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Autor: Hunziker, Walter

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On the Spectra of Schrödinger Multiparticle Hamiltonians

by **Walter Hunziker**

Seminar für Theoretische Physik der ETH Zürich

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Abstract. For two-body potentials which are locally square-integrable and vanish (arbitrarily slowly) at infinity, the spectrum of the N-particle Hamiltonian in the center-of-mass frame is shown to have the form which is commonly expected: it consists of a continuum σ_c (generated by states in which the system is broken up into independent parts, and characterized, therefore, in terms of the Hamiltonians of these parts), and, in the complement of σ_c, of eigenvalues only, which are of finite multiplicity and can accumulate at most at the lower end of σ_c. Some properties of the corresponding bound-state wave-functions are derived, and a problem is posed concerning the generalization of Faddeev's equations to $N > 3$.

1. Introduction

For the purposes of time-independent scattering theory, S. WEINBERG [1][1]) and C. VAN WINTER [2] have independently derived a functional equation for the N-particle Green's function which is superior, in many respects, to the usual resolvent equations. In the present paper, we want to apply this functional equation to the simpler task of discussing the bound states below the continuum. Some results in this direction have also been obtained by C. VAN WINTER [2, 3], but since our objective here is not scattering theory, we can manage with less restrictive assumptions on the potentials. Essentially, we deal with potentials which are locally square-integrable and vanish arbitrarily slowly at infinity (for the precise hypothesis see Theorem 2). The existence of the continuum, which can be inferred from time-dependent scattering theory in the case of short-range forces, is established as a consequence of the spatial cluster-decomposition properties of the Hamiltonian.

2. The Spectrum of *H*

Using customary notation, we first recall some properties of the N-particle Hamiltonian

$$H = \sum_{i=1}^{N} \frac{P_i^2}{2\, m_i} + \sum_{\alpha} V_\alpha(x_\alpha) = H_0 + V \tag{1}$$

on the Hilbertspace $\mathfrak{H} = L^2(R^{3N})$. For simplicity, we only treat the case of two-body forces. α labels the $\binom{N}{2}$ pairs of different particles, and x_α denotes the relative coordinates of the pair α. The potentials V_α are always supposed to be real.

[1]) Numbers in brackets refer to References, page 462.

Theorem 1 (Kato [4])

For any α, let

$$V_\alpha(\cdot) \in L^2(R^3) + L^\infty(R^3)\,^{2)}\,. \tag{2}$$

Then $D(V_\alpha) \supset D(H_0)$ and, for any $a > 0$, there exists $b < \infty$ such that

$$||\,V_\alpha\,u\,|| \leq a\,||\,H_0\,u\,|| + b\,||\,u\,|| \tag{3}$$

for all α and all $u \in D(H_0)$. $H_0 + V$, defined on $D(H_0)$, is self-adjoint and bounded from below.

If the potentials $V_\alpha(x_\alpha)$ vanish as $|\,x_\alpha\,| \to \infty$, we expect that the N-particle system can break up into parts which are independent, if they are far separated from each other. To formulate this, let $D_k = (C_1 \ldots C_k)$ be a partition of the set of particles $(1 \ldots N)$ into k subsets (clusters), and let I_{D_k} be the sum of all two-body potentials which link particles in different clusters. We define

$$H_{D_k} = H - I_{D_k}\,, \tag{4}$$

so that the Hamiltonian H_{D_k} describes the system of noninteracting clusters $C_1 \ldots C_k$. Let u be a state of the N-particle system in which the clusters are far separated from each other, then $H\,u = H_{D_k}\,u$, approximately. Therefore, we expect

$$\sigma(H_{D_k}) \subset \sigma(H)\,, \tag{5}$$

where $\sigma(A)$ denotes the spectrum of A. From here on, we fix the center-of-mass of the N-particle system without changing the notation: H_0, H, and H_{D_k} are now Hamiltonians in the center-of-mass frame, acting on a Hilbertspace $\mathfrak{H} = L^2(^{3N-3})$. Except for the explicit form (1) of H, everything said so far applies word for word in this modified situation. If governed by the Hamiltonian H_{D_k}, the clusters $C_1 \ldots C_k$ $(k \geq 2)$ can still move freely relative to each other, hence $\sigma(H_{D_k})$ is still continuous, extending from some real number ε_{D_k} to $+\infty$. Defining

$$\sigma_c = \bigcup_{D_k, k \geqslant 2} \sigma(H_{D_k}) = [\min_{D_k, k \geqslant 2} \varepsilon_{D_k}, +\infty)\,, \tag{6}$$

we can now state our main result, which will be proved in Sections 4–6:

Theorem 2

For any α, let

$$V_\alpha(\cdot) \in L^2(R^3) + L^\infty(R^3)\,, \tag{7}$$

such that the L^∞-component of V_α can be chosen arbitrarily small, in the sense of the L^∞-norm. Then $\sigma_c \subset \sigma(H)$, and the part of $\sigma(H)$ in the complement of σ_c consists of eigenvalues only, which are of finite multiplicity and can accumulate at most at the lower end of σ_c.

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[2]) $f \in L^p + L^q$ means that there exists an $f_p \in L^p$ and an $f_q \in L^q$ such that $f(x) = f_p(x) + f_q(x)$ almost everywhere.

The hypothesis of this theorem is satisfied, for example, if, for any α,

$$V_\alpha(\cdot) \in L^2(R^3) + L^P(R^3)\,, \quad 2 \leq p < \infty\,,$$

or if $V_\alpha(x_\alpha)$ is locally square-integrable and vanishes (arbitrarily slowly) as $|x_\alpha| \to \infty$. In particular, this covers the case of Coulomb-interactions.

Among other things, Theorem 2 provides a basis for perturbation theory. The usual perturbation formalism for bound states has been justified to a large extent, notably by F. RELLICH [5] and T. KATO [6], but the underlying assumption is always that one has to deal with an isolated eigenvalue of finite multiplicity.

3. Properties of Bound-State Wave-Functions

In this section, we want to apply Theorem 2 to obtain some information about the behavior of bound-state wave-functions at infinity (in configuration space).

In the center-of-mass frame, we describe the positions of the N particles by relative coordinates $x_1 \ldots x_m$, $m = 3N - 3$, which are taken as linear combinations of the cartesian coordinates of the N particles. For any multiindex $n = (n_1 \ldots n_m)$, n_i integer ≥ 0, let x^n denote the operator of multiplication by the monomial $x_1^{n_1} \ldots x_m^{n_m}$, defined on all elements of $L^2(R^m)$ for which this product is again in $L^2(R^m)$, and define

$$D_n = \bigcap_{\substack{k \leqslant n \\ l \leqslant |n| - |k|}} D(x^k H^l)\,,$$

where $k \leq n$ means $k_i \leq n_i$, $i = 1 \ldots m$, and $|n| = n_1 + n_2 + \cdots + n_m$. On D_n we introduce the norm

$$||u||_n = \sup_{\substack{k \leqslant n \\ l \leqslant |n| - |k|}} ||x^k H^l u||\,. \tag{8}$$

Since H^l and x^k are closed, D_n, normed by $||\ ||_n$, is complete.

Theorem 3 [7]

Under the hypothesis of Theorem 1, D_n is invariant under e^{-iHt}, $-\infty < t < +\infty$. For any $u \in D_n$, $e^{-iHt} u$ is continuous in t in the sense of the norm $||\ ||_n$, and there exists a constant c_n such that

$$||e^{-iHt} u||_n \leq c_n (1 + |t|)^{|n|} ||u||_n\,.$$

Now let ϱ be a compact isolated subset of $\sigma(H)$, and let P denote the spectral projection corresponding to ϱ: $P = \int \varrho\, dE(\lambda)$, where $H = \int \lambda\, dE(\lambda)$ is the spectral representation of H.

Lemma 1

(a) For any n, D_n is invariant under P: $P D_n \subset D_n$.

(b) If P is of finite rank and D_n dense in $\mathfrak{H}$, then the range of P is contained in D_n: $P \mathfrak{H} \subset D_n$.

Proof:

(a) By hypothesis, there exists a C^∞-function $f(\lambda)$ of compact support such that

$$f(\lambda) = 1 \text{ for } \lambda \in \varrho\,,$$
$$f(\lambda) = 0 \text{ for } \lambda \notin \varrho\,, \quad \lambda \in \sigma(H)\,.$$

Its Fourier-Transform

$$\tilde{f}(t) = (2\pi)^{-1} \int_{-\infty}^{+\infty} d\lambda\, e^{i\lambda t} f(\lambda)$$

belongs to the space $\mathfrak{S}$ of rapidly decreasing functions, and, for any $u \in \mathfrak{H}$, the projection $P\,u$ can be expressed by

$$P\,u = \int_{-\infty}^{+\infty} dt\, e^{-iHt} \tilde{f}(t)\, u\,.$$

Now let $u \in D_n$. Then, by Theorem 3, the integrand is a function of t with values in D_n, and, in the sense of the norm (8), this function is continuous in t and vanishes faster than any inverse power of t as $|t| \to \infty$. Since D_n is complete, this implies that the integral (as the limit of Riemann sums) converges in D_n. Hence $u \in D_n$ implies $P\,u \in D_n$.

(b) If P is of finite rank and D_n dense in $\mathfrak{H}$, we have

$$P\,\mathfrak{H} = P\,\overline{D_n} \subset \overline{P\,D_n} = P\,D_n \subset D_n\,.$$

Combining this lemma with Theorem 2, we obtain

Theorem 4

Under the hypothesis of Theorem 2, the eigenvectors of H corresponding to eigenvalues in the complement of σ_c, belong to $D(x^n)$ – for any multi-index n for which D_n is dense in $\mathfrak{H}$.

For $|n| = 1$, D_n is dense in $\mathfrak{H}$. It is somewhat annoying that, under the hypothesis of Theorem 1, we are not able to assert the same for arbitrary n. However, if, for example, each $V_\alpha(x_\alpha)$ is a C^∞-function on an open set of R^3 whose complement has measure 0, then $\overline{D_n} = \mathfrak{H}$ for all n (7). Thus, in all the cases commonly considered in physics, the bound states below the continuum belong to $D(x^n)$ for all n, or, roughly speaking, the bound-state wave-functions decrease faster than any inverse power of $|x|$ as $|x| \to \infty$. If the potentials V_α are C^∞-functions on all of R^3, vanishing for $|x_\alpha| \to \infty$ and having bounded derivatives to all orders, the wave-functions of the bound states belong to $\mathfrak{S}(R^m)$ [7].

4. The Continuous Spectrum

This section is devoted to the proof that $\sigma_c \subset \sigma(H)$. The argument given in Section 2 can be made rigorous by aid of the following simple lemma, which we state without proof:

Lemma 2

Let H be a self-adjoint operator on a Hilbertspace $\mathfrak{H}$. Then $\lambda \in \sigma(H)$ if and only if there exists a sequence $\{u_n\} \subset D(H)$ with $\| u_n \| = 1$ and $\lim_{n\to\infty} \| (\lambda - H) u_n \| = 0$.

In our case, the states u_n will be constructed as states in which the clusters of a given decomposition are more and more separated from each other. First, we want to investigate in what sense the interaction between the clusters vanishes in this limit.

Let $x_1 \ldots x_N$ be the cartesian coordinates of the N particles. In (relative) configuration space R^{3N-3}, we can choose, for any pair i, j $(i \neq j)$, the $3N - 3$ independent coordinates $x = x_i - x_j$ and x_h $(h \neq i, j)$, which shall be denoted by (x, x_h). The norm on $\mathfrak{H} = L^2(R^{3N-3})$ is then given by

$$\| u \|^2 = c_{ij} \int | u(x, x_h) |^2 \, d^3x \prod_{h \neq i,j} d^3x_h ,$$

where $c_{ij} > 0$ is the modulus of a jacobian depending only on the masses and on the choice of the pair i, j.

Let $D_k = (C_1 \ldots C_k)$ be a decomposition of $(1 \ldots N)$ into k clusters $(k \geq 2)$. For any set $s = (s_1 \ldots s_k)$ of k 3-vectors we define the operator $T(s)$ on $\mathfrak{H}$ by

$$\big(T(s)u\big)\,(x, x_h) = u\,(x + t_i - t_j, x_h + t_h) ,$$

where $t_l = s_m$ for $l \in C_m$. In words: $T(s)$ is the unitary operator which represents a translation of each cluster C_m, as a whole, by s_m. Let $| s | = \min_{n \neq m} | s_n - s_m |$. As $| s | \to \infty$, the clusters become more and more separated from each other, and the interaction between them vanishes in the following sense:

Lemma 3

Under the hypothesis of Theorem 2,

$$\lim_{|s|\to\infty} \| I_{D_k} T(s)\, u \| = 0 , \tag{9}$$

for all $u \in D(H_0)$.

Proof:

We only consider a single term V_α of I_{D_k}. For any $\varepsilon > 0$, we can choose a splitting $V_\alpha(x) = V_{\alpha,2}(x) + V_{\alpha,\infty}(x)$ (a.e) such that $V_{\alpha,p}(\cdot) \in L^p(R^3)$ and that $\| V_{\alpha,\infty}(\cdot) \|_\infty < \varepsilon$. Then $\| V_{\alpha,\infty}\, T(s)\, u \| \leq \varepsilon \| u \|$ for all s, hence it suffices to prove (9) under the hypothesis that $V_\alpha(\cdot) \in L^2(R^3)$. $T(s)$ is unitary and commutes with H_0 on $D(H_0)$. Therefore, by (3),

$$\| V_\alpha\, T(s)\, u \| \leq a \, \| H_0\, u \| + b \, \| u \|$$

for all s and all $u \in D(H_0)$. Providing $D(H_0)$ with the norm $\| H_0\, u \| + \| u \|$, we see that $V_\alpha\, T(s)$ is a bounded operator from $D(H_0)$ into $\mathfrak{H}$, bounded uniformly in s. It is sufficient, therefore, to prove (9) on a dense set in $D(H_0)$, for example on $\mathfrak{S}(R^{3N-3})$. We can even restrict ourselves to states u of the form

$$u(x, x_h) = f(x)\, g(x_h) ,$$

with $f \in \mathfrak{S}(R^3)$ and $g \in \mathfrak{S}(R^{3N-6})$, since the finite linear combinations of such products are dense in $\mathfrak{S}(R^{3N-3})$. Choosing the coordinates (x, x_h) for the pair $i, j = \alpha$, we then obtain

$$|| V_{ij} T(s) u ||^2 = c_{ij} ||g||_2^2 \int d^3x \, | V_{ij}(x) f(x + t_i - t_j) |^2 .$$

Now i and j belong to different clusters, so that $| t_i - t_j | \to \infty$ as $| s | \to \infty$. Since $V_\alpha(\cdot) \in L^2(R^3)$ and $f \in \mathfrak{S}(R^3)$, the last integral vanishes as $| s | \to \infty$, by the dominated convergence theorem.

It is now a simple matter to show that $\sigma_c \subset \sigma(H)$. Let $\lambda \in \sigma(H_{D_k})$. By Lemma 2, there exists a sequence $\{u_n\} \subset D(H_0)$ with $|| u_n || = 1$ and

$$\lim_{n\to\infty} || (\lambda - H_{D_k}) u_n || = 0 . \tag{10}$$

On the other hand, the translation operators $T(s)$ corresponding to the decomposition D_k commute with H_{D_k} on $D(H_0)$, hence

$$|| (\lambda - H) T(s) u_n || \leq || (\lambda - H_{D_k}) u_n || + || I_{D_k} T(s) u_n || .$$

By (10) and Lemma 3, the right hand side can be made arbitrarily small by first choosing n and then $| s |$ large enough. Since $T(s) u_n \in D(H_0)$ and $|| T(s) u_n || = 1$, Lemma 2 applies and we conclude that $\lambda \in \sigma(H)$.

In the case of short-range forces, time-dependent scattering theory gives a stronger result:

Theorem 5 (Hack [8])

For any α, let

$$V_\alpha(\cdot) \in L^2(R^3) + L^p(R^3) , \quad 2 \leq p < 3 . \tag{11}$$

Then, for any decomposition D_k, the two strong limits

$$\underset{t\to\pm\infty}{s\text{-lim}} \, e^{iHt} e^{-iH_{D_k}t} = \Omega^{\pm}_{D_k}$$

exist on all of $\mathfrak{H}$.

(In his proof, Hack assumes that $V_\alpha(\cdot) \in L^2(R^3)$ and he establishes strong convergence only on a subspace of so-called channel states. It is not difficult, however, to extend his proof so that it covers Theorem 5). It follows that the operators $\Omega^{\pm}_{D_k}$ are isometric and satisfy

$$e^{-iHt} \Omega^{\pm}_{D_k} = \Omega^{\pm}_{D_k} e^{-iH_{D_k}t} ,$$

hence the ranges of $\Omega^{\pm}_{D_k}$ reduce the group e^{-iHt} and the parts of H in these invariant subspaces are unitarily equivalent to H_{D_k}. This is much more, of course, than $\sigma_c \subset \sigma(H)$ only.

5. The Functional Equation for $G(z)$

The main tool for the proof of the second part of Theorem 2 is the functional equation of S. Weinberg and C. van Winter for the resolvent $G(z) = (z - H)^{-1}$.

In this section, we rederive this equation by analyzing perturbation graphs – a method familiar to physicists, which can easily be justified in the present case.

Let $G_0(z) = (z - H_0)^{-1}$. By Theorem 1, $V_\alpha G_0(z)$ is defined on all of $\mathfrak{H}$ for $z \notin \sigma(H_0)$. For Re $z < 0$ we have $\| G_0(z) \| = | z |^{-1}$ and, by (3),

$$\| V_\alpha G_0(z) \| \leq 2\,a + \frac{b}{|z|} .$$

Since a can be chosen arbitrarily small, we conclude that

$$\lim_{\mathrm{Re}\, z \to -\infty} \| V_\alpha G_0(z) \| = 0 . \tag{12}$$

Let $M < 0$ be such that $\| V_\alpha G_0(z) \| < \binom{N}{2}^{-1}$ for all α if Re $z < M$. Thus, for Re $z < M$, the iteration solution of the resolvent equation

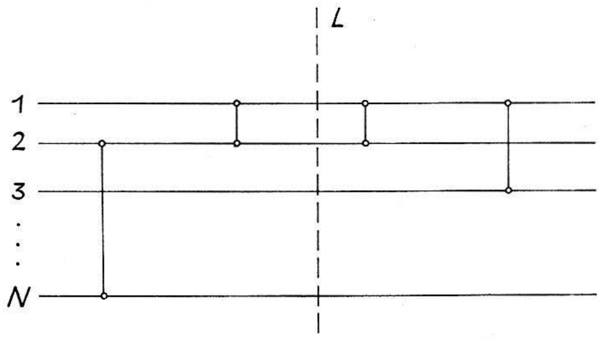
$$G(z) = G_0(z) + G_0(z)\, V\, G(z)$$

exists and is given by the series

$$G(z) = \sum_{n=0}^{\infty} \sum_{\alpha_1 \ldots \alpha_n} G_0(z)\, V_{\alpha_1}\, G_0(z)\, V_{\alpha_2} \ldots G_0(z)\, V_{\alpha_n}\, G_0(z) , \tag{13}$$

which converges absolutely, in the sense of the operator norm. (This also exhibits that H is bounded from below). Following S. WEINBERG [1], we define the graph representing the term $G_0\, V_{\alpha_1}\, G_0\, V_{\alpha_2} \ldots G_0\, V_{\alpha_n}\, G_0$ of the series (13) to consist of N horizontal lines (‚particles') and n vertical lines (‚interactions') linking just the pairs of particles $\alpha_1 \ldots \alpha_n$ from left to right.

For example, the graph



represents the term $G_0\, V_{2N}\, G_0\, V_{12}\, G_0\, V_{12}\, G_0\, V_{13}\, G_0$. Now we turn to a classification of these graphs:

(a) Each graph G consists of a certain number k of connected parts $(1 \leq k \leq N)$ – only the endpoints of the interactions counting as connections – and thus defines a cluster-decomposition $D_k(G)$: two particles belong to the same cluster if their lines belong to the same connected part of G.

(b) $D_l \subset D_k$ means that D_k is obtained by further partitioning the clusters of D_l $(l < k)$. A graph G is called D_l-disconnected if $D_l \subseteq D_k(G)$, i.e. if none of the interactions of G link different clusters of D_l. Identifying graphs with terms in the series (13), we find, for Re $z < M$,

$$\sum (\text{all } D_l\text{-disconnected graphs}) = (z - H_{D_l})^{-1} = G_{D_l}(z).$$

(c) Let us cut a graph by a vertical line L (see Figure), and denote by $D(L)$ the decomposition defined by the subgraph to the left of L. First, let L be to the left of all interactions, then $D(L) = D_N = (1)\ (2) \ldots (N)$. If we now shift L gradually to the right through the whole graph, $D(L)$ assumes the sequence of values

$$S = (D_N, D_{N-1}, \ldots D_k)$$
$$D_{i+1} \supset D_i\,, \qquad N \geq k \geq 1\,, \tag{14}$$

where D_k is the decomposition corresponding to the whole graph according to (a). In this way, each graph G uniquely defines a sequence $S(G)$ of type (14). Conversely, for a given S of type (14), we define the class S to consist of all graphs G with $S(G) = S$. Any graph of class S has the form

$$G_0 \prod_{i=N-1}^{k} \left[\begin{pmatrix}\text{any interaction linking different}\\ \text{clusters of } D_{i+1} \text{ but not of } D_i\end{pmatrix} \begin{pmatrix}\text{any } D_i\text{-disconnected}\\ \text{graph}\end{pmatrix}\right] \tag{15}$$

where the 'factors' are ordered from left to right as i decreases.

Since the series (13) is absolutely convergent for $\operatorname{Re} z < M$, we can rearrange it in the following way: first, we sum over all graphs of a fixed class S. By (15), this yields

$$\begin{aligned} G_S(z) &= \sum (\text{all graphs of class } S) \\ &= G_0(z) \prod_{i=N-1}^{k} \left[\begin{pmatrix}\text{sum of all potentials linking different}\\ \text{clusters of } D_{i+1} \text{ but not of } D_i\end{pmatrix} \begin{pmatrix}\text{sum of all } D_i\text{-discon-}\\ \text{nected graphs}\end{pmatrix}\right] \\ &= G_{D_N}(z)\ V_{D_N D_{N-1}}\ G_{D_{N-1}}(z)\ V_{D_{N-1} D_{N-2}} \cdots V_{D_{k+1} D_k}\ G_{D_k}(z)\,, \end{aligned} \tag{16}$$

where $V_{D_i D_{i-1}} = I_{D_i} - I_{D_{i-1}}$, I_{D_k} being defined by (4).

The remaining finite sum over classes is carried out in two steps: first, we sum over all $S = (D_N \ldots D_k)$ with $k \geq 2$. This is the sum of all disconnected graphs and defines the disconnected part $D(z)$ of $G(z)$:

$$D(z) = \sum_{\substack{\text{all } S\\ \text{with } k \geqslant 2}} G_S(z)\,. \tag{17}$$

Similarly, we obtain the connected part $C(z)$ by summing over all S with $k = 1$ (sum of all connected graphs):

$$C(z) = \sum_{\substack{\text{all } S\\ \text{with } k = 1}} G_S(z)\,. \tag{18}$$

Noting that each term in the last sum ends with a factor $G_{D_1}(z) = G(z)$, we finally arrive at

$$G(z) = D(z) + C(z) = D(z) + I(z)\ G(z)\,, \tag{19}$$

$$I(z) = \sum_{\substack{\text{all } S\\ \text{with } k = 1}} G_{D_N}(z)\ V_{D_N D_{N-1}}\ G_{D_{N-1}}(z) \ldots G_{D_2}(z)\ V_{D_2 D_1}\,. \tag{20}$$

Note that $D(z)$ and $I(z)$ are defined for all $z \notin \sigma_c$, but that the functional Equation (19) for $G(z)$ is established, so far, only for $\operatorname{Re} z < M$. As it stands, $I(z)$ is defined on the intersection of the domains $D(V_\alpha)$ only. However, it has a unique bounded extension, since (20) implies

$$I(\bar{z})^* = \sum_{\substack{\text{all } S \\ \text{with } k=1}} V_{D_2 D_1} G_{D_2}(z) \ldots V_{D_N D_{N-1}} G_{D_N}(z) \tag{21}$$

which, by Theorem 1, is bounded for $z \notin \sigma_c$. In the following, $I(z)$ denotes the unique bounded extension of the operator defined by (20).

Lemma 4

(a) $I(\bar{z})$ and $D(z)$ are holomorphic in z for $z \notin \sigma_c$.

(b) $\lim\limits_{\operatorname{Re} z \to -\infty} \| I(z) \| = 0$.

Proof:

If $I(\bar{z})^*$ is holomorphic in z for $z \notin \sigma_c$, so is $I(z)$, hence it suffices to prove assertions (a) and (b) for the operator $V_\alpha G_D(z)$, where α is any pair and D any decomposition of $(1 \ldots N)$ into at least two clusters. Let $D(H_D)$ be normed by $\| u \|_D = \| H_D u \| + \| u \|$. For $z \notin \sigma_c$, $G_D(z)$ is a bounded operator from $\mathfrak{H}$ onto $D(H_D)$, and holomorphic in z for $z \notin \sigma_c$. By Theorem 1, the norm $\| u \|_D$ is equivalent to the norm $\| H_0 u \| + \| u \|$ on $D(H_0) = D(H_D)$, hence V_α is a bounded operator from $D(H_D)$ into $\mathfrak{H}$. This proves (a). To prove (b), we note that an estimate of type (3) also holds if H_0 is replaced by H_D, thus (b) follows in the same way as (12).

By Lemma 4(a), the functional equation

$$G(z) = D(z) + I(z)\, G(z)\,, \tag{22}$$

with $D(z)$ and $I(z)$ defined by (16) (17) (20), extends by analyticity from $\operatorname{Re} z < M$ to all $z \notin \sigma_c$, $z \notin \sigma(H)$. This is the final step in the derivation based on the perturbation series (13). Of course, (22) can also be obtained from the resolvent equations which link the various resolvents $G_{D_k}(z)$ [2].

6. The Discrete Spectrum

Lemma 5

If the potentials V_α satisfy the hypothesis of Theorem 2, then $I(z)$ is a compact operator for $z \notin \sigma_c$.

Proof:

For each α, there exists a sequence $V_{\alpha,n}(\cdot) \in L^2(R^3)$ such that $V_{\alpha,n}(\cdot) - V_\alpha(\cdot) \in L^\infty(R^3)$ and, for $n \to \infty$, $\| V_{\alpha,n}(\cdot) - V_\alpha(\cdot) \|_\infty \to 0$. Let $I_n(z)$ be defined by (20), with the proviso that in the terms $V_{D_{k+1} D_k}$ (but not in the resolvents G_{D_k}) the potentials V_α are to be replaced by $V_{\alpha,n}$. Clearly, for $z \notin \sigma_c$,

$$\lim_{n \to \infty} \| I_n(z) - I(z) \| = 0\,.$$

On the other hand, it was shown by C. VAN WINTER [2] and also by the author [9], that each term in the sum (20) is a Hilbert-Schmidt operator, for $z \notin \sigma_c$, provided that all the potentials occurring explicitely (i. e. in the terms $V_{D_{k+1} D_k}$) are square-integrable. Hence $I(z)$, being the uniform limit of a sequence of Hilbert-Schmidt operators, is compact for $z \notin \sigma_c$.

Combining Lemmas 4 and 5 we conclude (see appendix), that $(1 - I(z))^{-1}$ is meromorphic in z for $z \in \sigma_c$, and that

$$G(z) = \big(1 - I(z)\big)^{-1} D(z)$$

for all $z \notin \sigma_c$ for which $(1 - I(z))^{-1}$ exists (it might happen that $G(z)$ exists but not $(1 - I(z))^{-1}$. This point is further discussed in Section 7), thus $G(z)$ too is meromorphic in the complement of σ_c. This proves that the part of $\sigma(H)$ in the complement of σ_c consists of eigenvalues only, which can accumulate at most at the lower end of σ_c.

To show the finite multiplicity, let z_0 be a pole of $G(z)$ (eigenvalue of H) in the complement of σ_c, and P the projection onto the corresponding subspace of eigenvectors:

$$P = \lim_{z \to z_0} (z - z_0)\ G(z)\ .$$

On the other hand, by (22),

$$(z - z_0)\ G(z) = (z - z_0)\ D(z) + I(z)\ (z - z_0)\ G(z)\ .$$

Since $D(z)$ and $I(z)$ are holomorphic in a neighborhood of z_0, we obtain, for $z \to z_0$:

$$P = I(z_0)\ P\ , \tag{23}$$

hence P is a compact projection and therefore of finite rank. This concludes the proof of Theorem 2.

7. Faddeev's Equations

(23) shows that, for $z \notin \sigma_c$,

$$H f = z f \text{ implies } f = I(z) f\ .$$

For $N > 2$, however, the reverse is not proved and probably false. Thus we cannot conclude that $(1 - I(z))^{-1}$ exists (for $z \notin \sigma_c$) whenever z is not an eigenvalue of H. From all we know, the poles of $(1 - I(z))^{-1}$ need not even be real.

In the case $N = 3$, this annoying feature is avoided by L. D. Faddeev [10] in the following way: he splits the Green's function into components

$$G(z) = G_0(z) + \sum_\alpha R_\alpha(z)$$

where $R_\alpha = G_0\ V_\alpha\ G$ or, in terms of graphs, $R_\alpha = \Sigma$ (of all graphs with V_α as the left-most interaction). Instead of (22), L. D. Faddeev treats a linked set of equations for the components R_α:

$$R_\alpha = (G_\alpha - G_0) + \sum_{\beta \neq \alpha} G_\alpha\ V_\alpha\ R_\beta$$

where $G_\alpha = (z - H_0 - V_\alpha)^{-1}$ is the disconnected part of R_α. This splitting into components has the effect that, for $z \notin \sigma_c$, the homogeneous equations

$$H f = z f \text{ and } f_\alpha = \sum_{\beta \neq \alpha} G_\alpha\ V_\alpha f_\beta$$

are equivalent, if connected by

$$f = \sum_{\alpha} f_{\alpha}, \quad f_{\alpha} = G_0 V_{\alpha} f.$$

The author has tried, unsuccessfully so far, to find a generalization of FADDEEV's equations to $N > 3$ which still has this property. The analysis of graphs suggests several ways of splitting $G(z)$ – or $C(z)$ – into components (an example is (18)), and linked sets of equations for these components can be derived along the lines followed in Section 5. The unsolved problem is to show whether or not nontrivial solutions of the homogeneous equations correspond to bound states.

Appendix

In this appendix we formulate and prove a theorem which seems to be known but which we couldn't find in the literature.

Theorem

(a) Let C be the closure, in the sense of the operator norm, of the set of operators of finite rank on a Banachspace B.

(b) Let $A(z)$ be a C-valued function of the complex variable z, which is holomorphic in an open, connected region G.

(c) Suppose that $(1 - A(z_0))^{-1}$ exists for some $z_0 \in G$. Then $(1 - A(z))^{-1}$ is meromorphic in z for $z \in G$, i.e. its only possible singularities in G are poles.

Remark:

If B is a Hilbertspace, then C is the set of all compact operators on B. To our knowledge, it is still open if the same is true for any Banachspace B.

Proof:

(a) The theorem holds if there exists a subspace $R \subset B$, $\dim R < \infty$, such that the range of $A(z)$ is contained in R for all $z \in G$. To show this, let $a(z)$ be the restriction of $A(z)$ to R. Then

$$(1 - A(z))^{-1} = 1 + (1 - a(z))^{-1} A(z), \tag{24}$$

whenever the right-hand side exists. In any basis of R, the determinant of $1 - a(z)$ is holomorphic in $z \in G$ and not identically zero, for $a(z_0) f = f$ implies $f = 0$, by (c). Hence $(1 - a(z))^{-1}$ and, by (24), $(1 - A(z))^{-1}$ are meromorphic in G.

(b) Now let $A(z)$ satisfy the hypothesis of the theorem. For any $z \in G$ there exists an operator $F(z)$ of finite rank such that $\| A(z) - F(z) \| < 1/2$, and, since $A(z)$ is continuous in z, *a* neighborhood $U(z) \subset G$ of z such that

$$\| A(z') - F(z) \| < 1 \text{ if } z' \in U(z). \tag{25}$$

Let z_1 be an arbitrary point in G. We have to show that $(1 - A(z))^{-1}$ is meromorphic in a neighborhood of z_1. By the Heine-Borel theorem, there exists a sequence of points $s_1 \dots s_N$ in G such that

$$z_0 \in U(s_1), \ z_1 \in U(s_N), \ U(s_i) \cap U(s_{i+1}) \text{ non-empty for } i = 1 \dots N - 1.$$

Let $e(z) = A(z) - F(s_1)$. $e(z)$ is holomorphic in G and $\| e(z) \| < 1$ for $z \in U(s_1)$, by (25). Thus $(1 - e(z))^{-1}$ is holomorphic in $U(s_1)$, and

$$1 - A(z) = [1 - F(s_1)\,(1 - e(z))^{-1}]\,(1 - e(z))\,.$$

For $z \in U(s_1)$, $F(s_1)\,(1 - e(z))^{-1}$ is of finite rank, with range contained in the range of $F(s_1)$, and holomorphic in z. Also, by (c), $[1 - F(s_1)\,(1 - e(z_0))^{-1}]^{-1}$ exists. Thus, by part (a) of the proof, $(1 - A(z))^{-1}$ is meromorphic in $U(s_1)$. Since $U(s_1) \cap U(s_2)$ is open and non-empty, there exists a new $z_0 \in U(s_2)$ for which $(1 - A(z_0))^{-1}$ exists. By the same analysis, we find that $(1 - A(z))^{-1}$ is meromorphic in $U(s_2) \dots U(s_N)$.

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