Zeitschrift: Helvetica Physica Acta
Band: 38 (1965)
Heft: III

Artikel: Some remarks on the boundary value problem in general relativity
Autor: Leutwyler, H. / Klauder, J.R.
DOI: https://doi.org/10.5169/seals-113592

Some Remarks on the Boundary Value Problem in General Relativity

by **H. Leutwyler***) and **J. R. Klauder**
Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey

(15. XII. 64)

Abstract: A qualitative discussion of one- and two-surface initial value problems, in the presence or absence of external sources, is given for general relativity, and compared with corresponding initial value problems in electromagnetism. In the one-surface form, the initial, freely specifiable functions are 12 in number, e.g., the six space-like components of the metric and their time derivatives. In the two-surface form, the presence of four time-dependent gauge functions allows for 20 initial, freely specifiable functions, the ten components of the metric on each surface, in the absence of external sources. Only under restrictive regularity conditions – which reduce the 20 functions to 12 – does the two-surface form possess an acceptable one-surface limit. In the presence of prescribed external sources, the gauge group is destroyed and the two-surface formulation has only 12 freely specifiable functions. This modification is in marked contrast to the electromagnetic case where a corresponding reduction of freely specifiable initial data does not take place in the presence of sources.

I. Introduction

The purpose of this note is to show that the boundary value problems associated with EINSTEIN's equations fall into four completely different classes. We shall not try to make our conclusions mathematically rigorous but intend rather to present a simple, intuitive picture.

EINSTEIN's equations in the presence of sources $\tau^{\mu\nu}$ read[1])

$$S^{\mu\nu} = \tau^{\mu\nu}\,, \quad S^{\mu\nu} = \sqrt{|g|}\left(R^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}\,R\right). \tag{1}$$

These hyperbolic differential equations for the ten components of the metric tensor $g_{\mu\nu}(x)$ lead to two natural boundary value problems, which are of physical interest in particular for the quantum theory of gravity.

Case 1. One surface boundary values[2])

Given an everywhere space-like surface σ' (which we shall for simplicity assume to be characterized by $x^0 = \sigma' =$ const.), specify a set of initial values

$$g'_{\mu\nu} = g_{\mu\nu}(\boldsymbol{x}, \sigma')$$

*) Permanent Adress: Institut für theoretische Physik, Universität Bern, Bern, Switzerland.

and

$$\partial_0 g'_{\mu\nu} = \partial_0 g_{\mu\nu}(\boldsymbol{x}, \sigma'),$$

such that Equation (1) admits a solution for $x^0 > \sigma'$.

Case 2. Two surface boundary values

Given two surfaces $\sigma'' > \sigma'$, specify a set of boundary values $g'_{\mu\nu} = g_{\mu\nu}(\boldsymbol{x}, \sigma')$ and $g''_{\mu\nu} = g_{\mu\nu}(\boldsymbol{x}, \sigma'')$ such that EINSTEIN's equations admit a solution between σ' and σ''.

We want to show that these two boundary value problems are essentially different. A different number of independent data may be specified for cases 1 and 2. Furthermore the situation is completely different in the absence and presence of sources, no matter how weak the sources are.

To understand why *the two-surface formulation is not equivalent to the one-surface formulation in the limit* $\sigma'' \to \sigma'$ we consider first the analogous situation in electrodynamics, which is very similar. Then we investigate a three-dimensional Riemannian space, where we can understand the difference easily.

On the other hand, the inequivalence of the boundary problems in the absence or presence of sources is due to the fact, that, in general, a nonvanishing source $\tau^{\mu\nu}$ destroys the gauge group associated with EINSTEIN's equations, because the source, which is a given function of x (and possibly $g_{\mu\nu}$), is generally not a gauge *in*variant quantity, but must be changed if the same physical situation is described in a different coordinate system. This distinguishes gravity from electromagnetism, where the source, i.e., the current, *is* a gauge invariant quantity, and therefore the gauge group remains even in the presence of sources.

Our principal results are summarized in Table 1. These results have been used in the Feynman quantization of gravity as developed by one of the authors[3]). In this formalism, the basic definition of the transition amplitude involves a 'thin' two-surface boundary problem, in which the fact that all boundary values $g'_{\mu\nu}$ and $g''_{\mu\nu}$ may be specified independently is of fundamental importance. Note that due to gauge invariance the thickness of the slice in coordinate space is irrelevant as long as it is finite. The limit as the thickness tends to zero leads to the class of boundary data considered by R. F. BAIERLEIN, D. H. SHARP and J. A. WHEELER[2]). Only in this limit are the regularity conditions formulated by these authors compatible with gauge invariance. In the treatment of Feynman quantization mentioned above, the boundary value problem for surfaces with finite separation is involved, and the application of the analysis given by BAIERLEIN, SHARP and WHEELER to this case would violate gauge invariance.

II. One-surface Formulation in the Absence of Sources. Summary of Previous Results. Analogy with Electromagnetism

The one-surface boundary problem or initial value problem has been analyzed by various authors[4]), and we recall briefly their main results.

It has been shown that, as in the case of electrodynamics, one is not allowed to specify the 20 quantities $g_{\mu\nu}$ and $\partial_0 g_{\mu\nu}$ independently. The equations

$$S^{\mu 0} = 0 \tag{2}$$

have the character of subsidiary conditions, corresponding to

$$\partial_0 \partial_i A^i - \partial_i \partial^i A_0 = 0 . \tag{3}$$

In fact (2) involves only $g_{\mu\nu}$ and $\partial_0 g_{ik}$, in the same way as (3) involves only A_μ and $\partial_0 A_i$. A natural way to find consistent initial values in electrodynamics is to look upon (3) as determining A_0, given $\partial_0 A_i$ and appropriate boundary conditions as $|\boldsymbol{x}| \to \infty$. Thus it is sufficient to specify A_i' [i.e., $A_i(\boldsymbol{x}, \sigma')$] and $\partial_0 A_i'$; A_0' can then be determined by means of (3) while $\partial_0 A_0'$ is left undetermined. Since MAXWELL's equations are gauge invariant, the specification of a consistent set of initial values is not sufficient to determine the history $A_\mu(x)$ for $x^0 > \sigma'$; there are a whole family of solutions generated by

$$\bar{A}_\mu = A_\mu + \partial_\mu \Lambda , \tag{4}$$

that satisfy MAXWELL's equations as well as the boundary values A_μ' and $\partial_0 A_i'$, provided $\Lambda' = \partial_0 \Lambda' = 0$. This degeneracy is the origin of the fact that the subsidiary condition (3) is compatible with any value of $\partial_0 A_0'$, simply because these gauge transformations change $\partial_0 A_0'$.

In complete analogy to this situation A. PERES and N. ROSEN[4]) as well as BAIERLEIN, SHARP and WHEELER[2]) interpret the initial value problem for general relativity in the following way: Specify the quantities g_{ik}' and $\partial_0 g_{ik}'$ independently, and determine $g_{\mu 0}'$ from the subsidiary conditions (2) as in electrodynamics. It turns out that g_{00}' is involved in a purely algebraic fashion, while one has to solve three second-order differential equations inside the surface σ' [analogous to (3)] in order to find g_{i0}'. Thus one is still free to specify boundary values for these differential equations.

The approach by A. LICHNEROWICZ[4]) and Y. FOURÈS-BRUHAT[4]) is based on a different interpretation of the initial value problem. These authors specify all ten $g_{\mu\nu}'$ as well as two of the six $\partial_0 g_{ik}'$. The remaining four terms are obtained by means of the subsidiary conditions (2).

III. One Surface Formulation with Sources

Subsidiary conditions still occur in EINSTEIN's and MAXWELL's equations even in the presence of external sources. The problem of finding consistent initial value data remains roughly the same as that in the absence of sources, although the precise character of the differential equations from which either A_0' or $g_{\mu 0}'$ are determined are of course changed. More significant is the effect of the integrability conditions that the external sources must fulfill in order for any solution to exist. It is in this aspect that MAXWELL's and EINSTEIN's equations are profoundly different.

The integrability conditions for these two cases are

$$\partial_\mu j^\mu = 0 , \tag{5}$$

$$\nabla_\mu \tau^{\mu\nu} = \partial_\mu \tau^{\mu\nu} + \Gamma^\nu_{\mu\lambda} \tau^{\mu\lambda} = 0 . \tag{6}$$

Equation (5) involves only the external current and is independent of A_μ. On the other hand, the appearance in (6) of the metric and its derivatives is a reflection of the loss of gauge invariance that occurs in (1), as already remarked in Section 1. With gauge invariance destroyed, Equation (6) might best be viewed as coordinate conditions, i.e., conditions on the metric, rather than on the source. Support for this view is found in the following example.

Suppose the source describes a fluid at rest without pressure. This situation is characterized by

$$\tau^{\mu\nu} = \begin{cases} 0\,; & \mu \text{ or } \nu \neq 0 \\ \varrho(\boldsymbol{x})\,; & \mu = \nu = 0\,, \end{cases} \tag{7}$$

where we have chosen a time scale such that ϱ is independent of x^0. If $\varrho(\boldsymbol{x}) \neq 0$ then the conservation law (6) implies $\Gamma^{\mu}_{00} = 0$, i.e.,

$$\left.\begin{aligned} \partial_0\, g_{00} &= 0\,, \\ \partial_0\, g_{i0} &= \frac{1}{2}\, \partial_i\, g_{00}\,. \end{aligned}\right\} \tag{8}$$

Equations (8) show that the integrability conditions are not conditions on the source, but conditions on the geometry $g_{\mu\nu}(x)$ in contradistinction to Equation (5), which is a restriction on the current. In other words we may specify the motion of the fluid as we like; the geometry resulting from EINSTEIN's equations will adjust itself in such a way that the elements of the fluid travel on geodesics.

The conditions (8) have the character of coordinate conditions. We recall that in the absence of sources, the quantities $g_{\mu 0}$ are not determined by EINSTEIN's equations; the $g_{\mu 0}$ only serve to tell us what coordinate system we have chosen. We now see that when matter is present, the integrability conditions (6) provide equations for $\partial_0 g_{\mu 0}$ by which $g_{\mu 0}$ is determined. This is as it should be, since there is no longer any gauge group wherever the fluid is present. Indeed, the solution of EINSTEIN's equation is uniquely determined, once we are given a consistent set of boundary values $g'_{\mu 0}$, $\partial_0 g'_{ik}$.

Not every source is as simple to analyze as a pressureless fluid, however, and sometimes the gauge group is not completely destroyed by the existence of the source. For example, consider a scalar field interacting with gravity, where in appropriate units we have

$$\tau^{\mu\nu} = \sqrt{|g|}\, \left\{\frac{1}{2}\, g^{\mu\nu}(\partial^{\alpha}\varphi\, \partial_{\alpha}\varphi - m^2\varphi^2) - \partial^{\mu}\varphi\, \partial^{\nu}\varphi\right\}. \tag{9}$$

It follows that

$$\nabla_{\mu}\, \tau^{\mu\nu} = -\sqrt{|g|}\, \partial^{\nu}\varphi(\Box\varphi + m^2\varphi)\,. \tag{10}$$

Thus, the integrability conditions require

$$\Box\varphi + m^2\varphi = 0\,, \tag{11}$$

which is only one condition on the derivatives of the metric. Therefore, we must have a degeneracy left. The degeneracy in this case may be clearly seen if we introduce coordinates such that the surfaces $\varphi =$ const. coincide with a coordinate hypersurface. Without loss of generality assume for simplicity that these are given by $x^0 =$ const., from which $\partial_i\varphi = 0$ follows. Moreover let us choose the x^0-scale such that $\partial_0\varphi =$ const., independent of x. This source indeed possesses an invariance group since the field distribution remains the same under the transformations

$$\bar{x}^0 = x^0 , \quad \bar{x}^i = f^i(x^k, x^0) . \tag{12}$$

We shall not try to give a general treatment of the source problem, but instead consider one special class. If $\tau^{\mu\nu}$ involves only the metric, but not its derivatives, then

$$\det \left| \frac{\partial \tau^{0\nu}}{\partial g_{0\mu}} + \frac{1}{2} g^{\mu\nu} \tau^{00} \right| \neq 0 \tag{13}$$

is a sufficient condition to guarantee that $\partial_0 g_{\mu 0}$ can be determined from the conservation law and hence that no gauge invariance remains[5]).

Summarizing we note that one may specify g'_{ik} and $\partial_0 g'_{ik}$ independently, and these determine the complete history $g_{\mu\nu}(x)$ provided the source is nondegenerate in the sense of (13).

IV. Two-surface Boundary Problem in the Absence of Sources

a. *Electromagnetism*

First consider again the boundary problem in electrodynamics, i.e., suppose we are given initial and final values A'_μ and A''_μ: Does there exist a solution of MAXWELL's equations that connects these boundary values? From the analysis of the initial value problem one might expect that he is not free to specify A'_0 and A''_0 independently. It is very easy to see, however, that this conjecture cannot be correct. Clearly A'_0 and A''_0 are independent of the other potentials, since they are affected by the slopes of the gauge transformation $\partial_0 \Lambda'$ and $\partial_0 \Lambda''$, which are clearly independent of $\partial_i \Lambda'$ and $\partial_i \Lambda''$. We want to show that we are in fact allowed to specify all eight boundary data A'_μ and A''_μ.

In order to show this let us first construct a particular solution of MAXWELL's equations $a_\mu(x)$ that satisfies $a_0(x) \equiv 0$ everywhere. In this case the equations of motion read

$$\Box\, a_i - \partial_i \partial^k a_k = 0 , \quad \partial_0 \partial^k a_k = 0 . \tag{14}$$

Furthermore, let us assume $\partial^k a''_k = \partial^k a'_k = 0$. Then these equations imply

$$\Box\, a_i = 0 . \tag{15}$$

Equation (15) may be solved for arbitrary boundary values, and leads to $\partial^k a_k \equiv 0$ because $\partial^k a''_k = \partial^k a'_k = 0$. Thus we find the result that apart from a vanishing divergence, a'_k and a''_k may be specified arbitrarily.

We define the sought-for general solution A_μ with the help of our auxiliary solution:

$$A_\mu = a_\mu + \partial_\mu \Lambda . \tag{16}$$

The potential A_μ already satisfies MAXWELL's equations of motion, and all that remains to be shown is that Λ and the boundary values a'_k and a''_k can be chosen such that A_μ assumes its prescribed boundary values. To achieve this result we put

$$\partial_0 \Lambda' = A'_0, \quad \partial_0 \Lambda'' = A''_0, \tag{17}$$

$$\partial^k \partial_k \Lambda' = \partial^k A'_k, \quad \partial^k \partial_k \Lambda'' = \partial^k A''_k. \tag{18}$$

These choices are compatible since the four functions Λ', Λ'', $\partial_0\Lambda'$ and $\partial_0\Lambda''$ are independent. Furthermore (18) insures that a'_k and a''_k have vanishing divergence. Consequently, we have shown that A'_μ and A''_μ can indeed be freely specified.

To understand what happens in the limit $\sigma'' \to \sigma'$, let us compare $\partial_0 A'_i$ with its average value in the slice, which is defined by

$$(\partial_0 A_i)_{av} = \frac{1}{\varepsilon} \int_{\sigma'}^{\sigma''} dx^0 \, \partial_0 A_i = \frac{1}{\varepsilon} (A''_i - A'_i) ; \quad \varepsilon = \sigma'' - \sigma' . \tag{19}$$

On the other hand,

$$\partial_0 A'_i = \partial_0 a'_i + \partial_i \partial_0 \Lambda' . \tag{20}$$

If one solves $\Box \, a_i = 0$ and expresses $\partial_0 \, a'_i$ in terms of a'_i and a''_i, he obtains in the limit $\varepsilon \to 0$

$$\left.\begin{aligned} \partial_0 a'_i &= \frac{1}{\varepsilon} (a''_i - a'_i) = \frac{1}{\varepsilon} \left\{A''_i - A'_i - \partial_i \int G(\boldsymbol{x} - \boldsymbol{x}') \, (\partial^k A''_k - \partial^k A'_k) \, d\boldsymbol{x}'\right\} \\ &\partial^k \partial_k G(\boldsymbol{x}) = \delta^{(3)}(\boldsymbol{x}) . \end{aligned}\right\} \tag{21}$$

Therefore, if we define α_i by

$$\alpha_i = \partial_0 A'_i - (\partial_0 A_i)_{av} , \tag{22}$$

then (20) and (21) lead to the expression

$$\Delta \alpha_i = \partial_i \left\{\Delta A'_0 - \frac{1}{\varepsilon} \partial^k (A''_k - A'_k)\right\} . \tag{23}$$

This shows that only if the boundary values A'_μ and A''_μ are chosen such that the expression in the bracket vanishes—which is of the form of the subsidiary condition (3)—will the solpe at σ' tend to the average of the slope in between σ' and σ''. If the boundary values do not satisfy this criterion, $\partial_0 A_i$ will vary rapidly within the slice and will not converge to a well-defined limit when $\varepsilon \to 0$. To secure a proper one-surface limit requires both

$$\partial_0 A'_i = (\partial_0 A_i)_{av} + 0(\varepsilon) , \quad \partial_0 A''_i = (\partial_0 A_i)_{av} + 0(\varepsilon) . \tag{24}$$

These conditions for regularity imply

$$A''_0 = A'_0 + 0(\varepsilon) , \quad \Delta A'_0 - \frac{1}{\varepsilon} \partial^k (A''_k - A'_k) = 0(\varepsilon) . \tag{25}$$

Only if these conditions are met does the two-surface boundary problem have a one-surface limit. In the limit $\varepsilon \to 0$, the two conditions (25) reduce the number of independently specifiable data from 8 to 6.

b. Three-dimensional Riemannian space

EINSTEIN's equations in three dimensions are particularly simple to solve. $S^{\mu\nu} = 0$ implies algebraically that all components of the curvature tensor vanish, which means that the space is flat. (In this subsection, μ, $\nu = 0, 1, 2$; while $i, k, = 1, 2$.) Therefore $g_{\mu\nu}(x)$ may be written as

$$g_{\mu\nu}(x) = \partial_\mu \Lambda^\alpha(x)\, \partial_\nu \Lambda^\beta(x)\, \eta_{\alpha\beta}\,, \tag{26}$$

where $\eta_{\alpha\beta}$ is the constant metric which characterizes the flat space in Cartesian coordinates. To establish the connection to the well-known theory of two-surfaces in Euclidean threespace, we choose the signature to be $(+++)$. The two-surface boundary value problem may now be viewed as follows: We give $g_{\mu\nu}(\boldsymbol{x}, \sigma')$ and $g_{\mu\nu}(\boldsymbol{x}, \sigma'')$, and determine $\Lambda^\alpha(x)$ such that $g_{\mu\nu}(x)$ takes on these boundary values at σ' and at σ''.

The quantities $g'_{\mu 0}$ and $g''_{\mu 0}$ are easily adjusted by choosing an appropriate slope $\partial_0 \Lambda^{\alpha\prime}$ and $\partial_0 \Lambda^{\alpha\prime\prime}$.

The remaining equations

$$g'_{ik} = \partial_i \Lambda^{\alpha\prime}\, \partial_k \Lambda^{\beta\prime}\, \eta_{\alpha\beta}\,, \quad g''_{ik} = \partial_i \Lambda^{\alpha\prime\prime}\, \partial_k \Lambda^{\beta\prime\prime}\, \eta_{\alpha\beta} \tag{27}$$

are the well-known relations between the metric of a two-surface and the functions $\Lambda^{\alpha\prime}(\boldsymbol{x})$, $\Lambda^{\alpha\prime\prime}(\boldsymbol{x})$ which determine the shape of this two-surface in the embedding space. The problem is to find the shape of the surface for a given metric. As is well known, this problem has always local solutions; any given two-geometry may be represented locally by a surface in Euclidean three space. The surface is unique if we specify appropriate boundary conditions to the differential Equations (27). Thus we conclude that we may indeed specify $g'_{\mu\nu}$ and $g''_{\mu\nu}$ independently.

This conclusion holds true no matter how small the time separation between the surfaces is. But what happens in the *limit* that the two-surfaces coincide?

Suppose $g'_{\mu\nu}$ and $g''_{\mu\nu}$ differ by order ε and furthermore that the boundary data for the differential Equations (27) are such that the surfaces σ' and σ'' are very close to each other. As we go to the limit $\varepsilon \to 0$, we recognize that the coordinate system $\boldsymbol{x}$ becomes singular *unless* the following condition, which is easily established from Figure 1, is satisfied: The directions of the curves $x^i =$ const. at both σ' and at σ''

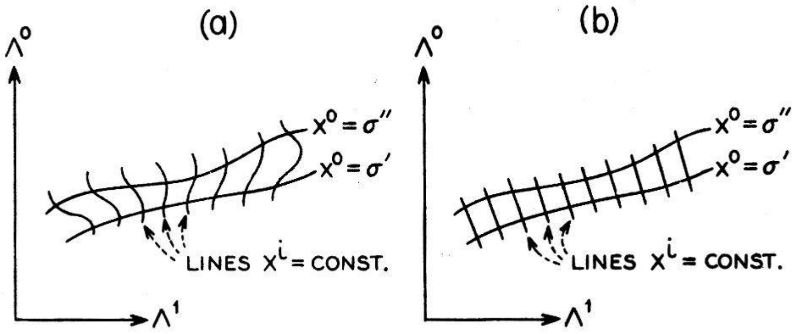

Fig. 1

Regular and singular one-surface limit of the two-surface boundary problems in a three-dimensional Riemannian space. *a* Two-surface boundary values that do not admit a regular one-surface limit. All 12 data independently specified. *b* Two-surface boundary values that admit a regular one-surface limit. Only the six data g'_{ik} and g''_{ik} independently specified: $g'_{\mu 0}$ and $g''_{\mu 0}$ determined by requirements of regularity.

must coincide with the geodesic connecting the two points (x^i, σ') and (x^i, σ''). Since the directions of the lines $x^i = \text{const.}$ are determined by $g'_{\mu 0}$ and $g''_{\mu 0}$, while the geodesic, which is a straight line in Λ-space, is determined by the shape of the surfaces, and therefore by g'_{ik} and g''_{ik}, this places a restriction on the boundary data which guarantees that the two-surface boundary values admit a regular one-surface limit. In the limit as $\varepsilon \to 0$ this amounts to drawing the lines $x^i = \text{const.}$ as straight lines in Λ-space. It is then clear that the shape of the surfaces σ' and σ'' determine the lengths of these straight lines as well as their angles with σ' and σ'', i.e., g'_{ik} and g''_{ik} determine $g'_{\mu 0}$ and $g''_{\mu 0}$. This is precisely what we did in the one-surface formulation: determine $g'_{\mu 0}$ from g'_{ik} and $\partial_0 g'_{ik}$.

Formally the above conditions read

$$\left.\begin{aligned} \partial_0 \Lambda^{\alpha'} &= \frac{1}{\varepsilon} (\Lambda^{\alpha''} - \Lambda^{\alpha'}) + 0(\varepsilon) , \\ \partial_0 \Lambda^{\alpha''} &= \frac{1}{\varepsilon} (\Lambda^{\alpha''} - \Lambda^{\alpha'}) + 0(\varepsilon) , \end{aligned}\right\} \tag{28}$$

which are the analogues of (24), and in the limit become six restrictions on the 12 boundary values $g'_{\mu\nu}$ and $g''_{\mu\nu}$ of the two-surface formulation. This leaves six independent quantities as in the one-surface formulation.

c. Two-surface formulation for gravity in the absence of sources

Our analogies with electromagnetism and three-dimensional Riemannian space imply that one should be able to specify $g'_{\mu\nu}$ and $g''_{\mu\nu}$ independently in the two-surface boundary value problem for gravity. A proof of this statement goes far beyond the scope of this note and must be expected to be considerably more difficult than the proof of the existence of solutions of the initial value problem.

In the case of the three-dimensional Riemannian space the two-surface problem was reduced to solutions of differential equations because we knew the explicit solution of the equations of motion. However, lacking an explicit solution of the equation $S^{ik} = 0$ in four dimensions, the problem cannot be reduced to a system of differential equations for $g''_{\mu\nu}$ and $g'_{\mu\nu}$. Therefore the two-surface boundary problem for gravity is essentially nondifferential in nature.

In order that our conclusion be not based entirely on analogies, we shall give three additional simple arguments to show that this conclusion is reasonable.

First let us count the number of independent degrees of freedom. Let us construct an implicit solution by the following procedure. As in the electromagnetic case, we begin with the construction of an auxiliary solution $a_{\mu\nu}(x)$, defined by

$$a'_{ik} = g'_{ik} , \tag{29}$$

$$a_{\mu 0} = \delta_{\mu 0} . \tag{30}$$

The four subsidiary conditions $S^{\mu 0} = 0$ restrict the six initial values of $\partial_0 a_{ik}$ such that only two of them may be specified freely[6]).

When we start with a compatible set of initial values, the solution $a_{\mu\nu}(x)$ is then unique by virtue of the coordinate conditions (30), and exists for a finite distance into the future. Given this auxiliary solution we obtain the general solution from

$$g_{\mu\nu}(x) = \partial_\mu \Lambda^\alpha(x) \, \partial_\nu \Lambda^\beta(x) \, a_{\alpha\beta}(\Lambda(x)) . \tag{31}$$

If we choose

$$\Lambda^i(\boldsymbol{x}, \sigma') = x^i , \quad \Lambda^0(\boldsymbol{x}, \sigma') = \sigma' , \tag{32}$$

and furthermore determine $\partial_0 \Lambda^\alpha(\boldsymbol{x}, \sigma')$ in such a way that $g_{\mu 0}(\boldsymbol{x}, \sigma')$ coincide with the prescribed boundary values $g'_{\mu 0}(\boldsymbol{x})$, then the boundary conditions at σ' are satisfied. On the other hand, we may adjust the eight quantities $\Lambda^\alpha(\boldsymbol{x}, \sigma'')$ and $\partial_0 \Lambda^\alpha(\boldsymbol{x}, \sigma'')$, together with the two degrees of freedom in the initial values for $\partial_0 a'_{ik}$, in order to satisfy the ten final boundary conditions

$$g_{\mu\nu}(\boldsymbol{x}, \sigma'') = g''_{\mu\nu}(\boldsymbol{x}) .$$

The unsatisfactory feature of this construction is of course the implicit appearance of the two degrees of freedom in the initial values, which are used to adjust the final boundary values.

The second argument is the trivial remark that if we only are allowed to specify g'_{ik} and g''_{ik}—the intrinsic geometries of the initial and final surfaces—then we are certainly allowed to specify $g'_{\mu 0}$ and $g''_{\mu 0}$ in addition, since these values may always be adjusted by a simple coordinate transformation which affects the angles between the lines $x^i =$ const. and the initial and final surfaces.

The third argument in favor of the claim that $g'_{\mu\nu}$ and $g''_{\mu\nu}$ may be specified independently is based on the investigation of the neighborhood of a given solution. We would like to show that for given initial values $g'_{\mu\nu}$ we have a ten-parameter family of solutions per point on the surface σ''. To show this we may construct neighboring solutions by successive approximations of flat space, $g'_{\mu\nu} = g''_{\mu\nu} = \eta_{\mu\nu}$. The linear approximation to EINSTEIN's equations is

$$\Box\, h^{(1)}_{\mu\nu} + \partial_{\mu\nu}\, h^{(1)}_{\sigma}{}^{\sigma} - \partial_{\mu\sigma}\, h^{(1)}_{\nu}{}^{\sigma} - \partial_{\nu\sigma}\, h^{(1)}_{\mu}{}^{\sigma} = 0 , \tag{33}$$

where $g_{\mu\nu} = \eta_{\mu\nu} + \lambda\, h^{(1)}_{\mu\nu} + \lambda^2\, h^{(2)}_{\mu\nu} + \ldots$. Since we hold $g'_{\mu\nu} = \eta_{\mu\nu}$ fixed, we have $h^{(1)\prime}_{\mu\nu} = 0$. It may be easily seen that in the approximation (33) the situation is exactly the same as for electromagnetism. The final values of $h^{(1)}_{\mu\nu}$ may indeed be prescribed arbitrarily. To complete the proof that we have indeed a ten-parameter family of solutions per point of the final surface, we would have to investigate the convergence of the approximation scheme, i.e., we would have to show that the expansion in λ is analytic in a finite domain of convergence. This goes again beyond the scope of this note.

V. Two-surface Formulation with Sources

We have already pointed out that the presence of sources destroys the invariance group. Consequently, for given, compatible initial values, the solution of the equations of motion is now, in general, unique. We cannot expect to be able to specify the same number of final boundary data as without a source, because the freedom of gauge transformations was essential in the source-free case.

A natural way to construct a solution is to start with some initial values g'_{ik} and $\partial_0\, g'_{ik}$. The subsidiary conditions $S^{\mu 0} = \tau^{\mu 0}$ may again be solved for the initial values $g'_{\mu 0}$. In addition, we may determine $\partial_0\, g'_{\mu 0}$ from the conservation law $\nabla_\mu\, \tau^{\mu\nu} = 0$, if we assume the source to be nondegenerate in the sense (13). This automatically provides a consistent set of initial values, which uniquely determines a solution of EINSTEIN's equations. In particular, g''_{ik} and $g''_{\mu 0}$ are determined. Finally we may adjust the initial values $\partial_0 g'_{ik}$ in such a way that g''_{ik} assumes prescribed values.

In the presence of sources, the transition to the one-surface formulation presents no difficulties.

Table 1

Comparison of required boundary data for MAXWELL's and EINSTEIN's equations, with or without sources for one- and two-surface formulations

	MAXWELL's Equations without source	with source	EINSTEIN's Equations without source	with source
Equations of motion	$\partial_\mu F^{\mu\nu} = 0$ $\partial_\mu F^{\mu\nu *} = 0$	$\partial_\mu F^{\mu\nu} = j^\nu$ $\partial_\mu F^{\mu\nu *} = 0$	$S^{\mu\nu} = 0$	$S^{\mu\nu} = \tau^{\mu\nu}$
Gauge group	$\bar{A}_\mu = A_\mu + \partial_\mu \Lambda$	$\bar{A}_\mu = A_\mu + \partial_\mu \Lambda$	$\bar{g}_{\mu\nu}(x)$ $= \partial_\mu \Lambda^\alpha \partial_\nu \Lambda^\beta g_{\alpha\beta}[\Lambda(x)]$	no gauge group in general†)
One-surface problem Freely specifiable boundary values and number of data (per point of the surface)	A'_i; $\partial_0 A'_i$ 6	A'_i; $\partial_0 A'_i$ 6	g'_{ik}; $\partial_0 g'_{ik}$ 12	g'_{ik}; $\partial_0 g'_{ik}$ 12
Number of data affected gauge group	2	2	8	—
Number of essential data	4	4	4	12
Two-surface formulation Freely specifiable boundary values	A'_μ; A''_μ 8	A'_μ; A''_μ 8	$g'_{\mu\nu}$; $g''_{\mu\nu}$ 20	g'_{ik}; g''_{ik} 12
Number of data affected gauge group	4	4	16	—
Number of essential data	4	4	4	12

†) For degenerate sources, e.g., a scalar field, part of the gauge group may remain. See the discussion in Section 3.

References

1) Notation: $\mu, \nu = 0, 1, 2, 3$; $i, k = 1, 2, 3$; $x^0 = ct$.

2) We adopt the terminology of R. F. BAIERLEIN, D. H. SHARP and J. A. WHEELER, Phys. Rev. *126*, 1864 (1962).

3) H. LEUTWYLER, *The Gravitational Field: Equivalence of Feynman Quantization and Canonical Quantization*, Phys. Rev. *134*, B 1155 (1964).

4) A. LICHNEROWICZ, J. Math. Pure Appl. *23*, 37 (1944); Helv. Phys. Acta. Suppl. *4*, 176 (1956); Y. FOURÈS-BRUHAT, Acta. Math. *88*, 141 (1952); Journ. Rat. Mech. Anal. *4*, 951 (1956); A. PERES and N. ROSEN, Nuovo Cim. *13*, 430 (1959); R. F. BAIERLEIN, D. H. SHARP and J. A. WHEELER, Phys. Rev. *126*, 1864 (1962).

5) Note that the energy tensor of the electromagnetic field admits a one-parameter gauge group of the type sketched for the scalar field.

6) A. LICHNEROWICZ and Y. FOURÈS-BRUHAT, reference 3.