperatures, $\beta\gamma \gg 1$, Equation (14) simplifies considerably:

$$\langle \prod_\nu \tau_\nu \rangle \cong \ll \tau_0 \gg \prod_{\nu \neq 0} \langle \mid \tau_\nu \mid \rangle . \qquad (15)$$

The procedure described here is to be compared with that of Abrikosov's second paper [7] while in his first paper [2], ABRIKOSOV uses an artificial hamiltonian which is obtained from (7) by the substitution $\gamma\,(S-\nu) \to \lambda$. In this case, Equation (14) goes over into

$$\langle \prod_\nu \tau_\nu \rangle = \frac{e^{\beta\lambda}+1}{2S+1} \sum_\nu \ll \tau_\nu \gg \prod_{\nu' \neq \nu} \langle \mid \tau_{\nu'} \mid \rangle . \qquad (16)$$

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References

[1] P. W. ANDERSON, in Solid State Physics, edited by F. SEITZ and D. TURNBULL, vol. *14*, p. 112 (1963). A representation for $S = 1/2$ by boson operators has been used by SCHWINGER in an unpublished paper of 1952 to descripe arbitrary angular momentum. See J. SCHWINGER, in *Quantum Theory of Angular Momentum*, edited by L. C. Biedenharn and H. van Dam (Academic Press, 1965), p. 229. For our purpose, Schwinger's representation has the disadvantage that the physical states for spin $S > 1/2$ are generated from the vacuum by a product of $2\,S$ creation operators instead of only one, i. e. in our notation

$$|\nu\rangle = [(2S-\nu)!\,\nu!]^{-1/2}\,(a_0^*)^{2S-\nu}(a_1^*)^\nu\,|\rangle$$

instead of Equation (4).

[2] A. A. ABRIKOSOV, Physics *2*, 5 (1965). It has been proposed independently by B. GIOVANNINI, see C. P. ENZ and B. GIOVANNINI, communications to the Bern meeting of the Swiss Physical Soc. (April 1966), Helv. phys. Acta *39*, 224 (1966).

[3] B. GIOVANNINI, Scientific Papers of the College of General Education, University of Tokyo *15*, 49 (1965).

[4] The representation of Lie algebras by boson or fermion operators has been used extensively by H. L. LIPKIN, in 'Lie Groups for Pedestrians' (North Holland, 1965).

[5] R. E. BEHRENDS, J. DREITLEIN, C. FRONSDAL, and B. W. LEE, Rev. Mod. Phys. *34*, 1 (1962).

[6] C. P. ENZ, Helv. phys. Acta *38*, 150 (1965). The derivation by taking the limit of infinite volume, as given, e.g., by A. A. ABRIKOSOV, L. P. GORKOV, and I. E. DZYALOSHINSKI, *Quantum Field Theory in Statistical Physics* (Prentice-Hall, 1963), is obviously not valid here.

[7] A. A. ABRIKOSOV, Physics *2*, 61 (1966).