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Intrinsic Superselection Rules of Algebraic Hilbert Space

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Abstract. The structure of a quantum theory described by a Hilbert space over an arbitrary finite algebra with unity quantity is studied. It is found that such a quantum theory, in which the observables are linear with respect to the quantities of the algebra, is isomorphic to a quantum theory described by a Hilbert space over a field in which there are superselection rules. GLEASON's theorem is shown to be applicable; the linear manifolds generated by the minimal ideals of the algebra give rise to the pure states. The minimal ideals therefore play a role analogous to the phases in a complex Hilbert space. Illustrations are given for the complex, quaternion and Cayley algebras.

1. Introduction

Since the work of VON NEUMANN[1]) in 1932 in establishing a rigorous and clear mathematical foundation for quantum mechanics, there has been considerable effort in attempting to find generalizations of the associated algebraic structure. These efforts have followed essentially two main lines of development. The first of these has consisted in the search for explicit algebraic generalizations of the operator calculus and the structure of the Hilbert space, while the second has concentrated on the characterization of quantum mechanics through the construction of an underlying lattice of propositions.

In this paper we show that an algebraic Hilbert space, i.e., a Hilbert space over a finite algebra with unity quantity (this includes all of the algebraic generalizations so far proposed[2])) satisfies the postulates of the propositional calculus of quantum mechanics. We furthermore show that *a quantum theory described by such a Hilbert space, in which the observables are linear with respect to the quantities of the algebra, is isomorphic to a quantum theory described by a Hilbert space over a field in which there are superselection rules.* This latter result indicates that algebraic Hilbert space provides a

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complete, mathematically consistent analytic model for a quantum theory with superselection rules, and thus the two main lines of development are shown to be related.

In the following paragraphs we review briefly the developments along the two main lines referred to above.

With the idea that only commutative multiplication can have direct physical interpretation, JORDAN[3]) introduced the commutative algebras characterized by quasi-multiplication, i.e., in which $A \cdot B = {}^1/_2\ (AB + BA)$. JORDAN, VON NEUMANN and WIGNER[4]) studied the so-called r number algebras satisfying the basic combinational rules of this characterization. They found, with the help of a theorem of ALBERT[5]), that essentially new results could be obtained only for the algebra of all three-rowed Hermitian matrices with elements in the real non-associative algebra of Cayley numbers.

More recently, JAUCH and collaborators[6]) have explored the structure of a quaternion quantum mechanics, i.e., a quantum mechanics described by a Hilbert space in which the quaternions play the role of scalar multipliers. In a subsequent work, EMCH[7]) studied the unitary group representations in quaternionic Hilbert spaces (the mathematical properties of such Hilbert spaces were first studied by STONE[8]) and TEICHMÜLLER[9])) and carried out a classification of elementary systems in this theory for the Poincaré group[10]).

ALBERT[11])[12]) has shown that the real field, the field of all complex numbers, the algebra of real quaternions, and the eight dimensional algebra of all real Cayley numbers [13])[14])[15]) are the only absolute-valued real algebras with unity quantity. To complete the exploration of Hilbert spaces over division algebras, GOLDSTINE and HORWITZ[16]) carried out an investigation of a Hilbert space theory in which the non-associative algebra of Cayley numbers was admitted as the algebra of scalars. They found that such a theory could be consistently worked out within the context of the non-associative algebra if sufficient care were taken in the construction of linear manifolds. In order to obtain Fourier series representations, however, they found that it was necessary to obtain the associative closure of the algebra, and were led in this way to consider Hilbert spaces over general (finite) associative algebras[2]).

We shall review the results of the latter in some detail in the next Section, since it forms part of the mathematical basis for our work.

Let us now turn to the second main line of development, that is, the attempts to find a complete, inclusive categorization of quantum mechanics – the propositional calculus. In 1936, BIRKHOFF and VON NEUMANN[17]) formulated the «logic of quantum mechanics» in terms of the propositional calculus, and compared it to the logic of classical mechanics. In this formulation, it is postulated that the measurement of any physical quantity can be reduced to a series of experiments in which only the truth or falsehood of quantitative statements, or propositions, is relevant. In ordinary quantum mechanics, these propositions are represented by projection operators on the linear manifolds of a complex Hilbert space; these operators have eigenvalues 1 or 0 corresponding to the truth or falsehood of the associated propositions. The structure of the calculus of propositions is, however, independent of its particular representation in the complex Hilbert space.

BIRKHOFF and VON NEUMANN found that classical and quantum mechanics were both associated with an orthocomplemented lattice, but that classical systems satisfy the distributive law $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ whereas quantum systems do not. PIRON[18]) has given an axiom for quantum systems which replaces the distributive law of classical systems. Having obtained such a formulation, the question of deciding upon the possible analytical structures in which to embed the propositional calculus becomes mathematically well-posed. Conversely, if we accept the propositional formulation as a direct reflection of the basic empirical relationships of quantum mechanics, then any generalization of the algebraic and analytic structure must contain an underlying propositional calculus which is in agreement with this formulation.

The propositions of ordinary quantum mechanics, of course, satisfy these requirements, but the axioms admit systems of greater generality. It is pointed out by JAUCH and PIRON[19]) that such a proposition system may, for example, admit superselection rules.

WICK, WIGHTMAN and WIGNER[20]) showed in 1954 that there exist superselection rules in nature, i.e., that there are subspaces of the Hilbert space of quantum mechanical state vectors which cannot be connected to each other by any observable, and that a selection rule operates between these states. In the presence of superselection rules, the relative phase of components of the state vector belonging to different subspaces cannot in principle be determined. Hence it is impossible to prepare a pure state by a non-trivial linear superposition of states lying in distinct superselection subspaces. They treated, for example, the superselection rule which exists between subspaces corresponding to integer and half-integer angular momenta, and showed thereby that the Hermitian spinor fields $\psi + \psi^*$ and $i(\psi - \psi^*)$ are not measurable.

JAUCH and PIRON[19]) showed how such phenomena can be realized in the structure of the lattice of propositions (in terms of its reducibility), a possibility which was not explicitly foreseen in the work of BIRKHOFF and VON NEUMANN.

In the next Section we show that the lattice of linear manifolds of algebraic Hilbert space – II[2]) – is consistent with the calculus of propositions postulated as characterizing quantum mechanics[17])[18]). In the Section following, we introduce the concept of a state as a (generalized) probability function on the lattice of propositions in the usual way. Following the application of GLEASON's theorem[21]) to our Hilbert space [this step provides the form of the most general state as a functional over the projection operators on the linear manifolds of the space], we discuss the properties of the pure states of a physical system. We find, in fact, that it is not the complete algebra over which the space is defined that plays a role analogous to the complex phase, but rather the minimal ideals of the algebra. The group of symmetry transformations analogous to the phase transformations of the complex Hilbert space is therefore much larger than that of the complex phase group.

In the succeeding Sections we discuss two types of orthonormal expansions for vectors in algebraic Hilbert space. One of these is made up of minimal linear manifolds which are algebraically closed (closed under operations of the algebra).

The other type of orthonormal set which we shall discuss provides a representation for the operators which is strictly isomorphic to that of a quantum theory with superselection rules described by a Hilbert space over a field*). It is made up of linear

*) A demonstration of the superselection rules will be given in terms of the spectral decompositions of bounded Hermitian operators, without use of orthonormal sets, in an Appendix.

manifolds closed only with respect to the *field* over which the algebra is defined, and the lattice of propositions corresponding to manifolds of this type is not the same as the lattice of propositions corresponding to manifolds which are algebraically closed. The former lattice is in fact reducible when the latter is not (provided that the irreducible representations of the algebra are of dimension greater than one).

2. Algebraic Hilbert Space and the Lattice of Propositions of Quantum Mechanics

In this Section we define and describe the structure of a Hilbert space over a finite algebra with unity quantity*) (cf. II) and show that the linear manifolds form a lattice that is consistent with the propositional calculus postulated to be valid for the description of any quantum mechanical system.

The work of II dealt explicitly with algebras defined over the real field, but the extension of the results to the complex field is straightforward. We will not specify in our general development whether our ground field Φ is real or complex, but the involution

$$(a^*)^* = a, \ (a\, b)^* = b^*\, a^*; \ a, b \in \mathfrak{A}$$

of our algebra $\mathfrak{A}$ over Φ will be assumed effective on Φ as the complex conjugate if Φ is complex.

Our Hilbert space $\mathcal{H}$ is closed under the usual distributive and associative operations of addition and multiplication by scalars. However, the quantities a of $\mathfrak{A}$ will play the role of «scalars», and since these quantities are not commutative, we must choose between the alternatives of left and right-handed multiplication onto the elements (vectors) f in $\mathcal{H}$. All of the theorems which we shall obtain can be proved with equal facility with either convention, and therefore our general theoretical framework is the same with either choice. However, the detailed algebraic structure of the results is very much dependent upon the choice of convention. For left multiplication of scalars, for example, an operator**) which is linear with respect to all of the quantities of the algebra $\mathfrak{A}$ must commute with the algebra. In a finite dimensional model for the vector space this implies that the matrix operators that are, as we shall term this property, *totally linear*, are constructed entirely of numbers in the field Φ. For right multiplication of scalars by contrast, the totally linear operators are *all* matrices with elements in $\mathfrak{A}$.

Operators which are linear over Φ we shall simply call *linear* (totally linear operators, are, of course, linear). In the case of left multiplication of scalars, linear matrix operators may not commute with the elements of $\mathfrak{A}$, but for right multiplication of scalars there will generally be no matrix representation because the requirement that the operator be not totally linear is equivalent to the imposition of non-associativity.

It is the totally linear operators which will be of primary interest to us, since the operators representing physical observables will be postulated to be in this class. We

*) Such algebras may be represented as matrix algebras [cf. (2.12)].

**) Operators will *always* act from the left on vectors as is conventional in Hilbert spaces. An operator is said to be *linear* with respect to a set of quantities if the order of application of the operator and any of these quantities to a vector is immaterial.

therefore choose to follow II in adopting right multiplication of scalars for our main text. For the sake of completeness, however, and to illustrate the differences in algebraic structure between these alternative conventions, we reconstruct all of our results using left multiplication of scalars in an Appendix.

To complete the description of our Hilbert space we define a scalar product

$$(f, g) = (g, f)^* \tag{2.1}$$

with values in $\mathfrak{A}$. It has the properties

$$(f + g, h) = (f, h) + (g, h) \tag{2.2}$$

and

$$(f, g\, a) = (f, g)\, a\,. \tag{2.3}$$

The quantity (f, f) is symmetric $[(f, f) = (f, f)^*]$ and positive definite in any representation of $\mathfrak{A}$. It vanishes if and only if $f = 0$.

It is still necessary for us to define a real-valued norm. To do this we define a mapping tr a of every element a of $\mathfrak{A}$ onto the ground field Φ which we call the trace. Our results are dependent only upon the properties of the trace which may be derived from its definition as a diagonal sum in a representation of $\mathfrak{A}$, and we shall not therefore specify whether it is to be taken over the regular representation of $\mathfrak{A}$ or over one or more irreducible representations. With the help of this mapping, we define the norm $\| f \|$ of a vector f by

$$\| f \|^2 = \operatorname{tr}(f, f)\,. \tag{2.4}$$

Since (f, f) is positive definite, $\| f \|$ is zero if and only if f is null. It is also convenient to define a norm on our algebra $\mathfrak{A}$, i.e.,

$$| a | = \operatorname{tr}(a\, a^*)^{1/2} = \operatorname{tr}(a^*\, a)^{1/2} \tag{2.5}$$

in which the positive square root is used. It then follows[2]) that the Schwarz inequality is valid in the form

$$| \operatorname{tr}(f, g) | \leqq \| f \| \cdot \| g \| \tag{2.6}$$

or equivalently

$$| (f, g) | \leqq \| f \| \cdot \| g \|, \tag{2.7}$$

where the vertical bars are taken to indicate the usual complex modulus in case they are used on quantities of Φ.

We finally postulate that our space is complete, i.e., every Cauchy sequence in $\mathcal{H}$ converges to a limit in $\mathcal{H}$, and we shall assume for the sake of convenience that $\mathcal{H}$ is separable.

In order to discuss the lattice of algebraically closed*) linear manifolds of $\mathcal{H}$, it will be necessary for us to discuss *projections* into these manifolds. However, for our later work we will need projections into linear manifolds which are *not* algebraically

*) An algebraically closed linear manifold M contains, along with f and g, $fa + gb$ and is closed (every Cauchy sequence in M converges to a limit in M).

closed, and we therefore demonstrate in the following a result which is general enough to serve our later needs as well.

Let M be a closed linear manifold, i.e., if f and g belong to M, then $f\alpha + g\beta$ are also in M for any $\alpha, \beta \in \Phi$. For any f in $\mathcal{H}$, there is then a unique decomposition

$$f = g_0 + h_0 \tag{2.8}$$

such that g_0 is in M and tr $(h_0, g) = 0$ for all g in M.

To see that this is true*), we follow a procedure due to MURRAY (cf. II). Suppose that f does not lie in the manifold M. Then $\| f - g \|$ has a greatest lower bound r_0 over g in M which is not zero, because if r_0 were zero there would be a sequence of g's over which $\| f - g \|$ goes to zero. Since M is closed, this would imply that f is in M. There is, however, a sequence g_n of g's for which the norm of $h_n = f - g_n$ approaches r_0. Furthermore $\| h_n + h_m \| = 2 \| f - (g_n + g_m)^1/_2 \| \geqslant 2 r_0$, and therefore $\| h_n - h_m \|^2 = 2(\| h_n \|^2 + \| h_m \|^2) - \| h_n + h_m \|^2 \leqslant 2(\| h_n \|^2 + \| h_m \|^2) - 4 r_0^2 \to 0$, i.e., the h_n form a Cauchy sequence and have a limit h_0 in $\mathcal{H}$. It then follows that g_n has a limit $g_0 = f - h_0$. To show that $\mathrm{tr}(h_0, g) = 0$ for all g in M, we first note that for real λ and any g in M, $\| h_0 - g\lambda \| = \| f - g_0 - g\lambda \| \geqslant r_0 = \| h_0 \|$; hence (we take into account the possibility of complex Φ)

$$|| h_0 ||^2 + \lambda^2 || g ||^2 - 2\lambda \, \mathrm{Re\,tr}\,(h_0, g) \geqq || h_0 ||^2 ,$$

i.e., Re $\mathrm{tr}(h_0, g) = 0$. Since g and gi belong to M if Φ is the complex field, it follows that $\mathrm{tr}(h_0, g) = 0$. Finally, our construction is unique because $f = g_0 + h_0 = g_0' + h_0'$ implies that $g_0 - g_0' = h_0' - h_0$ is of norm zero. ■

We define the g_0 of (2.8) as the *projection* of f on M, i.e.,

$$g_0 = P_M f \, . \tag{2.9}$$

In case the closed linear manifold is not algebraically closed the decomposition $fa = g_0 a + h_0 a$ is not necessarily that of (2.8), i.e., $g_0 a$ may not be in M. Hence, in general, $P_M(fa) \neq (P_M f)a$. Such projections into non-algebraically closed manifolds are therefore not totally linear operators and satisfy only $\mathrm{tr}\,(f, P_M g) = \mathrm{tr}\,(P_M f, g)$. If, on the other hand, M is algebraically closed, then along with $\mathrm{tr}\,(h_0, g) = 0$, $\mathrm{tr}[(h_0, g)a] = 0$ and hence $(h_0, g) = 0$. Furthermore, $g_0 a$ is in M and therefore $P_M(fa) = (P_M f)a$, for all f in $\mathcal{H}$ and a in $\mathfrak{A}$. Such P_M are therefore totally linear, and satisfy $(f, P_M g) = (P_M f, g)$.

In II it is shown that a bounded Hermitian operator A, i.e., a linear operator satisfying

$$(f, A\, g) = (A\, f, g) \tag{2.10}$$

and defined everywhere, has a spectral resolution of the form

$$A = \int \lambda \, dP(\lambda) \, , \tag{2.11}$$

where the $P(\lambda)$ are totally linear projections. It is easy to see, in fact, that linear operators satisfying (2.10) for all f, g in $\mathcal{H}$ are totally linear.

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*) We use the symbol ■ at the end of a paragraph to indicate the conclusion of a formal proof.

To prove this assertion, it is convenient to introduce the basis[22][25] $\varrho_{\mu\nu}$ for our algebra $\mathfrak{A}$, where*)

$$\begin{gathered}\varrho_{\mu\nu}\,\varrho_{\alpha\beta} = \delta_{\nu\alpha}\,\varrho_{\mu\beta}\\ \varrho_{\mu\mu} = e_\mu \text{ (primitive idempotent)}\\ \varrho^*_{\mu\nu} = \varrho_{\nu\mu}\,.\end{gathered} \tag{2.12}$$

In terms of this basis, any

$$(f, g) = \sum_{\mu\nu} \varrho_{\mu\nu} \operatorname{tr}\,[\varrho_{\nu\mu}(f, g)] = \sum_{\mu\nu} \varrho_{\mu\nu} \operatorname{tr}\,(f\,\varrho_{\mu\nu}, g)\,, \tag{2.13}$$

and hence

$$(f, A\,g) = \sum_{\mu\nu} \varrho_{\mu\nu} \operatorname{tr}\,((f\,\varrho_{\mu\nu}), A\,g) = \sum_{\mu\nu} \varrho_{\mu\nu} \operatorname{tr}\,(A(f\,\varrho_{\mu\nu}), g) =$$
$$(A\,f, g) = \sum_{\mu\nu} \varrho_{\mu\nu} \operatorname{tr}\,((A\,f)\,\varrho_{\mu\nu}, g)\,.$$

The expansions over $\varrho_{\mu\nu}$ must be equal term by term, and therefore $\operatorname{tr}(A(f\,\varrho_{\mu\nu}), g) = \operatorname{tr}((Af)\,\varrho_{\mu\nu}, g)$. Since this equality is valid for all g, it follows that $A(f\,\varrho_{\mu\nu}) = (Af)\varrho_{\mu\nu}$ for all of the $\varrho_{\mu\nu}$, i.e., A is totally linear. ∎

It is, of course, also possible to consider a theory in which the Hermitian operators satisfy the weaker condition

$$\operatorname{tr}\,(A\,f, g) = \operatorname{tr}\,(f, A\,g)\,. \tag{2.14}$$

A «weakly Hermitian» operator satisfying (2.14) is also Hermitian, i.e., satisfies (2.10), if and only if it is totally linear. It was shown in II that a Hilbert space $\mathcal{H}_\Phi$ in which the scalar product is taken to be $\operatorname{tr}(f, g)$, and for which therefore the Hermitian operators satisfy (2.14), is isomorphic to the usual Hilbert space over Φ. The linear manifolds relevant to a quantum theory in which the weakly Hermitian operators are admitted as observables are therefore closed over Φ. The properties of such linear manifolds are well-known.

In what follows, we assume the validity of the postulate

Q. The operators which represent physical observables are contained in the class of totally linear operators on $\mathcal{H}$.

Hence, not all of the Hermitian operators on the coherent Hilbert space over Φ are admitted as observables.

We are now in a position to discuss the propositional calculus associated with the algebraically closed (physically meaningful according to postulate Q) linear manifolds of our Hilbert space. These linear manifolds in fact constitute a complete, weakly modular, orthocomplemented *lattice* with properties consistent with the calculus of propositions postulated[17][18] as characterizing quantum mechanics. The lattice is furthermore *atomic*, i.e., there exists a minimal non-null algebraically closed linear manifold within every algebraically closed linear manifold. Therefore, according to PIRON[18], the system of propositions can be embedded in a Hilbert space over a field. We shall explicitly construct the isomorphism between the present Hilbert space and a Hilbert space over Φ in a later Section. For the proof of the other assertions, we follow essentially the demonstration given by PETIT[23].

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*) In terms of a matrix representation, the $\varrho_{\mu\nu}$ may be taken to be 1 in the μ^{th} row, ν^{th} column, and zero elsewhere.

To say that the algebraically closed linear manifolds of $\mathcal{H}$ form a lattice means that there must be a partial ordering, a greatest lower bound (intersection) and a least upper bound (union). These properties follow directly from the properties of the projections uniquely associated with the algebraically closed linear manifolds; these latter properties are, as shown in II, the usual ones. These projections do not generally commute (i. e., they correspond to incompatible propositions), and the projection for the intersection of two manifolds M, N must therefore be calculated according to*)

$$P_{M \cap N} = \lim_{n \to \infty} (P_M P_N)^n . \tag{2.15}$$

It is clear that the null manifold and the entire space correspond to the absurd (0) and trivial (I) propositions and furthermore that the lattice is orthocomplemented, i.e.,

$$(M')' = M ,$$
$$M \cup (\mathcal{H} - M) = \mathcal{H} ,$$
$$M \cap (\mathcal{H} - M) = 0 ,$$

and

$$(M \cup N)' = M' \cap N' . \tag{2.16}$$

The complement M' of a linear manifold M is the usual orthogonal complement $M' = \mathcal{H} - M$. The union of two linear manifolds is the closed linear manifold spanned (over $\mathfrak{A}$) by the totality of elements of both, and the intersection is composed of the closed linear manifold of all elements common to both.

Our lattice is furthermore weakly modular, i.e.,

$$M \subseteq N \text{ implies } (M \cup N') \cap N = M , \tag{2.17}$$

since N' is in the orthogonal complement of M, and among the elements of the union of M and N' those in common with N are in M.

Two propositions M and N are said to be *compatible* if they satisfy the symmetrical relation

$$(M \cap N') \cup N = (N \cap M') \cup M .$$

The axiom of PIRON[18]) which replaces the distributive law of the lattice of propositions of classical mechanics can be stated as follows:

P. If $M \subseteq N$, then M and N are compatible.

Since M and N are algebraically closed linear manifolds, they are uniquely described by the properties of their totally linear projections. If $M \subseteq N$, then the projections commute and the compatibility relation follows directly.

Finally, to discuss the atomic property of our lattice, we will need some results from the theory of rings[24]), i.e., those which deal with *ideals. A* set ξ of elements of $\mathfrak{A}$ is called a right ideal if it is *not identical to* $\mathfrak{A}$ and $x \in \xi$, $a \in \mathfrak{A}$ implies that $xa \in \xi$, i.e., ξ is closed under multiplication on the right by any of the elements of $\mathfrak{A}$. A right

*) We use the same symbols in the following for the linear manifolds and the propositions to which they correspond.

ideal of $\mathfrak{A}$ is said to be maximal if it is not contained in any other right ideal of $\mathfrak{A}$. It is true [24]) for the algebras $\mathfrak{A}$ with unity quantity that every right ideal can be extended to a maximal right ideal. Left ideals can of course be analogously defined.

A right ideal ξ generates on an element f of $\mathcal{H}$ a manifold which we shall call an ideal linear manifold $f\xi$; clearly an ideal linear manifold is algebraically closed. In the following, we show that the existence of a maximal ideal in $\mathfrak{A}$ implies the existence of a (non-null) minimal algebraically closed linear manifold in the algebraically closed linear extension $f\mathfrak{A}$ of every element f of $\mathcal{H}$, i.e., the corresponding lattice of propositions is *atomic*.

The structure of the usual complex Hilbert space of quantum mechanics is much simpler in this respect, but not by any means trivial. The algebraically closed linear extension of a (normalized) vector ψ of the complex Hilbert space is $\{e^{i\varphi}\psi\}$ for all real φ. These «rays» are the minimal algebraically closed linear manifolds which generate the pure states in a phase-independent way (we shall discuss this point further in the next Section). It is just this freedom of phase which permits, for example, the introduction of electromagnetic interactions in a gauge invariant way.

To prove the assertion that our lattice is atomic, we note that the relation $f\xi \supseteq f\eta$, for ξ, η two ideals, implies that for every $y \in \eta$ there exists an $x \in \xi$ such that $fx = fy$, i.e., $f(x - y) = 0$. Hence $x - y$ belongs to the annihilator ideal ζ_f of f (ζ_f contains all $a \in \mathfrak{A}$ such that $fa = 0$), and therefore*)

$$f\eta \subseteq f\xi \text{ implies } \eta \subseteq (\xi \cup \zeta_f) \,. \tag{2.18}$$

There exists, however, a maximal ideal ξ_{max} in $\mathfrak{A}$, and therefore a manifold $M_{max} = f\xi_{max}$ in $f\mathfrak{A}$. If ξ_{max} differs from $\mathfrak{A}$ by ζ_f, then $f\xi_{max} = f\mathfrak{A}$. We show below (at the conclusion of this Section) that for every f a ξ_{max} exists for which the last equality does *not* hold, and we assume in what follows that ξ_{max} is chosen in this way. It is easy to show that $f\mathfrak{A}$ is closed (Cauchy sequences have limits in $f\mathfrak{A}$) and therefore that M_{max} has an orthogonal complement $M'_{max} \neq 0$ *in* $f\mathfrak{A}$, i.e., $M'_{max} \cup M_{max} = f\mathfrak{A}$. Suppose that there exists an $M \subset M'_{max}$ which is neither M'_{max} nor null. Then $M \cup M_{max}$ is neither $f\mathfrak{A}$ nor M_{max}; to this manifold corresponds an ideal ξ such that $M \cup M_{max} = f\xi$, and according to (2.18), $\xi \cup \zeta_f \supseteq \xi_{max}$. If $\xi \cup \zeta_f = \xi_{max}$, then $M \cup M_{max}$ is $f\xi_{max}$, contrary to hypothesis. But then we have found an ideal larger than ξ_{max} which is not all of $\mathfrak{A}$ (if $\xi \cup \zeta_f = \mathfrak{A}$, then $f\xi = f(\xi \cup \zeta_f) = f\mathfrak{A}$), and have therefore arrived at a contradiction. The manifold M'_{max} is therefore a minimal algebraically closed linear manifold. ∎

It is clear from the above that our lattice satisfies the covering law [18]) for the minimal manifolds $M'_{max} \equiv M_{min}$, i.e., $N \subseteq M \subseteq N \cup M_{min}$ implies that $M = N$ or $M = N \cup M_{min}$.

Let us state explicitly what these minimal manifolds are in terms of the *minimal* ideals of our algebra $\mathfrak{A}$ in a form that will be useful for our later work. Every ideal is generated [25]) by an idempotent of the algebra $\mathfrak{A}$, and, in particular, these minimal ideals are generated by the primitive idempotents defined in (2.12). Hence every M_{min} is of the form

$$M_{min} = \{f e_\mu a\}$$

for f and μ fixed and a ranging over $\mathfrak{A}$.

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*) We freely use $\cup$ and $\cap$ among the subsets of $\mathfrak{A}$ as well as the manifolds of $\mathcal{H}$.

To complete the discussion of this Section, let us prove the assertion: there always exists an M_{max} which is not $f\,\mathfrak{A}$. There is a minimal right ideal $e_\mu\,\mathfrak{A}$ for which $M_{min} \neq 0$; if $M_{min} = 0$ for every choice of μ, then $f e_\mu = 0$ for every μ. Since $\sum_\mu e_\mu$ is the unity quantity of $\mathfrak{A}$, this implies that $f = 0$. ■

3. States and the Minimal Ideals

In the preceding Section, we showed that the lattice of propositions (algebraically closed linear manifolds) associated with our algebraic Hilbert space is consistent with the conditions imposed upon a lattice which is capable of the description of a quantum mechanical system. To understand the role played by the algebra $\mathfrak{A}$ in the description of such a physical system, it will be necessary for us to introduce the notion of a *state*, and to find an explicit expression for it in terms of the quantities of the Hilbert space. A constructive procedure of this type has been carried out for the usual complex Hilbert space (for example, Ref. [26]), and we review this case first.

A state is defined as a function $w(M)$ on the set of all propositions of the lattice, with the properties [19])

$$0 \leqq w(M) \leqq 1\,;$$
$$w(\phi) = 0, \quad w(I) = 1\,;$$

$$\begin{aligned}&\text{if } M \text{ is compatible with } N,\ w(M) + w(N) = w(M \cap N) + w(M \cup N)\,;\\ &\text{if } w(M) = w(N) = 1 \text{ then } w(M \cap N) = 1\,;\\ &\text{if } M \neq 0, \text{ there exists a state } w \text{ such that } w(M) \neq 0.\end{aligned} \tag{3.1}$$

Two states are different if there exists a proposition M such that $w_1(M) \neq w_2(M)$. If w_1 and w_2 are two different states then $w(M) = \lambda_1\, w_1(M) + \lambda_2\, w_2(M)$, with $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_1 + \lambda_2 = 1$, defines a new state. A state which can be represented in this way with two different states is called a *mixture*; a state which is not a mixture is called *pure*.

Gleason [21]) has shown that for an irreducible system of propositions (a lattice with no element but ϕ and I compatible with all elements; called *coherent* by Jauch and Piron [19])) realized by the projections P_M of a separable Hilbert space, real or complex but of dimension $\geqslant 3$, there exists a density matrix ϱ for every continuous*) state such that

$$w(M) = \mathrm{Tr}\,(\varrho\, P_M)\,, \tag{3.2}$$

where we use «*Tr*» to distinguish the trace over the entire Hilbert space from our algebraic trace «*tr*». For a Hilbert space over a field, it is easy to show that there exists one and only one state $w(M)$ such that $w(\hat{M}) = 1$ for a given $\hat{M}$ which is a *minimal* closed linear manifold.

We have already stated in Section 2 that our algebraic Hilbert space is isomorphic to the usual Hilbert space over a field Φ with respect to the scalar product $\mathrm{tr}(f, g)$. It is this description which is appropriate when all of the weakly Hermitian operators are admitted as potentially observable. We use it in the following to prove the assertion made above.

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*) Cf. Ref. [21]). It has been pointed out to us that this condition can be weakened to the following: let P_{M_0} be the limit of the ordered sequence $P_{M_1} \leq P_{M_2} \leq \ldots$ Then $\omega(M_i) \to \omega(M_0)$.

For $\mathcal{H}_\Phi$, (3.2) can be written[26])

$$w(M) = \sum_i \gamma_i \operatorname{tr}(f_i, P_M f_i) , \tag{3.3}$$

where $\gamma_i > 0$ and $\sum_i \gamma_i = 1$. We wish to emphasize that the linear manifolds we consider here are closed only over Φ, and the corresponding projections are not totally linear operators. Hence our postulate Q is not satisfied for these projections. What we are therefore dealing with is precisely the coherent lattice corresponding to the general complex (or real) Hilbert space.

Let $\hat{P}$ be the projection [in the sense of (2.8) and (2.9)] on the minimal manifold $\hat{M} = \{f_0 \lambda\}$ over $\lambda \in \Phi$. For the state $w(M)$ for which $w(\hat{M}) = 1$,

$$0 = \sum_i \gamma_i \operatorname{tr}(f_i, (I - \hat{P}) f_i) . \tag{3.4}$$

The projection $\hat{P}$ has the weak Hermitian property (2.14) and is idempotent, hence each term of (3.4) is non-negative, i.e.,

$$\operatorname{tr}\left((I - \hat{P}) f_i, (I - \hat{P}) f_i\right) = \| (I - \hat{P}) f_i \|^2 = 0 . \tag{3.5}$$

It therefore follows that

$$f_i = P f_i = f_0 \lambda_i . \tag{3.6}$$

If $\| f_0 \|^2 = \| f_i \|^2 = 1$, it follows that $| \lambda_i |^2 = 1$. Hence (3.3) takes the following form

$$w(M) = \sum_i \gamma_i \, | \lambda_i |^2 \operatorname{tr}(f_0, P_m f_0) = \operatorname{tr}(f_0, P_m f_0) . \ \blacksquare \tag{3.7}$$

Each of the elements f_0 of $\mathcal{H}$ therefore defines a pure state through its closed linear extension over the field Φ, and the applicability of the word «state» to the vectors of $\mathcal{H}$ is in this sense justified.

The states (3.7) defined over the non-totally linear P_M are not, in general, *physical* states in view of our postulate Q. However, we assume that the requirements (3.1) for a state function $w(M)$ over the algebraically closed linear manifolds M are valid, since these refer to the lattice of propositions and not to the particular analytical framework in which it is embedded.

In the next Section we show that, with respect to the totally linear operators, our Hilbert space $\mathcal{H}$ and its transformations are isomorphic to the direct sum of a finite sequence of Hilbert spaces over Φ. The isomorphism is constructed as follows. Under the totally linear operators on $\mathcal{H}$, the subspaces $\mathcal{H}_\mu = \mathcal{H}\, e_\mu$, where e_μ is a primitive idempotent, are invariant. These subspaces are not *algebraically* closed, but each is isomorphic to a Hilbert space over Φ which is invariant under the totally linear operators. The totally linear operators are represented completely by the set of all scalar products of the kind $\operatorname{tr}(f, g)$, between elements of the same subspace, from all of these subspaces.

In each of these subspaces we may write the most general function of the manifolds in $\mathcal{H}_\mu$ (closed over Φ) satisfying (3.1), as

$$w_\mu (M_\Phi) = \sum_i \gamma_i \operatorname{tr}(f_i^\mu, P_{M\Phi} f_i^\mu) , \tag{3.8}$$

where the $f_i^\mu = f_i e_\mu$ belong to $\mathcal{H}_\mu$ and the M_Φ are linear manifolds, closed over Φ, in $\mathcal{H}_\mu$. For each M_Φ, however, there exists an *algebraically* closed linear manifold M in $\mathcal{H}$ such that

$$P_M f_i^\mu = P_{M_\Phi} f_i^\mu \, . \tag{3.9}$$

Suppose that M_Φ is spanned by $g_1 e_\mu, g_2 e_\mu, \ldots$ over Φ. Let M be the algebraically closed linear extension of M_Φ, i.e., M is spanned over $\mathfrak{A}$ by the same elements. Hence, according to (2.8),

$$f_i = g_1 e_\mu a_1 + g_2 e_\mu a_2 + \ldots + h \, , \tag{3.10}$$

where

$$(h, g_1 e_\mu) = (h, g_2 e_\mu) = \ldots = 0 \, . \tag{3.11}$$

Furthermore,

$$f_i e_\mu = f_i^\mu = g_1 e_\mu (a_1)_{\mu\mu} + g_2 e_\mu (a_2)_{\mu\mu} + \ldots + h e_\mu \, , \tag{3.12}$$

where $(a_1)_{\mu\mu}, (a_2)_{\mu\mu}, \ldots$ are in Φ, and with the help of (3.11) we obtain

$$\mathrm{tr}\,(h e_\mu, g_1 e_\mu) = \mathrm{tr}\,(h e_\mu, g_2 e_\mu) = \ldots = 0 \, .$$

Since (3.10) and (3.12) are unique decompositions of the form (2.8), it follows that (3.9) is valid. ■

The result (3.9) enables us to replace M_Φ by M in the expression (3.8), i.e., we have that

$$w_\mu(M) = \sum_i \gamma_i \, \mathrm{tr}\,(f_i^\mu, P_M f_i^\mu) \tag{3.13}$$

for the manifolds M which correspond to the algebraically closed linear extensions of the M_Φ in $\mathcal{H}_\mu$. However, *every* M is the algebraically closed linear extension of an M_Φ in $\mathcal{H}_\mu$. It suffices to verify this for an M generated by a single element f of $\mathcal{H}$. Let $M = \{fa\}$ over $a \in \mathfrak{A}$. Then $\{fa\} e_\mu = \{\sum_\nu f \alpha_{\nu\mu} \varrho_{\nu\mu}\}$, with $\alpha_{\nu\mu} \in \Phi$, is a linear manifold closed over Φ in $\mathcal{H}_\mu$. Its algebraically closed linear extension is $\{\sum_\nu f \alpha_{\nu\mu} \varrho_{\nu\mu} \cdot b\}$ over $b \in \mathfrak{A}$, i.e., $\{\sum_{\nu\lambda} f \alpha_{\nu\mu} \beta_{\mu\lambda} \varrho_{\nu\lambda}\}$ with $\alpha_{\nu\mu}, \beta_{\mu\lambda} \in \Phi$. Since $\alpha_{\nu\mu} \beta_{\mu\lambda}$ is arbitrary in Φ, this last is identical with $\{fa\}$. Hence $w_\mu(M)$ is defined over all of the algebraically closed linear manifolds M of $\mathcal{H}$ and for these it is the most general function in $\mathcal{H}_\mu$ satisfying (3.1). ■

Since the $\mathcal{H}_\mu$ are orthogonal in $\mathcal{H}_\Phi$ and invariant under the operations of the class of totally linear operators on $\mathcal{H}$, the general state $w(M)$ is (cf., Ref.[26]), p. 135)

$$w(M) = \sum_\mu \lambda_\mu w_\mu(M) \, , \tag{3.14}$$

where $\sum_\mu \lambda_\mu = 1$, $\lambda_\mu \geqslant 0$. Hence (set $\lambda_\mu \gamma_i = \gamma_i^\mu$)

$$w(M) = \sum_{i\mu} \gamma_i^\mu \, \mathrm{tr}\,(f_i^\mu, P_M f_i^\mu) = \sum_{i\mu\nu} \sqrt{\gamma_i^\mu} \sqrt{\gamma_i^\nu} \, \mathrm{tr}\,(f_i^\mu, P_M f_i^\nu) \, ,$$

since $\mathrm{tr}(f_i^\mu, P_M f_i^\nu) = 0$ for $\mu \neq \nu$. After normalizing the vectors $\sum_\mu \sqrt{\gamma_i^\mu} f_i^\mu$ we obtain that (3.3) is valid as the most general form for a function satisfying (3.1) and (3.14), and defined over the algebraically closed linear manifolds of $\mathcal{H}$. ■

In the remaining paragraphs of this Section we discuss some of the properties of the states (3.3).

According to our postulate Q, only the totally linear projections are admissible as physical observables. In order to construct a *pure* state, we must therefore choose a minimal algebraically closed linear manifold of $\mathcal{H}$, and require that $w(M)$ assume the value unity when M is that manifold.

To illustrate this construction, let $\hat{P}$ be the (totally linear) projection on the minimal algebraically closed linear manifold $\hat{M} = \{ f_0 e_\mu a \}$ over $a \in \mathfrak{A}$. For the state $w(M)$ for which $w(\hat{M}) = 1$,

$$0 = \sum_i \gamma_i \operatorname{tr} (f_i (I - \hat{P}) f_i) ,$$

is in (3.4), and therefore

$$f_i = \hat{P} f_i = f_0 e_\mu a_i . \tag{3.15}$$

The normalization of f_i implies that

$$\| f_i \|^2 = \operatorname{tr} (f_0 e_\mu a_i, f_0 e_\mu a_i) = \operatorname{tr} [e_\mu a_i a_i^* e_\mu (f_0, f_0)] = (a_i a_i^*)_{\mu\mu} (f_0, f_0)_{\mu\mu} = 1 . \tag{3.16}$$

We take $(f_0, f_0)_{\mu\mu} = 1$; it then follows that

$$(a_i a_i^*)_{\mu\mu} = 1 . \tag{3.17}$$

If we now substitute (3.15) into (3.3), we obtain

$$\begin{aligned} w(M) &= \sum_i \gamma_i \operatorname{tr} (f_0 e_\mu a_i, P_M f_0 e_\mu a_i) \\ &= \sum_i \gamma_i \operatorname{tr} [e_\mu (a_i a_i^*) e_\mu (f_0, P_M f_0)] \\ &= \sum_i \gamma_i \operatorname{tr} e_\mu (f_0, P_M f_0) \\ &= \operatorname{tr} e_\mu (f_0, P_M f_0) . \end{aligned} \tag{3.18}$$

The pure states are therefore of the form (3.18), which has an analogy to the form of a mixed state in which the density matrix has been replaced by a projection (e_μ). If we had taken for $\hat{M}$ the linear extension of f_0 over the whole algebra $\mathfrak{A}$ instead of the ideal $e_\mu \mathfrak{A}$, the density matrix of which e_μ is an extreme form would have occurred in (3.18).

To see this, let $M = \{ f_0 a \}$ over $a \in \mathfrak{A}$. For the $w(M)$ for which $w(\hat{M}) = 1$ we then obtain, from (3.3),

$$f_i = \hat{P} f_i = f_0 a_i . \tag{3.19}$$

The normalization of f_i implies that

$$\operatorname{tr} (f_0 a_i, f_0 a_i) = \operatorname{tr} [a_i a_i^* (f_0, f_0)] = 1, \tag{3.20}$$

and substituting (3.19) into (3.3) we have

$$w(M) = \sum_i \gamma_i \operatorname{tr} [a_i a_i^* (f_0, P_M f_0)] = \operatorname{tr} [\varrho_{\mathfrak{A}} (f_0, P_M f_0)] , \tag{3.21}$$

where

$$\varrho_{\mathfrak{A}} = \sum_i \gamma_i \, a_i \, a_i^* \tag{3.22}$$

is analogous to a density matrix in the algebra $\mathfrak{A}$. It satisfies

$$\mathrm{tr}\,[\varrho_{\mathfrak{A}}\,(f_0, f_0)] = 1 \; . \; \blacksquare \tag{3.23}$$

In the case of a quantum mechanics described by a Hilbert space over a field Φ, the elements $f \in \mathcal{H}$, multiplied by the elements of unit modulus in Φ (a ray) generate the pure states through the procedure typified by (3.4)–(3.7). The *apparent* analogue to the ray of a Hilbert space over Φ, i.e., the algebraically closed linear extension $\{ f_0 \, a \}$ of a vector f_0, on the other hand, generates a *mixed* state of the form (3.21). The mixture has its source, of course, in the fact that the manifold $\{ f_0 \, a \}$ is not minimal; its presence indicates that our formal structure may be useful in describing systems with superselection rules (cf., Ref. [26]), p. 135).

The pure states, as we have seen, are generated through the procedure typified by (3.4)–(3.7) by the *minimal* algebraically closed linear extensions of elements of $\mathcal{H}$, i.e., by the manifolds $\{ f_0 \, e_\mu \, a \}$ over $a \in \mathfrak{A}$. Hence it is the (normalized) elements $f_0 \, e_\mu \, a$ which are in fact analogous to the elements $f_0 \, e^{i\varphi}$ of a complex Hilbert space, and therefore the quantities $e_\mu \, a$ of the minimal right ideals $e_\mu \, \mathfrak{A}$ of $\mathfrak{A}$ ($(aa^*)_{\mu\mu} = 1$) play the role of a *generalized phase*. Since

$$e_\mu \, a = \sum_\nu \alpha_\nu \, \varrho_{\mu\nu} \tag{3.24}$$

and the normalization implies that [for $(f_0, f_0)_{\mu\mu} = 1$]

$$\sum_\nu | \, \alpha_\nu \, |^2 = 1 \; , \tag{3.25}$$

the «phase» freedom of a minimal manifold in the algebraic Hilbert space consists of the unitary transformations in an n dimensional Euclidean vector space, where n is the number of primitive idempotents of $\mathfrak{A}$.

We note in passing that this vector space (generated by a primitive idempotent) transforms under the operations of $\mathfrak{A}$ according to the irreducible representation associated with the idempotent [25]). If the connecting element $\varrho_{\mu\nu}$ is not zero, then the representations associated with e_μ and e_ν are equivalent; the distinct irreducible representations are therefore associated with the (two-sided) ideals which are not connected by such equivalence mappings. This symmetry is reflected in the structure of the pure states in the following way. Let $w(M)$ be the pure state $\mathrm{tr}\; e_\mu (f, P_M f)$ and $w'(M)$ the pure state $\mathrm{tr}\; e_\nu \, (g, P_M g)$. If f and g are related by $g e_\nu = f \varrho_{\mu\nu}$, then

$$w'(M) = \mathrm{tr}\,[e_\nu \, (f \varrho_{\mu\nu}, P_M f \varrho_{\mu\nu})] = \mathrm{tr}\,[\varrho_{\mu\nu} \, (f, P_M f) \, \varrho_{\mu\nu}] = \mathrm{tr}\; e_\mu \, (f, P_M f) \; ,$$

i.e., $w'(M) = w(M)$; the transition probabilities [27]) are therefore preserved under this transformation.

It is clear from what has been discussed so far that we are dealing, in effect, with *two* lattices. One of these is the lattice of propositions corresponding to the linear manifolds closed over Φ; it is reducible [19]) with respect to the *primitive* idempotents of the algebra $\mathfrak{A}$, and the representations of the totally linear operators on the Hilbert space $\mathcal{H}_\Phi$ over Φ are therefore also reducible. The other lattice of propositions corresponds to the linear manifolds closed over $\mathfrak{A}$, and is reducible only over the idempotents of $\mathfrak{A}$ which generate the two-sided ideals (inequivalent irreducible representations). For $\mathfrak{A}$ an algebra with only one irreducible representation, the first kind of lattice is reducible if the dimension of the representation is greater than 1, while the second is not. Since the usual representation of quantum mechanics is given by a Hilbert space $\mathcal{H}_\Phi$, it is the reducibility of the first kind of lattice which corresponds to superselection rules in this context.

In the next Section we shall construct sets of basis vectors in $\mathcal{H}$ which make explicit the reducibility of the observables in both kinds of lattices.

4. Complete Orthonormal Sets and Representations of the Linear Operators

There are two essentially different ways of constructing an orthonormal set in $\mathcal{H}$, corresponding to the use of manifolds closed over Φ or over $\mathfrak{A}$. Since, as we have shown, the pure states are generated by the minimal algebraically closed linear manifolds $\{ f e_\mu a \}$ over $a \in \mathfrak{A}$, it is of interest first to construct an orthonormal set which spans $\mathcal{H}$ with coefficients in the minimal ideals of $\mathfrak{A}$. We then consider the construction of a basis which spans $\mathcal{H}$ over Φ. A strict isomorphism is obtained between $\mathcal{H}$ and its operator calculus and a (reducible) Hilbert space over Φ.

Suppose that the sequence $f_1, f_2 \ldots$ is dense in the (separable) Hilbert space, i.e., any element g is the limit of a subsequence of this sequence. If f_2 is not entirely contained in $\{ f_1\, a \}$ over $a \in \mathfrak{A}$, then there is [according to (2.8)] a unique decomposition

$$f_2 = f_1\, a_2 + h_2\,, \tag{4.1}$$

where $(h_2, f_1) = 0$ and $h_2 \neq 0$. If f_2 is contained in $\{ f_1 a \}$, then we proceed in the sequence until a member is found which is not; we then call *this* element f_2 and carry out the procedure described above. In the same way, we find an element f_3 which is not contained entirely in the algebraically closed linear manifold spanned by $\{ f_1 a \}$ and $\{ h_2 a \}$. Let $f_1 \equiv \varphi_1$ and $\{ \varphi_1 a \} \equiv M_1$, $h_2 \equiv \varphi_2$ and $\{ \varphi_2 a \} \equiv M_2$. Then

$$f_3 = \varphi_1\, a_3 + \varphi_2\, a_3' + h_3\,, \tag{4.2}$$

where $(h_3, \varphi_1) = (h_3, \varphi_2) = 0$ and we set $h_3 \equiv \varphi_3$. This process can be continued until $\mathcal{H}$ is completely spanned by the manifolds $M_1, M_2, \ldots$

To show that the set

$$\varphi_1, \varphi_2, \ldots ;\ (\varphi_i, \varphi_j) = 0 \text{ for } i \neq j \tag{4.3}$$

is complete, we note that at each step of the procedure, all of the sequence $f_1, f_2, \ldots$ that has been used in the construction is included in the union of the closed linear manifolds so far generated. For example, f_1 is certainly contained in M_1; f_2, as can be seen from its decomposition (4.1), is included in $M_1 \cup M_2$ $(h_2 = \varphi_2)$; f_3, as can be seen from (4.2) is in $M_1 \cup M_2 \cup M_3$, etc. Since $f_1, f_2, \ldots$ is dense in $\mathcal{H}$, we have therefore spanned all of $\mathcal{H}$ with the algebraically closed linear extensions of the elements of the sequence (4.2). ■

We now turn to the decomposition of the manifolds M_i into orthogonal *minimal* algebraically closed linear manifolds. What we will show is that each vector φ_i of the set (4.3) can be further decomposed into $\psi_1^{(i)}, \psi_2^{(i)}, \ldots$ where the $\psi_\mu^{(i)}$ are equal in number to the primitive idempotents of $\mathfrak{A}$ (if none of the $\psi_\mu^{(i)}$ vanish accidentally) and satisfy

$$(\psi_\mu^{(i)}, \psi_\nu^{(j)}) = e_\mu \, \delta_{\mu\nu} \, \delta_{ij} \,. \tag{4.4}$$

Any vector f in $\mathcal{H}$ then has the expansion

$$f = \sum_{\mu i} \psi_\mu^{(i)} \, a_{\mu i} \tag{4.5}$$

where it suffices for $a_{\mu i}$ to be a quantity in the minimal ideal $e_\mu \mathfrak{A}$. To construct the set $\psi_\mu^{(i)}$ associated with φ_i, let $\psi_1^{(i)} = \varphi_i e_1$, and $M_1 = \{ \psi_1^{(i)} a \}$. Since $\psi_1^{(i)} e_1 = \psi_1^{(i)}$, it suffices that a range over the minimal ideal $e_1 \mathfrak{A}$. We may then decompose $\varphi_i e_2$ into a part $\psi_1^{(i)} a_2$ in M_1 and a part h_2 satisfying $(h_2, \psi_1^{(i)}) = 0$. Since $\varphi_i e_2 \cdot e_2 = \varphi_i e_2$ and the decomposition is unique, it follows that $h_2 \, e_2 = h_2$ (and $e_1 \, a_2 \, e_2 = a_2$). We therefore call $h_2 = \psi_2^{(i)}$, and proceed in this way until all of the idempotents of $\mathfrak{A}$ have been used. Clearly the manifolds $\{ \psi_\mu^{(i)} \, a \}$ over μ and $a \in \mathfrak{A}$ span $\{ \varphi_i a \}$, and are therefore complete in $\mathcal{H}$ over i, μ and $a \in \mathfrak{A}$. Since $\psi_\mu^{(i)} \, e_\mu = \psi_\mu^{(i)}$, the quantity $(\psi_\mu^{(i)}, \psi_\mu^{(i)})$ is proportional to e_μ with a positive real multiple, and (unless $\psi_\mu^{(i)}$ is zero) the normalization to the form (4.4) can easily be carried out. ∎

Since every f has the expansion (4.5), the vector $A \, \psi_\mu^{(i)}$ is also of the form (4.5), i.e.,

$$A \, \psi_\mu^{(i)} = \sum_{\nu j} \psi_\nu^{(j)} \, a_{\nu j, \mu i} \,. \tag{4.6}$$

If we now operate with B, we obtain

$$B(A \, \psi_\mu^{(i)}) = \sum_{\nu j} B \, (\psi_\nu^{(j)} \, a_{\nu j, \mu i}) \,. \tag{4.7}$$

Since $a_{\nu j, \, \mu i} \in e_\nu \mathfrak{A}$, we may write it as

$$a_{\nu j, \mu i} = \sum_\lambda{}' \alpha_{\nu j, \mu i}^{(\lambda)} \, \varrho_{\nu \lambda} \,, \tag{4.8}$$

with $\alpha_{\nu j, \, \mu i}^{(\lambda)}$ in Φ. If B is linear over Φ, the factors $\alpha_{\nu j, \, \mu i}^{(\lambda)}$ can be extracted from the parentheses on the right side of (4.7). However, the relation

$$B \, \psi_\nu^{(j)} = \sum_{\nu' k} \psi_{\nu'}^{(k)} \, b_{\nu' k, \nu j} \,, \tag{4.9}$$

does not define $B(\psi_\nu^{(j)} \, \varrho_{\nu\lambda})$. Hence it is not possible to obtain a representation of the (non-totally) linear operators on $\mathcal{H}$ if the $\psi_\mu^{(i)}$ are used as a basis.

However, if the operators A and B are totally linear, then the $a_{\nu j, \, \mu i}$ occurring in (4.6) are of the form $\alpha_{\nu j, \, \mu i} \, \varrho_{\nu\mu}$ where $\alpha_{\nu j, \, \mu i} \in \Phi$, and, similarly, in (4.9), $b_{\nu' k, \, \nu j} = \beta_{\nu' k, \, \nu j} \, \varrho_{\nu' \nu}$. Hence

$$B \, A \, \psi_\mu^{(i)} = \sum_{\nu' k, \nu j} \psi_{\nu'}^{(k)} \, \beta_{\nu' k, \nu j} \, \varrho_{\nu' \nu} \, \alpha_{\nu j, \mu i} \, \varrho_{\nu\mu} = \sum_{\nu' k} \psi_{\nu'}^{(k)} \, \varrho_{\nu' \mu} \sum_{\nu j} \beta_{\nu' k, \nu j} \, \alpha_{\nu j, \mu i} \tag{4.10}$$

and the coefficients $\alpha_{\nu j, \, \mu i}$, $\beta_{\nu' k, \, \nu j}$ form a matrix representation of A and B over Φ. Such matrix representations over Φ define a Hilbert space over Φ in which the operator calculus of totally linear operators on $\mathcal{H}$ may be embedded, but this Hilbert space and its operator calculus is not isomorphic to $\mathcal{H}$ and its operator calculus since the vectors $\psi_\mu^{(i)}$ are not carried into a sum of vectors with coefficients in Φ, i.e., we have the additional transformation $\varrho_{\nu\mu}$ in

$$A\,\psi_\mu^{(i)} = \sum_{\nu j} \psi_\nu^{(j)}\,\varrho_{\nu\mu}\,\alpha_{\nu j,\mu i}\,. \tag{4.11}$$

We note, however, that $\alpha_{\nu j,\,\mu i} = 0$ if e_μ and e_ν correspond to *inequivalent* irreducible representations of $\mathfrak{A}$, and therefore the Hilbert space over Φ referred to above (for totally linear operators) is reducible with respect to the minimal *two-sided* ideals of $\mathfrak{A}$. This decomposition corresponds to the reducibility of the lattice of manifolds in which the centre includes $\{f e_r\}$ over all $f \in \mathcal{H}$, where the e_r are the idempotents (generally not primitive) which distinguish the inequivalent irreducible representations.

We now turn to the construction of a basis which spans $\mathcal{H}$ over Φ.

Consider again the sequence $f_1, f_2 \ldots$ which is dense in $\mathcal{H}$. Let M_1 be the manifold $\{\sum_\mu f_1\, e_\mu\, \lambda_\mu)$ over all μ, $\lambda_\mu \in \Phi$. Then (if f_2 is not entirely in M_1)

$$f_2 = \sum_\mu f_1\, e_\mu\, \lambda_\mu^{(2)} + h_2$$

where $\mathrm{tr}[(h_2, f_1)\, e_\mu] = 0$ for all of the μ. We then call $f_1 = \chi_1$, $h_2 = \chi_2$ and decompose f_3 as follows:

$$f_3 = \sum_\mu \chi_1\, e_\mu\, \lambda_\mu^{(3)} + \sum_\mu \chi_2\, e_\mu\, \lambda_\mu^{'(3)} + h_3\,.$$

It then follows that $h_3 = \chi_3$ satisfies

$$\mathrm{tr}\,[(\chi_3, \chi_1)\, e_\mu] = \mathrm{tr}\,[(\chi_3, \chi_2)\, e_\mu] = 0$$

for all of the μ. If this process is continued until the sequence $f_1, f_2 \ldots$ is exhausted, we obtain a set $\chi_i e_\mu$ of vectors satisfying (after normalization)

$$\mathrm{tr}\,(\chi_i\, e_\mu, \chi_j\, e_\nu) = \delta_{\mu\nu}\,\delta_{ij} \tag{4.12}$$

where the linear combinations of $\chi_i\, e_\mu$ over Φ span $\mathcal{H}$, i.e., any f has the expansion

$$f = \sum_{\mu i} \chi_i\, e_\mu\, \alpha_{\mu i}\,. \tag{4.13}$$

The coefficients $\alpha_{\mu i}$ are easily obtained with the help of (4.12):

$$\alpha_{\mu i} = \mathrm{tr}\,(\chi_i\, e_\mu, f)\,. \tag{4.14}$$

The operators A, B on $\mathcal{H}$ which are linear over Φ have a consistent representation in terms of the coefficients in Φ induced on this basis. Hence the Hilbert space over Φ, which is defined by the representation matrices, supports the (non-observable, non-totally) linear operators on $\mathcal{H}$ as well as the totally linear operators. To see this, we proceed as in (4.6) and (4.7):

$$A\,\chi_i\, e_\mu = \sum_{\nu j} \chi_j\, e_\nu\, \alpha_{\nu j,\,\mu i} \qquad B\,\chi_i\, e_\mu = \sum_{\nu j} \chi_j\, e_\nu\, \beta_{\nu j,\,\mu i}\,, \tag{4.15}$$

where $\alpha_{\nu j,\,\mu i}$ and $\beta_{\nu j,\,\mu i}$ are in Φ. If B is linear over Φ,

$$B\,A\,\chi_i\, e_\mu = \sum_{\nu j} (B\,\chi_j\, e_\nu)\,\alpha_{\nu j,\,\mu i}$$

$$= \sum_{\lambda k,\nu j} \chi_k\, e_\lambda\, \beta_{\lambda k,\,\nu j}\,\alpha_{\nu j,\,\mu i}\,. \;\blacksquare \tag{4.16}$$

Since, according to (4.13)-(4.15),

$$\alpha_{\nu j,\,\mu i} = \mathrm{tr}\,(\chi_j\, e_\nu, A\,\chi_i\, e_\mu)\,, \tag{4.17}$$

and $A(\chi_i e_\mu) \neq (A\chi_i)e_\mu$ in general, there is no restriction on the indices $_{\nu j,\ \mu i}$ of the coefficients.

If, however, A is *totally linear*, then

$$\alpha_{\nu j,\ \mu i} = \delta_{\mu\nu}\, \alpha_{j\,i}^{(\mu)} = \delta_{\mu\nu}\, \mathrm{tr}\, e_\mu\, (\chi_j,\, A\, \chi_i)\ . \tag{4.18}$$

Hence the representations of the totally linear operators in the orthonormal basis $\{\chi_i\, e_\mu\}$ are reduced to block form with respect to the indices referring to the primitive idempotents of $\mathfrak{A}$. For the totally linear operators, the Hilbert space over Φ defined by the matrices (4.18) is therefore reducible with respect to the primitive idempotents of $\mathfrak{A}$. This decomposition corresponds to the reducibility of the lattice of manifolds in which the centre includes the $\{fe_\mu\}$ over all $f \in \mathcal{H}$*).

These linear manifolds are not algebraically closed, but they are invariant under the totally linear operators and are orthogonal in $\mathcal{H}_\Phi$ (the Hilbert space with $\mathrm{tr}(f, g)$ for scalar product). Furthermore, the representations (4.17) of the linear operators on $\mathcal{H}$ and the correspondences established by (4.16) imply that the relation between $\mathcal{H}$ and its operator calculus and the Hilbert space over Φ and its operator calculus defined by the representations is that of an isomorphism. ■

This completes the proof of our principal result as stated in Section 1. A proof will also be given in Appendix 1 which does not utilize an orthonormal set. It is shown there that a spectral resolution can be constructed for a bounded totally linear operator in each of the subspaces $\mathcal{H}_\mu = \{fe_\mu\}$ over all $f \in \mathcal{H}$.

The manifolds $\{\psi_\mu^{(i)}\, a\}$ are minimal and therefore generate pure states. The $\psi_\mu^{(i)}$ may therefore be interpreted as «state vectors» in the same sense as in the usual mathematical description of quantum mechanics, i.e., in both cases it is the «rays» or minimal closed manifolds which are in one-to-one correspondence with the pure states.

It is also of interest to try to understand the significance of the closed linear extensions of the elements $\chi_i\, e_\mu$, since these form an alternative basis for $\mathcal{H}$. The $\{\chi_i\, e_\mu\}$ were constructed to be orthogonal in the sense of the trace scalar product, and the manifolds $\{\chi_i\, e_\mu\, \alpha\}$ over $\alpha \in \Phi$ completely span $\mathcal{H}$. These manifolds generate those pure states which, according to our postulate Q, are not physical, since the projection into a manifold closed only over Φ is not a totally linear operator. To generate a physical state, it is necessary to construct manifolds containing $\chi_i\, e_\mu\, a$ for all $a \in \mathfrak{A}$. The presence of e_μ assures that such a manifold is minimal, and by construction the associated projection is totally linear. Hence the $\{\chi_i\, e_\mu\, a\}$ generate pure states. However, the manifolds $\{\chi_i\, e_\mu\, a\}$ and $\{\chi_j\, e_\nu\, a\}$ are not necessarily orthogonal in the stronger sense.

What we show in the following is that these manifolds generate states depending on the $\chi_i\, e_\mu$ alone, and that, when these states are distinct the manifolds *do not intersect*. The projections associated with these manifolds therefore do not in general commute, i.e., the corresponding propositions are incompatible.

Since $\mathrm{tr}(\chi_i\, e_\mu\, a,\, \chi_i\, e_\mu\, a) = (aa^*)_{\mu\mu}\, (\chi_i,\, \chi_i)_{\mu\mu}$ implies that $(aa^*)_{\mu\mu} = 1$ if $(\chi_i,\, \chi_i)_{\mu\mu} = 1$, it follows from our previous discussion that the state determined by $\{\chi_i\, e_\mu\, a\}$ is

$$w(M) = \mathrm{tr}\, e_\mu\, (\chi_i,\, P_M\, \chi_i)\ . \tag{4.19}$$

*) The corresponding superselection rules may be said to be *intrinsic* to the algebraic Hilbert space since they follow directly from the structure of the algebra.

Suppose that $\{\chi_i e_\mu a\}$ has a non-vanishing projection into $\{\chi_j e_\nu a\}$, i.e., for some $a_1 \in \mathfrak{A}$,

$$\chi_i e_\mu a_1 = \chi_j e_\nu b_1 + h_1 , \tag{4.20}$$

where $(h_1, \chi_j e_\nu) = 0$. If $h_1 = 0$, then the closed linear extensions of $\chi_i e_\mu$ and $\chi_j e_\nu$ are identical, and the states (4.19) corresponding to each of these are also the same. Such symmetry properties of the states were discussed also in the previous Section and lead to no contradictions. What we are concerned with here is the relation between these closed linear extensions when the states (4.19) are in fact *different*. We therefore assume that $h_1 \neq 0$. Taking the norm of (4.20), one obtains

$$(a_1 a_1^*)_{\mu\mu} = (b_1 b_1^*)_{\mu\mu} + || h_1 ||^2 ,$$

i.e., in the strict sense of the inequality,

$$(a_1 a_1^*)_{\mu\mu} > (b_1 b_1^*)_{\mu\mu} . \tag{4.21}$$

Furthermore,

$$\chi_j e_\nu b_1 = \chi_i e_\mu a_2 + h_2 ,$$

where $(h_2, \chi_i e_\mu) = 0$ and therefore (by our previous argument $h_2 \neq 0$)

$$(b_1 b_1^*)_{\mu\mu} > (a_2 a_2^*)_{\mu\mu} . \tag{4.22}$$

If the cycle (4.21)–(4.22) is repeated indefinitely, it is clear that $(a_n a_n^*)_{\mu\mu}$ and $(b_n b_n^*)_{\mu\mu} \to 0$. Hence the product of the projections into these two manifolds taken to the n^{th} power vanishes as $n \to \infty$. According to (2.15), this implies that the closed linear extensions of two vectors $\chi_i e_\mu$ and $\chi_j e_\nu$ which generate distinct states have no intersection. ■

In the above demonstration no special use was made of the fact that $\chi_i e_\mu$ and $\chi_j e_\nu$ belong to an orthonormal set, and therefore the result holds in general for the manifolds $\{f e_\mu a\}$ and $\{g e_\nu a\}$ and the states which they determine.

5. Some Illustrations

To illustrate the isomorphism between algebraic Hilbert spaces and (reducible) Hilbert spaces over a field, we discuss the structure of some of the Clifford algebras (cf., Ref.[25]), p. 267 ff., for example).

Consider, for example, the algebra of quaternions E, I, J, K over the complex field. For simplicity, we choose to discuss the Clifford algebra $n = 2$ in which K is dependent upon I and J as $K = IJ$ (in the Clifford algebra $n = 3$, K is independent and there are additional idempotents).

Since $I = -I^*$, $J = -J^*$ and $IJ + JI = 0$, there are two orthogonal self-conjugate idempotents:

$$e_1 = \frac{1}{2}(E - i\,I\,J) , \qquad e_2 = \frac{1}{2}(E + i\,I\,J) . \tag{5.1}$$

Since $e_1 I e_2 = \alpha \varrho_{12}$, α complex, we find that

$$\varrho_{12} = \frac{1}{2}(I - i\,J) \qquad \varrho_{21} = -\frac{1}{2}(I + i\,J) \tag{5.2}$$

and $\alpha = 1$. The coefficients in the representation

$$a = \sum_{\mu\nu} \alpha_{\mu\nu}\, \varrho_{\mu\nu} \tag{5.3}$$

for a (a quaternion) can easily be calculated, with the result that

$$\begin{aligned} E &= e_1 + e_2 \\ I &= \varrho_{12} - \varrho_{21} \\ J &= i\,(\varrho_{12} + \varrho_{21}) \\ K &= i\,(e_1 - e_2)\,. \end{aligned} \tag{5.4}$$

(Clearly iI, $-iJ$, $-iK$ are the usual Pauli spin operators.)

The two minimal ideals are of the form

$$e_1\,a = \mu_1\,e_1 + \mu_2\,\varrho_{12} \tag{5.5}$$

$$e_2\,a = \mu_1\,e_2 + \mu_2\,\varrho_{21}\,.$$

and under right multiplication by I, J and K, the two-dimensional vectors with components μ_1, μ_2 transform in the first of (5.5) with, respectively, representations for I, J, K as given by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{5.6}$$

and in the second

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{5.7}$$

The matrices of (5.7) can be obtained from those of (5.6) by an equivalence transformation with $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$.

A quantum mechanics described by a Hilbert space over the quaternion algebra [6]) is therefore isomorphic to a quantum mechanics with a superselection rule described by a complex Hilbert space in which the subspaces $\{fe_1\}$ and $\{fe_2\}$ of $\mathcal{H}$ are invariant under the action of the totally linear operators*). The pure states are determined by minimal algebraically closed linear manifolds of the form $\{fe_1a\}$ and $\{fe_2a\}$ for fixed f and a ranging over the quaternion algebra.

As a second example, we wish to consider the Hilbert space over the complex numbers as an algebraic Hilbert space; it is isomorphic in the same sense as described above to a reducible real Hilbert space [28]). The complex algebra is a proper subalgebra of the quaternions, and consists of $E = e_1 + e_2$, $I = \varrho_{12} - \varrho_{21}$, and all real linear combinations. The transformation group on the minimal ideals (5.5) is of course much smaller, since only the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ (in addition to the unit matrix) is available; it corresponds in fact to the usual complex phase group.

The Dirac algebra, i.e., the algebra corresponding to the $n = 4$ and 5 Clifford algebras, is a proper subalgebra of what has been called in I the Cayley ring [16]), i.e., the Clifford algebras for $n = 6, 7$. This ring is obtained as the group algebra of the group of associative operators which is the closure of the non-associative algebra of the Cayley numbers. We refer to I for a complete discussion of the construction of the idempotents for the Cayley ring. All of the idempotents can be constructed of real linear combinations of the group elements, including the idempotent which splits the sixteen-dimensional representation of the case $n = 7$ into two inequivalent irreducible

*) Cf. Ref. [7]), p. 761 ff., and the discussion of the symplectic representation in Ref. [6]) for related constructions.

eight-dimensional representations (each appropriate to $n = 6$); this decomposition is analogous to that of the quaternion $n = 3$ case, for which $e_{\pm} = {}^1/_2(E \pm IJK)$ splits the four-dimensional representation into two two-dimensional representations, each appropriate to $n = 2$. In each of the $n = 6$ subspaces, there are eight submanifolds of $\mathcal{H}$ which are invariant under the action of the totally linear operators. It is remarkable that the idempotents reduce operators of the Cayley ring to elements of the Cayley algebra. Since the Cayley algebra, as mentioned earlier, has a modulus, expressions such as $e_\mu a a^* e_\mu$, which occur in the analysis of Fourier expansions and in the construction of the pure states, are just of the form $| a |^2 e_\mu$, where $| a |^2$ is the modulus squared. The minimal linear manifolds with elements of the form $f e_\mu a$ are therefore products of vectors $f e_\mu$ with Cayley numbers a of (when normalized) modulus unity in the eight-dimensional Cayley algebra.

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Appendix 1

The Decomposition of $\mathcal{H}$ over Φ

The decomposition of $\mathcal{H}$ into subspaces invariant under the action of the totally linear operators was given above in terms of a discrete orthonormal set appropriate to the treatment of $\mathcal{H}$ as a Hilbert space over Φ. It is also possible to exhibit this decomposition in the form of the spectral theorem.

The spectral resolution (2.11) of a bounded Hermitian operator contains projections on algebraically closed linear manifolds. In the treatment of $\mathcal{H}$ as a Hilbert space over Φ, it is more appropriate to use projections into manifolds closed only over Φ.

Let $\mathcal{H}_\mu = \{f e_\mu\}$ for $f \in \mathcal{H}$ and e_μ a primitive idempotent of $\mathfrak{A}$; then

$$\mathcal{H} = \sum_\mu \mathcal{H}_\mu . \tag{A1.1}$$

If M_μ is a linear manifold (closed over Φ) in $\mathcal{H}_\mu$, and M_ν another in $\mathcal{H}_\nu$ ($\nu \neq \mu$) then

$$P_{M_\mu} P_{M_\nu} = 0 . \tag{A1.2}$$

To see that this is true, we note that since these are projections onto linear manifolds closed over Φ, they are (weakly) Hermitian in the trace scalar product, i.e.,

$$\operatorname{tr}(P_{M_\mu} f, P_{M_\nu} g) = \operatorname{tr}(f, P_{M_\mu} P_{M_\nu} g) . \tag{A1.3}$$

However, $P_{M_\mu} f = h e_\mu \in \mathcal{H}_\mu$ and $P_{M_\nu} g = h' e_\nu \in \mathcal{H}$, and therefore

$$\operatorname{tr}(P_{M_\mu} f, P_{M_\nu} g) = \operatorname{tr}[e_\mu (h, h') e_\nu] = 0 \quad (\mu \neq \nu) .$$

Hence, in (A 1.3) $\operatorname{tr}(f, P_{M_\mu} P_{M_\nu} g) = 0$ for all $f, g \in \mathcal{H}$, and therefore (A 1.2) is proved. ∎

According to II, the function $f_\lambda(A)$ of a bounded Hermitian operator A may be defined, where

$$f_\lambda(x) = max\,(x - \lambda, 0)$$

for x, λ real. We then define the linear manifold

$$M^{(\mu)}(A, \lambda) = \{f \mid f_\lambda(A) f = 0 , f \in \mathcal{H}_\mu\} . \tag{A1.4}$$

By the same procedure used in II it can easily be shown that

$$M^{(\mu)}(A, \lambda) = \begin{cases} \mathcal{H}_\mu & \lambda > C \\ \phi & \lambda < -C \end{cases} \tag{A1.5}$$

where C is the bound of A.

According to (A 1.1), the unity operator is then given by

$$\sum_\mu P_{M^{(\mu)}(A, \lambda)}$$

for $\lambda > C$, and again following the procedure outlined in II we find that

$$\mathrm{tr}\,(f, A f) = \sum_\mu \int \lambda \, d \, \mathrm{tr}\,(f, P_{M^{(\mu)}(A, \lambda)} f) . \tag{A1.6}$$

Replacing f by $f \pm g$ and using the weak Hermitian property of the $P_{M^{(\mu)}(A, \lambda)}$, one obtains

$$\mathrm{tr}\,(f, A g) = \sum_\mu \int \lambda \, d \, \mathrm{tr}\,(f, P_{M^{(\mu)}(A, \lambda)} g) . \tag{A1.7}$$

Since (A 1.7) is valid for all f, g, we conclude that an alternative spectral form for A is

$$A = \sum_\mu \int \lambda \, d \, P_{M^{(\mu)}(A, \lambda)} . \tag{A1.8}$$

The sum $\sum_\mu P_{M^{(\mu)}(A, \lambda)}$ is a projection since each term is a projection and (A 1.2) is valid. The sum is furthermore totally linear, since it corresponds to the projection into a manifold $\{f\}$ satisfying $f_\lambda(A) f = 0$, where f is otherwise unrestricted in $\mathcal{H}$ (and A is totally linear). Since the resolution (2.11) is unique, it then follows that

$$P(\lambda) = \sum_\mu P_{M^{(\mu)}(A, \lambda)} .$$

The form (A 1.8), however, explicitly exhibits the reduction of the totally linear Hermitian operators in $\mathcal{H}$ over Φ.

Appendix 2

Left Multiplication

In the preceding, we have taken as a convention the right multiplication of «scalars» on vectors f. We show in what follows that a convention of left multiplication of scalars (but left multiplication of operators as usual) results in a theory of identical content, as far as the principal theorems are concerned, but that the detailed algebraic structure has a very different form.

For the convention of left multiplication, we say that if $f, g \in \mathcal{H}$ then $af + bg \in \mathcal{H}$, $a, b \in \mathfrak{A}$. The scalar product (f, g) has all of the properties given in Section 2, but (2.3) is replaced by

$$(a f, g) = a\,(f, g) . \tag{A2.1}$$

Every f in $\mathcal{H}$ has a unique decomposition into a part in a manifold M and a part orthogonal in the following sense:

$$f = g_0 + h_0 \tag{A 2.2}$$

where $g_0 \in M$ and $\mathrm{tr}(h_0, g) = 0$ for all $g \in M$. If M is algebraically closed, ag is in M along with g, hence $\mathrm{tr}(h_0, ag) = \mathrm{tr}[(h_0, g)a^*] = 0$, i.e., $(h_0, g) = 0$. If M is closed only over Φ, orthogonality in the trace scalar product is all that is available. Projections can be defined in the same way as in (2.9). If the manifold M is algebraically closed, then $af = ag_0 + ah_0$ is the unique decomposition (A 2.2) for af.

It therefore follows that

$$P_M\, a = a\, P_M \tag{A 2.3}$$

when M is algebraically closed. We define an operator with the property

$$A(a\,f) = a(A\,f) \tag{A 2.4}$$

for all $f \in \mathcal{H}$ and a in $\mathfrak{A}$ as totally linear. (It is clear that commutativity has replaced associativity in changing the convention on scalar multiplication.)

If the Hilbert space is realized with a collection of sequence vectors, the operators A are matrices. Then (A 2.4) implies that the elements of A are in the centre of $\mathfrak{A}$. This requirement is not imposed in case the convention of right multiplication is used.

A Hermitian operator satisfying $(Af, g) = (f, Ag)$ is necessarily, as in the discussion following (2.10), totally linear. To see this, we use again the basis (2.12) and note that

$$\begin{aligned}(f, g) &= \sum_{\mu\nu} \varrho_{\mu\nu}\, \mathrm{tr}\, \varrho_{\nu\mu}\, (f, g) \\ &= \sum_{\mu\nu} \varrho_{\mu\nu}\, \mathrm{tr}\, (f, \varrho_{\mu\nu}\, g)\,.\end{aligned} \tag{A 2.5}$$

The Hermitian property then implies that $\mathrm{tr}(Af, \varrho_{\mu\nu}\, g) = \mathrm{tr}(f, \varrho_{\mu\nu}\, Ag)$, i.e., $A\, \varrho_{\mu\nu} = \varrho_{\mu\nu}\, A$, and hence A commutes with $a \in \mathfrak{A}$.

The lattice of propositions corresponding to the left algebraically closed linear manifolds is identical to that of the right algebraically closed linear manifolds; the minimal manifolds are, however, of the form $\{ae_\mu f\}$ over $a \in \mathfrak{A}$. These latter generate the pure states $w(M) = \mathrm{tr}\, e_\mu\, (f, P_M f)$ in precisely the same way as given in (3.15)–(3.18), and the (non-minimal) manifolds $\{af\}$ over $a \in \mathfrak{A}$ generate mixed states as in (3.23).

Proceeding with the construction of orthonormal sets, the vectors analogous to the $\psi_\mu^{(i)}$ satisfy (4.4) with $e_\mu \psi_\mu^{(i)} = \psi_\mu^{(i)}$. From the form of the expansion $A\psi_\mu^{(i)} = \sum_{\nu j} a_{\mu i,\, \nu j}\, \psi_\nu^{(j)}$, it is clear that operators which are not totally linear do not have a consistent representation among the $a_{\mu i,\, \nu j}$, and that when A is totally linear, $A\psi_\mu^{(i)} = \sum_{\nu j} \alpha_{\mu i,\, \nu j}\, \varrho_{\mu\nu} \psi_\nu^{(j)}$ for $\alpha_{\mu i,\, \nu j} \in \Phi$.

An orthonormal set $\{e_\mu\, \chi_i\}$ can be constructed as for (4.12), (4.13), satisfying

$$\mathrm{tr}\, (e_\mu\, \chi_i, e_\nu\, \chi_j) = \delta_{\mu\nu}\, \delta_{ij} \tag{A 2.6}$$

and for any $f \in \mathcal{H}$,

$$f = \sum \alpha_{\mu i}\, e_\mu\, \chi_i, \tag{A 2.7}$$

with $\alpha_{\mu i} \in \Phi$. It follows from (A 2.7) that all linear operators (sufficiently well behaved for discrete representation) are represented by matrices of the form (in Φ)

$$\alpha_{\mu i,\nu j} = \mathrm{tr}\,(A\, e_\mu\, \chi_i, e_\nu\, \chi_j)\ .$$

Hence, if A is totally linear $\alpha_{\mu i,\, \nu j} = 0$ for $\mu \neq \nu$.

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