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# On the Clebsch-Gordan Series of Semisimple Lie Algebras

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*Abstract.* Starting from a formula of STEINBERG, we derive a simple representation theorem for the highest weights in the decomposition of a tensor product of irreducible modules into irreducible constituents which is valid for arbitrary split semisimple Lie algebras (over a field of characteristic 0). Furthermore we use the formula of STEINBERG to evaluate the multiplicities of the irreducible modules corresponding to these highest weights for special Lie algebras.

## Introduction

In the physical literature, there exist quite a few papers about the Clebsch-Gordan series of  $SU_3$ <sup>1)</sup>. But it seems that it has been overlooked that Steinberg<sup>2)</sup> has given a formula for the decomposition of a tensor product of irreducible modules into irreducible constituents which is valid for arbitrary split semisimple Lie algebras over a field of characteristic 0. The formula of Steinberg expresses the multiplicities of the irreducible constituents by a double sum over the Weyl group  $W$ . Hence to determine the multiplicities, one only has to know the root system.

In § 1 we discuss briefly the formula of Steinberg. Starting from this formula, we prove a general representation theorem for the highest weights in the decomposition of the tensor product in § 2. With the help of this theorem, we can easily determine the multiplicities of the irreducible modules corresponding to these highest weights for special Lie algebras. This is carried out in § 3 for the algebras  $A_2$ ,  $G_2$ , and  $A_3$ .

## § 1. The Formula of Steinberg

Let  $\mathfrak{M}_{A'}$  and  $\mathfrak{M}_{A''}$  be two finite dimensional irreducible modules with the highest weights  $A'$  and  $A''$  of a finite dimensional semisimple Lie algebra  $\mathfrak{L}$  over a field of characteristic 0. Further we assume that  $\mathfrak{L}$  has a splitting Cartan subalgebra  $\mathfrak{H}$  (the characteristic roots of every  $ad(h)$ ,  $h \in \mathfrak{H}$ , are in the base field). If the base field is algebraically closed, any finite dimensional Lie algebra is of course split.

The tensor product  $\mathfrak{M}_{A'} \otimes \mathfrak{M}_{A''}$  is, according to a general theorem, completely reducible (this is the case for arbitrary finite dimensional Lie algebras over a field of characteristic 0). Let

$$\mathfrak{M}_{A'} \otimes \mathfrak{M}_{A''} = \bigoplus_A m_A \mathfrak{M}_A \quad (1)$$

be its decomposition into irreducible modules with the multiplicities  $m_A$ , then the formula of Steinberg reads

$$m_A = \sum_{S, T \in W} \det(S T) P [S(\Lambda' + \delta) + T(\Lambda'' + \delta) - (\Lambda + 2\delta)]. \quad (2)$$

The sum on the right hand side of (2) extends over the Weyl group  $W$ . This group is finite and is generated by the reflections at the simple roots (hence  $\det(ST) = \pm 1$ ).  $\delta$  is one half of the sum of all positive roots:  $\delta = \frac{1}{2} \sum_{\alpha > 0} \alpha$ .  $P[M]$  is the number of solutions of  $\sum_{\alpha > 0} k_{\alpha} \alpha = M$ , where the  $k_{\alpha}$  are non-negative integers. From this definition follows that  $P[M]$  is different from zero only if  $M$  is an integral linear function over the Cartan algebra  $\mathfrak{H}^*$ .

It is perhaps useful to see how one can immediately obtain from (2) the usual Clebsch-Gordan series for the Lie algebra  $A_1$ . Let  $\alpha$  be the only positive root;  $\lambda = \alpha/2$  is the fundamental dominant weight;  $\delta = \alpha/2$ . The Weyl group consists simply of  $I$  and  $S_{\alpha}$  ( $S_{\alpha}$ : reflection at the root  $\alpha$ ), i.e.  $W$  is the cyclic group  $Z_2$ . We put  $\Lambda' = m' \lambda$ ,  $\Lambda'' = m'' \lambda$ ,  $\Lambda = m \lambda$ ;  $m, m', m''$  non-negative integers. If we assume that  $m' \geq m''$ , then the only terms which contribute to the sum of the right hand side of (2) are  $(S, T) = (1, 1)$  and  $(S, T) = (1, S)$ . We obtain

$$m_A = P \left[ \frac{m' + m'' - m}{2} \alpha \right] - P \left[ \frac{m' - m'' - m - 2}{2} \alpha \right]$$

which means:  $m_A = 1$  for  $m = m' + m'', m' + m'' - 2, \dots, m' - m''$  and  $m_A = 0$  in all other cases.

## § 2. A Representation Theorem for $\Lambda$ in (1)

In this paragraph we prove the following

*Theorem.* The highest weights in (1) necessarily have the form

$$\Lambda = \Lambda' + \Lambda'' - \sum_{j=1}^l n_j \alpha_j$$

with non-negative integers  $n_j$  and the simple system of roots  $\pi = (\alpha_1, \alpha_2, \dots, \alpha_l)$ .

*Proof:* To prove this theorem, we need the following

\*)  $M$  is an integral linear function over  $\mathfrak{H}$  if  $M \in \mathfrak{H}^*$  ( $\mathfrak{H}^*$  dual space of  $\mathfrak{H}$ ) has the property  $M(h_i)$  integer for  $i = 1, 2, \dots, l$  ( $l$  = rank of  $\mathfrak{L}$ ). Here the  $h_i$  are those elements of the Cartan algebra which belong to the set of canonical generators. They are defined in the following way: Let  $\pi = (\alpha_1, \dots, \alpha_l)$  be a simple system of roots with the characteristic property that every root  $\alpha = \sum_{i=1}^l k_i \alpha_i$ ,  $\alpha_i \in \pi$ , where the  $k_i$  are all either non-negative or non-positive integers. To every linear function  $\alpha_i \in \mathfrak{H}^*$  we attribute the vector  $h_{\alpha_i} \in \mathfrak{H}$  such that  $\alpha_i(h) = (h_{\alpha_i}, h)$  for all  $h \in \mathfrak{H}$  (scalar product = Killing form); then  $h_i = 2 h_{\alpha_i} / (\alpha_i, \alpha_i)$ . The integral linear functions form a lattice with the fundamental dominant weights (defined by the property  $\lambda_i(h_j) = \delta_{ij}$ ) as a basis. There is a 1 : 1 correspondence between the isomorphism classes of finite dimensional irreducible modules for  $\mathfrak{L}$  and the set of dominant integral linear functions of  $\mathfrak{H}$  ( $\Lambda$  dominant integral function if  $\Lambda(h_i) \geq 0$ ).

*Lemma.* For  $S \in W$  and  $S \neq I$ ,  $\delta - S\delta$  is a non zero sum of distinct positive roots\*).

This lemma can be found in <sup>3)</sup>. For reasons of completeness repeat the short proof:

Since the Weyl group simply permutes the roots,  $S\delta = \delta - \sum \beta$ , where the summation is taken over the  $\beta = -S\alpha > 0$ . If there would be no such  $\beta$ , then  $S\alpha > 0$  for all  $\alpha$ . Then simple roots would be carried over into simple ones (compare footnote pag. 57), i.e.  $S\pi = \pi$ . According to a well known theorem<sup>4)</sup>, we could conclude  $S = I$ , contrary to hypothesis.

Since also the weights are simply permuted under the Weyl group, especially  $S\Lambda$  is a weight if  $\Lambda$  is the highest weight (dominant integral linear function on  $\mathfrak{H}$ ) of an irreducible module. According to a well known theorem it can be represented as

$$S\Lambda = \Lambda - \sum_{j=1}^l k_j \alpha_j \text{ with non-negative integers } k_j.$$

Hence, using the lemma, we get for  $S \neq I$

$$S(\Lambda + \delta) = \Lambda + \delta - \sum k_j \alpha_j - (\delta - S\delta) = \Lambda + \delta - \sum \kappa_j \alpha_j$$

where the  $\kappa_j$  are non-negative integers which do not vanish simultaneously.

The general argument of  $P$  in (3), which we simply denote with  $X_{S,T}$ , is for  $(S, T) \neq (1, 1)$  therefore of the form

$$X_{S,T} = \Lambda' + \Lambda'' - \Lambda - \sum w_j \alpha_j ;$$

$w_j$  non-negative integers, not all = 0.

From this one easily concludes, that a necessary condition for  $m_\Lambda \neq 0$  is

$$P[\Lambda' + \Lambda'' - \Lambda] \neq 0. \quad (4)$$

In order to translate this condition into an explicit form, we put

$$\Lambda' = \sum m'_s \lambda_s, \Lambda'' = \sum m''_s \lambda_s, \Lambda = \sum m_s \lambda_s,$$

$\lambda_s$ ;  $s = 1, \dots, l$  are the fundamental dominant weight (compare footnote pag. 57). If we expand the  $\lambda_s$  in terms of the simple roots, the condition  $\lambda_j(h_i) = \delta_{ij}$  immediately shows that the expansion matrix is the inverse Cartan matrix, i.e.

$$\lambda_i = \sum (A^{-1})_{ji} \alpha_i, \quad (5)$$

where

$$A_{ij} \stackrel{\text{Def.}}{=} \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = \alpha_j(h_i), \quad (6)$$

hence

$$\Lambda' + \Lambda'' - \Lambda = \sum \alpha_j \left( \sum_s (A^{-1})_{js} \Delta m_s \right),$$

with

$$\Delta m_s = m'_s + m''_s - m_s$$

---

\* ) In the subspace  $\mathfrak{H}_0^* \subset \mathfrak{H}^*$  over the rationals with basis  $\alpha_1 \dots \alpha_l$ , we introduce the usual ordering:  $\alpha = \sum \lambda_i \alpha_i > 0$  if  $\lambda_1 = \dots = \lambda_h = 0$ ,  $\lambda_{h+1} > 0$ ,  $h < l$ .  $\alpha > \beta$  if  $\alpha - \beta > 0$ . The simple roots then can not be written as a sum of positive roots.

(4) requires that

$$\sum_s (A^{-1})_{js} \Delta m_s = n_j ,$$

with non-negative  $n_j$ . From this we get

$$m_s = m'_s + m''_s - \sum A_{sj} n_j , \quad (7)$$

or

$$\Delta = \Delta' + \Delta'' - \sum A_{sj} n_j \lambda_s$$

and with (5)

$$\Delta = \Delta' + \Delta'' - \sum n_j \alpha_j \quad (8)$$

what we intended to prove.

We remark that not for every  $\Delta$  of the form (8) (with  $\Delta$  dominant)  $m_\Delta$  has to be different from zero. Indeed, one easily finds counter examples. On the other hand, the weight  $\Delta = \Delta' + \Delta''$  always appears with multiplicity one. For practical purposes the formula (7) is more useful. Of course, the  $n_j$  are restricted by the condition

$$\sum A_{sj} n_j \leq m'_s + m''_s .$$

### § 3. Evaluation of Steinberg's Formula for Special Lie algebras

To decompose the tensor product (1), we can now, according to the theorem of § 2, restrict ourselves to dominant weights  $\Delta$  of the form (8). For the calculation of the multiplicities  $m_\Delta$ , we have to know explicitly the  $X_{S,T}$ , i.e. we have to determine expressions of the type  $S(\Delta + \delta)$  ( $\Delta$  = highest weight,  $S \in W$ ).

We first derive a generally valid recursion formula which is useful for this purpose. A reflection  $S_i$  at a simple root  $\alpha_i$  is given by

$$S_i \alpha_j = \alpha_j - A_{ij} \alpha_i . \quad (9)$$

Now, the following equation holds:  $S_i \delta = \delta - \alpha_i$ . This is due to the fact that  $S_i \alpha > 0$  if  $\alpha > 0$ , except for  $\alpha = \alpha_i$ , where of course  $S_i \alpha_i = -\alpha_i$  (compare <sup>5</sup>)). Hence

$$S_i \delta = \frac{1}{2} \sum_{\substack{\alpha > 0 \\ \alpha \neq \alpha_i}} \alpha - \frac{1}{2} \alpha_i = \delta - \alpha_i .$$

From this we get

$$S_i (\Delta + \delta) = \sum_{s,j} m_s (A^{-1})_{js} S_i \alpha_j + \delta - \alpha_i$$

or with (9)

$$S_i (\Delta + \delta) = \Delta + \delta - (m_i + 1) \alpha_i . \quad (10)$$

Now we put for  $S \in W$

$$S (\Delta + \delta) - (\Delta + \delta) = - \sum_i \sigma_j (S) \alpha_j$$

then we get from (10) the following recursion formula

$$\sum_j \sigma_j (S_i S) \alpha_j = (m_i + 1) \alpha_i + \sum_j \sigma_j (S) \alpha_j - \sum_j \sigma_j (S) A_{ij} \alpha_i. \quad (11)$$

We turn now to special Lie algebras.

### 1. Example: $A_2$

Let  $\alpha_1$  and  $\alpha_2$  be the two simple roots for  $A_2$ . The Cartan matrix is

$$(A_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

(7) reads in this case

$$\begin{aligned} m_1 &= m'_1 + m''_1 - (2 n_1 - n_2) \\ m_2 &= m'_2 + m''_2 - (2 n_2 - n_1) \end{aligned}$$

$n_1, n_2$  non-negative integers.

The Weyl group consists of the following six elements:  $W = \{1, S_1, S_2, S_1 S_2, S_1 S_2 S_1, (S_1 S_2)^2\}$ . The defining relation (beside  $S_1^2 = S_2^2 = 1$ ) is  $S_2 = (S_1 S_2)^2 S_1$ . We also remark here that the Weyl group for  $A_1$  is isomorphic to the symmetric group  $S_{l+1}^6$ . With the help of the recursion formula (11), we obtain now for the multiplicities the following explicit expression

$$m_A = \sum_{S, T \in W} \det(S, T) P \left[ \sum_i (n_i - \sigma'_i(S) - \sigma''_i(T)) \alpha_i \right] \quad (13)$$

$\sigma'_i(S)$  and  $\sigma''_i(S)$  can be read off in table 1 (substitute for  $m_j$  in  $\sigma_i(S)$  respectively  $m'_i$  and  $m''_i$ ).

Table 1

| $S$           | $\sigma_1(S)$   | $\sigma_2(S)$   |
|---------------|-----------------|-----------------|
| 1             | 0               | 0               |
| $S_1$         | $1 + m_1$       | 0               |
| $S_2$         | 0               | $1 + m_2$       |
| $S_1 S_2$     | $1 + m_1$       | $2 + m_1 + m_2$ |
| $S_1 S_2 S_1$ | $2 + m_1 + m_2$ | $2 + m_1 + m_2$ |
| $(S_1 S_2)^2$ | $2 + m_1 + m_2$ | $1 + m_2$       |

For concrete examples the sum in (13) is carried out immediately. We illustrate this for the tensor product  $(1,1) \otimes (3,0)$ . (For a more general example compare the appendix). The possible  $n$ -values in (12) are:  $n \equiv (n_1, n_2) = (3,2), (2,1), (2,0), (1,1), (1,0), (0,0)$ . For  $n = (3,2)$  the following terms contribute in (13):  $(S, T) = (1,1), (1, S_2), (S_1, 1), (S_2, 1), (S_1, S_2)$  and one gets

$$\begin{aligned} m_{A(0,0)} &= P [3 \alpha_1 + 2 \alpha_2] - P [3 \alpha_1 + \alpha_2] - P [\alpha_1 + 2 \alpha_2] - \\ &- P [3 \alpha_1] + P [\alpha_1 + \alpha_2] = 3 - 2 - 2 - 1 + 2 = 0. \end{aligned}$$

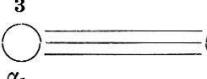
Still easier one sees that  $m_{A(0,3)} = 0$  (corresponding to  $n = (2,0)$ ), while in all other cases  $m_A = 1$ . Thus we get the well known decomposition

$$(1,1) \otimes (3,0) = (1,1) \oplus (3,0) \oplus (2,2) \oplus (4,1)$$

or

$$8 \otimes 10 = 8 \oplus 10 \oplus 27 \oplus 35$$

### 2. Example: $G_2$

From the Dynkin diagram:  one can read off the Cartan matrix

$$A_{ij} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

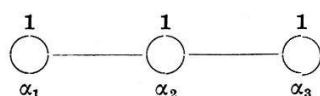
The Weyl group and  $\sigma_i(S)$   $i = 1, 2$  are given in table 2. For low dimensional representations only few terms contribute in (13).

Table 2

| $S$                     | $\sigma_1(S)$ | $\sigma_2(S)$   |
|-------------------------|---------------|-----------------|
| 1                       | 0             | 0               |
| $S_1$                   | $(m_1+1)$     | 0               |
| $S_2$                   | 0             | $m_2+1$         |
| $S_2 S_1$               | $m_1+1$       | $3m_1+m_2+4$    |
| $S_1 S_2 S_1$           | $3m_1+m_2+4$  | $3m_1+m_2+4$    |
| $(S_2 S_1)^2$           | $3m_1+m_2+4$  | $3(2m_1+m_2+3)$ |
| $S_1 (S_2 S_1)^2$       | $4m_1+2m_2+6$ | $3(2m_1+m_2+3)$ |
| $(S_2 S_1)^3$           | $4m_1+2m_2+6$ | $6m_1+4m_2+10$  |
| $S_1 (S_2 S_1)^3$       | $3m_1+2m_2+5$ | $6m_1+4m_2+10$  |
| $(S_2 S_1)^4$           | $3m_1+2m_2+5$ | $3m_1+3m_2+6$   |
| $S_1 (S_2 S_1)^4$       | $m_1+m_2+2$   | $3m_1+3m_2+6$   |
| $(S_2 S_1)^5 = S_1 S_2$ | $m_1+m_2+2$   | $m_2+1$         |

### 3. Example: $A_3$

Because the Lie algebra  $A_3$  is possibly of physical interest, we give here the explicit expressions for this example. From the Dynkin diagram



one obtains for the Cartan matrix

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

(8) reads here

$$\begin{aligned} m_1 &= m'_1 + m''_1 - (2n_1 - n_2) \\ m_2 &= m'_2 + m''_2 - (2n_2 - n_1 - n_3) \\ m_3 &= m'_3 + m''_3 - (2n_3 - n_2). \end{aligned}$$

The construction of the Weyl group from the reflections at the simple roots is here somewhat tedious. Beside  $S_j^2 = 1$ ,  $j = 1, 2, 3$ , the defining relations of this group are:

$$S(13) = S(31), S(121) = S(212), S(232) = S(323), \text{ where for example}$$

$$S(231) \equiv S_2 S_3 S_1.$$

The different elements of the Weyl group and  $\sigma_i(S)$   $i = 1, 2, 3$  are given in table 3. For given  $n = (n_1, n_2, n_3)$  only those terms contribute of course to  $m_A$  for which the inequalities  $\sigma'_j(S) + \sigma''_j(T) \leq n_j$ ,  $j = 1, 2, 3$ , are fulfilled. This condition restricts the summation over the Weyl group in many cases to a few terms only.

Table 3

| $S$         | $\sigma_1(S)$         | $\sigma_2(S)$           | $\sigma_3(S)$         |
|-------------|-----------------------|-------------------------|-----------------------|
| 1           | 0                     | 0                       | 0                     |
| $S_1$       | $m_1 + 1$             | 0                       | 0                     |
| $S_2$       | 0                     | $m_2 + 1$               | 0                     |
| $S_3$       | 0                     | 0                       | $m_3 + 1$             |
| $S(12)$     | $m_1 + m_2 + 2$       | $m_2 + 1$               | 0                     |
| $S(21)$     | $m_1 + 1$             | $m_1 + m_2 + 2$         | 0                     |
| $S(13)$     | $m_1 + 1$             | 0                       | $m_3 + 1$             |
| $S(23)$     | 0                     | $m_2 + m_3 + 2$         | $m_3 + 1$             |
| $S(32)$     | 0                     | $m_2 + 1$               | $m_2 + m_3 + 2$       |
| $S(121)$    | $m_1 + m_2 + 2$       | $m_1 + m_2 + 2$         | 0                     |
| $S(123)$    | $m_1 + m_2 + m_3 + 3$ | $m_2 + m_3 + 2$         | $m_3 + 1$             |
| $S(231)$    | $m_1 + 1$             | $m_1 + m_2 + m_3 + 3$   | $m_3 + 1$             |
| $S(132)$    | $m_1 + m_2 + 2$       | $m_2 + 1$               | $m_2 + m_3 + 2$       |
| $S(321)$    | $m_1 + 1$             | $m_1 + m_2 + 2$         | $m_1 + m_2 + m_3 + 3$ |
| $S(232)$    | 0                     | $m_2 + m_3 + 2$         | $m_2 + m_3 + 2$       |
| $S(1231)$   | $m_1 + m_2 + m_3 + 3$ | $m_1 + m_2 + m_3 + 3$   | $m_3 + 1$             |
| $S(3121)$   | $m_1 + m_2 + 2$       | $m_1 + m_2 + 2$         | $m_1 + m_2 + m_3 + 3$ |
| $S(1232)$   | $m_1 + m_2 + m_3 + 3$ | $m_2 + m_3 + 2$         | $m_2 + m_3 + 2$       |
| $S(2321)$   | $m_1 + 1$             | $m_1 + m_2 + m_3 + 3$   | $m_1 + m_2 + m_3 + 3$ |
| $S(2312)$   | $m_1 + m_2 + 2$       | $m_1 + 2 m_2 + m_3 + 4$ | $m_2 + m_3 + 2$       |
| $S(12321)$  | $m_1 + m_2 + m_3 + 3$ | $m_1 + m_2 + m_3 + 3$   | $m_1 + m_2 + m_3 + 3$ |
| $S(12312)$  | $m_1 + m_2 + m_3 + 3$ | $m_1 + 2 m_2 + m_3 + 4$ | $m_2 + m_3 + 2$       |
| $S(21321)$  | $m_1 + m_2 + 2$       | $m_1 + 2 m_2 + m_3 + 4$ | $m_1 + m_2 + m_3 + 3$ |
| $S(123121)$ | $m_1 + m_2 + m_3 + 3$ | $m_1 + 2 m_2 + m_3 + 4$ | $m_1 + m_2 + m_3 + 3$ |

### Final Remarks

In the derivation of STEINBERG's formula, an explicit formula of KONSTANT<sup>7)</sup> for the multiplicities  $n_M$  of the weights  $M$  in the irreducible module with highest weight

$\Lambda$  is essential ( $n_M$  = dimensionality of the weight space if  $M$  is a weight, and  $n_M = 0$  if  $M$  is not a weight). This formula too is very useful also for practical purposes. Because KONSTANT's formula has not yet been used in the physical literature, we give it here

$$n_M = \sum_{S \in W} \det(S) P[S(\Lambda + \delta) - (M + \delta)] .$$

With the earlier formulas, we can immediately evaluate the right hand side for special Lie algebras. For a weight  $M = \Lambda - \sum_{j=1}^r n_j \alpha_j$  we obtain

$$n_M = \sum_{S \in W} \det(S) P \left[ \sum_i (n_i - \sigma_i(S) \alpha_i) \right]$$

with the same tables for  $\sigma_i(S)$ .

Finally, we would like to remark that the algebraic theory of characters for Lie algebras<sup>8)</sup> certainly gives simple formulae (which only contain the root system) for the following problem: Let  $\mathfrak{L}'$  be a sub-algebra of  $\mathfrak{L}$  and let be given an irreducible module for  $\mathfrak{L}$ . This module is then completely reducible for  $\mathfrak{L}'$  (for semisimple  $\mathfrak{L}'$ ). One can now ask for the irreducible constituents with respect to  $\mathfrak{L}'$ . This question will be discussed in a future paper.

### Acknowledgements

It is a pleasure to acknowledge the value of conversations with Dr. KAMBER. I wish to express also my gratitude to the Swiss National Fonds for financial aid.

### Appendix

To demonstrate the power of the method which we have presented in this paper, we show in detail how one can immediately decompose the tensor product  $(m_1, m_2) \otimes (1,1)$  of a general irreducible representation  $(m_1, m_2)$  of  $A_2$  with the eightdimensional representation  $(1,1)$ . The possible  $n$ -values in (12) are  $n \equiv (n_1, n_2) = (0,0), (0,1), (1,0), (1,1), (1,2), (2,1), (2,2), (2,3), (2,4) \dots$ . If both  $m_1, m_2 > 1$  it is easy to see from table 1, that for the above first seven  $n$ 's only the following term contribute in (13):  $(S, T) = (1,1), (1, S_1), (1, S_2)$ . Furthermore, the corresponding multiplicities are respectively:

$$\begin{aligned} m = P[0], P[\alpha_2], P[\alpha_1], P[\alpha_1 + \alpha_2], P[\alpha_1 + 2\alpha_2] - \\ - P[\alpha_1], P[2\alpha_1 + \alpha_2] - P[\alpha_2], P[2\alpha_1 + 2\alpha_2] - P[\alpha_1] - P[\alpha_2], \end{aligned}$$

i.e.  $m_A = 1$  except for  $n = (1,1)$ , where  $m_A = 2$ . But these seven irreducible constituents give the complete decomposition as one can see for instance by comparing the dimensions. We remember that the dimension of an irreducible module  $\mathfrak{M}_{A(\mu_2, \mu_1)}$  with the highest weight  $\Lambda = (\mu_1, \mu_2)$  is given by

$$\dim \mathfrak{M}_{A(\mu_1, \mu_2)} = (\mu_1 + 1) (\mu_2 + 1) \left[ 1 + \frac{\mu_1 + \mu_2}{2} \right].$$

The special cases where  $m_1$  and  $m_2$  are not both larger than one are easily discussed. For example, in the case  $n = (1,1)$ , we get for  $(m_1, m_2) \neq (0,0)$   $m_A = P[\alpha_1 + \alpha_2] = 2$ . If  $m_1 = 0, m_2 \neq 0$  one obtains  $m_A = P[\alpha_1 + \alpha_2] - P[\alpha_2] = 1$ ; the same holds for  $m_1 \neq 0, m_2 = 0$ , while for  $m_1 = m_2 = 0$  we get  $m_A = 0$ .

Thus we have the following result:

$$(m_1, m_2) \otimes (1,1) = (1) \oplus (2) \oplus \dots \oplus (7),$$

where

- (1) =  $(m_1 + 2, m_2 - 1)$  with  $m_A = 0$ , except for  $m_2 = 0$ .
- (2) =  $(m_1 - 1, m_2 - 1)$  with  $m_A = 0$ , except for  $m_1$  or  $m_2 = 0$ .
- (3) =  $(m_1 - 2, m_2 + 1)$  with  $m_A = 0$ , except for  $m_1 = 0, 1$ .
- (4) =  $(m_1 + 1, m_2 + 1)$  with  $m_A = 0$ .
- (5) =  $(m_1 - 1, m_2 + 2)$  with  $m_A = 0$ , except for  $m_1 = 0$ .
- (6) =  $(m_1 + 1, m_2 - 2)$  with  $m_A = 0$ , except for  $m_2 = 0, 1$ .
- (7) =  $(m_1, m_2)$  with  $m_A = 2$  for  $(m_1, m_2) \neq (0,0)$ .

$$m_A = 1 \text{ for } m_1 = 0, m_2 \neq 0 \text{ or } m_1 \neq 0, m_2 = 0.$$

$$m_A = 0 \text{ for } m_1 = m_2 = 0.$$

## References

- <sup>1)</sup> J. J. DE SWART, Revs. Mod. Phys. 35, 916 (1963).
- <sup>2)</sup> N. JACOBSON, *Lie Algebras*, pag. 259, Interscience Tracts in Pure and Applied Mathematics Nr. 10.
- <sup>3)</sup> N. JACOBSON, op. cit. Lemma 2., pag. 248.
- <sup>4)</sup> N. JACOBSON, op. cit. Theorem 2., pag. 242.
- <sup>5)</sup> N. JACOBSON, op. cit. Lemma 1, pag. 241.
- <sup>6)</sup> N. JACOBSON, op. cit. pag. 226.
- <sup>7)</sup> N. JACOBSON, op. cit. pag. 261.
- <sup>8)</sup> N. JACOBSON, op. cit. chapter VIII. See also: Séminaire «Sophus Lie», 1re année 1954/55, Théorie des Algèbres de Lie, Ecole Normale Supérieure, Paris; exposés n° 18 et n° 19.