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Causality and One-Particle Singularities of the S-Matrix*)

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Abstract. The existence of one-particle singularities located in the physical region of a three body scattering amplitude is shown to result from suitable requirements of macrocausality.

1. Introduction

By macrocausality we mean the set of conditions an S-matrix has to satisfy in order to describe correctly the macroscopic space-time development of the processes we observe. Professor STUECKELBERG's scrutiny led him to recognize[1]), at a very early stage, the importance of the causality principle in a non-lagrangian construction of the S-matrix. Therefore, we found it appropriate to honour his sixtieth birthday with a note concerning the role of causality in a non-perturbative approach of the S-matrix.

In contrast with other requirements, as unitarity and relativistic invariance, macrocausality cannot be expressed in a general, adequate, and technically useful way. We recall only one of the difficulties encountered in such a formulation. Macrocausality should deal with correlations of macroscopically separated events. However, it appears very difficult to make a clear-cut distinction between macroscopic and microscopic space-time distances. One is therefore forced to restrict oneself to statements about events separated by infinite space-time distances. In other words, one formulates conditions of asymptotic causality instead of macrocausality; the consequences of such conditions may be very weak[2]).

Nevertheless, recently J. H. CRICHTON and E. H. WICHMAN[3]) established a property as important as the cluster decomposition property, or vacuum structure of the S-matrix on the basis of asymptotic conditions. Another structure, the one-particle structure, i. e. the existence of singularities of the type

$$\left[\frac{1}{\pi}\, P\, \frac{1}{p^2 - m^2} - i\, \delta\, (p^2 - m^2)\right],$$

has been deduced from the axioms of field theory[4]). It is natural to ask if this structure follows also from suitable asymptotic conditions. The purpose of this note is to show that the answer to this question is affirmative, in the sense that, if a transition amplitude has one-particle singularities in its physical domain, these singularities do follow from asymptotic causality. This is due to the fact that such singularities describe an exchange of a particle which can be real. As real particles may propagate over infinite distances, contributions due to their exchange survive, and dominate, in some suitably chosen limits. It follows from stability conditions that only amplitudes describing processes with more then two particles in both final and initial states have one-

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particle singularities in their physical region. Therefore, asymptotic causality cannot predict the one-particle structure of the amplitudes related with two body collisions. The one-particle singularities of these amplitudes describe the exchange of virtual particles, which cannot travel over macroscopic distances.

In the following two sections we show how asymptotic causality leads to one-particle singularities in the case of a transition between three initial and three final particles. This is the simplest transition exhibiting one-particle singularities in a physical domain. Our analysis could easily be extended to more intricate transitions. In a recent work, D. BRANSON discusses also three body scattering. He proves that unitarity and vacuum structure imply the existence of a singularity of the type $\delta(p^2 - m^2)$. In order to show that a $P(1/p^2 - m^2)$-type singularity combines with the previous one, BRANSON uses some continuation to the amplitude outside its physical domain[5]). Our considerations avoid such an artifice and involve only observable quantities.

2. One-Particle Singularities of a Three Body Scattering Amplitude

We consider the three body process:

$$A_1 + A_2 + A_3 \to B_1 + B_2 + B_3 . \tag{2.1}$$

All particles are supposed to have spin zero. In order to get a localized process, we assign to the initial and final particles normalized wave packets $\varphi_i(x)$ and $\chi_i(x)$ $(i = 1,2,3)$:

$$\varphi_i(x) = \int (dp)^4 e^{-i(p,x)} \bar{\varphi}_i(p) \ , \qquad \bar{\varphi}_i(p) = \theta(p_0)\, \delta\, (p^2 - m_i^2)\, \varphi_i(\boldsymbol{p})\, 2\, p_{i\,0} ,$$

$$\chi_i(x) = \int (dq)^4 e^{-i(q,x)} \bar{\chi}_i(q) , \qquad \bar{\chi}_i(q) = \theta(q_0)\, \delta\, (q^2 - m_{3+i}^2)\, \tilde{\chi}_i(\boldsymbol{q})\, 2\, q_{i\,0} . \tag{2.2}$$

We use the metric $(a, b) = a_0 b_0 - (\boldsymbol{a}, \boldsymbol{b})$; m_i is the non-vanishing mass of particle A_i, m_{3+i} that of particle B_i. $\tilde{\varphi}_i(\boldsymbol{p})$ and $\tilde{\chi}_i(\boldsymbol{q})$ are supposed to be infinitely differentiable and to have compact supports. Let $T_{3,3}[\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3/\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3]$ be the connected part of the transition amplitude $\langle \boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3 \mid T \mid \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle$. (In our notation, T_{n_f, n_i} describes a process with n_i initial particles and n_f final particles.) The contribution of this connected part to the amplitude of process (2.1) is:

$$T_C = \int \Pi\, (dq_i)^4\, (dp_i)^4\, g(p_3, q_1, q_2)\, f(q_3, p_1, p_2) ,$$
$$T_{3\,3}\left[\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3 \,/\, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\right] \delta\, (q_1 + q_2 + q_3 - p_1 - p_2 - p_3) . \tag{2.3}$$

We introduced the following notation, for further convenience:

$$f(q_3, p_1, p_2) = \bar{\chi}_3^*(q_3)\, \bar{\varphi}_1(p_1)\, \bar{\varphi}_2(p_2),\ \ g(p_3, q_1, q_2) = \bar{\varphi}_3(p_3)\, \bar{\chi}_1^*(q_1)\, \bar{\chi}_2^*(q_2) . \tag{2.4}$$

The substitution:

$$g(p_3, q_1, q_2) \to g(p_3, q_1, q_2) \exp\left[i\, (p_3 - q_1 - q_2, a)\right] \tag{2.5}$$

produces a space-time displacement of the wave packets φ_3, χ_1, and χ_2 of particles A_3, B_1, and B_2, characterized by the four-vector a $(\varphi_3(x) \to \varphi_3(x - a), \ldots)$. Performing this substitution in (2.3), T_C becomes a function $T(a)$ of a:

$$T(a) = \int (dk)^4\, \bar{T}(k)\, e^{-i(k,a)} \ , \qquad \bar{T}(k) = \int \Pi\, (dq_i)^4\, (dp_i)^4\, g(p_3, q_1, q_2)\, f(q_3, p_1, p_2) \times ,$$
$$\times\, T_{3,3}\left[\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3 \,/\, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\right] \delta\, (k - p_1 - p_2 + q_3)\, \delta\, (k - q_1 - q_2 + p_3) . \tag{2.6}$$

We insist on the fact that $T(a)$ is a physical transition amplitude; the integral in (2.6) extends over the physical region of $T_{3,3}$. Therefore any property of $T(a)$ has a direct physical meaning.

Imagine that the wave packets of particles A_3, B_1, and B_2 overlap in a space-time region $M(a)$, whereas the wave packets of particles A_1, A_2, and B_3 overlap in a region N. As $a_0 \to +\infty$, the region $M(a)$ moves into the infinitely remote future of the region N. We postulate that, in this limit, the transition (2.1) is due to the double-scattering (figure 1):

$$A_1 + A_2 \to B_3 + X \; ; \quad X + A_3 \to B_1 + B_2 \, . \tag{2.7}$$

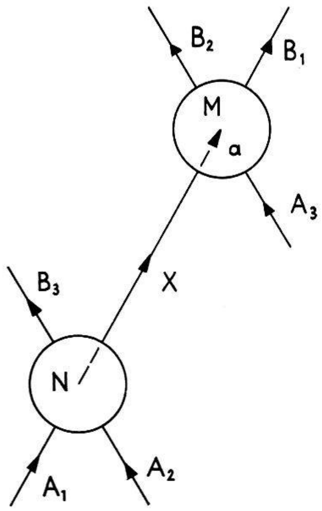

Figure 1

In other words, we assume that, once M and N are separated by a positive macroscopic time interval, the connection between these two space-time regions is due to the propagation of the stable free particle X. As a matter of fact, a transition like (2.1) is always realized experimentally under such conditions that M and N are macroscopically disjoint, and the experiment is analyzed as a double scattering. Furthermore, we have assumed that selection rules exclude the exchange of any other single particle state than X. It would be easy to relax this latter restriction.

Let $\psi_1(x)$ be the wave packet of the particle X produced in N. This wave packet is determined by the wave packets of particles A_1, A_2, and B_3 and by the amplitude $T^{(1)}_{2,2}$ of the reaction $A_1 + A_2 \to B_3 + X$:

$$\tilde{\psi}_1(\boldsymbol{k}) = \frac{1}{2\,\omega(\boldsymbol{k})} \int (dp_1)^4 \, (dp_2)^4 \, (dq_3)^4 f(q_3, p_1, p_2) \times$$
$$\times \; T^{(1)}_{2,2} \left[\boldsymbol{q}_3, \boldsymbol{k} \,/\, \boldsymbol{p}_1, \boldsymbol{p}_2\right] \delta\left(\boldsymbol{k} + \boldsymbol{q}_3 - \boldsymbol{p}_1 - \boldsymbol{p}_2\right) \times \delta\left(\omega(\boldsymbol{k}) + q_{3\,0} - p_{1\,0} - p_{2\,0}\right) , \tag{2.8}$$

$\omega(\boldsymbol{k}) = (|\,\boldsymbol{k}\,|^2 + \mu^2)^{1/2}$, μ = mass of particle X. Similarly, the wave packet $\psi_2(x)$ of the particle X absorbed in M is determined by:

$$\tilde{\psi}_2{}^*(\boldsymbol{k}) = \frac{1}{2\,\omega(k)} \int (dq_1)^4 \, (dq_2)^4 \, (dp_3)^4 \, g(p_3, q_1, q_2) \times$$
$$\times \; T^{(2)}_{2,2} \left[\boldsymbol{q}_1, \boldsymbol{q}_2 \,/\, \boldsymbol{k}, \boldsymbol{p}_3\right] \delta\left(\boldsymbol{k} + \boldsymbol{p}_3 - \boldsymbol{q}_1 - \boldsymbol{q}_2\right) \times$$
$$\times \; \delta\left(\omega(\boldsymbol{k}) + p_{3\,0} - q_{1\,0} - q_{2\,0}\right) \exp\left[-\,i\left(\omega(\boldsymbol{k}) a_0 - (\boldsymbol{k}, \boldsymbol{a})\right)\right] . \tag{2.9}$$

$T^{(2)}_{2,2}$ is the amplitude of the reaction $X + A \to B_1 + B_2$.

The scalar product of the wave packets ψ_1 and ψ_2 gives the amplitude $F(a)$ of the double scattering (2.7):

$$F(a) = i(\psi_2, \psi_1) = \int (dk)^3\, \tilde{F}(\boldsymbol{k}) \exp\left[- i\left(\omega(\boldsymbol{k})\, a_0 - (\boldsymbol{k}, \boldsymbol{a})\right)\right] ,$$

$$\tilde{F}(\boldsymbol{k}) = i\, 2\, \omega(\boldsymbol{k})\, \tilde{\psi}_2^*(\boldsymbol{k})\, \tilde{\psi}_1(\boldsymbol{k}) \exp\left[+ i\left(\omega(\boldsymbol{k})\, a_0 - (\boldsymbol{k}, \boldsymbol{a})\right)\right] . \qquad (2.10)$$

As $a_0 \to - \infty$, $M(a)$ recedes to the infinitely remote past of N. Macrocausality excludes the propagation of a particle backward in time, over macroscopic time intervals. Therefore, the contribution due to the double-scattering (2.7) must disappear in the limit $a_0 \to - \infty$. In this limit, macro causality would allow an exchange due to the emission of a free particle, call it Y, in M, and its absorption in N:

$$A_3 \to B_1 + B_2 + Y \,; \quad Y + A_1 + A_2 \to B_3 \,.$$

However, the assumed stability of particles A_3 and B_3 excludes such a process.

We are led to the decomposition:

$$T(a) = G(a) + H(a) \,; \quad G(a) = \theta(a_0)\, F(a) \,. \qquad (2.11)$$

The step-function $\theta(a_0)$ could be replaced by any function approaching sufficiently rapidly the value $+ 1$ for $a_0 \to + \infty$ and the value 0 for $a_0 \to - \infty$. We use $\theta(a_0)$ for simplicity; as $F(a)$ is a bounded function of a, the multiplication by a step-function introduces no ambiguity. As $T(a)$ and $F(a)$ are bounded functions of a, the same is true for $H(a)$. Equation (2.11) becomes more then a definition of $H(a)$ if we require some asymptotic behavior for this function. We shall postulate that:

$$\int (da)^4 \mid H(a) \mid^2 < \infty \,. \qquad (2.12)$$

As $\int (da)^3 \mid G(a) \mid^2 = \text{cst.}\ \theta(a_0)$, $G(a)$ is not square integrable over R^4. Therefore, $H(a)$ has an overall stronger decrease then $G(a)$ for $a_0 \to + \infty$; (2.12) warrants thereby the dominance of the double-scattering (2.7) in this limit. Condition (2.12) leads to:

$$\int (da)^4\, \theta(- a_0) \mid T(a) \mid^2 < \infty \,,$$

this should express the impossibility of one-particle exchanges in the limit $a_0 \to - \infty$. Finally, the decrease of $H(a)$ in space-like directions implied by (2.12) is ensured by the short range of the forces at work. A more detailed justification of (2.12) will be given in the following section.

The Fourier transform of (2.11) reads:

$$\overline{T}(k) = \overline{G}(k) + \overline{H}(k) \,. \qquad (2.13)$$

According to (2.11) and (2.10):

$$\overline{G}(k) = - \frac{1}{2} \left[\frac{1}{\pi}\, P \frac{1}{k_0 - \omega(\boldsymbol{k})} - i\, \delta\left(k_0 - \omega(\boldsymbol{k})\right)\right] \tilde{F}\,(\boldsymbol{k}) \,. \qquad (2.14)$$

Condition (2.12) implies:

$$\int (dk)^4 \mid \overline{H}(k) \mid^2 < \infty \,. \qquad (2.15)$$

Equation (2.14) shows that $\overline{G}(k)$ has a singularity at $k_0 = \omega(\boldsymbol{k})$ which is not square integrable. From (2.15) it follows that $\overline{H}(k)$ is less singular then $\overline{G}(k)$ and cannot compensate the singularity of the latter function. Therefore, $\overline{T}(k)$ is certainly singular at $k_0 = \omega(\boldsymbol{k})$, its leading singularity being that of $\overline{G}(k)$.

Finally, we have to make sure that the singularity of $\overline{T}(k)$ is due to a singularity of $T_{3,3}$. To this end, we eliminate the Dirac-distributions appearing in the integral (2.6) defining $\overline{T}(k)$. In a first step we get:

$$\begin{aligned}\overline{T}(k) = \int (dq_1)^3 \, (dq_2)^3 \, (dp_1)^3 \, (dp_2)^3 \, \tilde{g}(\boldsymbol{q}_1 + \boldsymbol{q}_2 - \boldsymbol{k}, \boldsymbol{q}_1, \boldsymbol{q}_2) \times \\ \times \tilde{f}(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{k}, \boldsymbol{p}_1, \boldsymbol{p}_2) \, T_{3,3}\,[\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{k} \,/\, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{q}_1 + \boldsymbol{q}_2 - \boldsymbol{k}] \times \\ \times \delta\left(k_0 - \omega_1(\boldsymbol{p}_1) - \omega_2(\boldsymbol{p}_2) + \nu_3\,(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{k})\right) \times \\ \times \delta\left(k_0 - \nu_1\,(\boldsymbol{q}_1) - \nu_2\,(\boldsymbol{q}_2) + \omega_3\,(\boldsymbol{q}_1 + \boldsymbol{q}_2 - \boldsymbol{k})\right). \end{aligned} \tag{2.16}$$

The functions $\tilde{g}$ and $\tilde{f}$ are obtained from the $\tilde{\varphi}$'s and $\tilde{\chi}$'s in the same way as f and g are obtained from the φ's and χ's in (2.4). In the last formula, we use the notation:

$$\omega_i(\boldsymbol{p}) = (|\,\boldsymbol{p}\,|^2 + m_i^2)^{1/2}\,, \quad \nu_i(\boldsymbol{q}) = (|\,\boldsymbol{q}\,|^2 + m_{3+i}^2)^{1/2}\,.$$

It is convenient to introduce the new variables:

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2\,, \quad \boldsymbol{p} = \boldsymbol{p}_1 - \boldsymbol{p}_2\,,$$
$$\boldsymbol{Q} = \boldsymbol{q}_1 + \boldsymbol{q}_2\,, \quad \boldsymbol{q} = \boldsymbol{q}_1 - \boldsymbol{q}_2\,,$$

and to define the vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ by their spherical polar coordinates with respect to a system of coordinates whose z-axis is parallel to $\boldsymbol{k}$. Let α and β be the polar angles of $\boldsymbol{P}$ and $\boldsymbol{Q}$, φ and ψ their azimuths. The Dirac-distributions remaining in (2.16) allow us to get rid of the angles α and β:

$$\begin{aligned}\overline{T}(k) = \frac{1}{64}\,\frac{1}{|\,\boldsymbol{k}\,|^2}\int_0^\infty dP\,P \int_0^\infty dQ\,Q \int_0^{2\pi} d\varphi \int_0^{2\pi} d\psi \int (dq)^3 \int (dp)^3 \times \\ \times\, g(k, Q, \psi, \boldsymbol{q})\,\tilde{f}(k, P, \varphi, \boldsymbol{p})\; T_{3,3}(k, Q, P, \varphi, \psi, \boldsymbol{q}, \boldsymbol{p}) \times \\ \times \left[k_0 - \omega_1\left(\frac{1}{2}\,(\boldsymbol{P} + \boldsymbol{p})\right) - \omega_2\left(\frac{1}{2}\,(\boldsymbol{P} - \boldsymbol{p})\right)\right] \left[k_0 - \nu_1\left(\frac{1}{2}\,(\boldsymbol{Q} + \boldsymbol{q})\right) - \right. \\ \left. - \nu_2\left(\frac{1}{2}\,(\boldsymbol{Q} - \boldsymbol{q})\right)\right] \;\theta(\Delta_1(k, P, \varphi, \boldsymbol{p}))\;\theta(\Delta_2(k, Q, \psi, \boldsymbol{q}))\,. \end{aligned} \tag{2.17}$$

The conditions $\Delta_i > 0$ make sure that α and β take on real values; they define the physical region of our process.

As $\tilde{g}(k, Q, \psi, \boldsymbol{q})$ and $\tilde{f}(k, P, \varphi, \boldsymbol{p})$ are bounded functions with compact support, they cannot be responsible for the infinitude of $\overline{T}(k)$ at $k_0 = \omega(\boldsymbol{k})$. Furthermore, as the location and nature of the singularity of $\overline{T}(k)$ are independent of the choice of $\tilde{g}$ and $\tilde{f}$, $T_{3,3}$ must have a singularity of the same type:

$$\begin{aligned} T_{3,3}[\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3 \,/\, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3] = \\ -\frac{1}{4\,\omega(\boldsymbol{k})}\left[\frac{1}{\pi}\,P\,\frac{1}{k_0 - \omega(\boldsymbol{k})} - i\,\delta\left(k_0 - \omega(\boldsymbol{k})\right)\right] \times \\ \times\, U\,[\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, /\, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3] + R\,[\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3 \,/\, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3]\,. \end{aligned} \tag{2.18}$$

In this formula, k_0 and $\boldsymbol{k}$ stand for:

$$k_0 = \omega_1(\boldsymbol{p}_1) + \omega_2(\boldsymbol{p}_2) - \nu_3(\boldsymbol{q}_3) = \nu_1(\boldsymbol{q}_1) + \nu_2(\boldsymbol{q}_2) - \omega_3(\boldsymbol{p}_3) ,$$

$$\boldsymbol{k} = \boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{q}_3 = \boldsymbol{q}_1 + \boldsymbol{q}_2 - \boldsymbol{p}_3 .$$

$\overline{G}(k)$ can be brought into a form similar to the expression (2.17) for $\overline{T}(k)$. A comparison of these two expressions gives for the "residue" U:

$$U[\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3 / \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3] = T^{(2)}_{2,2} [\boldsymbol{q}_1, \boldsymbol{q}_2 / \boldsymbol{k}, \boldsymbol{p}_3] \times T^{(1)}_{2,2} [\boldsymbol{q}_3, \boldsymbol{k} / \boldsymbol{p}_1, \boldsymbol{p}_2] . \tag{2.19}$$

The "regular" term R verifies the condition:

$$\int (dk)^4 \Big| \int (dq_1)^4 \int (dq_2)^4 \int (dp_1)^4 \int (dp_2)^4 \, g(q_1 + q_2 - k, q_1, q_2) f(p_1 + p_2 - k, p_1, p_2) \times R [\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{k} / \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{q}_1 + \boldsymbol{q}_2 - \boldsymbol{k}] \Big|^2 < \infty \tag{2.20}$$

as a consequence of (2.15). Therefore, R is less singular at $k_0 = \omega(\boldsymbol{k})$ then the first term in (2.18). For example, it could behave like $(k_0 - \omega(\boldsymbol{k}))^{-\alpha}$ with $\alpha < 1/2$. On the other hand, R can have stronger singularities in the remaining variables; these are smeared out by the q- and p-integrations in (2.20). Such singularities cannot be excluded since R contains the possible one-particle singularities arising in other channels then the k-channel considered so far. Some of these singularities may appear in the physical domain of process (2.1).

If we assume that $T_{3,3}$ is an analytic function of k^2, (2.18) can be brought into the more familiar form:

$$T_{3,3} = - \frac{1}{2\pi} \frac{1}{k^2 - m^2 + i\varepsilon} U\big((q_1 + q_2)^2, (p_1 + p_2)^2, (q_1 - p_3)^2, (q_3 - p_1)^2\big) + R_1(k^2, \ldots) \tag{2.21}$$

with:

$$R_1 = - \frac{1}{4\pi} \frac{1}{\omega(\boldsymbol{k}) \, (k_0 + \omega(\boldsymbol{k}))} U + R . \tag{2.22}$$

R_1 has no pole at $k_0 = \omega(\boldsymbol{k})$. Therefore, we succeeded in showing that conditions (2.11) and (2.12) imply the existence of a one-particle pole in $T_{3,3}$. The fact that we cannot prove that R_1 is regular at $k_0 = \omega(\boldsymbol{k})$ is due to the weakness of condition (2.12).

3. Further Discussion of Condition (2.12)

The condition (2.12) played a crucial role in the discussion presented in the preceding section. It is this condition which allowed us to separate the one-particle pole we were interested in. We devote the present section to a justification of the behavior of $H(a)$ as $a_0 \to \infty$ implied by (2.12).

Beside the one-particle exchange considered so far, possible exchanges of real two-particle states (figure 2) may contribute to the transition (2.1) in the limit $a_0 \to \infty$. We shall investigate the behavior of such a contribution. Its amplitude is given by:

$$K(a) = \int (dk)^4 \, \overline{K}(k) \, e^{-i(k,a)} \tag{3.1}$$

with:

$$\overline{K}(k) = \int \Pi \, (dp_i)^4 \, (dq_i)^4 f(q_3, p_1, p_2) \, g(p_3, q_1, q_2) \times \times \delta\,(q_1 + q_2 - p_3 - k) \, \delta\,(p_1 + p_2 - q_3 - k) \int (dk_1)^4 \int (dk_2)^4 \times \tag{3.2}$$

$$\times T^{(2)}_{2,3} [\boldsymbol{q}_1, \boldsymbol{q}_2 / \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{p}_3] \, \delta_+ (k_1^2 - \mu_1^2) \, \delta_+ (k_2^2 - \mu_2^2) \, T^{(1)}_{3,2} [\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{q}_3 / \boldsymbol{p}_1, \boldsymbol{p}_2] \, \delta\,(k_1 + k_2 - k) .$$

$T^{(1)}_{3,2}$ is the amplitude of the production process $A_1 + A_2 \to X_1 + X_2 + B_3$, while $T^{(2)}_{2,3}$ is the amplitude of the process $X_1 + X_2 + A_3 \to B_1 + B_2$.

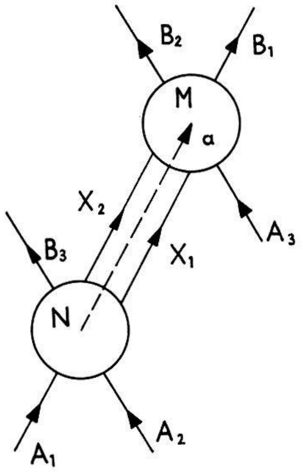

Figure 2

As the functions f and g have compact supports, the same is true for the function $\bar{K}(k)$. We shall further show that $\bar{K}(k)$ is bounded. It is convenient to perform the following change of variables in (3.2):

$$p_i \to p_i' = \Lambda(k)\, p_i,\quad q_i \to q_i' = \Lambda(k)\, q_i,\quad k_i \to k_i' = \Lambda(k)\, k_i,$$

where $\Lambda(k)$ is a Lorentz transformation bringing the system of particles X_1 and X_2 to rest:

$$\Lambda(k)\, k = k^*,\quad k^* = (\sqrt{k^2}, 0, 0, 0)\,.$$

If:

$$f_\Lambda(q_3, p_1, p_2) = f(\Lambda^{-1} q_3, \Lambda^{-1} p_1, \Lambda^{-1} p_2)\,,$$

$$g_\Lambda(p_3, q_1, q_2) = g(\Lambda^{-1} p_3, \Lambda^{-1} q_1, \Lambda^{-1} q_2)\,,$$

$\bar{K}(k)$ becomes:

$$\bar{K}(k) = \int \Pi\, (dp_i)^4\, (dq_i)^4\, f_\Lambda(q_3, p_1, p_2)\, g_\Lambda(p_3, q_1, q_2) \times$$

$$\times\, \delta\,(q_1 + q_2 - p_3 - k^*)\, \delta\,(p_1 + p_2 - q_3 - k^*)\, L(q_i, p_i, k^*)\,. \tag{3.3}$$

L is given by:

$$L(q_i, p_i, k^*) = \int (du)^4\, T^{(2)}_{2,3}\,[q_1, q_2 \,/\, \tfrac{1}{2} k^* + u,\ \tfrac{1}{2} k^* - u, p_3] \times$$

$$\delta_+\Big(\Big(\frac{1}{2} k^* + u\Big)^2 - \mu_1^2\Big)\, \delta_+\Big(\Big(\frac{1}{2} k^* - u\Big)^2 - \mu_2^2\Big) \times \tag{3.4}$$

$$\times\, T^{(1)}_{3,2}\,[\tfrac{1}{2} k^* + u, \tfrac{1}{2} k^* - u, q_3 \,/\, p_1, p_2] = \frac{|\boldsymbol{u}|}{2\sqrt{k^2}}\, \theta\big(\sqrt{k^2} - \mu_1 - \mu_2\big) \int d\Omega_{\boldsymbol{u}}\, T^{(2)}_{2,3}\, T^{(1)}_{3,2}\,.$$

In the last expression, $u_0 = (\mu_1^2 - \mu_2^2)/2\, k^2$ and

$$|\,\boldsymbol{u}\,|^2 = \frac{1}{4\, k^2}\, (k^2 - (\mu_1 + \mu_2)^2)\, (k^2 - (\mu_1 - \mu_2)^2)\,.$$

The amplitudes $T^{(1)}_{3,2}$ and $T^{(2)}_{2,3}$ have no one-particle singularities in their physical domains. Therefore, we may assume that these amplitudes are bounded functions in these domains. This implies the boundedness of L. The elimination of some Dirac-distributions in (3.3) gives:

$$\bar{K}(k) = \int (dp_1)^3 \int (dp_2)^3 \int (dq_1)^3 \int (dq_2)^3 \tilde{f}_A(\boldsymbol{p}_1 + \boldsymbol{p}_2, \boldsymbol{p}_1, \boldsymbol{p}_2) \times$$

$$\times g_A(\boldsymbol{q}_1 + \boldsymbol{q}_2, \boldsymbol{q}_1, \boldsymbol{q}_2)\, L\, \delta\,(\omega_1(\boldsymbol{p}_1) + \omega_2(\boldsymbol{p}_2) - \nu_3\,(\boldsymbol{p}_1 + \boldsymbol{p}_2) - k_0^*) \times$$

$$\times\, \delta\,(\nu_1(\boldsymbol{q}_1) + \nu_2(\boldsymbol{q}_2) - \omega_3(\boldsymbol{q}_1 + \boldsymbol{q}_2) - k_0^*)\,. \tag{3.5}$$

The remaining Dirac-distributions disappear after integration over the angles formed by $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. Once this has been done, $\bar{K}(k)$ appears as the integral of a bounded integrand extended over a compact domain. This circumstance establishes the boundedness of $\bar{K}(k)$.

Now, as $\bar{K}(k)$ is a bounded function with compact support, we have:

$$\int (da)^4 \mid K(a) \mid^2 = (2\,\pi)^4 \int (dk)^4 \mid \bar{K}(k) \mid^2 < \infty\,. \tag{3.6}$$

It may be worthwhile to notice that, because of the discontinuities appearing in the derivatives of $T^{(1)}_{3,2}$ and $T^{(2)}_{2,3}$, $\bar{K}(k)$ is not infinitely continously differentiable. As a consequence, it is impossible to get more informations about the asymptotic behavior of $K(a)$ than those implied by its square integrability (3.6) without introducing very specific assumptions about the discontinuities of $T^{(1)}_{3,2}$ and $T^{(2)}_{2,3}$.

As mentioned before, $H(a)$ may contain terms which behave asymptotically, for $a_0 \to \infty$, like $K(a)$. If we assume that the possible two-particle exchange terms govern the behavior of $H(a)$ as $a_0 \to \infty$, we are justified, from (3.6), to postulate:

$$\int^{\infty} da_0 \int (da)^3 \mid H(a) \mid^2 < \infty\,. \tag{3.7}$$

This is precisely that part of condition (2.12) we wished to make plausible.

Note added in proof: After completion of this work, the author received a preprint of an interesting article entitled "S-Matrix Theory and Double Scattering", due to D. Iagolnitzer (Saclay). It is shown there that the existence of one particle poles is consitent with assumptions about transition probabilities whereas our proof starts from assumptions about transition amplitudes.

References

[1]) E. C. G. Stueckelberg and D. Rivier, Helv. Phys. Acta *23*, 215 (1950).
[2]) G. Wanders, Il Nuovo Cimento *14*, 168 (1959).
[3]) J. H. Crichton and E. H. Wichman, Phys. Rev. *132*, 2788 (1963).
[4]) W. Zimmermann, Il Nuovo Cimento *13*, 503 (1959).
[5]) D. Branson, Phys. Rev. *135*, 1255 (1964).