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Acoustic Scattering from an Inhomogeneity of the Medium*)

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Abstract. The wave equation is written in its dimensionless form, which makes it possible to compute separately the effects of an inhomogeneity of density or of compressibility in a small region, on the pressure and velocity of the region where irradiated by an incident plane wave. Specific formulas are given for these quantities, and for the resulting scattered wave from a sphere of radius $a \ll \lambda$, when the compressibility and density inside the sphere differ by factors of any size from the values outside.

In discussing the scattering of sound waves by a region in the medium, in which the density and compressibility have different values than in the rest of the medium, it is usual to mingle the two effects by writing the wave equation as

$$\Delta^2 p + \omega^2 \varrho \varkappa p = 0 \tag{1}$$

where $(\omega/2\pi)$ is the frequency, ϱ the density and $\varkappa$ the compressibility of the medium. If, within region R, $\varrho = \varrho_s$ and $\varkappa = \varkappa_s$, different from the constant ϱ and $\varkappa$ of the rest of the medium, then the difference $\omega^2 (\varrho_s \varkappa_s - \varrho \varkappa)$ can be taken to the right-hand side of the equation and treated as a scattering term.

However, it is more instructive to set up the equation in dimensionless form by starting from the two basic equations relating fluid velocity with pressure

$$((\partial \boldsymbol{u}/\partial t) = -(1/\varrho) \operatorname{grad} p; \; \varkappa (\partial p/\partial t) = -\operatorname{div} \boldsymbol{u}$$

and writing

$$\frac{1}{\varrho \omega^2} \Delta^2 p + \varkappa p = \begin{cases} 0 \text{ outside } R \\ (\varkappa - \varkappa_s) p + \frac{1}{\omega^2} \operatorname{div}\left[\left(\frac{1}{\varrho} - \frac{1}{\varrho_s}\right) \operatorname{grad} p\right] \text{ inside } R. \end{cases} \tag{2}$$

Now multiplication by $\varrho \omega^2 = (k^2/\varkappa)$ produces

$$\Delta^2 p + k^2 p = \begin{cases} 0 \text{ outside } R \\ -\delta_k p + \operatorname{div}[\delta_p \operatorname{grad} p] \text{ inside } R, \end{cases} \tag{3}$$

*) This note is in memory of pleasant and informative discussions on wave scattering with Professor E. C. G. Stueckelberg, in Princeton and in Munich, some 35 years ago.

where

$$\delta_k = \frac{\varkappa_3 - \varkappa}{\varkappa} \text{ and } \delta_p = \frac{\varrho_s - \varrho}{\varrho_s} .$$

This equation can now be solved so that the effects of change of compressibility are distinguishable from the effects of change of density.

For example, the integral equation for the scattering of sound from region R, in unbounded space, is

$$p(r) = A\, e^{i\boldsymbol{k}_i \cdot \boldsymbol{r}} + \int_R [k^2\, \delta_k\, p(\boldsymbol{r}_0)\, g(\boldsymbol{r} \mid \boldsymbol{r}_0) + \delta_\varrho\, (\mathrm{grad}_0\, p) \,.\, (\mathrm{grad}_0\, g)]\, dv_0 \,, \tag{4}$$

where g is the usual Green's function $[e^{ik|\boldsymbol{r}-\boldsymbol{r}_0|}/4\,\pi \mid \boldsymbol{r} - \boldsymbol{r}_0 \mid]$. At large distances from region R the scattered wave becomes $A\,(e^{ikr}/r)\, F_s\,(\boldsymbol{k}_i \cdot \boldsymbol{k}_s)$

$$F_s = \frac{1}{4\,\pi} \int_R [k^2\, \delta_k\, p(\boldsymbol{r}_0) + k_p\, c\, \boldsymbol{k}_s \cdot \boldsymbol{u}(\boldsymbol{r}_0)]\, e^{-i\boldsymbol{k}_s \cdot \boldsymbol{r}_0}\, dv_0 \,, \tag{5}$$

where $\boldsymbol{k}_s$ is the vector with magnitude k in the direction of scattering. Thus at long wave-lengths the first term in the integral is a spherically symmetric wave produced by the difference in compressibility of the medium in R, from that outside R, whereas the second term represents the dipole radiation caused by the fact that a region of differing density moves at a different rate than does the surrounding medium. In this form of the solution the two effects are clearly distinguished.

This formulation can be extended to include the scattering effect of turbulence, when the fluid velocity within R is $\boldsymbol{U} + \boldsymbol{u}$, $\boldsymbol{u}$ being the velocity of the sound wave and $\boldsymbol{U}$ the velocity of turbulent motion. The additional term entering on the right-hand side of (3) is then $-\,2\,\Sigma\,[\partial^2\,(\varrho\, u_n\, U_m)/\partial x_n\, \partial x_m]$, which adds another term

$$-\,(\varrho/2\,\pi) \int [\boldsymbol{k}_s \cdot \boldsymbol{U}_\omega(\boldsymbol{r}_0)]\, [\,\boldsymbol{k}_s \cdot \boldsymbol{u}(\boldsymbol{r}_0)]\, e^{-i\boldsymbol{k}_s \cdot \boldsymbol{r}_0}\, dv_0 \tag{6}$$

to the integral of (5), where $\boldsymbol{U}_w$ is the Fourier transform of $\boldsymbol{U}$ for frequency $(\omega/2\,\pi)$. This represents quadrupole radiation from the turbulent region.

If the differential quantities $\delta_\varkappa$ and δ_ϱ are small (and $\boldsymbol{U} = 0$) then at long wave-lengths the Born approximation indicates that the monopole and dipole strengths of the scattered wave are

$$S_s \simeq \frac{i\,k\,\langle p \rangle}{\varrho\, c} \int_R \delta_\varkappa\, dv; \ \boldsymbol{D}_s \simeq \langle \boldsymbol{u} \rangle \int_R \delta_p\, dv \,, \tag{7}$$

where $\langle p \rangle$ and $\langle \boldsymbol{u} \rangle$ are the mean acoustic pressure and velocity, caused by the incident wave, near region R. If the δ's are not small, but region R is a sphere of radius a, small compared to the wavelength, then separation into spherical harmonic components demonstrates that the scattering strengths are

$$S_s \simeq \frac{i\,k\,\langle p \rangle}{\varrho\, c} \int \left(\frac{\varkappa_s - \varkappa}{\varkappa}\right) dv; \ \boldsymbol{D}_s \simeq \langle \boldsymbol{u} \rangle \int_R \left(\frac{3\,\varrho - 3\,\varrho_s}{2\,\varrho_s + \varrho}\right) dv \,. \tag{8}$$

In this case, for an incident plane wave in the z direction, of amplitude A, the pressure and radial velocity at the surface of the sphere, the scattered pressure at large distances and the scattering and absorption cross sections, at long wavelengths, are:

$$p\,(a,\vartheta) \simeq A\left[1 + i\,k\,a\,\frac{3\,\varrho_s}{2\,\varrho_s+\varrho}\cos\vartheta\right] \quad u_r\,(a,\vartheta) \simeq (A/\varrho\,c)\left[\frac{1}{3}\,i\,k\,a\,\frac{\varkappa_s}{\varkappa} + \frac{3\,\varrho}{2\,\varrho_s+\varrho}\cos\vartheta\right]$$

$$p_s\,(r,\vartheta) \to A\,(e^{ikr}/r)\,F_s(\vartheta)$$

$$F_s(\vartheta) \simeq \frac{1}{3}\,k^2\,a^3\left[\left(\frac{\varkappa_s-\varkappa}{\varkappa}\right) + \frac{1}{3}\left(\frac{3\,\varrho_s-3\,\varrho}{2\,\varrho_s+\varrho}\right)\cos\vartheta\right] \tag{9}$$

$$Q_s \simeq \frac{4}{\varrho}\,\pi\,a^2\,(k\,a)^4\left[\left|\frac{\varkappa_s-\varkappa}{\varkappa}\right|^2 + \frac{1}{3}\left|\frac{3\,\varrho_s-3\,\varrho}{2\,\varrho_s+\varrho}\right|^2\right]$$

$$Q_a \simeq \frac{4}{3}\,\pi\,a^2\,(k\,a)\,\boldsymbol{Im}\left[\frac{\varkappa_s-\varkappa}{\varkappa} + \frac{3\,\varrho_s-3\,\varrho}{2\,\varrho_s+\varrho}\right]$$

where we have assumed that $\varkappa_s$ and/or ϱ_s may be complex quantities, representing energy loss.

For a heavy, incompressible sphere ($\varkappa_s \ll \varkappa$, $\varrho_s \gg \varrho$) the angle-distribution of the scattered intensity has the factor $(-1 + {}^3/_2 \cos\vartheta)^2$, with some forward scattering, though with more backward scattering. (This is in contrast to the Born approximation, which would predict no scattering at $\vartheta = 0$.) In this case the radial velocity of the surface is zero and the pressure there is $A\,(1 + {}^3/_2\, i\,k\,a\cos\vartheta)$. For a light, compressible sphere ($\varkappa_s \gg \varkappa$, $\varrho_s \ll \varrho$) the scattered intensity has the factor $[(\varkappa_s/\varkappa) - 3\cos\vartheta]^2$, again differing from the Born approximation. The pressure at $r = a$ is A, the pressure of the incident wave, and the radial velocity at $r = a$ is $(A/\varrho\,c)\,[i\,k\,a\,(\varkappa_s/\varkappa) + 3\cos\vartheta]$ which can be quite large. The sphere moves back and forth along the z axis with three times the velocity of the surrounding fluid.

Thus separation of the effects of compressibility from the effects of density, by treating the wave equation in its dimensionless form, enables one to see more clearly the separate effects, on the motion and pressure of region R and on the resulting scattered wave, of the two kinds of inhomogeneities.