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Generalized and Reducible Free Fields in Fock Space

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Summary: In the Fock space $\mathcal{H}_{m,0}$ generated from the vacuum by a mass m, scalar free field not all the free fields are irreducible (and hence equivalent). We construct the reducible and the generalized covariant free fields in $\mathcal{H}_{m,0}$ and discuss some of their properties.

§ 1. Introduction

In a future quantum theory of local interacting fields one would expect the latter to operate in the Fock space of asymptotic states [1] [1]) and to transform like the asymptotic free fields under the unitary representation $U(a, \lambda)$ of the inhomogeneous Lorentz group [2]).

Restricting our interest to the case of only one kind of particles (mass m, spin zero) we have in $\mathcal{H} \equiv \mathcal{H}_{m,0}$ irreducible free fields $A(x)$:

$$(\Box + m^2)\, A(x) = 0\,, \qquad [A(x)\,,\, A(y)] = i\,\Delta\,(m\,,\, x - y)\,,$$

$$U(a\,,\lambda)\, A(x)\; U^{+}(a\,,\lambda) = A\;(\lambda\, x + a)\,, \qquad \text{(1a, b, c)}$$

$$[C\,,\, A(x)] = 0 \;\Rightarrow\; C = \alpha\, \mathbf{1}. \qquad (2)$$

For any two of them there is a unitary operator in $\mathcal{H}_{m,0}$ which transforms one into the other and commutes with $U(a, \lambda)$ [3].

The local functions [4] of $A(x)$ (Borchers classes [5]) are further examples of local covariant fields in Fock space. In this paper we shall consider the following two questions:

(1) Are the equations (1) sufficient to characterize the irreducible free fields in the Fock space $\mathcal{H}_{m,0}$ (of identical mass m, spin zero particles)? Or are there also fields $\phi(x)$ in $\mathcal{H}_{m,0}$ obeying equations (1) like $A(x)$ which are reducible and do not generate $\mathcal{H}_{m,0}$ from the vacuum?

(2) Are there generalized free fields in $\mathcal{H}_{m,0}$?

All such fields would lie in Borchers classes different from those of the irreducible free fields. – The second question is of a certain interest in connection with models where use is made of 'Greenberg fields' for the description of resonances or more general scattering states [6].

In §2 reducible and generalized free fields in $\mathcal{H}_{m,0}$ are constructed explicitely, in §3 we will show this construction to be exhaustive, §4 serves to summarize the results.

[1]) Numbers in brackets refer to References, page 70.

[2]) For examples of local fields whose automorphisms $\phi(x) \to \phi\;(\lambda\, x + \alpha)$ are not implemented by unitary operators cf. ref. 2.

§ 2. Construction of Reducible and Generalized Free Fields in the Fock Space $\mathcal{H}_{m,0}$

Theorem 1: Let $\mathcal{H}_{m,0}$ be the Fock space of identical, mass m, spin zero particles with the unitary representation $U(a, \lambda)$ of the inhomogeneous Lorentz group; and let

$$\varrho_A(\varkappa) = c\,\delta\,(\varkappa - m) + \Theta\,(\varkappa - 3\,m)\,\varrho_A(\varkappa)\,, \tag{A}$$

or

$$\varrho_B(\varkappa) = \Theta\,(\varkappa - 2\,m)\,\varrho_B(\varkappa), \tag{B}$$

with $0 < c < \infty$ and non-negative integrable functions

$$\Theta\,(\varkappa - 3\,m)\,\varrho_A(\varkappa),\ \Theta\,(\varkappa - 2\,m)\,\varrho_B(\varkappa).$$

Then there are in $\mathcal{H}_{m,0}$ (generalized) free fields $\phi_A(x), \phi_B(x)$ with [3])

$$[\phi(x), \phi(y)] = i\,\Delta'\,(x - y) = i \int d\varkappa\,\varrho(\varkappa)\,\Delta\,(\varkappa, x - y)\,, \tag{3}$$

$$U(a, \lambda)\,\phi(x)\,U^+(a, \lambda) = \phi\,(\lambda\,x + a)\,. \tag{4}$$

ϕ_B is always reducible, ϕ_A is irreducible or reducible depending on its construction; in the latter case with

$$\varrho_A(\varkappa) = \delta\,(\varkappa - m)$$

it is a reducible free field.

Proof: Let $\varphi(x)$ with the commutation relations (3), (A) or (B) be a Greenberg field in the Hilbert space $\mathcal{H}_1$ it generates cyclically from the vacuum [7]. The representation

$$U_1(a, \lambda)\,\varphi(x)\,U_1^+(a, \lambda) = \varphi\,(\lambda\,x + a) \tag{5}$$

of the inhomogeneous Lorentz group in $\mathcal{H}_1$ is isomorphic to

$$1 \cdot \mathbf{1} \oplus 1 \cdot \int_0^\infty{}^{\oplus} d\varkappa\,\varrho(\varkappa)\,[\varkappa, 0] \oplus \int_{2M}^\infty{}^{\oplus} d\varkappa \sum_{l=0}^{\infty}{}^{\oplus} v(\varkappa, l) \cdot [\varkappa, l] \oplus \infty \cdot \int_{3M}^\infty{}^{\oplus} d\varkappa \sum_{l=0}^{\infty}{}^{\oplus} [\varkappa, l]\,. \tag{6}$$

$[\varkappa, l]$ denotes the irreducible representations (mass $\varkappa$, angular momentum l) of the inhomogeneous Lorentz group [8]

$$M = \underline{\text{fin}}\ \text{supp}\ \varrho(\varkappa)\,,$$

$$\text{i.e.}\quad M_A = m\,,\qquad M_B \geqslant 2\,m$$

and the multiplicities

$$v_A(\varkappa, l) = \delta_{l, 2n}\,,\qquad n = 0, 1, 2, 3, \ldots\,,\qquad v_B(\varkappa, l) = \infty\,.$$

On the other hand $U(a, \lambda)$ in $\mathcal{H}_{m,0}$ is equivalent to

$$1 \cdot \mathbf{1} \oplus 1 \cdot [m, 0] \oplus 1 \cdot \int_{2m}^\infty{}^{\oplus} d\varkappa \sum_{l=0}^{\infty}{}^{\oplus} [\varkappa, 2\,l] \oplus \infty \cdot \int_{3m}^\infty{}^{\oplus} d\varkappa \sum_{l=0}^{\infty}{}^{\oplus} [\varkappa, l]\,. \tag{7}$$

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[3]) The indices A, B will be omitted in the following as long as the formulas hold true for A and for B.

Now let $U_2^A(a, \lambda)$, $U_2^B(a, \lambda)$ be unitary representations of the inhomogeneous Lorentz group in Hilbert spaces $\mathcal{H}_2^A$, $\mathcal{H}_2^B$ with the decompositions

$$U_2^A \cong 1 \cdot \mathbf{1} \oplus \int_{3m}^{\infty}{}^{\oplus} d\mu(\varkappa)\, \sigma(\varkappa, l) \cdot [\varkappa, l]\,, \tag{8}$$

$$U_2^B \cong U_2^A \oplus 1 \cdot [m, 0] \oplus 1 \cdot \int_{2m}^{3m}{}^{\oplus} d\varkappa\, (1 - \Theta_\varrho(\varkappa))\, [\varkappa, 0]$$
$$\oplus\, 1 \cdot \int_{2m}^{3m}{}^{\oplus} d\varkappa \sum_{l=1}^{\infty}{}^{\oplus} [\varkappa, 2l] \oplus \infty \cdot \int_{3m}^{m+M_B}{}^{\oplus} d\varkappa \sum_{l=0}^{\infty}{}^{\oplus} [\varkappa, l] \tag{9}$$

with $d\mu(\varkappa)$ absolutely continuous with respect to the measure $d\varkappa$ in (7), with arbitrary multiplicities $\sigma(\varkappa, l) = \infty, 0, 1, 2, \ldots$ and with $\Theta_\varrho(\varkappa)$ denoting the characteristic function of the support of $\varphi(\varkappa)$

$$\Theta_\varrho(\varkappa) = \begin{cases} 1 & \varkappa \in \operatorname{supp} \varrho \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

In the tensor product space

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \tag{11a}$$

the inner Kronecker product [9]

$$U_1(a, \lambda) \otimes U_2(a, \lambda) \tag{11b}$$

provides a representation of the inhomogeneous Lorentz group. Its decomposition into irreducible representations is the same as that of U. From this it follows [8] that the two representations are equivalent: there is a unitary operator V with

$$\mathcal{H}_{m,0} = V\, (\mathcal{H}_1 \otimes \mathcal{H}_2)\,,$$

$$U(a, \lambda) = V\, (U_1(a, \lambda) \otimes U_2(a, \lambda))\, V^+, \tag{12}$$

and

$$\phi(x) = V\, (\varphi(x) \otimes \mathbf{1})\, V^+ \tag{13}$$

obeys the equations (3) (4) of our theorem.

The operators G of the form

$$G = V\, (\mathbf{1} \otimes g)\, V^+\,, \qquad g \in \mathfrak{L}(\mathcal{H}_2)$$

obviously commute with $\phi(x)$. Now $\mathfrak{L}(\mathcal{H}_2) = \{\alpha\, \mathbf{1}\}$ iff $\mathcal{H}_2$ is one dimensional, which is possible only in case A and only if

$$\int^{\oplus} d\mu(\varkappa)\, \sigma(\varkappa, l) \cdot [\varkappa, l] \equiv 0\,,$$

i.e. if $\sigma(\varkappa, l) = 0$ μ-almost-everywhere. Then and only then the field $\phi_A(x)$ so constructed is irreducible.

§ 3. Characterization of all (Generalized) Covariant Free Fields in the Fock Space $\mathcal{H}_{m,0}$

Theorem 2: Let R_ϕ be the von-Neumann-algebra of a (generalized) free field ϕ in $\mathcal{H}_{m,0}$ with $R_\phi \wedge R'_\phi = \{\alpha\, \mathbf{1}\}$ [10].

Then there exist Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and a unitary operator V with

$$V^+ \mathcal{H}_{m,0} = \mathcal{H}_1 \otimes \mathcal{H}_2 ,$$

$$V^+ R_\phi V = \mathfrak{L}(\mathcal{H}_1) \otimes \mathbf{1} ,$$

$$V^+ \phi(x) V = \varphi(x) \otimes \mathbf{1} .$$

$\varphi(x)$ is a (generalized) free field in the space $\mathcal{H}_1$ it generates cyclically, the spectral functions of $\phi(x)$ and $\varphi(x)$ obey equation A or B of theorem 1.

Proof: Let $\mathcal{H}_0 \subset \mathcal{H}_{m,0}$ denote the subspace $\phi(x)$ generates from the vacuum, P_0 the projector onto $\mathcal{H}_0$; then $P_0 \in R'_\phi$.

Since R'_ϕ is a factor, the central support of P_0 in R'_ϕ is equal to one which in turn implies [11] that the restriction R_0 of R_ϕ onto $\mathcal{H}_0$ is an isomorphism $R_\phi \overset{1:1}{\longleftrightarrow} R_0$. Furthermore $R'_0 = \{\alpha \mathbf{1}\}$ in $\mathcal{H}_0$, so that R_0 and R_ϕ are factors of type I [12], and there is an isomorphism V with [13] [4])

$$\begin{aligned} \mathcal{H}_{m,0} &= V (\mathcal{H}_1 \otimes \mathcal{H}_2) , \\ R_\phi &= V (\mathfrak{L}(\mathcal{H}_1) \otimes \mathbf{1}) V^+ , \\ R'_\phi &= V (\mathbf{1} \otimes \mathfrak{L}(\mathcal{H}_2)) V^+ . \end{aligned} \tag{18}$$

Since all automorphisms of a type I von Neumann algebra are inner ones [14], R_ϕ contains a unitary representation of the inhomogeneous Lorentz group

$$U_\phi(a, \lambda) = V (U_1(a, \lambda) \otimes \mathbf{1}) V^+ \in R_\phi \tag{19}$$

with $$U(a, \lambda) F U^+(a, \lambda) = U_\phi(a, \lambda) F U^+_\phi(a, \lambda) \quad \forall F \in R_\phi .$$

Obviously $$U^+_\phi(a, \lambda) U(a, \lambda) \in R'_\phi$$

and in consequence may be represented as follows:

$$\begin{aligned} U^+_\phi(a, \lambda) U(a, \lambda) &= V (\mathbf{1} \otimes U_2(a, \lambda)) V^+ , \\ U &= V (U_1 \otimes U_2) V^+ . \end{aligned} \tag{20}$$

$\varphi(x) \otimes \mathbf{1} = V^+ \phi(x) V$ is a factor representation of the (generalized) free field commutations relations in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Since $U_\phi \in R_\phi$, $\varphi(x)$, operating in $\mathcal{H}_1$, is in its Fock representation [5]); and U_1, U_ϕ both have the decomposition (6) with $M \geqslant 0, v = 1$ or $v = \infty$. To provide the decomposition (7) for U, the identity representation must be contained once in U_2:

$$U_2 \cong 1 \cdot \mathbf{1} \oplus \dots ,$$

$$U \cong U_1 \otimes \mathbf{1} \oplus \dots .$$

[4]) For another proof of such decompositions cf. ARAKI [10].

[5]) E. C. G. SUDARSHAN's proof [2] for this consists in showing that the infinitesimal generators of the Lorentz group have a unique form in terms of oscillator variables and that they majorize the number operator which is finite only in the Fock representation. Oscillator variables for the Greenberg fields $\phi(x)$ are obtained by expanding $\phi(x)$ in terms of a complete orthonormal set of positive frequency functions whose momentum space support is that of the Greenberg weight function $\varrho(p^2)$. – The so defined oscillator variables allow a direct generalization of SUDARSHAN's proof.

Furthermore in the decomposition of $U_1\ \varrho(\varkappa)\,d\varkappa$ must be absolutely continuous with respect to the measure in (7) which defines the direct integral decomposition of U:

$$\varrho(\varkappa) = c\,\delta\,(\varkappa - m) + \Theta\,(\varkappa - 2\,m)\,\varrho(\varkappa)\,. \tag{21}$$

Case A: $c \neq 0$, i.e. $M = m$. In this case $\varrho(\varkappa) = 0$ almost everywhere in the interval $2\,m \leqslant \varkappa < 3\,m$, since otherwise the irreducible representations contained in

$$\int_{2m}^{3m}{}^{\oplus} d\varkappa\,\varrho(\varkappa)\,[\varkappa,0] \neq 0$$

would appear twice, in the second and in the third term of equation (6), whereas from (7) we see that they should only have the multiplicity one. (Or to put it more intuitively, there would be, at a given energy below the three particle threshold, two orthogonal s-states, one being generated by applying the discrete part of ϕ twice and one by applying the continuous part of ϕ once to the vacuum. But there is only one such s-state in $\mathcal{H}_{m,0}$.)

With this restriction
$$U \cong U_1 \otimes \mathbf{1}$$
and if we further want
$$U \cong U_1 \otimes U_2 = U_1 \otimes \mathbf{1} \oplus U_1 \otimes (U_2 \ominus \mathbf{1})$$
the additional term $U_1 \otimes (U_2 \ominus \mathbf{1})$ may not change the multiplicities of the direct integral decomposition; it may only contribute to the last term in (7), i.e.
$$U_2 \cong U_2^A \text{ of equation (8)}\,.$$
Case B: $c = 0$, $M \geqslant 2\,m$.

Here U_2 must provide the discrete representation $[m, 0]$
$$U_2 \cong 1 \cdot \mathbf{1} \oplus 1 \cdot [m,0] \oplus \ldots .$$
$U_1 \otimes [m, 0]$ contributes to the direct integrals of equation (7) only above $m + M \geqslant 3\,m$. Below $m + M$, the difference of the decompositions of U and U_1 must be contained in U_2:

$$U_2 \cong 1 \cdot \mathbf{1} \oplus 1 \cdot [m,0] \oplus 1 \cdot \int_{2m}^{3m}{}^{\oplus} d\varkappa\,(1 - \Theta_\varrho(\varkappa))\,[\varkappa, 0]$$
$$\oplus\, 1 \cdot \int_{2m}^{3m}{}^{\oplus} d\varkappa \sum_{l=1}^{\infty}{}^{\oplus}\,[\varkappa, 2\,l] \oplus \infty \cdot \int_{3m}^{m+M}{}^{\oplus} d\varkappa \sum_{l=0}^{\infty}{}^{\oplus}\,[\varkappa, l] \oplus \ldots .$$

What was said for $U_2 \ominus \mathbf{1}$ in case A, holds true for the rest of U_2 here, so that
$$U_2 \cong U_2^B \text{ of equation (9).}$$

§ 4. Summary

For the existence of covariant (generalized) free fields in the Fock space $\mathcal{H}_{m,0}$, equation A or B, or equivalently the existence of a decomposition as in equations (12), (6)–(9), is sufficient (Th. 1) and necessary (Th. 2).

We have seen that in $\mathcal{H}_{m,0}$ local (Greenberg) fields may be constructed which destroy and create two particle s-wave states.

As long as there is no bound state the Lagrange formalism (in terms of Greenberg fields) given by Thirring [7] for the Zachariasen model may be extended from the two particle sector to the whole Fock space $\mathcal{H}_{m,0}$ of the particles described by it. –

A final question to be answered here is whether reducible free fields might serve to define a unitary nontrivial scattering operator via the equation

$$A_{out}(x) - A_{in}(x) = \int \Delta\ (x - x')\ j(x')\ dx' \tag{22}$$

where $j(x)$ is given as a (local or non local) finite order Wick polynomial of some free field $A(x)$. Smeared out with an appropriate test function f on the forward mass shell in momentum space and multiplied by the projector E_n onto states with an energy less than $n + 1$ particle masses, equation (22) reads

$$a_{out}(f)\ E_n - a_{in}\ (f)\ E_n = -i\ j\ (f)\ E_n\,. \tag{23}$$

For $j = j[A_{irr}]$ given as a finite Wick polynomial of an irreducible free field $A_{irr}\ (x)$ equation (23) will not hold since for large enough n the norm of the rhs will become larger than that of the lhs [15]. In the case of reducible $A_{red}\ (x)$ we observe that

$$\|\ j\ [A_{red}]\ E_n\ \| \geqslant \|\ j\ [A_{red}]\ E_n\ P_0\ \| = \|\ j\ [A_{irr}]\ E_n \|\,,$$

the equality following from the fact that the restriction of $A_{red}(x)$ to $P_0\,\mathcal{H}_{m,0}$ is unitarily equivalent to $A_{irr}\ (x)$.

We conclude that the argument ruling out finite Wick polynomials as current operators is valid for reducible as well as for irreducible fields.

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