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# On the Theory of Multiple Coulomb Excitation

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*Abstract.* The theory of multiple Coulomb excitation is extended. This paper deals mainly with the investigation of deviations from the sudden approximation, the computation of the multiple excitation of an asymmetric rotator and an improved method for the treatment of coupled rotational bands. Furthermore formulae for the angular distribution of deexcitation gamma rays are derived. Numerical results of sample calculations are discussed.

## 1. Introduction

The theory of multiple Coulomb excitation was developed by K. Alder and A. Winther in the semiclassical approximation (Reference [1]) where the projectile is moving on a classical orbit in the Coulombfield of the target nucleus. Except for the excitation of vibrational states their calculations were made in the sudden approximation, thus neglecting the finite excitation energy of the nuclear levels. The excitation amplitudes have been evaluated so far for the surface vibrational model and the rotational model of axial symmetric nuclei. Transitions between different rotational bands have also been discussed (References [1], [2]).

One of the main objects here is to investigate the deviations from the sudden approximation. Besides pure rotational and vibrational states we further calculate the multiple excitation of an asymmetric rotator and present new methods for the treatment of coupled rotational bands. In addition we derive formulae for the angular distribution of the deexcitation gamma rays.

At the end of this work we give some characteristic illustrations for the theoretical predictions on the multiple Coulomb excitation. In order to facilitate the extraction of important information from experiments we will give a more complete tabulation of our numerical results separately (Reference [12]). All the calculations have been performed without using the $\chi_{eff}(\vartheta)$ or $\mu = 0$ approximation (References [1,2]).

## 2. General Methods

### A. *Definitions*

In the semiclassical approximation the interaction Hamiltonian $H_E(t)$, if expanded into multipoles*), can be written in the following form:

$$H_E(t) = \sum_{\lambda} \frac{4 Z_1 e}{v a^{\lambda}} \sqrt{\frac{\pi}{2\lambda+1}} \frac{(\lambda-1)!}{(2\lambda-1)!!} \sum_{\mu=-\lambda}^{\lambda} \overline{R}_{\lambda\mu}(t) M^*(E\lambda, \mu) \tag{2.1}$$

*) For the angular momentum algebra the notation of reference 3 is used.

where $M(E\,\lambda,\,\mu)$ denotes the electric multipole operator of the target nucleus. The parameter $a$ is half the distance of closest approach in a head-on collision and is given by

$$a = \frac{Z_1 Z_2 e^2}{m\,v^2} \tag{2.2}$$

where $Z_1$ and $Z_2$ are the charge numbers of projectile and target nucleus respectively, while $v$ is the relative velocity and $m$ the reduced mass. The orbital functions $\bar{R}_{\lambda\mu}$ are defined by

$$\bar{R}_{\lambda\mu}(t) = \frac{v\,a^\lambda}{r_p(t)^{\lambda+1}} \sqrt{\frac{\pi}{2\,\lambda+1}}\ \frac{(2\,\lambda-1)\,!!}{(\lambda-1)\,!}\ Y_{\lambda\mu}(\vartheta_p(t),\,\varphi_p(t)) \tag{2.3}$$

where $(r_p(t),\,\vartheta_p(t),\,\varphi_p(t))$ are the polar coordinates of the projectile moving on a classical orbit. The corresponding orbital integrals $R_{\lambda\mu}(\vartheta,\,\xi)$ are given by

$$R_{\lambda\mu}(\vartheta,\,\xi) = \int_{-\infty}^{\infty} e^{i\,\xi\,(v/a)\,t}\ \bar{R}_{\lambda\mu}(t)\,dt \tag{2.4}$$

where $\vartheta$ is the scattering angle in the center of mass system. With the normalization chosen in Equation (2.3) the $R_{\lambda\mu}$ are dimensionless and obey the relation

$$\sum_\mu |\,R_{\lambda\mu}(\vartheta,\,\xi)\,|^2 \leqslant 1\,. \tag{2.5}$$

Equality holds for the special case $\vartheta = \pi$ and $\xi = 0$. We further note the symmetry relation

$$R_{\lambda\mu}(\vartheta,\,-\xi) = (-1)^\mu\,R_{\lambda-\mu}(\vartheta,\,\xi)^*\,. \tag{2.6}$$

The excitation amplitude $a_f^i$ from the initial state $|\,i\,\rangle$ to the final state $|\,f\,\rangle$ of the nucleus is conveniently expressed in the interaction representation by

$$a_f^i = \langle f\,|\,T\,e^{-(i/\hbar)\int_{-\infty}^{\infty}\tilde{H}(t)dt}\,|\,i\,\rangle \tag{2.7}$$

with the transformed Hamiltonian

$$\tilde{H}(t) = e^{(i/\hbar)\,H_0 t}\,H_E(t)\,e^{-(i/\hbar)\,H_0 t}\,. \tag{2.8}$$

The symbol $T$ stands for the time ordering, while $H_0$ is the nuclear Hamiltonian.

We now introduce the characteristic parameter $\xi$,

$$\xi_{n\,m} = \frac{a}{v}\,\frac{E_m - E_n}{\hbar} \tag{2.9}$$

where $E_k$ denotes the energy of the nucleus in the state $|\,k\,\rangle$. If the relation

$$\xi_{n\,m}\,\sin^{-1}\frac{\vartheta}{2} \ll 1 \tag{2.10}$$

holds for all nuclear states involved in the excitation process we can use the sudden approximation which is expressed by the relation $\xi_{n\,m} = 0$. The time ordering in Equation (2.7) can then be dropped.

### B. *Multiple Excitation for* $\xi \neq 0$

In order to take full account of the multiplicity of the excitation process even for finite differences in the nuclear energy, it has been proposed (References [4], [5]) to approximate the time ordered product in Equation (2.7) by an expansion in the exponential function.

Starting from the Ansatz

$$T\, e^{-(i/\hbar)\int_{-\infty}^{t} \tilde{H}(t')\,dt'} = e^{\sum_{\nu=1}^{\infty} (-i/\hbar)^{\nu} G_{\nu}(t)} \tag{2.11}$$

and expanding both sides of this equation, we obtain the $G_\nu$ by equating the coefficients of each power of $(-\,i/\hbar)$:

$$\begin{aligned}
G_1(t) &= \int_{-\infty}^{t} dt'\, \tilde{H}(t') \\
G_2(t) &= \frac{1}{2} \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' \left[\tilde{H}(t'), \tilde{H}(t'')\right] \\
G_3(t) &= \frac{1}{3} \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' \int_{-\infty}^{t''} dt''' \left[\tilde{H}(t'), [\tilde{H}(t''), \tilde{H}(t''')]\right] \\
&\quad - \frac{1}{12} \int_{-\infty}^{t} \int_{-\infty}^{t} dt'\, dt'' \int_{-\infty}^{t''} dt''' \left[\tilde{H}(t'), [\tilde{H}(t''), \tilde{H}(t''')]\right], \\
&\;\;\vdots
\end{aligned} \tag{2.12}$$

Cutting off this expansion we can calculate the excitation amplitudes by the diagonalization method (Reference [1]). The matrix elements of $G_\nu$ can then be expressed in terms of quantities used for the perturbation treatment (Reference [6]).

### C. *Treatment of an Additional Small Interaction*

In this section we will study the case where $\tilde{H}(t)$ consists of two parts

$$\tilde{H}(t) = \tilde{H}_1(t) + \tilde{H}_2(t) \tag{2.13}$$

of which only one ($\tilde{H}_1$) gives rise to multiple excitations whereas the other ($\tilde{H}_2$) can be treated as a perturbation. Starting from the Schrödinger equation in the interaction representation

$$i\,\hbar \frac{\partial \varphi}{\partial t} = \tilde{H}(t)\, \varphi \tag{2.14}$$

we introduce the transformed Hamiltonian $\tilde{\tilde{H}}(t)$,

$$\tilde{\tilde{H}} = \left[T\, e^{(i/\hbar)\int_{-\infty}^{t} \tilde{H}_1(t')\,dt'}\right] \tilde{H}_2(t)\; T\, e^{-(i/\hbar)\int_{-\infty}^{t} \tilde{H}_1(t')\,dt'}, \tag{2.15}$$

and a new wave function $\psi(t)$ by

$$\varphi(t) = \left[ T e^{-(i/\hbar) \int_{-\infty}^{t} \tilde{H}_1(t') dt'} \right] \psi(t) . \tag{2.16}$$

The Schrödinger equation can then be rewritten as

$$i \hbar \frac{\partial \psi}{\partial t} = \tilde{\tilde{H}} \psi . \tag{2.17}$$

The formal integral of Equation (2.17), transformed according to Equation (2.16), reads

$$\varphi(t) = \left[ T e^{-(i/\hbar) \int_{-\infty}^{t} \tilde{H}_1(t') dt'} \right] T e^{-(i/\hbar) \int_{-\infty}^{t} \tilde{\tilde{H}}(t') dt'} \varphi(-\infty) . \tag{2.18}$$

This leads to the perturbation expansion

$$a_f^i = \sum_{\nu \geqslant 0} \langle f \mid T_\nu \mid i \rangle \tag{2.19}$$

with

$$T_0 = T e^{-(i/\hbar) \int_{-\infty}^{\infty} \tilde{H}_1(t) dt}$$

$$T_1 = -\frac{i}{\hbar} \int_{-\infty}^{\infty} dt \, T e^{-(i/\hbar) \int_{t}^{\infty} \tilde{H}_1(t') dt'} \tilde{H}_2(t) \, e^{-(i/\hbar) \int_{-\infty}^{t} \tilde{H}_1(t') dt} \tag{2.20}$$

etc.

For the evaluation of the matrix elements one has to use the methods of Section 2.B and perform the time integration numerically. It leads however to a considerable simplification if $\tilde{H}_1$ can be treated in the sudden approximation.

It is possible to get a simpler expression for the excitation amplitudes if $\tilde{H}_1(t)$ commutes with $\tilde{H}_2(t)$. In this case we have

$$\tilde{\tilde{H}}(t) = \tilde{H}_2(t) \tag{2.21}$$

and Equation (2.18) yields

$$a_f^i = \langle f \mid \left[ T e^{-(i/\hbar) \int_{-\infty}^{\infty} \tilde{H}_1(t) dt} \right] T e^{-(i/\hbar) \int_{-\infty}^{\infty} \tilde{H}_2(t) dt} \mid i \rangle . \tag{2.22}$$

## D. *Choice of the Frame of Reference*

We now introduce a specific frame of reference: The $x$-axis lies in the direction of the vector-product $\boldsymbol{v}_i \times \boldsymbol{v}_f$ of the initial and final velocity of the projectile and the positive $z$-axis bisects the angle between $-\boldsymbol{v}_i$ and $\boldsymbol{v}_f$ (Reference [1])). This coordinate system leads to several relations for the excitation amplitudes and the orbital integrals $R_{\lambda\mu}$.

The invariance of $H_E(t)$ with respect to reflexion in the scattering plane yields

$$a_{f,-M_f}^{i,-M_i} = (-1)^{\Delta\pi + \Delta I}\, a_{f,M_f}^{i,M_i} \tag{2.23}$$

and

$$\overline{R}_{\lambda -\mu} = \overline{R}_{\lambda\mu} \tag{2.24}$$

where $\Delta\pi$ denotes the parity difference of the initial and final state. We further note that $\Delta M = 0$ for $\vartheta = \pi$ and $(-1)^{\Delta M} = 1$ for $\xi = 0$. The corresponding rules for the orbital integrals are: $R_{\lambda\mu}(\pi, \xi) = 0$ for $\mu \neq 0$ and $R_{\lambda\mu}(\vartheta, 0) = 0$ for odd $\mu$. From

$$\overline{R}_{\lambda\mu}(-t) = (-1)^{\mu}\, \overline{R}_{\lambda\mu}(t) \tag{2.25}$$

we obtain

$$R_{\lambda\mu}(\vartheta, -\xi) = (-1)^{\mu} R_{\lambda\mu}(\vartheta, \xi) \tag{2.26}$$

which means, according to Equations (2.6) and (2.24), that the orbital integrals are real.

With the above choice of the frame of reference $R_{\lambda\mu}(\vartheta, \xi)$ is given in terms of the tabulated classical integrals $I_{\lambda\mu}(\vartheta, \xi)$ (see References [6]) and [7])) by

$$R_{\lambda\mu}(\vartheta, \xi) = \frac{(2\lambda-1)!!}{2(\lambda-1)!} \sum_{\mu'} D^{\lambda}_{\mu\mu'}\left(0, \frac{\pi}{2}, 0\right) D^{\lambda}_{0\mu'}\left(0, \frac{\pi}{2}, 0\right) I_{\lambda\mu'}(\vartheta, \xi)\,. \tag{2.27}$$

For the special case $\lambda = 2$ we find in particular

$$\begin{aligned}
R_{20}(\vartheta, \xi) &= \frac{3}{8}\left(I_{20} + \frac{3}{2} I_{22} + \frac{3}{2} I_{2-2}\right)\\
R_{2\pm1}(\vartheta, \xi) &= \frac{-3}{8}\sqrt{\frac{3}{2}}\left(I_{22} - I_{2-2}\right)\\
R_{2\pm2}(\vartheta, \xi) &= \frac{-3}{8}\sqrt{\frac{3}{2}}\left(I_{20} - \frac{1}{2} I_{22} - \frac{1}{2} I_{2-2}\right).
\end{aligned} \tag{2.28}$$

The orbital integrals for $\xi = 0$ can be expressed in terms of elementary functions (Reference [6])). They are compiled for $\lambda = 2$ in the appendix.

### E. *Angular Distribution of Deexcitation Gamma Rays*

The angular distribution of the deexcitation gamma rays is given in terms of the density matrices by (Reference [8]))

$$w(\boldsymbol{q}) = \sum_{M, M'} \varrho(M, M')\, \varrho_{\gamma}(M, M') \tag{2.29}$$

with

$$\varrho(M, M') = \frac{1}{2I_i+1} \sum_{M_i} a_{IM'}^{I_i M_i *}\, a_{IM}^{I_i M_i} \tag{2.30}$$

and

$$\varrho_{\gamma}(M, M') = 2 \sum_{\text{even } k} \sqrt{\frac{2k+1}{2I+1}}\, (-1)^{I+M} \begin{pmatrix} I & I & k \\ -M & M' & \varkappa \end{pmatrix}$$
$$\times D^{k*}_{\varkappa 0}(\boldsymbol{z} \to \boldsymbol{q}) \sum_{\lambda\lambda'} F_k(\lambda, \lambda', I_f, I)\, \delta_{\lambda}\, \delta_{\lambda'}\,. \tag{2.31}$$

The factors $F_k$ are the usual geometrical coefficients known from the theory of $\gamma$-$\gamma$-correlation*) and $\delta_\lambda$ denotes the amplitude of the $2^\lambda$-pole $\gamma$-transition from the excited state with spin $I$ to a state with spin $I_f$. The $D$-function in Equation (2.31) describes a rotation of the frame of reference which turns the $z$-axis into the direction of the emitted $\gamma$-quant. This leads to the following expression for the angular distribution:

$$w(\vartheta_\gamma, \varphi_\gamma) = \sum_{\substack{\text{even } k \\ \lambda, \lambda'}} F_k(\lambda, \lambda', I_f, I)\, \delta_\lambda\, \delta_{\lambda'} \sum_{\varkappa'} g_{k\varkappa'}\, D^{k*}_{\varkappa' 0}(\varphi_\gamma, \vartheta_\gamma, 0) \tag{2.32}$$

where the coefficient $g_{k\varkappa'}$ is given by

$$g_{k\varkappa'} = 2 \sqrt{\frac{2k+1}{2I+1}}\, \frac{(-1)^{I+M}}{(2I_i+1)} \sum_{\varkappa, M, M'} \begin{pmatrix} I & I & k \\ -M & M' & \varkappa \end{pmatrix}$$

$$\times D^{k*}_{\varkappa\varkappa'}\left(\frac{\pi}{2}, \frac{\pi+\vartheta}{2}, 0\right) \sum_{M_i} a^{I_i M_i *}_{I M'}\, a^{I_i M_i}_{I M}\,. \tag{2.33}$$

The direction $\varphi_\gamma$, $\vartheta_\gamma$ of the $\gamma$-emission is described in a frame of reference with the $x$-axis in the scattering plane and the $z$-axis in the direction of the incoming particle. For the case where only the direction of $\gamma$-emission is measured, multiplication of $w(\vartheta_\gamma, \varphi_\gamma)$ by the differential Rutherford cross section and integration over $\varphi_\gamma$ and $\vartheta$ leads to

$$W(\vartheta_\gamma) = \sigma_I \sum_{\text{even } k} A_k\, P_k(\cos\vartheta_\gamma) \sum_{\lambda, \lambda'} F_k(\lambda, \lambda', I_f, I)\, \delta_\lambda\, \delta_{\lambda'} \tag{2.34}$$

where $\sigma_I$ is the total excitation cross section (Equation (4.2)). The particle parameter $A_k$ is defined by

$$A_k = \frac{G_k}{G_0}\,,$$

$$G_k = \sqrt{2k+1} \sum_{\varkappa, M, M'} (-1)^{I+M} \begin{pmatrix} I & I & k \\ -M & M' & \varkappa \end{pmatrix} \int_0^\pi \sin\vartheta\, d\vartheta\, \sin^{-4}\frac{\vartheta}{2} \tag{2.35}$$

$$\times D^{k*}_{\varkappa 0}\left(\frac{\pi}{2}, \frac{\pi+\vartheta}{2}, 0\right) \sum_{M_i} a^{I_i M_i *}_{I M'}\, a^{I_i M_i}_{I M}\,. \tag{2.36}$$

It must be noted here that the so called $\chi_{eff}(\vartheta)$ or $\mu = 0$ approximation (References [1], [2]) cannot be used in this case because of the appearance of interference terms.

## 3. Application to Nuclear Models

### A. *Vibrational States of Spherical Symmetric Nuclei*

In Reference [1]) is shown that the multiple Coulomb excitation for vibrational states can be treated for arbitrary $\xi$. We want to derive here the results of Reference [1]) in another way which throws some light on the method described in Section 2.B.

*) For a tabulation see e.g. Reference [9]).

In the case of quadrupole vibrations the interaction Hamiltonian $\tilde{H}(t)$ can be expressed in terms of creation and annihilation operators $b_{2\mu}^{+}$ and $b_{2\mu}$ by

$$\tilde{H}(t) = \chi \hbar \sum_{\mu} \overline{R}_{2\mu}(t) \left(e^{-i\omega t} b_{2\mu} + (-1)^{\mu} e^{i\omega t} b_{2-\mu}^{+}\right) \tag{3.1}$$

where $\chi$ is a coupling parameter defined by

$$\chi = \frac{Z_1 Z_2 e^2 R_0^2}{v a^2 \sqrt{10 \pi \hbar \sqrt{B C}}}. \tag{3.2}$$

The nuclear frequency $\omega$ is given by

$$\omega = \frac{v}{a} \xi = \sqrt{\frac{C}{B}}. \tag{3.3}$$

The parameters $B$ and $C$ denote the inertial parameter and the restoring force respectively and $R_0$ is the nuclear radius. Since $[\tilde{H}(t), \tilde{H}(t')]$ is a pure imaginary $c$-number, it is easy to verify by differentiation that $G_3$ and all higher terms in Equation (2.11) vanish. Introducing the real number $g$ by

$$\left(\frac{-i}{\hbar}\right)^2 G_2(\infty) = i g \tag{3.4}$$

we obtain for the excitation amplitudes

$$a_{n_\mu}^0 = e^{ig} \langle n_\mu \mid e^{-i\chi \sum_{\mu} (-1)^{\mu} R_{2\mu}(\vartheta, \xi) \left(b_{2\mu}^{+} + b_{2\mu}\right)} \mid 0 \rangle \tag{3.5}$$

where $\mid n_\mu \rangle = \mid n_2 n_1 n_0 n_{-1} n_{-2} \rangle$ denotes a state with $N$ phonons,

$$N = \sum_{\mu} n_\mu . \tag{3.6}$$

The exponential function in Equation (3.5) can be decomposed into a product:

$$e^{-i\chi \sum_{\mu} (-1)^{\mu} R_{2\mu}^{+}(\vartheta, \xi) \left(b_{2\mu}^{+} + b_{2\mu}\right)}$$
$$= e^{-i\chi \sum_{\mu} (-1)^{\mu} R_{2\mu} b_{2\mu}^{+}} \, e^{-i\chi \sum_{\mu} (-1)^{\mu} R_{2\mu} b_{2\mu}} \, e^{-(\chi^2/2) \sum_{\mu} | R_{2\mu} |^2} . \tag{3.7}$$

This can be verified by expanding the exponential functions and comparing the coefficients of $\chi^n$. Using the state function

$$\mid n_\mu \rangle = \prod_{\mu} \frac{(b_{2\mu}^{+})^{n_\mu}}{\sqrt{n_\mu !}} \mid 0 \rangle \tag{3.8}$$

we then find

$$a_{n_\mu}^0 = e^{ig} e^{-(\chi^2/2) \sum_{\mu} | R_{2\mu}(\vartheta, \xi) |} (-i\chi)^N \prod_{\mu} \frac{[(-1)^{\mu} R_{2\mu}(\vartheta, \xi)]^{n_\mu}}{\sqrt{n_\mu !}} . \tag{3.9}$$

In order to obtain the excitation amplitudes $a_{NIM}^0$ for the states $\mid N I M \rangle$ with a defined spin $I$ and $N > 1$, the $\mid N I M \rangle$ must be expanded in terms of the $\mid n_\mu \rangle$.

For $N = 2$ this leads to

$$a_{2IM}^{0} = \sqrt{\frac{2I+1}{2}}\,(-i\chi)^2\,e^{ig}\,e^{-(\chi^2/2)\sum_{\mu}|R_{2\mu}(\vartheta,\xi)|^2} \times \sum_{\mu\mu'} \begin{pmatrix} 2 & 2 & I \\ \mu & \mu' & -M \end{pmatrix} R_{2\mu}(\vartheta,\xi)\,R_{2\mu'}(\vartheta,\xi)\,. \tag{3.10}$$

## B. *Coupled Rotational Bands*

For axially symmetric nuclei transitions between rotational bands of different intrinsic structure are treated in References [1]) and [2]). The methods used there are here generalized to include bands with different intrinsic quadrupole moments. For equal intrinsic quadrupole moments a formula is given for the multiple transition between the bands.

For pure rotational bands the Hamiltonian of the nucleus is given by

$$H_0 = H_{rot} + H_{int} \tag{3.11}$$

where $H_{int}$ and $H_{rot}$ are the Hamiltonians of the intrinsic structure and the rotational energy, respectively. The eigenfunctions of $H_0$ can be expressed as a product of a rotational wave function $|\,I\,K\,M\,\rangle$ and an intrinsic wave function $|\,n\,K\,\rangle$:

$$|\,I\,K\,M\,\rangle\,|\,n\,K\,\rangle = \sqrt{\frac{2I+1}{4\pi}}\,D^{I}_{MK}(\alpha,\beta,0)\,|\,n\,K\rangle\,. \tag{3.12}$$

The actual state functions of pure rotational bands are obtained from the eigenfunctions (3.12) by symmetrization with respect to $\pm K$. The excitation amplitudes $a^{i,K_i}_{f,K_f}$ of the symmetrized state functions can, however, be expressed directly in terms of the amplitudes $b^{i,K_i}_{f,K_f}$ defined by

$$b^{i,K_i}_{f,K_f} = \langle I_f\,K_f\,M_f\,|\,\langle n_f\,K_f\,|\,T\,e^{-(i/\hbar)\int_{-\infty}^{\infty}\tilde{H}(t)\,dt}\,|\,n_i\,K_i\,\rangle\,|\,I_i\,K_i\,M_i\,\rangle\,. \tag{3.13}$$

The following relations are found (see also Reference [2])):

$$\begin{aligned}
a^{i,0}_{f,0} &= b^{i,0}_{f,0} \\
a^{i,K_i}_{f,K_f} &= \sqrt{2}\,b^{i,K_i}_{f,K_f} \qquad \text{for } K_i \text{ or } K_f = 0 \\
a^{i,K_i}_{f,K_f} &= b^{i,K_i}_{f,K_f} + (-1)^{I_f+K_f}\,b^{i,K_i}_{f,-K_f} = b^{i,K_i}_{f,K_f} + (-1)^{I_i+K_i}\,b^{i,-K_i}_{f,K_f} \\
&\qquad \text{for } K_i \text{ and } K_f \neq 0\,.
\end{aligned} \tag{3.14}$$

Equation (3.13) can be evaluated by the methods given in Section 2.C., where $H_1$ and $H_2$ denote the diagonal and offdiagonal part of $H_E(t)$ respectively, with respect to the intrinsic states. Thus we have

$$\langle\,n'\,K'\,|\,H_1\,|\,n\,K\rangle = \langle\,n'\,K'\,|\,H_E(t)\,|\,n\,K\,\rangle\,\delta_{nn'}\,\delta_{KK'} \tag{3.15}$$

and

$$\langle\,n'\,K'\,|\,H_2\,|\,n\,K\rangle = \langle\,n'\,K'\,|\,H_E(t)\,|\,n\,K\rangle\,(1-\delta_{nn'}\,\delta_{KK'})\,. \tag{3.16}$$

The operator $H_E(t)$ is here given by

$$H_E(t) = \sum_\lambda \frac{4 Z_1 e}{v a^\lambda} \sqrt{\frac{\pi}{2\lambda+1}} \frac{(\lambda-1)!}{(2\lambda-1)!!} \sum_{\mu\mu'} \overline{R}_{\lambda\mu}(t)\, D^{\lambda *}_{\mu\mu'}(\alpha, \beta, 0)\, M^*_{int}(E\lambda, \mu') \tag{3.17}$$

where $M_{int}(E\lambda, \mu)$ is the operator of the electric multipole moment in the body fixed system.

Using the relation

$$\langle n K \mid M_{int}(E\lambda, 0) \mid n K\rangle = \sqrt{\frac{5}{16\pi}}\, Q_0\, \delta_{\lambda, 2} \tag{3.18}$$

which also defines the intrinsic quadrupole moment $Q_0$, we can express the matrix elements of $H_1$ by

$$\langle n_r K_r \mid H_1 \mid n_r K_r\rangle = H_r'(t) = \frac{4}{3} q^{(r)} \hbar \sum_\mu \overline{R}_{2\mu}(t)\, D^{2*}_{\mu 0}(\alpha, \beta, 0) \tag{3.19}$$

with a coupling parameter $q$ defined by

$$q = \frac{Z_1 e Q_0}{4 \hbar v a}\,. \tag{3.20}$$

The evaluation of $T_0$ (Equation (2.20)) now yields the excitation amplitudes $b^{i,K}_{f,K}$ within the ground state band. In the sudden approximation they are given by (Reference [1]))

$$\begin{aligned} b^{i,K}_{f,K} &= \langle I_f K M_f \mid e^{-(i/\hbar)\int_{-\infty}^{\infty} H'(t)\,dt} \mid I_i K M_i\rangle \\ &= \sum_I (-1)^{M_i - K} \sqrt{(2 I_i + 1)(2 I + 1)(2 I_f + 1)} \\ &\times \begin{pmatrix} I_f & I_i & I \\ -M_f & M_i & M \end{pmatrix} \begin{pmatrix} I_f & I_i & I \\ -K & K & 0 \end{pmatrix} a_{IM}(\vartheta, q) \end{aligned} \tag{3.21}$$

where $a_{IM}$ are the excitation amplitudes for $I_i = 0$,

$$\begin{aligned} a_{IM}(\vartheta, q) &= \langle I\, 0\, M \mid e^{-(i/\hbar)\int_{-\infty}^{\infty} H'(t)\,dt} \mid 0\,0\,0\rangle \\ &= \delta_{I,even}\, \delta_{M,even} \sqrt{2 I + 1} \sqrt{\frac{(I+M)!}{(I-M)!}}\, \frac{\Gamma\left(\frac{I-M+1}{2}\right)(-1)^{(I-M)/2}}{2^{M+1}\, \Gamma\left(I + \frac{3}{2}\right)\left(\frac{M}{2}\right)!} \\ &\times e^{-(4i/3) q R_{20}(\vartheta, 0)} \sum_{m,n} \frac{\left(\frac{I+M}{2} + 1\right)_m \left(-\frac{I-M}{2} - m\right)_{2n}}{\left(I + \frac{3}{2}\right)_m \left(\frac{M}{2} + 1\right)_n m!\, n!} \\ &\times [2 i q R_{20}(\vartheta, 0)]^{(I/2)+m} \left[\frac{-R_{22}(\vartheta, 0)}{\sqrt{6}\, R_{20}(\vartheta, 0)}\right]^{(M/2)+2n}. \end{aligned} \tag{3.22}$$

Here we used the notation

$$z_m = \frac{\Gamma(z+m)}{\Gamma(z)} = z\,(z+1)\ldots(z+m-1)\,. \tag{3.23}$$

For backward scattering Equation (3.22) reduces to (Reference [1]))

$$a_{I\,M}(\pi, q) = \delta_{M,0}\sqrt{2\,I+1}\,(-\,2\,i\,q)^{I/2}\,e^{-(4\,i/3)\,q}$$
$$\times \frac{\Gamma\left(\frac{I+1}{2}\right)}{2\,\Gamma\left(I+\frac{3}{2}\right)}\,{}_1F_1\left(\frac{I+2}{2},\,I+\frac{3}{2},\,2\,i\,q\right) \tag{3.24}$$

where ${}_1F_1$ is a confluent hypergeometric function.

For small transition probabilities between the bands we can treat $H_2(t)$ to first order, i.e. evaluate only $T_1$ (Equation (2.20)). Using

$$\tilde{H}'(t) = e^{(i/\hbar)\,H_{rot}\,t}\,H'(t)\,e^{-(i/\hbar)\,H_{rot}\,t} \tag{3.25}$$

we obtain for the amplitudes $b_f^i$ under the condition $|\,n_i\,k_i\,\rangle \neq |\,n_f\,k_f\,\rangle$

$$b_f^i = \frac{-\,i}{\hbar}\,\langle I_f\,K_f\,M_f\,|\int\limits_{-\infty}^{\infty} dt\;T\,e^{-(i/\hbar)\,(1-(p/2))\int\limits_t^\infty \tilde{H}'(t')\,dt'}$$
$$\times\,\langle\,n_f K_f\,|\,\tilde{H}_2(t)\,|\,n_i\,K_i\rangle\,e^{-(i/\hbar)\,(1+p/2))\int\limits_{-\infty}^t \tilde{H}'(t')\,dt'}\,|\,I_i\,M_i\,K_i\rangle\,. \tag{3.26}$$

An average coupling parameter $q$ and a relative difference $p$ of $q^{(i)}$ and $q^{(f)}$ have been introduced. They are defined by

$$q = \frac{q^{(i)}+q^{(f)}}{2} \tag{3.27}$$

and

$$p = \frac{q^{(i)}-q^{(f)}}{q}\,. \tag{3.28}$$

If we again neglect the differences in the rotational energy, $\tilde{H}_2$ reduces to

$$\tilde{H}_2(t) = e^{(i/\hbar)\,H_{int}\,t}\,H_2(t)\,e^{-(i/\hbar)\,H_{int}\,t} \tag{3.29}$$

and, since $H'$ commutes with this operator, we find

$$b_f^i = \frac{-\,i}{\hbar}\,\langle\,I_f\,K_f\,M_f\,|\int\limits_{-\infty}^{\infty} dt\,\langle n_f\,K_f\,|\,\tilde{H}_2(t)\,|\,n_i\,K_i\rangle$$
$$\times\,e^{-(i\,p/\hbar)\int\limits_0^t H'(t')\,dt'}\;e^{(i\,p/2\hbar)\int\limits_0^\infty \{H'(t')-H'(-t')\}\,dt'}\;e^{-(i/\hbar)\int\limits_{-\infty}^\infty H'(t')\,dt'}\,|\,I_i\,K_i\,M_i\rangle\,. \tag{3.30}$$

The matrix element of the last term of this equation is the amplitude $b_{I\,K_i\,M}^{I_i\,K_i\,M_i}$ given by Equations (3.21) and (3.22).

Introducing the parameter $\chi^{(\lambda)}$ which describes the strength of the $2^\lambda$-pole coupling between two bands by

$$\chi^{(\lambda)} = \frac{\sqrt{16\pi}\, Z_1 e}{\hbar v a^\lambda} \frac{(\lambda-1)!}{(2\lambda-1)!!} \langle n_i K_i \mid M_{int}(E\lambda, \mu') \mid n_f K_f \rangle \tag{3.31}$$

with $\mu' = K_i - K_f$, we obtain

$$\langle n_f K_f \mid \tilde{H}_2(t) \mid n_i K_i \rangle = e^{i(v/a)\xi t} \hbar \sum_{\lambda,\mu} \sqrt{2\lambda+1}\, \chi^{(\lambda)} \bar{R}_{\lambda\mu}(t) D^{\lambda *}_{\mu\mu'} . \tag{3.32}$$

Equation (3.30) can now be expanded in powers of $p$. In the lowest order (for small values of $p$) one finds

$$b_f^i = -i \sum_{I,M,\lambda} (-1)^{M-K_i} \sqrt{(2I_f+1)(2I+1)(2\lambda+1)}\, \chi^{(\lambda)}$$

$$\times \begin{pmatrix} I & I_f & \lambda \\ -K_i & K_f & \mu' \end{pmatrix} R_{\lambda\mu}(\vartheta, \xi) \begin{pmatrix} I & I_f & \lambda \\ -M & M_f & \mu \end{pmatrix} b^{I_i K_i M_i}_{I K_i M}(\vartheta, q) . \tag{3.33}$$

Since in our frame of reference we have

$$H'(t) - H'(-t) = \frac{8}{3} q \hbar \bar{R}_{21}(t) \{D^{2*}_{10} + D^{2*}_{-10}\} , \tag{3.34}$$

Equation (3.30) can be considerably simplified in the special case of backward scattering. The matrix elements of $\exp(-i p/\hbar \int_0^t H'(t')\, dt')$ can then be expressed directly by Equation (3.24) where $q$ is substituted by $p q \int_0^t \bar{R}_{20}(t')\, dt'$.

The orbital integrals for $\xi = 0$ used here are listed in the appendix. Table 2 shows that even for deflection angles $\vartheta \neq \pi$ the term given in Equation (3.34) contributes not much and can therefore be treated in a low order.

Multiple transitions between the bands are easily treated if the intrinsic quadrupole moments $Q_0$ can be assumed to be equal for all bands involved in the excitation process. Equation (2.22) then yields

$$b_f^i = \sum_{IM} \langle I_f K_f M_f \mid \langle n_f K_f \mid T e^{-(i/\hbar)\int_{-\infty}^{\infty} \tilde{H}_2(t)\, dt} \mid n_i K_i \rangle \mid I K_i M \rangle$$

$$\times \langle I K_i M \mid e^{-(i/\hbar)\int_{-\infty}^{\infty} H'(t)\, dt} \mid I_i K_i M_i \rangle . \tag{3.35}$$

The second matrix element of Equation (3.35) is again given by Equations (3.21) and (3.22) whereas the evaluation of the first part depends on the nature of the intrinsic states. A first order treatment leads again to Equation (3.33). If the intrinsic structure can be described by the vibrational model ($\beta$- or $\gamma$-vibrations) the methods discussed in Section 3.A can be applied.

### C. *Asymmetric Rotator*

In order to describe the lower energy states of some even-even nuclei the model of an asymmetric rotator was proposed (References [10]), [11])). If we restrict ourselves to rotational states, the state functions $| N\ I\ M \rangle$ are given by

$$| N I M \rangle = \sum_{K=0}^{I} A_{IK}^{N} \frac{\sqrt{2I+1}}{4\pi\sqrt{1+\delta_{K,0}}} (D_{MK}^{I} + (-1)^{I} D_{M-K}^{I}) \tag{3.36}$$

where the summation goes only over even $K$. By $N$ we number the ascending eigenvalues of the energy $E_I^N$ for a given spin $I$. The coefficients $A_{IK}^N$ and the energy $E_I^N$ depend on the parameters $\beta$ and $\gamma$ which describe the shape of the nucleus. The interaction Hamiltonian for rotational states is given by

$$H'(t) = \frac{4}{3} q \hbar \sum_{\mu} \overline{R}_{2\mu}(t) \{D_{\mu 0}^{2*} \cos\gamma + \frac{\sin\gamma}{\sqrt{2}} (D_{\mu 2}^{2*} + D_{\mu -2}^{2*})\} \tag{3.37}$$

where $q$ is defined by Equation (3.20).

In the sudden approximation the excitation amplitudes $a_{IM}^N$ from the ground state $| 100 \rangle$ to a state $| N I M \rangle$ take the form

$$a_{IM}^{N} = a_{NIM}^{100} = \sum_{K=0}^{I} A_{IK}^{N} B_{MK}^{I} \sqrt{2 - \delta_{K,0}}\,, \tag{3.38}$$

where the integral $B_{MK}^I$ is defined by

$$B_{MK}^{I} = \frac{\sqrt{2I+1}}{8\pi^2} \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} D_{MK}^{I}(\varphi, \vartheta', \psi)\, e^{-(i/\hbar)\int_{-\infty}^{\infty} H'(t)\,dt} d\varphi \sin\vartheta'\, d\vartheta'\, d\psi\,. \tag{3.39}$$

The following symmetry relations have been used:

$$B_{MK}^{I} = (-1)^{I} B_{M-K}^{I} = (-1)^{I} B_{-MK}^{I} = B_{-M-K}^{I}\,. \tag{3.40}$$

It is easy to see that $B_{MK}^I$ disappears for odd $M$ or $K$.

In the special case of backward scattering Equation (3.39) is similar to the corresponding relation of the symmetric rotator. We have (see Equation (3.22))

$$B_{MK}^{I} = B_{K}^{I} \delta_{M,0} \delta_{I,\,even} \qquad \text{for } \vartheta = \pi \tag{3.41}$$

with

$$B_K^I = \sqrt{2I+1} \sqrt{\frac{(I+K)!}{(I-K)!}} \frac{\Gamma\left(\frac{I-K+1}{2}\right)(-1)^{I/2}}{2^{K+1}\Gamma\left(I+\frac{3}{2}\right)\left(\frac{K}{2}\right)!} e^{-(4i/3)q\cos\gamma}$$

$$\times \sum_{m,n} \frac{\left(\frac{I+K}{2}+1\right)_m \left(-\frac{I-K}{2}-m\right)_{2n}}{\left(I+\frac{3}{2}\right)_m \left(\frac{K}{2}+1\right)_n m!\, n!} (2iq\cos\gamma)^{(I/2)+m} \left(\frac{\mathrm{tg}\gamma}{2\sqrt{3}}\right)^{(K/2)+2n}. \tag{3.42}$$

The integral $B_{MK}^I$ (Equation (3.39)) can be computed for arbitrary scattering angles by the following method: The interaction Hamiltonian $H'$ (Equation (3.37)) is a sum of nine terms. Thus the exponential function in the integral (3.39) can be written as a product of nine exponential functions. These are expanded after having extracted the constant factor $\exp(-(4i/3)\, q\, R_{20} \cos\gamma)$. The integration over $\varphi$ and $\psi$ leads then to two relations for the indices of summation and the integral over $\vartheta'$ can be evaluated after the substitution $\cos^2(\vartheta/2) = x$ according to

$$\int_0^1 dx\,(1-x)^n\, x^m = \frac{n!\,m!}{(n+m+1)!}\,. \tag{3.43}$$

An appropriate rearrangement of the summations leads to

$$B^I_{MK} = e^{-(4i/3)\,q\,R_{20}(\vartheta,0)\cos\gamma} \sum_{\alpha \geqslant I/2;\, m,k \geqslant 0} (8\, i\, q\, R_{20}(\vartheta,0)\cos\gamma)^\alpha \times \left(\frac{-R_{22}(\vartheta,0)}{\sqrt{6}\, R_{20}(\vartheta,0)}\right)^{(M/2)+2m} \left(\frac{-\mathrm{tg}\gamma}{2\sqrt{3}}\right)^{(K/2)+2k} S^{IMK}_{\alpha m k} \tag{3.44}$$

where $S$ is defined by

$$S^{IMK}_{\alpha m k} = (-1)^I\, \frac{\sqrt{(2I+1)\,(I+M)!\,(I-M)!\,(I+K)!\,(I-K)!}}{(I+2\alpha+1)!}$$
$$\times \sum_{\sigma,\mu,\nu,\varrho,\lambda} \frac{(-1)^{\sigma+\mu+\nu+\lambda+\varrho}\left(\frac{M+K}{2}+\alpha+\sigma+\varrho+\nu-\lambda-\mu\right)!}{(M+K+\sigma)!\,(I-M-\sigma)!\,(I-K-\sigma)!\,\sigma!}$$
$$\times \frac{\left(I-\frac{M+K}{2}+\alpha-\sigma-\varrho-\nu+\lambda+\mu\right)!}{\left[\alpha+\varrho+\nu+\mu+\lambda-2(m+k)-\frac{M+K}{2}\right]!\left(m+\frac{M}{2}-\varrho-\lambda\right)!}$$
$$\times \frac{1}{\left(k+\frac{K}{2}-\varrho-\mu\right)!\,(m-\nu-\mu)!\,(k-\nu-\lambda)!\,\varrho!\,\nu!\,\mu!\,\lambda!}\,. \tag{3.45}$$

The summation should be carried out in such a way that no factorial has a negative argument. This implies for the indices $m$ and $k$

$$\mathrm{Max}\left(0, -\frac{M}{2}\right) \leqslant m \leqslant \frac{1}{2}\left(\alpha - \frac{M}{2}\right)$$
$$\mathrm{Max}\left(0, -\frac{K}{2}\right) \leqslant k \leqslant \frac{1}{2}\left(\alpha - \frac{K}{2}\right). \tag{3.46}$$

The additional rule $\alpha \geqslant I/2$ (see Equation (3.44)) can be derived by partial integration if in Equation (3.39) the Rodriguez formula for $D^I_{MK}(0, \vartheta', 0)$ (Reference [3])) is used after integration over $\varphi$ and $\psi$*).

The coefficients $S^{IMK}_{\alpha m k}$ have the following symmetry properties:

$$S^{I\ M\ K}_{\alpha\ m\ k} = S^{I\ K\ M}_{\alpha\ k\ m} = S^{I\ \ -M\ \ K}_{\alpha\ m+(M/2)\ k}$$
$$= S^{I\ M\ \ -K}_{\alpha\ m\ k+(K/2)} = S^{I\ \ -M\ \ -K}_{\alpha\ m+(M/2)\ k+(K/2}\,. \tag{3.47}$$

---

*) This can also be seen from selection rules in the perturbation expansion.

For the two special cases of backward scattering and axial symmetry ($\sin\gamma = 0$) we have $m = M = 0$ and $k = K = 0$ respectively. This reduces the multiple summation in Equation (3.45) to the simple one over $\sigma$. With the help of the formula (see Equation (3.23))

$$\sum_{\sigma} (-1)^{\sigma} \binom{L}{\sigma} \frac{(I+r-L+\sigma)!\,(I+r-\sigma)!}{(I-L+\sigma)!\,(I-\sigma)!}$$

$$= 2^L \left(\frac{1}{2}\right)_{(L/2)} (-r)_{(L/2)} \frac{\left(I+r-\frac{L}{2}\right)!\,(I+r-L)!}{I!\left(I-\frac{L}{2}\right)!} \quad \text{for even } L \tag{3.48}$$

we get thus again Equations (3.22) and (3.42). Equation (3.48) can be proved by formal induction starting from the result for $L = 0$. The sum vanishes for odd $L$, and this implies that only even spin values occur.

## 4. Numerical Results

### A. *General Remarks*

Using methods and formulas given in the preceding sections we have calculated excitation probabilities $P_f$

$$P_f = \frac{1}{2\,I_i+1} \sum_{M_i, M_f} |\,a_f^i\,|^2\,, \tag{4.1}$$

total cross sections $\sigma_f$ in units of $a^2$ (Equation (2.2))

$$\frac{\sigma_f}{a^2} = \frac{1}{4} \int_0^{\pi} \frac{\sin\vartheta\, d\vartheta}{\sin^4(\vartheta/2)} P_f \tag{4.2}$$

and the coefficients $A_k$ for the angular distribution of the deexcitation $\gamma$-rays (see Equations (2.35) and (2.36)). However, because of the variety of the parameters entering the calculations, it is not possible to present here all numerical results. We will therefore give only typical examples which exhibit the nature of the models and approximations used. For a detailed comparison with experiments a more complete tabulation of results have been printed separately (Reference [12]). Some results have been published previously (see References [13] and [14]).

For the interpretation of the results for finite $\xi$ it has to be noted that, for a given target and projectile, $\xi$ decreases strongly with increasing coupling parameter $q$ or $\chi$, i.e.

$$\xi \sim \frac{1}{q^3} \sim \frac{1}{\chi^3}\,. \tag{4.3}$$

Almost all results show the common feature: They do not depend so much on the details of the chosen models but vary strongly with increasing $\xi$. Especially the angular distribution of the deexcitation $\gamma$-rays is very sensitive to a variation in $\xi$ (see Figure 2), since such a variation changes the distribution of the magnetic substates.

### B. *Vibrational States*

In Figures 1 and 2 typical examples of the variation of total cross sections and of the coefficients $A_k$ as functions of $\xi$ are shown. It may be noted that the cross sections $\sigma_{N I}$ fulfil the following relation (Reference [1]) for $\xi = 0$ and $N = 2$:

$$5\,\sigma_{20} = \frac{7}{2}\,\sigma_{22} = \frac{35}{18}\,\sigma_{24} = \sigma_2 \tag{4.4}$$

where the total excitation cross section $\sigma_2$ of states with principal quantum number $N = 2$ is given by

$$\sigma_2 = \sum_I \sigma_{2I}\,. \tag{4.5}$$

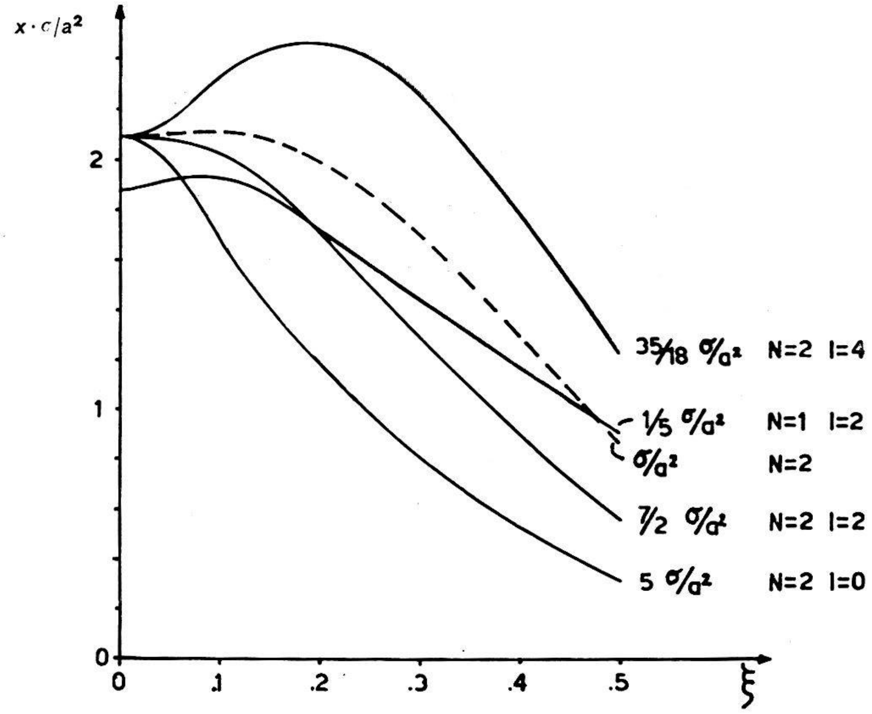



Figure 1

Vibrational states of spherical symmetric nuclei. Total cross sections $\sigma_{N I}$ are shown as functions of $\xi$ for $\chi = 2.4$ and principal quantum numbers $N = 1$ and 2. The dashed line represents the total cross section $\sigma_2$ (Equation (4.5)).

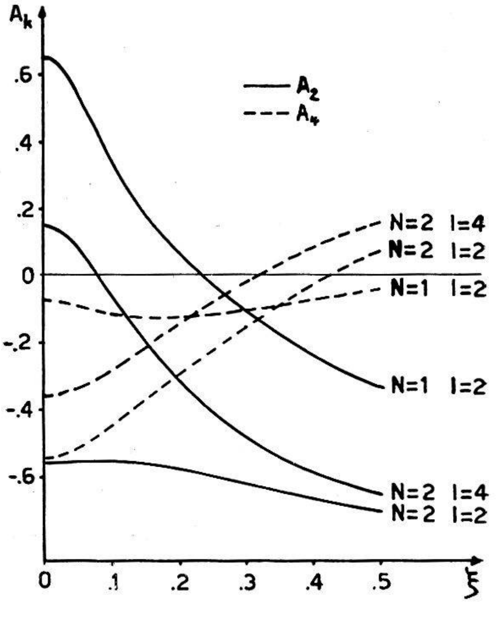



Figure 2

Vibrational states of spherical symmetric nuclei. The coefficients $A_2$ (fulldrawn lines) and $A_4$ (dashed lines) of the angular distribution of the deexcitation $\gamma$-rays are shown as functions of $\xi$ for $\chi = 2.4$ and Principal quantum numbers $N = 1$ and 2.

Table 1

Symmetric rotator. Total cross sections $\sigma_I$ in the form $\sigma_I/a^2\, q^I$ and the coefficients $A_k$ (Equations (2.35) and (2.36)) for the angular distribution of the deexcitation $\gamma$-rays are listed for even-even nuclei and $\xi = 0$ as a function of $q$ (Equation (3.20)). The excitation amplitudes from the initial state $|\,000\,\rangle$ to the final state $|\,I\,0\,M\,\rangle$ have been calculated according to Equation (3.22). The total cross sections are given in form of a number followed (in parenthesis) by the power of ten by which it should by multiplied.

| | $I = 2$ | | | $I = 4$ | | |
|---|---|---|---|---|---|---|
| $q$ | $\sigma_2/a^2\, q^2$ | $A_2$ | $A_4$ | $\sigma_4/a^2\, q^4$ | $A_2$ | $A_4$ |
| 0.0 | .1425 (+1) | .03240 | .00083 | .2846 (−1) | −.56848 | .11379 |
| 0.5 | .1402 (+1) | .04284 | −.00446 | .2783 (−1) | −.56401 | .10873 |
| 1.0 | .1335 (+1) | .07394 | −.01975 | .2602 (−1) | −.55035 | .09345 |
| 1.5 | .1236 (+1) | .12473 | −.04311 | .2329 (−1) | −.52676 | .06763 |
| 2.0 | .1118 (+1) | .19293 | −.07109 | .1997 (−1) | −.49202 | .03094 |
| 2.5 | .9947 (+0) | .27412 | −.09872 | .1645 (−1) | −.44454 | −.01677 |
| 3.0 | .8791 (+0) | .36141 | −.12015 | .1307 (−1) | −.38268 | −.07480 |
| 3.5 | .7779 (+0) | .44628 | −.13020 | .1008 (−1) | −.30538 | −.14081 |
| 4.0 | .6939 (+0) | .52092 | −.12645 | .7610 (−2) | −.21342 | −.20965 |
| 4.5 | .6260 (+0) | .58101 | −.11038 | .5697 (−2) | −.11093 | −.27279 |
| 5.0 | .5713 (+0) | .62681 | −.08653 | .4284 (−2) | −.00608 | −.31957 |
| 5.5 | .5264 (+0) | .66188 | −.06010 | .3274 (−2) | .09070 | −.34140 |
| 6.0 | .4883 (+0) | .69040 | −.03481 | .2561 (−2) | .17148 | −.33675 |
| 6.5 | .4554 (+0) | .71523 | −.01212 | .2050 (−2) | .23488 | −.31243 |
| 7.0 | .4264 (+0) | .73743 | .00819 | .1673 (−2) | .28562 | −.27904 |
| 7.5 | .4010 (+0) | .75691 | .02697 | .1383 (−2) | .33006 | −.24502 |

| | $I = 6$ | | | $I = 8$ | | |
|---|---|---|---|---|---|---|
| | $\sigma_6/a^2\, q^6$ | $A_2$ | $A_4$ | $\sigma_8/a^2\, q^8$ | $A_2$ | $A_4$ |
| 0.0 | .4737 (−3) | −.75879 | .30772 | .5293 (−5) | −.85104 | .45250 |
| 0.5 | .4640 (−3) | −.75680 | .30447 | .5197 (−5) | −.85000 | .45047 |
| 1.0 | .4360 (−3) | −.75073 | .29459 | .4918 (−5) | −.84685 | .44431 |
| 1.5 | .3930 (−3) | −.74024 | .27771 | .4484 (−5) | −.84144 | .43381 |
| 2.0 | .3398 (−3) | −.72476 | .25318 | .3939 (−5) | −.83353 | .41858 |
| 2.5 | .2818 (−3) | −.70344 | .22010 | .3333 (−5) | −.82276 | .39806 |
| 3.0 | .2244 (−3) | −.67509 | .17734 | .2714 (−5) | −.80858 | .37148 |
| 3.5 | .1716 (−3) | −.63814 | .12363 | .2127 (−5) | −.79029 | .33782 |
| 4.0 | .1264 (−3) | −.59076 | .05786 | .1604 (−5) | −.76691 | .29580 |
| 4.5 | .8998 (−4) | −.53101 | −.02030 | .1165 (−5) | −.73717 | .24388 |
| 5.0 | .6232 (−4) | −.45757 | −.10927 | .8144 (−6) | −.69948 | .18039 |
| 5.5 | .4240 (−4) | −.37092 | −.20396 | .5500 (−6) | −.65202 | .10386 |
| 6.0 | .2872 (−4) | −.27503 | −.29453 | .3603 (−6) | −.59302 | .01380 |
| 6.5 | .1965 (−4) | −.17808 | −.36761 | .2306 (−6) | −.52156 | −.08785 |
| 7.0 | .1375 (−4) | −.09018 | −.41178 | .1459 (−6) | −.43897 | −.19484 |
| 7.5 | .9906 (−5) | −.01815 | −.42469 | .9248 (−7) | −.35025 | −.29546 |

### C. *Rotational States of Axially Symmetric Nuclei*

Results are given for the excitation of nuclei with ground state spin $I_i = 0$ (even-even nuclei).

In Table 1 we give total cross sections and the coefficients $A_k$ for excitation of the ground state band ($K = 0$) calculated in the sudden approximation. The total cross sections $\sigma_{I_f,K}^{I_i,K}$ for $I_i \neq 0$ (also for half integer values) can be derived easily from the tabulated $\sigma_I = \sigma_{I,0}^{0,0}$ according to (Reference [1]) and Equation (3.21))

$$\sigma_{I_f,K}^{I_i,K} = (2\,I_f + 1) \sum_I \begin{pmatrix} I_f & I_i & I \\ -K & K & 0 \end{pmatrix}^2 \sigma_I \qquad \text{for } \xi = 0. \tag{4.6}$$

The results of calculations with finite $\xi$ as outlined in Section 2.C are shown in Figure 3. As a general parameter $\xi$ we introduced $\xi_{02}$ (Equation (2.9)) for the transition from the ground state ($I = 0$) to the first excited state ($I = 2$). We have computed the excitation probabilities $P_f$ only for backward scattering ($\vartheta = \pi$).

Some examples of transitions to other bands are presented in Figures 4 and 5 (see also Reference [14])).

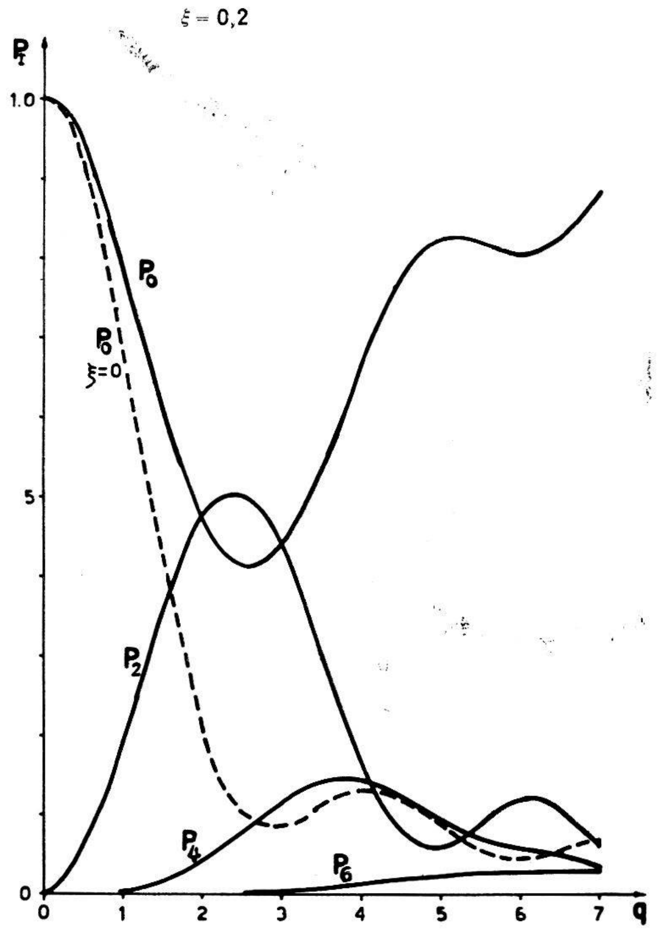
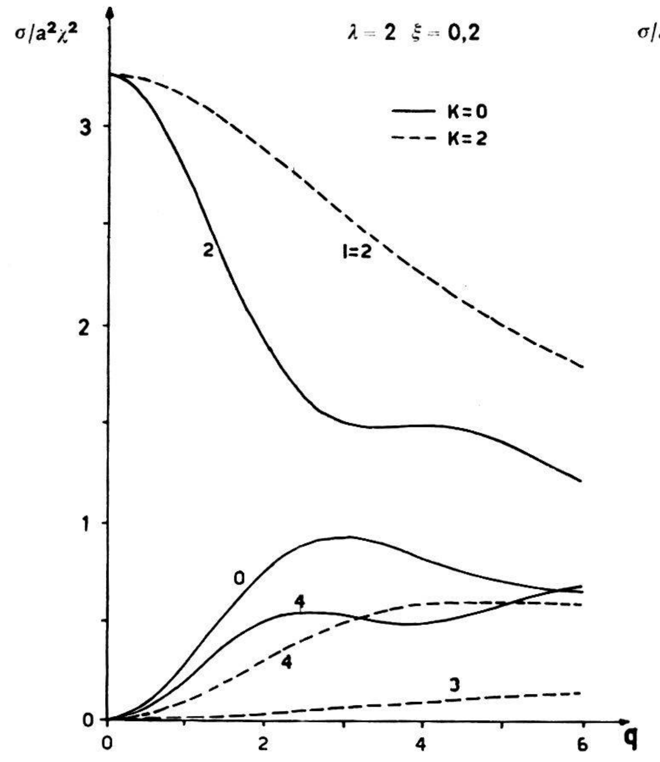
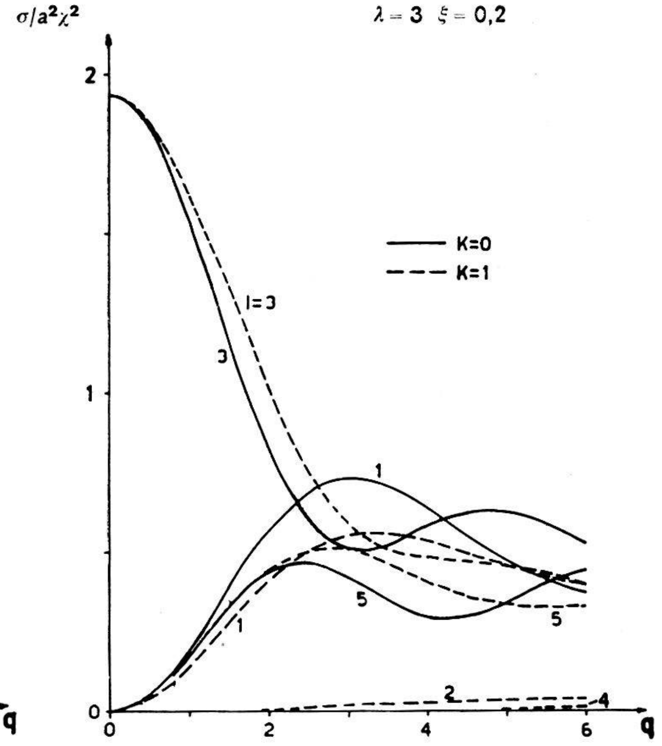



Figure 3

Excitation of the groundstate band of axially symmetric even-even nuclei ($I_i = 0$) for finite $\xi$. The excitation probabilities $P_I$ for backward scattering ($\vartheta = \pi$) are shown as functions of $q$ for $\xi = 0.2$. For comparison we also show $P_0$ for $\xi = 0$ (dashed line).




Figures 4 and 5

Coupled rotational bands of even-even nuclei ($I_i = 0$). Total cross sections in the form of $\sigma_I/a^2\,\chi^2$ are shown as functions of $q$ for $\xi = 0.2$. For $K \neq 0$ the cross sections have to be multiplied by a factor 2 because they were calculated with the amplitudes $b^i_f$ instead of $a^i_f$ (Equation (3.14)). The excitation amplitudes $b^i_f$ were calculated by use of Equation (3.33) for pure $2^\lambda$-pole transitions between the bands.

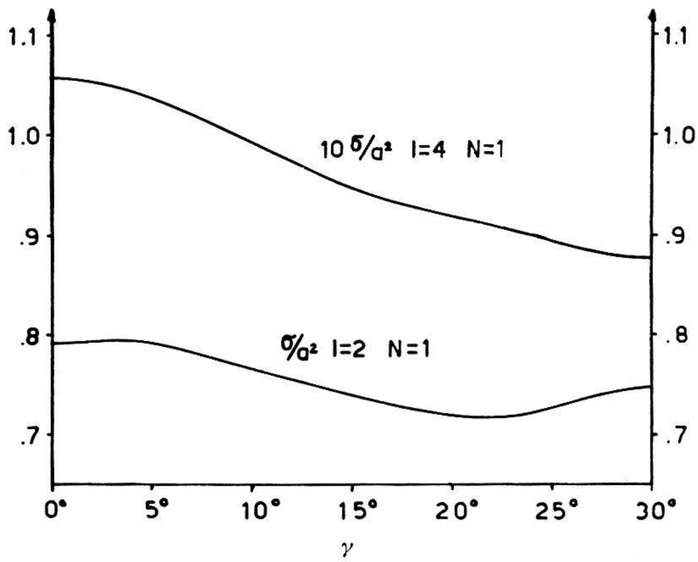



Figure 6

Asymmetric rotator. Total excitation cross sections for states with spin $I = 2$ and 4 and $N = 1$ are given in the form of $\sigma^N_I/a^2\,q^I$ as functions of the asymmetry parameter $\gamma$ for $q = 3$. The amplitudes $a^N_{IM}$ were calculated in the sudden approximation by use of Equations (3.38), (3.44), and (3.45).

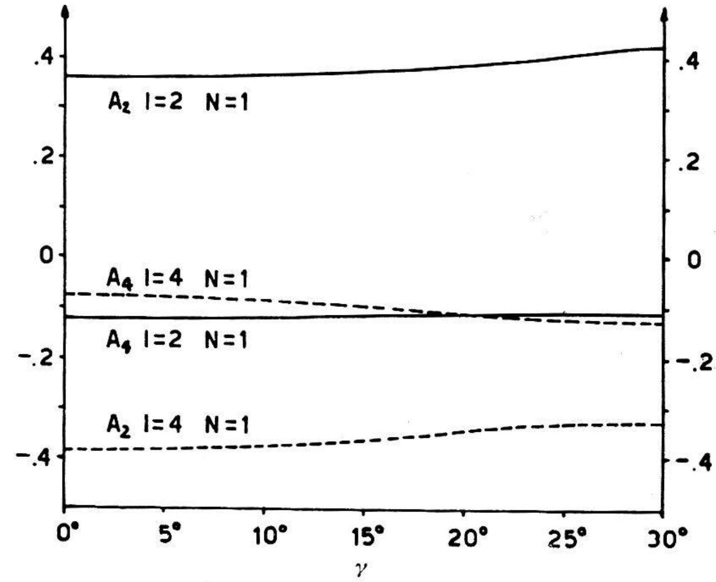



Figure 7
Asymmetric rotator. The coefficients $A_k$ for the angular distribution of the deexcitation $\gamma$-rays are shown for the same parameter values as in Figure 6.

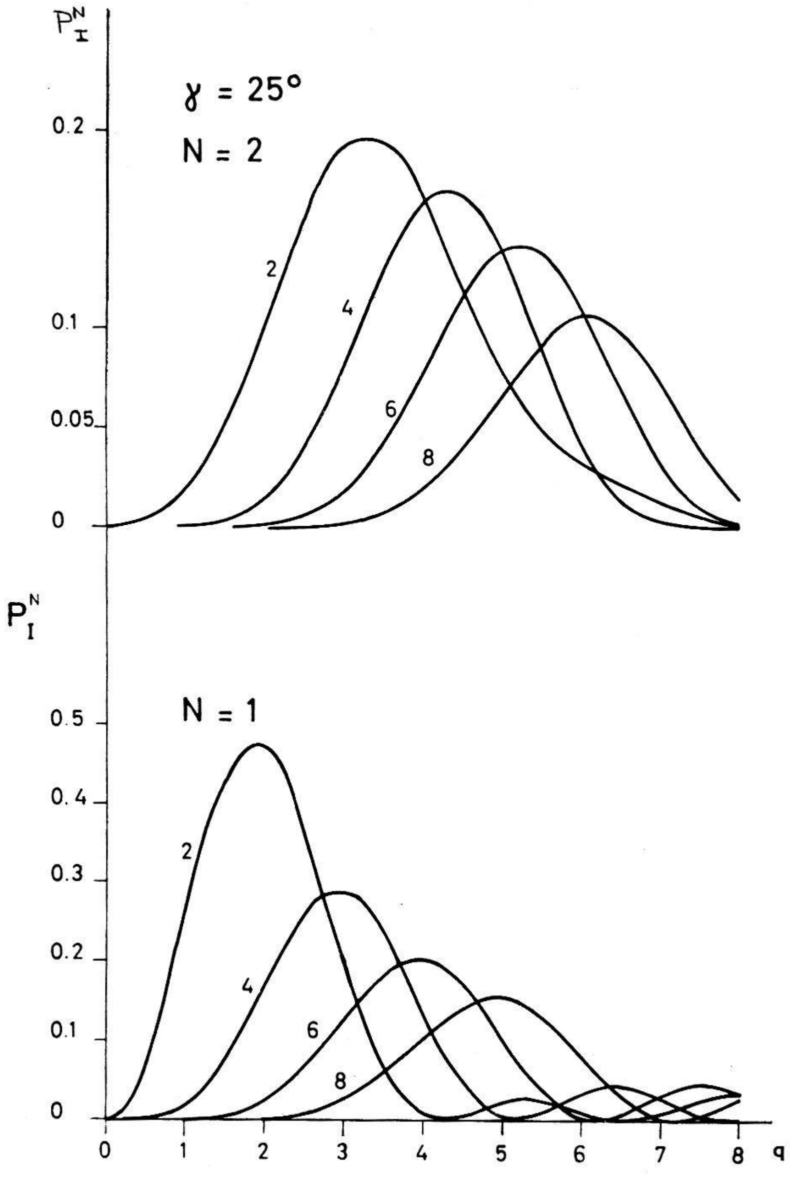



Figure 8
Asymmetric rotator. The excitation probabilities for backward scattering ($\vartheta = \pi$) – calculated in the sudden approximation according to Equations (3.38), (3.41), and (3.42) – are shown for $\gamma = 25°$ and $N = 1$ and 2.

### *D. Asymmetric Rotator*

Figures 6 and 7 show that the results for the lowest states ($N = 1$) belonging to a given even spin do not depend strongly on the parameter $\gamma$ which describes the deviation of the nuclear shape from axial symmetry. It has to be noted, however, that for a non-axial nuclear shape other states ($N \geqslant 2$ or odd spins), which do not exist for an axial shape can be excited considerably. In Figures 8 excitation probabilities of states with different quantum numbers $N$ can be compared for $\gamma = 25°$ in the case of backward scattering and $\xi = 0$.

## 5. Summary

In this work we have given formulas and numerical results for the multiple Coulomb excitation of rotational and vibrational states, for an asymmetric rotator and for coupled rotational bands. Furthermore we have developed a formalism to overcome the sudden approximation and to take into account finite $\xi$-values to any desired order.

We have only calculated the multiple excitation of rotational states for even-even nuclei, but all the computations can also be made for odd nuclei. In the discussion of coupled rotational bands we have not mentioned band mixing effects which play an important role also in $K$-allowed transitions.

The calculations have been carried out on a IBM 1620 computer at the University of Basel. We like to thank the staff of the 'Rechenzentrum' for their help and advice.

## Appendix

The orbital integrals $R_{\lambda\mu}$ for $\xi = 0$ can be expressed in terms of elementary functions. The resulting formulas are given as functions of the parameter $w$ defined in Reference [6]).

For $\lambda = 2$ we have

$$\int_{t_1}^{t_2} \overline{R}_{20}(t)\, dt = -\frac{3}{8} \operatorname{tg}^3 \frac{\vartheta}{2} \left[ \frac{s}{\sin(\vartheta/2)} (c\, b - 2) + \varphi \right] \Bigg|_{w_1}^{w_2} \tag{A.1}$$

$$\int_{t_1}^{t_2} \overline{R}_{2\pm 1}(t)\, dt = -\frac{3\, i}{8} \sqrt{\frac{2}{3}} \operatorname{tg}^3 \frac{\vartheta}{2} \frac{c^2\, b}{\sin(\vartheta/2)} \Bigg|_{w_1}^{w_2} \tag{A.2}$$

$$\int_{t_1}^{t_2} \overline{R}_{2\pm 2}(t)\, dt = -\frac{3}{8} \sqrt{\frac{3}{2}} \operatorname{tg}^3 \frac{\vartheta}{2} \left[ \frac{s}{3 \sin(\vartheta/2)} (c\, b + 2) - \varphi \right] \Bigg|_{w_1}^{w_2} \tag{A.3}$$

with

$$\varphi = \operatorname{arc\,tg} \frac{y_p}{z_p} = \operatorname{arc\,tg} \left( \frac{\sqrt{\varepsilon^2 - 1} \sinh w}{\cosh w + \varepsilon} \right) \tag{A.4}$$

$$c = \cos\varphi = \frac{z_p}{r_p} = \frac{\cosh w + \varepsilon}{\varepsilon \cosh w + 1} \tag{A.5}$$

$$s = \sin\varphi = \frac{y_p}{r_p} = \frac{\sqrt{\varepsilon^2 - 1}\,\sinh w}{\varepsilon\cosh w + 1} \tag{A.6}$$

$$b = \frac{3}{\varepsilon} - 2c = \frac{\cosh w - \varepsilon\,(2 - (3/\varepsilon^2))}{\varepsilon\cosh w + 1}. \tag{A.7}$$

The excentricity $\varepsilon$ of the hyperbolic orbit is connected with the scattering angle $\vartheta$ by

$$\varepsilon = \sin^{-1}\frac{\vartheta}{2}. \tag{A.8}$$

In particular we find

$$\begin{aligned} t &= \pm\infty \to w = \pm\infty,\ \varphi = \pm\frac{\pi - \vartheta}{2} \\ t &= 0 \quad\ \to w = 0, \quad\ \varphi = 0 \end{aligned} \tag{A.9}$$

which leads to the following expression for the classical orbital integrals:

$$\int_{-\infty}^{\infty} \bar{R}_{20}(t)\,dt = R_{20}(\vartheta, 0) = \frac{3}{4}\,\mathrm{tg}^2\frac{\vartheta}{2}\left[1 + \cos^2\frac{\vartheta}{2} - \frac{\pi - \vartheta}{2}\,\mathrm{tg}\frac{\vartheta}{2}\right] \tag{A.10}$$

$$\int_{-\infty}^{\infty} \bar{R}_{2\pm2}(t)\,dt = R_{2\pm2}(\vartheta, 0) = -\frac{3}{4}\sqrt{\frac{3}{2}}\,\mathrm{tg}^2\frac{\vartheta}{2}\left[\frac{2 + \sin^2(\vartheta/2)}{3} - \frac{\pi - \vartheta}{2}\,\mathrm{tg}\frac{\vartheta}{2}\right] \tag{A.11}$$

Table 2

The orbital integrals for $\xi = 0$ (Equations (A.10), (A.11), and (A.12) of the Appendix) and the ratios $R_{22}/R_{20}$ and $R'_{21}/R_{20}$ are listed as functions of $\vartheta$. The entries are given in the form of a number followed (in parenthesis) by the power of ten by which it should be multiplied.

| $\vartheta$ | $R_{20}(\vartheta, 0)$ | $R_{22}(\vartheta, 0)$ | $R'_{21}(\vartheta, 0)$ | $R_{22}/R_{20}$ | $R'_{21}/R_{20}$ |
|---|---|---|---|---|---|
| 10 | .106927 (−1) | −.379251 (−2) | −.409160 (−2) | −.354682 ( 0) | −.382654 ( 0) |
| 20 | .401927 (−1) | −.122952 (−1) | −.143480 (−1) | −.305907 ( 0) | −.356980 ( 0) |
| 30 | .852013 (−1) | −.223075 (−1) | −.282408 (−1) | −.261821 ( 0) | −.331460 ( 0) |
| 40 | .142908 ( 0) | −.317582 (−1) | −.437671 (−1) | −.222228 ( 0) | −.306260 ( 0) |
| 50 | .210765 ( 0) | −.393861 (−1) | −.593289 (−1) | −.186872 ( 0) | −.281493 ( 0) |
| 60 | .286350 ( 0) | −.445195 (−1) | −.736570 (−1) | −.155472 ( 0) | −.257227 ( 0) |
| 70 | .367298 ( 0) | −.469181 (−1) | −.857619 (−1) | −.127738 ( 0) | −.233494 ( 0) |
| 80 | .451270 ( 0) | −.466559 (−1) | −.949017 (−1) | −.103388 ( 0) | −.210299 ( 0) |
| 90 | .535951 ( 0) | −.440313 (−1) | −.100560 ( 0) | −.821553 (−1) | −.187629 ( 0) |
| 100 | .619072 ( 0) | −.394953 (−1) | −.102428 ( 0) | −.637977 (−1) | −.165453 ( 0) |
| 110 | .698439 ( 0) | −.335932 (−1) | −.100389 ( 0) | −.480975 (−1) | −.143734 ( 0) |
| 120 | .771976 ( 0) | −.269146 (−1) | −.945065 (−1) | −.348646 (−1) | −.122422 ( 0) |
| 130 | .837767 ( 0) | −.200528 (−1) | −.850043 (−1) | −.239360 (−1) | −.101465 ( 0) |
| 140 | .894101 ( 0) | −.135686 (−1) | −.722512 (−1) | −.151757 (−1) | −.808088 (−1) |
| 150 | .939513 ( 0) | −.796142 (−2) | −.567408 (−1) | −.847399 (−2) | −.603938 (−1) |
| 160 | .972822 ( 0) | −.364468 (−2) | −.390695 (−1) | −.374650 (−2) | −.401610 (−1) |
| 170 | .993161 ( 0) | −.927292 (−3) | −.199129 (−1) | −.933678 (−3) | −.200500 (−1) |
| 180 | 1.000000 | .000000 | .000000 | .000000 | .000000 |

and

$$\int_0^\infty \overline{R}_{2\pm1}(t)\, dt = i\, R'_{2\pm2}(\vartheta, 0) = \frac{3\,i}{8} \sqrt{\frac{2}{3}}\, \mathrm{tg}^3 \frac{\vartheta}{2} \left(2 + \cos^2 \frac{\vartheta}{2} - \frac{2}{\sin(\vartheta/2)}\right). \quad \text{(A.12)}$$

Numerical values of these integrals are given in Table 2 as a function of $\vartheta$. There we have also listed the ratios $R_{22}/R_{20}$ and $R'_{21}/R_{20}$ which are a mesure of how the term with $\mu = 0$ dominates in the excitation.

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