Zeitschrift: Helvetica Physica Acta
Band: 38 (1965)
Heft: VII

Artikel: Multiple excitation of collective nuclear states by inelastic scattering
Autor: Bierter, Willy
DOI: https://doi.org/10.5169/seals-113618

Multiple Excitation of Collective Nuclear States by Inelastic Scattering

by **Willy Bierter** *)

Institute of Theoretical Physics, University of Basel, Basel

(8. VII. 65)

An explicit WKB-solution of a system of coupled differential equations is presented where the adiabatic approximation has been assumed to be valid. The optical model is extended to include nonspherical potentials so that inelastic scattering to low-lying collective nuclear states are induced which are strongly coupled to the nuclear ground state. Calculations are given for the multiple excitation of collective quadrupole vibrational states by inelastic scattering of alpha-particles. In all calculations the effects of Coulomb excitation have been taken into account.

I. Introduction

The inelastic scattering of medium-energy protons or alpha-particles is a valuable tool to investigate direct nuclear reactions. Very often the inelastic scattering can be looked on as a nuclear surface phenomenon. Therefore a rapidly varying oscillatory angular distribution may be expected. Furthermore the collective excitations, as e.g. vibrations and rotations of the surface, are strongly excited. It should be particularly interesting to study these excitations of the low-lying collective states which are easily observed due to their strong enhancement.

The distorted-wave Born-approximation can often be applied to calculate the transition amplitude. In these cases the elastic scattering is assumed to be the dominant process and the inelastic transitions can be treated in first-order perturbation theory. In the case of weak coupling the distorted-wave Born-approximation is formally equivalent to the first-order adiabatic method[1]). However, if the interaction becomes so strong that many levels – strongly coupled to the nuclear ground state – are actively involved in the excitation process a set of coupled differential equations has to be solved. Recently, this has been done numerically for the case of inelastic proton scattering[2])[3]).

The aim of this paper is to show under which assumptions an explicit form for the solution of this set can be obtained in the WKB-approximation. The most important assumption is the adiabatic approximation which neglects the energy loss of the projectile due to the excitation of nuclear levels and treats the elastic and inelastic scattering as a similar process. Thus the problem of computing the partial cross section for inelastic scattering can be reduced to a calculation of the elastic scattering amplitude as a function of the target coordinates.

In the present work the main emphasis lies in the use of a generalized optical potential with an appropriate nonspherical shape in order to determine the transition amplitude for the multiple excitation of collective states.

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*) Pressent adress: Institute of Theoretical Physics, University of Heidelberg, Heidelberg.

II. Theory

In this section the WKB-approximation of the system of coupled differential equations is considered in detail in the adiabatic approximation. The transition amplitude for inelastic scattering of charged incident projectiles will be calculated. Throughout this work spin-orbit effects are neglected, although the formalism could be easily extended to include them.

As we work in the center-of-mass system, only reduced masses and relative velocities and momenta appear in the equations.

The Hamiltonian for the interacting system of incident particle and target nucleus is assumed to have the following form

$$H = H_p(\boldsymbol{r}) + U(r) + H_T(\alpha) + H_{int}(\boldsymbol{r}, \alpha) , \tag{1}$$

where $H_p(\boldsymbol{r})$ is the sum of the kinetic energy operator for the relative motion of particle and target nucleus and the Coulomb potential V_c which is given for a uniformly charged sphere of radius R_c by

$$\frac{2m}{\hbar^2} V_c = \begin{cases} \dfrac{2k\eta}{r} & \text{if } r > R_c \\ \dfrac{k\eta}{R_c}\left[3 - \left(\dfrac{r}{R_c}\right)^2\right] & \text{if } r \leqslant R_c . \end{cases}$$

The Coulomb parameter η is as usually defined by

$$\eta = \frac{Z_1 Z_2 e^2}{\hbar v} ,$$

where v is the relative velocity of projectile and target nucleus, Z_1 the charge number of the target and Z_2 that of the projectile.

The spherical optical potential $U(r)$ is used to describe the elastic scattering and $H_{int}(\boldsymbol{r}, \alpha)$ is the interaction potential between projectile and nucleus. The relative coordinates of the particle and the internal coordinates of the target nucleus are denoted, respectively, by $\boldsymbol{r} = (r, \vartheta, \varphi)$ and α. The target Hamiltonian $H_T(\alpha)$ defines a complete set of state vectors expressed by

$$H_T(\alpha)\, |\Phi_I(\alpha)\rangle = E_I |\Phi_I(\alpha)\rangle , \tag{2}$$

where I specifies the eigenvalues of a complete set of commuting observables. The exact wave function of the scattering process $|\psi(\boldsymbol{r}, \alpha)\rangle$ satisfies the Schrödinger equation

$$H\,|\psi(\boldsymbol{r}, \alpha)\rangle = (E + E_I)|\psi(\boldsymbol{r}, \alpha)\rangle \tag{3}$$

and may be separated as follows

$$|\psi(\boldsymbol{r}, \alpha)\rangle = |k(\boldsymbol{r}, \alpha)\rangle\, |\Phi_I(\alpha)\rangle + \Delta(\boldsymbol{r}, \alpha) . \tag{4}$$

Here $|k(\boldsymbol{r}, \alpha)\rangle$ is a partial adiabatic wave function corresponding to a projectile of energy E which is defined by the equation

$$[H_p(\boldsymbol{r}) + U(r) + H_{int}(\boldsymbol{r}, \alpha)]\, |k(\boldsymbol{r}, \alpha)\rangle = E\, |k(\boldsymbol{r}, \alpha)\rangle , \tag{5}$$

and has to be solved for fixed target variables α. The adiabatic approximation neglects the correction term $\Delta(\boldsymbol{r}, \alpha)$ [4]) and the energy loss of the projectile due to the excitation. Thus all nuclear states are considered to be degenerate.

We expand the wave function $| k >$ according to spherical harmonics

$$|k\rangle = \sqrt{4\pi} \sum_{l,\mu} i^l e^{i\sigma_l} \sqrt{2l+1}\; Y_{l\mu}(\vartheta, \varphi) \frac{1}{k r} F_l(k r)$$

$$+ \sum_{l',\mu'} a_{l'} Y_{l'\mu'}(\vartheta, \varphi) \frac{1}{k r} G_{l'}(k r) . \tag{6}$$

The first term describes the pure elastic scattering and thus satisfies the radial Schrödinger equation outside the interaction region

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{2m}{\hbar^2} V_c(r) - \frac{l(l+1)}{r^2} - \frac{2m}{\hbar^2} U(r)\right] F_l(k r) = 0 . \tag{7}$$

The reduced mass of the colliding pair is denoted by m. The radial function $F_l(k r)$ vanishes at $r = 0$ and behaves asymptotically for large values of r as

$$F_l(k r) = \frac{i}{2} [H_l^*(k r) - \bar{\eta}_l H_l(k r)] e^{i\sigma_l} , \tag{8}$$

where $H_l = g_l + i f_l$ is the outgoing-wave Coulomb function [5]) in terms of the regular and irregular radial Coulomb functions, which behave as

$$g_l \to \cos\alpha_l$$

$$f_l \to \sin\alpha_l$$

where

$$\alpha_l = k r - \frac{1}{2} l\pi - \eta \ln 2 k r + \sigma_l$$

$$\sigma_l = \arg \Gamma(l + 1 + i\eta) .$$

The quantity $\bar{\eta}_l$ is the diagonal S-matrix element which for a complex optical potential U can be given in terms of the phase shift as $\bar{\eta}_l = \exp(2 i \delta_l)$.

The second term of equation (6) gives rise to a set of coupled differential equations which can be written in the form

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{2m}{\hbar^2} V_c(r) - \frac{l(l+1)}{r^2} - \frac{2m}{\hbar^2} U(r) - \varphi_{l\mu;l\mu}(r, \alpha)\right] G_l(k r)$$

$$= \sum_{l',\mu'} (1 - \delta_{ll'}\delta_{\mu\mu'}) \varphi_{l\mu;l'\mu'}(r, \alpha) G_{l'}(k r) , \tag{9}$$

where the matrix elements $\varphi_{l\mu;l'\mu'}$ of the interaction potential are defined by

$$\varphi_{l\mu;l'\mu'}(r, \alpha) = \frac{2m}{\hbar^2} < Y_{l\mu} | H_{int}(\boldsymbol{r}, \alpha) | Y_{l'\mu'} > . \tag{10}$$

This interaction potential may conveniently be written in the form of a multipole expansion

$$H_{int}(\boldsymbol{r}, \alpha) = \sum_{\lambda,\mu} M_{\lambda\mu}(r, \alpha) Y^*_{\lambda\mu}(\vartheta, \varphi) , \tag{11}$$

where (ϑ, φ) are the polar angles of the vector $\boldsymbol{r}$. Since H_{int} is a scalar quantity, the moments $M_{\lambda\mu}$ are transformed under rotation in the same way as the spherical harmonics $Y_{\lambda\mu}$ and have therefore the parity $(-1)^{\lambda}$.

For a specific multipole order λ the integration over the angular variables in equation (10) yields for the matrix element φ

$$\varphi_{l\mu;l'\mu'} = \frac{2m}{\hbar^2}\left[\frac{(2l+1)\,(2\lambda+1)\,(2l'+1)}{4\pi}\right]^{1/2}(-1)^{\mu'} \times \begin{pmatrix} l & \lambda & l' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & \lambda & l' \\ -\mu & \mu-\mu' & \mu' \end{pmatrix} M_{\lambda\mu-\mu'}\,. \tag{12}$$

For the differential equation

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{2m}{\hbar^2}V_c(r) - \frac{l(l+1)}{r^2} - \frac{2m}{\hbar^2}U(r) - \varphi_{l\mu;l\mu}(r,\alpha)\right] u_l(kr) = 0\,, \tag{13}$$

where we have included the diagonal part of the interaction potential, the following solution in the WKB-approximation can be found outside the classical turning point r_l

$$u_l(kr) = \left(\frac{k}{K(r)}\right)^{1/2}\sin\left(\int_{r_l}^{r} K(r)\,dr + \frac{\pi}{4}\right) \quad \text{if } E - V > 0$$

$$u_l(kr) = \frac{1}{2}\left(\frac{k}{|K(r)|}\right)^{1/2}\exp\left(-\left|\int_{r_l}^{r} K(r)\,dr\right|\right) \text{ if } E - V < 0\,,$$

where the expressions $K(r)$ and V are defined by

$$K(r) = \left[k^2 - \frac{2m}{\hbar^2}V_c(r) - \frac{(l+1/2)^2}{r^2} - \frac{2m}{\hbar^2}U(r) - \varphi_{l\mu;l\mu}(r,\alpha)\right]^{1/2}$$

$$V \equiv V_c(r) + U(r) + \frac{\hbar^2}{2m}\left(\varphi_{l\mu;l\mu} + \frac{l(l+1)}{r^2}\right).$$

The asymptotic behaviour of the function $u_l(kr)$ for large values of r is given by

$$u_l(kr) \to \sin\left(kr - \frac{1}{2}l\pi - \eta\ln 2kr + \sigma_l + \eta_l\right),$$

with the following expression for the phase η_l

$$\eta_l = \int_{r_l}^{r} K(r)\,dr + \frac{\pi}{4} - kr + \frac{1}{2}l\pi + \eta\ln 2kr - \sigma_l\,.$$

The system of coupled differential equations may be solved by the ansatz

$$G_l(kr) = u_l(kr)\exp\left(-i\,\xi\,(kr)\right). \tag{14}$$

Substituting this expression into equation (9) we obtain

$$u_l^2\frac{d\xi}{dr} + \int u_l^2\left(\frac{d\xi}{dr}\right)^2 dr = \sum_{l',\mu'}(1 - \delta_{ll'}\,\delta_{\mu\mu'})\int u_l\,\varphi_{l\mu;l'\mu'}\,u_{l'}\,dr\,. \tag{15}$$

Using the WKB-approximation the second term on the left side of equation (15) has to be neglected. A first-order differential equation results for the function ξ. Since $d\xi/dr$ must be bounded at $r = 0$ and $u_l(r)$ behaves like r^{l+1} for small values of r, the lower limit of integration must be zero. Thus we see that

$$\frac{d\xi}{dr} = [u_l(k\,r)]^{-2} \sum_{l',\mu'} (1 - \delta_{ll'}\,\delta_{\mu\mu'}) \int_0^r u_l\,\varphi_{l\mu;l'\mu'}\,u_{l'}\,dr .$$

Since the integral on the right converges the asymptotic limit becomes

$$\frac{d\xi}{dr} = [\sin\gamma_l]^{-2} \sum_{l',\mu'} (1 - \delta_{ll'}\,\delta_{\mu\mu'}) \int_0^\infty u_l\,\varphi_{l\mu;l'\mu'}\,u_{l'}\,dr , \tag{16}$$

where the phase γ_l is defined by

$$\gamma_l = k\,r - \frac{1}{2}\,l\,\pi - \eta \ln 2\,k\,r + \sigma_l + \eta_l .$$

Carrying out the integration in equation (16) we obtain

$$\xi(k\,r) = -\frac{1}{k}\,[\operatorname{ctg}\gamma_l - 1] \sum_{l',\mu'} (1 - \delta_{ll'}\,\delta_{\mu\mu'})\,\sigma_{l\mu;l'\mu'} , \tag{17}$$

where

$$\sigma_{l\mu;l'\mu'} = \frac{1}{k} \int_0^\infty u_l\,\varphi_{l\mu;l'\mu'}\,u_{l'}\,d(k\,r) .$$

If we choose the direction of the incident beam as the polar axis of the space-fixed coordinate system equation (6) can be expressed in the asymptotic limit as follows

$$\begin{aligned} |k\rangle = \sum_l \Big[& i^l\,e^{i(\sigma_l+\delta_l)}\,(2\,l+1)\,P_l(\cos\vartheta)\,\frac{1}{k\,r}\sin\Big(k\,r - \frac{1}{2}\,l\,\pi + \sigma_l + \delta_l\Big) \\ & + a_l\,Y_{l0}(\vartheta)\exp\Big[-\frac{i}{k}\sum_{l',\mu'}(1 - \delta_{ll'}\,\delta_{0\mu'})\,\sigma_{l0;l'\mu'}\Big] \\ & \times \frac{1}{k\,r}\exp\Big(i\Big\{k\,r - \frac{1}{2}\,l\,\pi + \sigma_l + \eta_l\Big\}\Big)\Big] . \end{aligned} \tag{18}$$

In the second term of equation (18) a linear combination of solutions $G_l(k\,r)$ may be chosen so that the complete wave function $|\,k\,\rangle$ represents asymptotically an elastic ingoing and scattered wave and in addition an extra outgoing inelastic scattered wave. In the arguments of the asymptotic expressions of the wave functions the term $\eta \ln 2\,k\,r$ and in the second expression the term $\operatorname{ctg}\gamma_l \sum_{l',\mu'} (1 - \delta_{ll'}\,\delta_{\mu'0})\,\sigma_{l0;l'\mu'}$ have been dropped. The total adiabatic scattering amplitude is defined by

$$f(\vartheta;\Lambda) = \lim_{r\to\infty} r\,e^{-ikr}\,[|k\rangle - e^{ikz}] , \tag{19}$$

where $\Lambda \equiv (\alpha\,\beta\,\gamma)$ denote the Eulerian angles which rotate the space-fixed coordinate system into the body-fixed system. If we choose for a_l the expression

$$a_l = \frac{1}{2\,i}\sqrt{4\pi}\,\sqrt{2\,l+1}\;i^l\,e^{i\sigma_l}$$

and make use of the asymptotic expansion of a plane wave given by

$$e^{ikz} = \sum_l i^l (2l+1)\, P_l(\cos\vartheta)\, \frac{1}{kr} \sin\left(kr - \frac{1}{2} l\pi\right), \tag{20}$$

the evaluation of the scattering amplitude yields

$$f(\vartheta; \Lambda) = \frac{1}{2ik} \sum_l (2l+1)\, P_l(\cos\vartheta) \left\{ \exp\left[2i(\sigma_l + \delta_l)\right] + \exp\left[2i\sigma_l + i\eta_l - \frac{i}{k} \sum_{l',\mu'} (1 - \delta_{ll'}\,\delta_{\mu' 0})\, \sigma_{l0;l'\mu'}\right] - 1 \right\}. \tag{21}$$

According to equation (21) the amplitude for inelastic scattering from the nuclear ground state $| I_0 M_0 \rangle$ to the excited final state $| I_f M_f \rangle$ is thus in the adiabatic approximation

$$f_{I_f M_f; I_0 M_0}(\vartheta) = \langle I_f M_f \Big| \frac{1}{2ik} \sum_l (2l+1)\, P_l(\cos\vartheta)\, e^{2i\sigma_l} \times \exp\left[-\frac{i}{k} \sum_{l',\mu'} \sigma_{l0;l'\mu'}\right] \Big| I_0 M_0 \rangle . \tag{22}$$

Here we have made use of the approximate relation

$$\eta_l \sim -\frac{1}{k}\, \sigma_{l0;l0} ,$$

which is valid in the asymptotic limit. The exact asymptotic value of η_l is overestimated by a term of the order of r_l^{-5}.

The differential cross section can be expressed in terms of the scattering amplitude $f(\vartheta)$ by

$$\frac{d\sigma}{d\Omega} = \frac{1}{2I_0+1} \sum_{M_0, M_f} |f_{I_f M_f; I_0 M_0}(\vartheta)|^2 , \tag{23}$$

where we sum over the final magnetic quantum numbers M_f and average over all initial M_0.

III. Collective Model

In collective excitations of low-lying nuclear states two different modes of excitation can be distinguished. The first are spherical shape oscillations, while the second are rotations in the shape of a permanently deformed nucleus. To interprete successfully single[2)6)] and multiple[7)] excitations of these collective states the optical model has to be extended as to include nonspherical potentials[8)].

We have also introduced an absorptive potential in order to take care of all reaction processes other than the ones treated explicitly by means of coupled channels.

A. Rotations

The interaction Hamiltonian can be derived from a deformed potential well under the assumption that the potential strength depends only on the distance $(r - R)$ from the assumed nonspherical surface,

$$R(\vartheta', \varphi') = R_0 \left[1 + \sum_{\lambda,\nu} \alpha_{\lambda\nu}\, Y_{\lambda\nu}(\vartheta', \varphi')\right] , \tag{24}$$

where the polar angles (ϑ', φ') are measured with respect to a body-fixed coordinate system defined by the principal axes of the nucleus. The deformation parameter β and the asymmetry parameter γ are connected with the quadrupole deformation[9]) by

$$\alpha_{20} = \beta \cos\gamma, \ \alpha_{2\pm 1} = 0, \ \alpha_{2\pm 2} = \beta \frac{\sin\gamma}{\sqrt{2}} .$$

If the parameter $\gamma = 0$ the nucleus shows an axial symmetry.

The deformed potential well may be described by a Saxon-Wood potential as follows

$$U(r - R(\vartheta', \varphi')) = - U_0(e^{(r - R(\delta', \varphi'))/a} + 1)^{-1} . \tag{25}$$

An expansion of equation (25) around the mean value of the radius R_0 gives

$$U(r - R) = U(r - R_0) - \frac{d}{dr} U(r - R_0) \, \delta R + \cdots , \tag{26}$$

where

$$\delta R = R_0 [\sum_{\lambda, \nu} \alpha_{\lambda\nu} Y_{\lambda\nu}(\vartheta', \varphi')] . \tag{27}$$

The first term on the right side of equation (26) can be identified with the spherical optical potential while the nonspherical terms yield the interaction (11). In the following we compute explicitly the transition amplitude to first order in the deformation. The contributions from the second-order interaction potential in the transition amplitude to second order in the deformation are not negligible[10]) as will be discussed in section V.

The nuclear shape expressed by equation (27) is measured in a body-fixed coordinate system. The transformation into a space-fixed system is given by

$$Y_{\lambda\nu}(\vartheta', \varphi') = \sum_{\mu} Y_{\lambda\mu}(\vartheta, \varphi) \, D^{\lambda *}_{\mu\nu}(\alpha \, \beta \, \gamma) , \tag{28}$$

where $(\alpha \, \beta \, \gamma)$ are the Eulerian angles from the body-fixed to the space-fixed axes[11]).

The comparison of the first-order interaction term in equations (26) and (11) yields

$$M_{\lambda\mu} = - R_0 \frac{d}{dr} U(r - R_0) \sum_{\nu} \alpha^*_{\lambda\nu} D^{\lambda}_{\mu\nu}(\alpha \, \beta \, \gamma) . \tag{29}$$

For an axially-symmetric deformation the multipole operator simplifies to

$$M_{\lambda\mu} = - R_0 \, \beta \sqrt{\frac{4\pi}{2\lambda + 1}} \frac{d}{dr} U(r - R_0) \, Y_{\lambda\mu}(\beta, \alpha) , \tag{30}$$

where (β, α) are the polar angles of the nuclear symmetry axis. The derivative of the spherical optical potential $U(r - R_0)$ is for the special case of a Saxon-Wood potential (see equation (25)) given by

$$\frac{d}{dr} U(r - R_0) = \frac{U_0}{a} \frac{e^y}{(e^y + 1)^2} , \qquad y = \frac{r - R}{a} . \tag{31}$$

The matrix elements $\varphi_{l0; l'\mu'}$ defined by equation (12) have to be computed explicity and may be written for a quadrupole deformation as

$$\varphi_{l0; l'\mu'} = - \frac{2m}{\hbar^2} \beta \, R_0 \left(\frac{dU}{dr}\right) [(2l + 1) \, (2l' + 1)]^{1/2} \times (-1)^{\mu'} \, Y_{2\mu'}(\beta, \alpha) \begin{pmatrix} l & 2 & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & 2 & l' \\ 0 & \mu' & -\mu' \end{pmatrix} . \tag{32}$$

The nuclear wave functions of a rotational band are given in the notation of EDMONDS[11]) by

$$|I\,M\,K> = \sqrt{\frac{2\,I+1}{4\,\pi}}\;D^{I}_{M\,K}(\alpha\,\beta\,0)\,.$$

Expanding the product of the two D-functions again into D-functions the transition amplitude in equation (22) then can be written in the following form:

$$f_{I_f M_f; I_0 M_0}(\vartheta) = \frac{1}{2\,i\,k}\sum_I \sqrt{2\,I_f+1}\;\sqrt{2\,I_0+1}\;(2\,I+1)$$
$$\times\,(-1)^{M_0-K}\begin{pmatrix} I_f & I_0 & I \\ -M_f & M_0 & M_f-M_0\end{pmatrix}\begin{pmatrix} I_f & I_0 & I \\ -K & K & 0\end{pmatrix} a_{I\,M_f-M_0}, \tag{33}$$

where

$$a_{I\,M} = [4\,\pi\,(2\,I+1)]^{-1/2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta\,\sin\beta\;Y_{I\,M}(\beta,\alpha)\sum_l (2\,l+1)\;P_l\,(\cos\vartheta)\;e^{2i\sigma_l}$$
$$\times\exp\left[i\,\frac{R_0\,\beta}{E\,a}\,(U_0+i\,W_0)\sqrt{2\,l+1}\sum_{l'}\sqrt{2\,l'+1}\begin{pmatrix} l & 2 & l' \\ 0 & 0 & 0\end{pmatrix}\Gamma_{l\,l'}\right.$$
$$\left.\times\sum_{\mu}(-1)^{\mu}\,Y_{2\,\mu}(\beta,\alpha)\begin{pmatrix} l & 2 & l' \\ 0 & -\mu & \mu\end{pmatrix}\right]. \tag{34}$$

An imaginary part W_0 with the same shape as the real part U_0 is included in the optical potential. The radial matrix elements $\Gamma_{l\,l'}$ are defined by

$$\Gamma_{l\,l'} = \int_0^{\infty} u_l(k\,r)\;F(r)\;u_{l'}\,(k\,r)\;d(k\,r)\,, \tag{35}$$

where the function $F(r)$ is given by

$$F(r) = e^{y}\,(e^{y}+1)^{-2}\,.$$

Finally the differential cross section (23) can be expressed by the amplitudes $a_{I\,M}$ through

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\,k^2}\,(2\,I_f+1)\sum_{I,M}(2\,I+1)\begin{pmatrix} I_f & I_0 & I \\ -K & K & 0\end{pmatrix}^2 |a_{I\,M}|^2\,. \tag{36}$$

B. Vibrations

Vibrations of spherical nuclei can be described by the same nonspherical interaction potential as for rotations. But we have now to treat the deformation parameters $\alpha_{\lambda\nu}$ in equation (24) as dynamical variables. In the usual way we can decompose $\alpha_{\lambda\,\mu}$ into operators $b_{\lambda\,\mu}$ and $b^{*}_{\lambda\,\mu}$ which annihilate and create a single 2^{λ}-pole phonon of vibration and satisfy the relations

$$b_{\lambda\,\mu}\,b^{*}_{\lambda\,\mu} = n_\mu + 1\,, \quad b^{*}_{\lambda\,\mu}\,b_{\lambda\,\mu} = n_\mu\,, \quad \sum n_\mu = N\,, \tag{37}$$

where N is the total number of phonons in a given state. Therefore the parameter α can be expressed by

$$\alpha_{\lambda\,\mu} = \sqrt{\frac{\hbar\,\omega\lambda}{2\,C_\lambda}}\;[b_{\lambda\,\mu} + (-1)^{\mu}\,b^{*}_{\lambda-\mu}] = (-1)^{\mu}\,\alpha^{*}_{\lambda-\mu}\,, \tag{38}$$

where C_λ is the restoring force and $\hbar\,\omega_\lambda$ the energy of one phonon.

As we consider only quadrupole deformations, the quadrupole phonon states can be classified according to the five vibrational quantum numbers n_μ

$$|n_\mu\rangle \equiv |n_{+2}\, n_{+1}\, n_0\, n_{-1}\, n_{-2}\rangle = \prod_\mu \frac{1}{\sqrt{n_\mu!}} (b^*_{2\mu})^{n_\mu} |0\rangle . \tag{39}$$

In analogy to the permanent deformation parameter β for rotational excitations we introduce here a dynamical deformation parameter defined as the root mean square deformation in the nuclear ground state

$$\beta_\lambda^2 = \langle 0| \sum_\mu |\alpha_{\lambda\mu}|^2 |0\rangle = (2\lambda + 1)\, \frac{\hbar\,\omega_\lambda}{2\, C_\lambda} . \tag{40}$$

Since the deformation is no longer static, the formulas (24), (27), and (29) can be regarded as being referred to space-fixed axes. The multipole operator $M_{\lambda\mu}$ can then be expressed to first order in the interaction by

$$M_{\lambda\mu} = -\, R_0 \frac{d}{dr} U(r - R_0)\, \alpha^*_{\lambda\mu} , \tag{41}$$

and thus the matrix elements $\varphi_{l0;\,l'\mu'}$, defined by equation (12), yield

$$\varphi_{l0;\,l'\mu'} = -\, \frac{2\,m}{\hbar^2} \frac{R_0\,\beta}{\sqrt{4\,\pi}} \left(\frac{dU}{dr}\right) [(2\,l + 1)\,(2\,l' + 1)]^{1/2}\, (-\,1)^\mu \times \begin{pmatrix} l & 2 & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & 2 & l' \\ 0 & -\mu' & \mu' \end{pmatrix} [b^*_{2\mu'} + (-\,1)^{\mu'}\, b_{2-\mu'}] . \tag{42}$$

The transition amplitude for scattering from the ground state $|\,0\,\rangle$ to the state $|\,n_\mu\,\rangle$ may be written in the form

$$\langle n_\mu|\, f(\vartheta)\, |0\rangle = \langle n_\mu|\, \frac{1}{2\,i\,k} \sum_l (2\,l + 1)\, P_l\,(\cos\vartheta)\, e^{2\,i\,\sigma_l} \times \exp\Big[\sum_\mu A_\mu\, (b^*_{2\mu} + (-\,1)^\mu\, b_{2-\mu})\Big]|\, 0\rangle , \tag{43}$$

where

$$A_\mu = \frac{i}{\sqrt{4\,\pi}} \frac{R_0\,\beta}{E\,a}\, (U_0 + i\, W_0)\, \sqrt{2\,l + 1} \sum_{l'} \sqrt{2\,l' + 1}\; (-\,1)^\mu \times \begin{pmatrix} l & 2 & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & 2 & l' \\ 0 & -\mu & \mu \end{pmatrix} \Gamma_{ll'} . \tag{44}$$

Again we have included an imaginary part in the optical potential and the radial matrix elements $\Gamma_{l\,l'}$ are the same as defined in equation (35).

The operator expression in equation (43) can be evaluated by the following method. The product of the two operators exp ($\varepsilon\, a$) and exp ($\varepsilon\, b$) may be expanded

$$e^{\varepsilon a}\, e^{\varepsilon b} = \exp\Big(\sum_\nu \varepsilon^\nu\, G_\nu\Big) , \tag{45}$$

where the operators a, b satisfy the commutation relation

$$[a, b] = 1 , \tag{46}$$

and where ε is an arbitrary c-number.

Expanding all exponential functions we obtain

$$1 + \varepsilon(a+b) + \varepsilon^2\left(\frac{a^2}{2} + a\,b + \frac{b^2}{2}\right) + \varepsilon^3\left(\frac{a^3}{6} + \frac{a^2\,b}{2} + \frac{a\,b^2}{2} + \frac{b^3}{6}\right) + \ldots$$

$$= 1 + \sum_\nu \varepsilon^\nu\, G_\nu + \frac{1}{2!}\,(\sum_\nu \varepsilon^\nu\, G_\nu)^2 + \frac{1}{3!}\,(\sum_\nu \varepsilon^\nu\, G_\nu)^3 + \ldots$$

A comparison of both sides to the same power of ε^ν yields the following relations for G_ν if $G_0 = 1$:

$$G_1 = a + b, \quad G_2 = \frac{1}{2}\,[a, b], \quad G_3 = \frac{1}{12}\,[(a-b), [a, b]]\,, \text{ etc.} \tag{47}$$

The commutation relation (46) enforces G_3 and all higher terms to vanish. The operator expression (45) therefore takes the form

$$e^{\varepsilon(a+b)} = e^{\varepsilon a}\, e^{\varepsilon b}\, e^{-1/2\,\varepsilon^2}. \tag{48}$$

Using this relation the amplitude in equation (43) simplifies to

$$\langle n_\mu\,|f(\vartheta)|\,0\rangle = \frac{1}{2\,i\,k}\sum_l (2\,l+1)\,P_l\,(\cos\vartheta)\,e^{2i\sigma_l}$$

$$\times\,\exp\left[-\frac{1}{2}\sum_\mu (A_\mu)^2\right]\prod_\mu \frac{(A_\mu)^{n_\mu}}{\sqrt{n_\mu\,!}}\,. \tag{49}$$

The considered nuclear vibrational states are degenerate. Instead of labelling these states by the quantum numbers n_μ, the total vibrational number N, the angular quantum numbers I and M are conveniently used in the following. Therefore a transformation to this new representation must be made

$$\langle N\,I\,M\,|f(\vartheta)|\,0\rangle = \sum_{n_\mu} \langle N\,I\,M\,|\,n_\mu\rangle\,\langle n_\mu\,|f(\vartheta)|\,0\rangle\,.$$

The differential inelastic cross section with the excitation of a vibrational state with spin I and phonon number N is now given by

$$\frac{d\sigma}{d\Omega} = \sum_M |\langle N\,I\,M|\,f(\vartheta)|0\,\rangle|^2\,.$$

IV. Coulomb Excitation

The inelastic scattering of charged particles is always accompanied by Coulomb excitation which may even be the main mode of excitation[12]). In this chapter we want to show how this Coulomb excitation may be included in the calculations. This can be done by modifying the interaction form factors. In the case where both nuclear and Coulomb forces contribute to the excitation, their amplitudes are coherent and interfere.

The Coulomb interaction between target nucleus and incident projectile is given in multipole components as

$$H_c = 4\,\pi\,Z_1\,e\sum_{\lambda,\mu}\frac{1}{2\,\lambda+1}\,r^{-\lambda-1}\,Y_{\lambda\mu}(\vartheta,\varphi)\,\mathfrak{M}^*(E\,\lambda,\mu)\,, \tag{50}$$

which is valid if the projectile remains outside the nucleus during the collision. The multipole operator for electric radiative transitions $\mathfrak{M}(E\,\lambda,\,\mu)$ is defined by

$$\mathfrak{M}(E\,\lambda,\,\mu) = \int r^{\lambda}\, Y_{\lambda\,\mu}(\vartheta,\,\varphi)\, \varrho_n(\boldsymbol{r})\, d\tau\,.$$

The nuclear charge density operator $\varrho_n(\boldsymbol{r})$ can also be expanded in multipole components

$$\varrho_n(\boldsymbol{r}) = Z_2\, e \sum_{\lambda,\,\mu} \varrho_{\lambda\,\mu}(r)\; Y^{*}_{\lambda\,\mu}(\vartheta,\,\varphi)\,.$$

A comparison of equation (50) with the expansion (11) yields

$$M^{c}_{\lambda\,\mu} = \frac{4\,\pi\, Z_1\, e}{2\,\lambda+1}\, r^{-\lambda-1}\, \mathfrak{M}(E\,\lambda,\,\mu)\,,$$

which holds outside the nucleus.

In the special case of a uniformly charged nucleus, an explicit expression for $M^{c}_{\lambda\,\mu}$ can be obtained from equation (26). In first order in the deformation we get

$$M^{c}_{\lambda\,\mu} = \frac{3\,Z_1\,Z_2\,e^2}{2\,\lambda+1}\,\alpha^{*}_{\lambda\,\mu} \times \begin{cases} R_c^{\lambda}/r^{\lambda+1} & \text{if } r \geqslant R_c \\ r^{\lambda}/R_c^{\lambda+1} & \text{if } r < R_c\,, \end{cases} \tag{51}$$

where R_c denotes the radius of the charge distribution.

For a permanently deformed nucleus the spherical tensor $\alpha_{\lambda\,\mu}$ can be expressed through the deformation parameter β_λ and the orientation of the nuclear symmetry axis. We obtain thus for $M^{c}_{\lambda\,\mu}$

$$M^{c}_{\lambda\,\mu} = \frac{3\,Z_1\,Z_2\,e^2}{2\,\lambda+1}\,\beta_\lambda \sqrt{\frac{4\,\pi}{2\,\lambda+1}}\; Y_{\lambda\,\mu}(\beta,\,\alpha) \times \begin{cases} R_c^{\lambda}/r^{\lambda+1} & \text{if } r \geqslant R_c \\ r^{\lambda}/R_c^{\lambda+1} & \text{if } r < R_c\,. \end{cases} \tag{52}$$

Coulomb excitation effects can thus easily be included by modifying the radial form factors. The radial matrix elements $\Gamma_{l\,l'}$ in equation (35) has to be replaced by the following expression:

$$\Gamma_{l\,l'} = \int_0^{\infty} u_l(k\,r)\left[F(r) - \frac{3\,Z_1\,Z_2\,e^2\,a\,R_c^2}{5\,R_0\,U_0}\,\frac{1}{r^3}\right] u_{l'}(k\,r)\; d(k\,r), \quad r \geqslant R_c\,.$$

V. Discussion of Results

According to equations (49) and (44) and for a quadrupole interaction computations of different angular distributions have been performed for the inelastic alpha-particle scattering of 64.3 MeV and 43 MeV for Ni^{58}. Calculations have been carried out for the multiple excitation of the first 2^+ state ($Q_{2^+} = -\,1.45$ MeV) and the 0^+, 2^+, 4^+ triplet ($Q_{4^+} = -\,2.47$ MeV) of the vibrational nucleus.

The main purpose of this paper is a study of the coupled channels predictions in the WKB-approximation which might be a very attractive method for optical model calculations[14]). The potential has a broad, flat bottom with a gradually tapering surface. The small radial derivatives of the potential allow for high energies a WKB-treatment. We have made use of sets of optical parameters obtained from previous

analyses of experimental data by P. DARRIULAT et al.[13]) for Ni^{58} (64.3 MeV) and by R. H. BASSEL et al.[8]) for Ni^{58} (43 MeV).

The optical potentials have been chosen in such a way that the elastic and the inelastic 2^+ ($N = 1$) differential cross sections for Ni^{58} (α, α') are in agreement with experiment. All parameters of the model are then fixed and total and differential cross sections of the triplet can be predicted. We did not make extensive studies of the effects of varying the parameters since this would have required a large amount of computer time.

In all our calculations we have chosen the same shape for the absorptive part of the optical potential as for the real part. The deformation of the real part alone is responsible for the excitation of collective states.

In the numerical examples of this section we used the following parameters for the Saxon-Wood potential for Ni^{58} for the two different projectile energies E (Table I).

Table I

U_0 (MeV)	47.6	44.99
W_0 (MeV)	13.8	20.91
a (f)	0.549	0.565
R_0 (f)	6.14	6.08
E (MeV)	43	64.3
β	0.18	0.15

The inelastic angular distributions for 64.3 MeV alpha-particles on Ni^{58} are shown in figure 1. The experimental values had been obtained at BERKELEY[13]). They show well-defined oscillations which are typical for surface reactions. This follows from a discussion of the behaviour of the integrands in the radial matrix elements $\Gamma_{ll'}$[15]). This expression (see equation (35)) has its maximum for values of l and l' which correspond approximately to the classical orbital momentum. Moreover very high partial waves do not contribute due to the centrifugal repulsion.

The angular distributions in figures 2 and 3 have the same parameter values as given in table I; here the process of Coulomb excitation of the vibrational states by the incident charged projectiles is explicitly included. This process interferes destructively, especially at forward scattering angles, with the pure nuclear excitation mechanism. As has been shown in section IV the computation of the contribution due to Coulomb excitation is straightforward within the framework of the present treatment. We assumed that the deformation of the charge distribution and of the potential is the same. In the Coulomb excitation process a much larger number of partial waves contribute to the excitation amplitude than in the purely nuclear interaction. As mentioned above the Coulomb excitation contributes considerably only for small scattering angles, otherwise the nuclear contributions are the most important ones. This is illustrated in figure 4.

Even though the multiple excitation of collective states could be solved exactly without making use of the adiabatic approximation, we assumed that the ingoing and outgoing momenta are equal. In order to estimate the effect on the diffraction structure due to the energy loss an average of the initial and final wave numbers

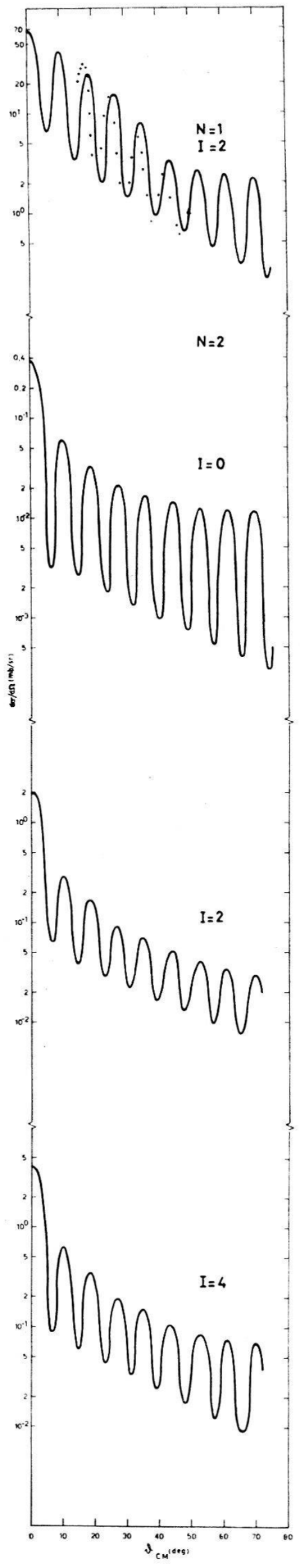

Figure 1

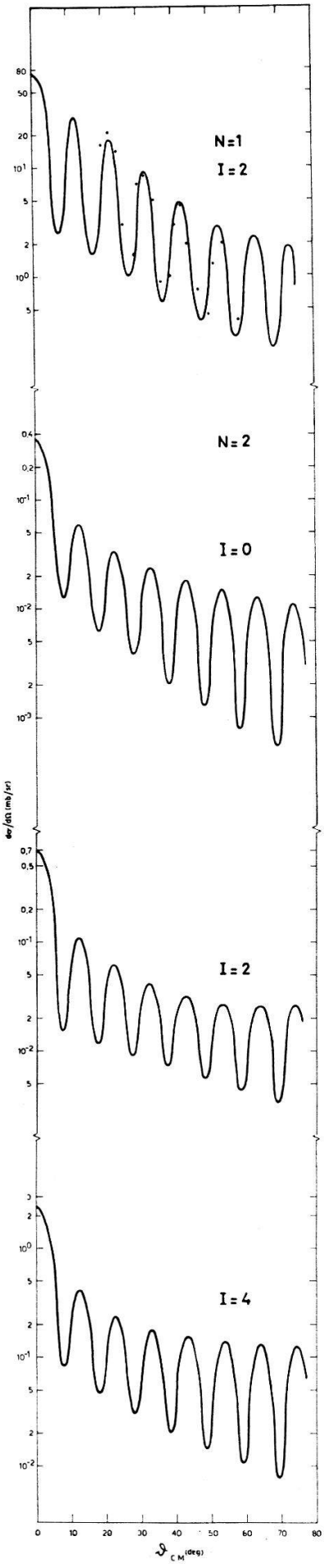

Figure 2

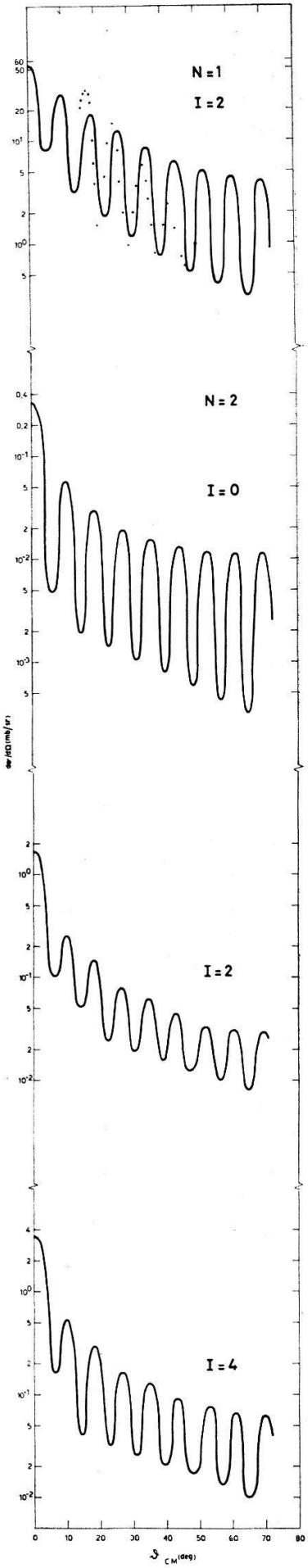

Figure 3

should be substituted for k. This effect is shown in the angular distribution for the first 2^+ level of Ni^{58} in figure 5. The most significant difference is a slight shift of the diffraction pattern to larger angles; however, the general shape of the pattern is not appreciably altered. Since in nonadiabatic transitions the overlap between initial and final wavefunctions is less perfect, a reduction in the intensity is observed. Furthermore the maxima and minima in the diffraction structure become less accentuated.

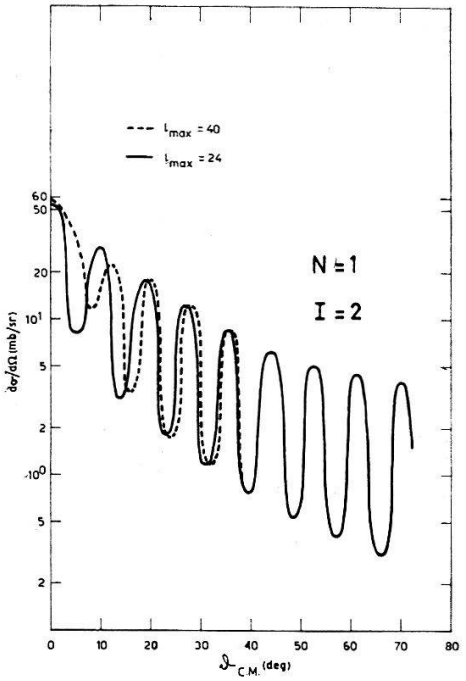

Figure 4

Effects of contribution of a larger number of partial waves to the angular distribution for 64.3 MeV alpha-particles on Ni^{58}, where Coulomb excitation is included. The dashed curve is calculated with $l_{max} = 40$. The solid curve is the same as in figure 3 with $l_{max} = 24$.

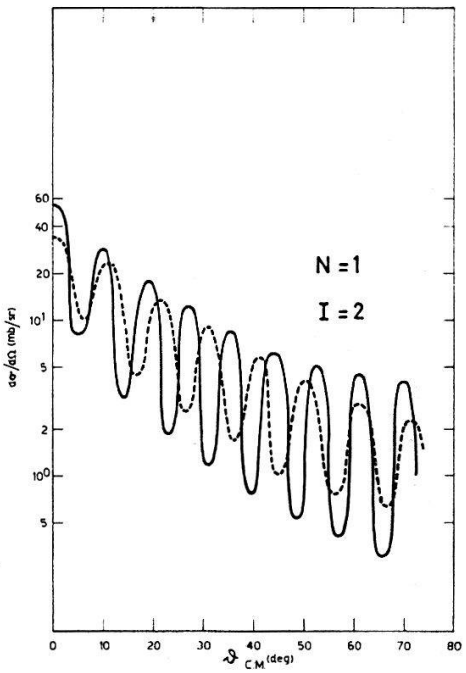

Figure 5

The dashed curve shows the effect on the diffraction structure for 64.3 MeV alpha-particles on Ni^{58} due to the energy loss during the scattering. Here we used for k the average of the initial and final wave numbers, $k = (1/2)(k_i + k_f)$. The solid curve is the same as in figure 3 with $k_i = k_f$.

Figure 1

Angular distributions for the scattering of 64.3 MeV alpha-particles from Ni^{58}. The optical-potential and nuclear deformation parameters used for the calculation are presented in table I. The experimental data are taken from reference 13. The reaction cross section for the first 2^+ state is 1.86 b.

Figure 2

Angular distributions for the scattering of 43 MeV alpha-particles from Ni^{58}, including the influence of Coulomb excitation. The optical-potential and nuclear deformation parameters used for the calculation are presented in table I. The experimental data are taken from reference 16. The reaction cross section for the first 2^+ state is 1.53 b.

Figure 3

Angular distributions for the scattering of 64.3 MeV alpha-particles from Ni^{58}, including the influence of Coulomb excitation. The same optical-potential and nuclear deformation parameters were used as for the solid curves in figure 1.

As can be seen in figure 2 the magnitude of the differential cross section of the 0^+ ($N = 2$) state is appreciably lower than that of the first 2^+ state (about a factor 200). It is also smaller than that of the triplet 2^+ and 4^+ states (about a factor 2 and 7, respectively). Comparing the cross sections of the projectile energies which have been calculated for 43 MeV (figure 2) and 64.3 MeV (figure 3) we see that in the latter case the differential cross section for the first 2^+ state is somewhat reduced (about a factor 1.4). It is approximately constant for the 0^+ state, whereas it increases for the triplet 2^+ and 4^+ states (about a factor 2.4 and 1.4, respectively). The predicted values for the reaction cross sections in case of inelastic alpha-particle scattering for the first 2^+ state are 1.53 barns for 43 MeV and 1.86 barns for 64.3 MeV. This indicates that the values of the inelastic cross section is growing with increasing projectile energy. It is pointed out in reference [8]) that they also rise with increasing charge number Z. A comparison of figures 1 and 3 shows that the inclusion of Coulomb excitation reduces somewhat the magnitude of all inelastic differential cross sections at forward scattering angles ($\sim$ 15%). For larger scattering angles the differential cross sections are pratically unchanged. This indicates that the multiple Coulomb excitation can be neglected for these bombarding energies. However, for projectile energies of about 20 MeV and for nuclei in the Ni mass region, the Coulomb excitation cross section may become comparable to the direct nuclear cross section. The same situation may be expected for bombarding energies of about 60 MeV in the mass region of the permanently deformed nuclei. The excited collective states will then be primarily of rotational character and the process is described by the equations (36) and (34).

As can be seen from the diagrams the computed inelastic differential cross sections fit the experimental datas reasonably well. There is a discrepancy for small scattering angles for the projectile energy of 64.3 MeV. This disagreement may be explained by a too small value of the deformation parameter β. It is interesting to compare the values of the nuclear deformations β derived from the inelastic scattering experiments with the results obtained by electromagnetic methods. The $B(E\,2)$ values extracted from Coulomb excitation and lifetime measurements are connected in first order with the deformation β by

$$\beta^2 = \frac{B(E\,2 : 0 \rightarrow 2)}{[(3/4\,\pi)\,Z\,e\,R_c^2]}\,, \tag{53}$$

where a uniform nuclear charge distribution with a sharp surface of an average radius $R_c = 1.2\,A^{1/3}\,f$ has been assumed. Taking into account second order effects we get slightly smaller values for β. We thus obtain for Ni^{58} a value of $\beta = 0.19$. This is in good agreement with the inelastic scattering result obtained for an energy of 43 MeV (see figure 2). It has been suggested that the deformabilities decrease with increasing projectile energies. Thus the discrepancy at 64.3 MeV can also be explained by the fact that during the interaction time the alpha-particle sees only a few oscillations of the nucleus (about 10 at 100 MeV), i.e. it does not see any stationary states at very high projectile energies for nuclei in this mass region.

It must be emphasized here that the magnitude of the cross section depends not only on the nuclear deformabilities β, but also on the details of the inelastic form factor $F(r)$ (see equation (35)).

The results of inelastic scattering processes determine the deformation parameters, provided the collective model is valid. These parameters agree with the results

obtained by other methods. The magnitude of the cross section is very sensitive to changes of the form factor parameters. This is again an argument in favour of the model. More detailed theoretical calculations and precise experimental data might show whether the form factor chosen in equation (31) is correct. Deviations from this expression are expected especially in the case of transitions between different single particle states in odd-A nuclei.

The results of figure 1 show that the angular distributions of the one and two phonon states are in phase, which is in accordance with the well-known phase rule[1]). This rule states that for even values of Δl the angular distribution is in phase, while for odd values of Δl it is out of phase. The conditions for the validity of this phase rule are fulfilled for (α, α') reactions at energies of about 40 MeV. It would be interesting to check this phase rule experimentally also for energies of some 60 MeV. Recent experiments[16]) have shown that the inelastic angular distributions with excitation of states with spin 4^+ in the nickel region disobey this phase rule. They were found to be almost exactly out of phase with the 2^+ distributions. This might be explained by second order effects in the deformation, thus leading to a direct two-phonon excitation of the 4^+ state[10]). Because the energy ratio for the 4^+ and 2^+ levels in Ni^{58} is different from 2, suggesting a breakdown of the vibrational model, an increasing contribution of the direct two-phonon excitation results. This can be explained by the fact that the first 4^+ state includes some admixture of the state containing a single 4^+ phonon. Therefore a comparison between experimental data and our theory should include this direct excitation mechanism, too (see also section VI).

VI. Summary and Conclusions

A coupled channel theory of direct interactions in the WKB-approximation is presented in this paper. The underlying basis of the used model is the adiabatic approximation. In the case of excitation of vibrational states it is possible to obtain exact results without the use of the adiabatic approximation. This is not the case for rotational states.

It is possible to extract reasonably good values for the nuclear deformabilities β from the inelastic scattering data. The agreement between the values of β obtained in this way with those derived from Coulomb excitation is satisfactory. This is an evidence for the validity of the collective model. However, at very high projectile energies further elaborate calculations are needed in order to be able to make definite statements. A direct solution of the set of coupled differential equations might be done on a fast computer. Such calculations have been carried out for the case of inelastic proton scattering[3]). Also numerical calculations with nonlocal operators in coordinate space might be of interest, since this takes into account the energy dependence of the optical potential at high projectile energies.

In the present treatment the inelastic scattering problem was solved only to first order in the interaction. In particular terms of order β^2 and higher were neglected. These terms are associated with the second derivative of the interaction potential which describes a direct two-phonon excitation of the 4^+ state. Including this term the actual angular distribution of the 4^+ state is then the combined result of both the multiple and direct excitation. Provided only pure quadrupole vibrations are used for the

description of higher collective states, the above mentioned admixture effect of the state containing a single 4^+ phonon must be included. This results in an apparently increased contribution of the direct two-phonon mechanism over the multiple mechanism.

Because of the sensitivity to interference between the various contributions, the inelastic scattering of alpha-particles, together with the theory developed here and the possible extensions, provide a very useful tool to study and interpret the higher collective states.

Acknowledgements

I wish to thank Professor K. ALDER for his help and his continued stimulating interest in the present work. The author is very grateful to Professor E. BAUMGARTNER for numerous discussions and suggestions. I am greatly indebted to Dr. W. SCHNEIDER for valuable comments and his help concerning the programming and executing the numerical calculations on the IBM 7044 in Stockholm. I wish also to thank Professor P. O. FRÖMAN for the hospitality I enjoyed at the Theoretical Institute in Uppsala. I should also like to thank for the financial support granted by the „Schweizerische Nationalfonds".

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