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# On the Analyticity Properties of the Scattering Amplitude in Relativistic Quantum Field Theory

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*Abstract.* The well-known analyticity properties of the 2-particle scattering amplitude are rigorously derived in the Wightman framework of a local relativistic field theory. No use is made of the LSZ asymptotic condition and of implicit 'technical assumptions'.

## § 0. Introduction

Dispersion relations<sup>1)</sup> have brought a fruitful new approach to the physics of strongly interacting particles. Since these relations are to a high degree model-independent, much work has been done in proving the necessary analyticity properties from a minimal set of mathematically well-defined postulates. The framework of LSZ<sup>15)</sup> or BOGOLIUBOV<sup>4)</sup> has usually been the starting point for these quite involved investigations.

Recently a physically satisfying relativistic scattering theory has been given by HAAG<sup>8)</sup> and RUELLE<sup>16)</sup>, which is based on the general set of axioms of WIGHTMAN<sup>21)</sup> for a local relativistic quantum field theory. On the other hand there has been some doubt<sup>6)</sup> as to whether a rigorous proof of dispersion relations is possible or not. The aim of this investigation is to show that, if one-particle states are created from the vacuum by WIGHTMAN fields and if certain mass-spectrum conditions are satisfied, the well-known analyticity properties of the 2-particle scattering amplitude can be rigorously proved. Therefore no further assumptions on the asymptotic behaviour of matrix elements of the interacting fields<sup>15)</sup> or on the existence and regularity of GREEN's functions (see <sup>19)</sup>) are necessary.

Needless to say our considerations are mainly technical, which is also reflected in the choice of a theory of only one kind of neutral scalar massive particles in self-interaction. Although the main idea of the proof is known to many workers in the field, it seemed desirable to fit this mosaic together, in order to clarify the interplay of locality, relativistic invariance and mass-spectrum conditions leading to dispersion relations in relativistic quantum field theory.

I am greatly indebted to many physicists in Zürich and Paris for stimulating discussions, especially to Professors R. JOST and A. S. WIGHTMAN and to Drs. M. FROISSART, J. LASCOUX, and R. STORA. I further wish to thank Dr. L. MOTCHANE for extending to me his kind hospitality at the Institut des Hautes Etudes Scientifiques.

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### § 1. Asymptotic Condition and Reduction Formulae

In this section we shall clarify the relation between the Haag-Ruelle asymptotic condition and the LSZ reduction formulae for the 2-particle scattering amplitude. For completeness we start by stating the general assumptions, which characterise the theory of a neutral scalar field in the axiomatic framework of WIGHTMAN<sup>21</sup>).

(A) A neutral scalar field  $A(x)$  is a tempered operator-valued distribution<sup>17</sup>). For all  $\varphi \in \mathfrak{S}(R^4)$  the linear operators  $A(\varphi) \equiv \int A(x) \varphi(x) dx$  are defined on a common invariant dense linear manifold  $D$  in a Hilbert space  $\mathfrak{H}$ .  $D$  and  $A(x)$  transform covariantly under a continuous unitary representation  $U(a, \Lambda)$  of the inhomogeneous Lorentz group  $i L_+^\uparrow$ :

$$U(a, \Lambda) A(x) U^{-1}(a, \Lambda) = A(\Lambda x + a), \quad U(a, \Lambda) D \subset D. \quad (1.1)$$

The spectrum of the energy-momentum operator  $P^\mu$  is assumed to lie in the forward light-cone  $V_+$ , except for a one-dimensional eigenspace spanned by the vacuum state  $\Omega$ , corresponding to the eigenvalue 0.  $\Omega$  is cyclic with respect to the algebra generated by  $\{A(\varphi) : \varphi \in \mathfrak{S}(R^4)\}$ . Finally the theory is local:

$$[A(x), A(y)] = 0 \text{ for } (x - y)^2 < 0. \quad (1.2)$$

In our investigation  $A(x)$  is to be the interpolating field for a relativistic scattering theory of particles of mass  $m$  and spin 0. The most natural way is to postulate that the one-particle states of the discrete irreducible representation  $[m, 0]$  of  $i L_+^\uparrow$  are generated by the application of  $A(x)$  to the vacuum, which we express by:

$$(B) \quad \langle A(x) A(y) \rangle_0 = i \Delta_m^+(x - y) + i \int_M^\infty d\varrho(\mu) \Delta_\mu^+(x - y), \quad (M > m). \quad (1.3)$$

Let  $\tilde{A}(p)$  be the Fourier transform of  $A(x)$ . We consider for  $\tilde{f} \in \mathfrak{S}(R^4)$  and  $\omega_p \equiv \sqrt{p^2 + m^2}$  the well-defined operator:

$$A(f, t) = \int dx A(x) f^*(x, t)$$

$$f(x, t) = (2\pi)^{-5/2} \int dp \tilde{f}(p) \left( \frac{p_0 + \omega_p}{2\omega_p} \right) e^{i(p_0 - \omega_p)t} e^{-i(p, x)}. \quad (1.4)$$

Let  $D_M \equiv \{p : (p, p) < M^2\}$ . Then it follows from (1.3) for  $f \in \mathfrak{S}(D_M)$  that  $A(f, t) \Omega = 0$  and that  $A^*(f, t) \Omega$  is a one-particle state  $\Phi_{\hat{f}} = |\hat{f}\rangle \in \mathfrak{H}_{[m, 0]}$  with a wave function of the form:

$$f(x) = (2\pi)^{-3/2} \int \frac{d^3 p}{2\omega_p} \hat{f}(p) e^{-i(\omega_p x^0 - p \cdot x)}, \quad \hat{f}(p) = \tilde{f}(\omega_p, p) \in \mathfrak{S}(R^3). \quad (1.5)$$

Under the assumptions (A) and (B) relativistic scattering states can be constructed, as has been shown by HAAG<sup>8</sup>) and RUELLE<sup>16</sup>) in a much more general version of the following theorem:

*Theorem 1.1:* Under the assumptions (A), (B) let  $\mathfrak{P}\{A^{(*)}(f_i, t)\}$  be an arbitrary polynomial in the  $A(f_i, t)$ ,  $A^*(f_j, t)$ ,  $\tilde{f}_i, \tilde{f}_j \in \mathfrak{S}(D_M)$ . Then one has strong convergence in  $\mathfrak{H}$  for  $t \rightarrow \pm \infty$  (ex = out, in):

$$s - \lim_{t \rightarrow \pm \infty} \mathfrak{P} \{A^{(*)}(f_i, t)\} \Omega = \Phi_{\mathfrak{P}}^{ex}. \quad (1.6)$$

In particular  $s - \lim_{t \rightarrow \pm \infty} \prod_{i=1}^m A^{*}(f_i, t) \Omega = |\hat{f}_1, \dots, \hat{f}_m^{ex}\rangle$  is an asymptotic  $m$ -particle state with wave functions (1.5)  $\hat{f}_i(\mathbf{p}) = \tilde{f}_i(\omega_p, \mathbf{p})$ .

This theorem can be proved by following closely RUELLE<sup>16)11)</sup>. Yet the use of the local expressions  $A(f, t)$  instead of the quasi-local operators of HAAG and RUELLE has to be paid for by a more careful study of the asymptotic behaviour of the truncated vacuum expectation values (TVEV)  $\langle \prod_{i=1}^k A^{*}(f_i, t) \rangle_0^T$  for  $k \geq 3$ . This is contained in the following:

*Lemma 1.1:* Under the assumption (A)

$$(1 + |t|)^{3/2(k-2)} \langle \prod_{i=1}^k A^{*}(f_i, t) \rangle_0^T \quad (1.7)$$

is bounded for all  $t$ , if  $\tilde{f}_i \in \mathfrak{S}(R^4)$  for  $1 \leq i \leq k$ ,  $k \geq 3$ .

*Proof:* Using (1.4) the TVEV (1.7) can be written as

$$\int d^4x_1 \dots d^4x_k d^3\mathbf{p}_1 \dots d^3\mathbf{p}_k \langle A(-x_1^0, \mathbf{x}_1) \dots A(-x_k^0, \mathbf{x}_k) \rangle_0^T \times \\ f'_1(x_1^0, \mathbf{p}_1) \dots f'_k(x_k^0, \mathbf{p}_k) \exp i(\pm(\omega_{p_1}t - \mathbf{p}_1 \mathbf{x}_1) \dots \pm(\omega_{p_k}t - \mathbf{p}_k \mathbf{x}_k)), \quad (1.8)$$

with  $f'_i \in \mathfrak{S}(R^4)$ . From the asymptotic behaviour of  $\langle A(x_1) \dots A(x_k) \rangle_0^T$  for large space-like separation of the arguments<sup>16)3)</sup>, it follows that for any integer  $M \geq 0$  there exists an integer  $N = N(M) \geq 0$  such that

$$\prod_{i=1}^k (1 + (x_i^0)^2)^{-N/2} \prod_{l=1}^{k-1} \prod_{j=1}^3 (1 + (x_{l+1}^j - x_l^j)^2)^{M/2} \langle A(-x_1^0, \mathbf{x}_1) \dots A(-x_k^0, \mathbf{x}_k) \rangle_0^T \quad (1.9)$$

is a bounded distribution in  $x_i^0$ ,  $\xi_l^j \equiv x_{l+1}^j - x_l^j$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq 3$ ,  $1 \leq l \leq k-1$ .

According to<sup>17)</sup> (vol. II, p. 57) (1.9) can be represented as  $\sum_{q=1}^Q D_q F_q(x_1^0, \dots, x_k^0, \xi_1, \dots, \xi_{k-1})$

with functions  $F_q \in L^\infty(x_1^0, \dots, x_k^0, \xi_1, \dots, \xi_{k-1})$  and monomials  $D_q$  in the derivatives  $\partial/\partial x_i^0$ ,  $\partial/\partial \xi_l^j$ . After partial integrations one can therefore transform (1.8) into

$$\sum_{r=1}^R \int d^4x_1 \dots d^4x_k \hat{F}_r(x_1^0, \dots, x_k^0, \xi_1, \dots, \xi_{k-1}) \prod_{l=1}^{k-1} \prod_{j=1}^3 (1 + (\xi_l^j)^2)^{-1} \times \\ \prod_{i=1}^k \int d\mathbf{p}_i e^{\pm i[\omega_{p_i}t - \mathbf{p}_i \mathbf{x}_i]} f'_i(x_i^0, \mathbf{p}_i), \quad (1.10)$$

with continuous bounded functions  $\hat{F}_r$  and with  $f'_i(x_i^0, \mathbf{p}_i) \in \mathfrak{S}(R^4)$ . Then we can majorize (1.8) by:

$$\sum_{r=1}^R c_r \int dx_1^0 \dots dx_k^0 d\mathbf{x}_k \max_{\mathbf{x}_1 \dots \mathbf{x}_{k-1}} \prod_{i=1}^k \left| \int d\mathbf{p}_i f'_i(x_i^0, \mathbf{p}_i) e^{\pm i[\omega_{p_i}t - \mathbf{p}_i \mathbf{x}_i]} \right|. \quad (1.11)$$

From the asymptotic behaviour of the solutions (1.5) of the Klein-Gordon equation<sup>16)2)</sup>, it follows that (1.11) is bounded for all  $t$  when multiplied by  $(1 + |t|)^{3/2(k-2)}$ ,  $k \geq 3$ . This proves theorem 1.1 as in<sup>8)</sup>,<sup>16)</sup>.



For a pure scattering theory of one kind of neutral scalar particles (see <sup>16</sup>) for the general case) it is reasonable to assume that the asymptotic states are dense in  $\mathfrak{H}$ , that is:

$$(C) \quad \mathfrak{H} = \mathfrak{H}_{ex} = \overline{L} \{ | \hat{f}_1, \dots, \hat{f}_m^{ex} \rangle : m = 0, 1, \dots, \hat{f}_i \in \mathfrak{S}(R^3) \}, \quad (1.12)$$

where  $\overline{L} \{ \dots \}$  denotes the closed linear hull of the vectors  $\{ | \hat{f}_1, \dots, \hat{f}_m^{ex} \rangle \}$  and 'ex' stands for 'in' or 'out' (equivalent due to the TCP theorem). Then the following weak convergence theorem can be proved:

*Corollary 1.1:* Under the assumptions (A), (B), (C) one has in the sense of weak convergence in  $\mathfrak{H}$ :

$$w - \lim_{t \rightarrow \pm \infty} \prod_{i=1}^m A^*(f_i, t) \Omega = | \hat{f}_1, \dots, \hat{f}_m^{ex} \rangle, \quad (1.13)$$

$$w - \lim_{t \rightarrow \pm \infty} \prod_{i=1}^m A(f_i, t) \prod_{j=1}^n A^*(g_j, t) \Omega = \prod_{i=1}^m A_{ex}(f_i) | \hat{g}_1, \dots, \hat{g}_n^{ex} \rangle, \quad (1.14)$$

$$\text{for } \tilde{f}_i \in \mathfrak{S}(R^4), \tilde{g}_j \in \mathfrak{S}(D_{2m}) \text{ with } \hat{f}_i(\mathbf{p}) = \tilde{f}_i(\omega_p, \mathbf{p}), \hat{g}_j(\mathbf{p}) = \tilde{g}_j(\omega_p, \mathbf{p}).$$

The proof of H. ARAKI<sup>2)</sup> for local rings of observables can be immediately translated into the Wightman framework<sup>9)</sup> using lemma 1.1 for the majorization of the higher TVEV.

Let  $\overline{D} \equiv \bigcap_{B \in \mathfrak{P}} D_{\overline{B}}$  be the intersection of the domains of the closures  $\overline{B} = B^{**}$  of all quasi-local operators  $B \in \mathfrak{P}$  (polynomials in the smeared-out fields). Then one can prove the following version of the LSZ asymptotic condition<sup>15)</sup>:

*Corollary 1.2:* Under (A) and (B) one has

$$(a) \quad | \hat{g}_1 \dots \hat{g}_n^{ex} \rangle \in \overline{D} \text{ for non-overlapping wave-packets } \hat{g}_i \in \mathfrak{D}(R^3) \text{ (supp } \hat{g}_i \cap \text{supp } \hat{g}_j = \emptyset \text{ for } i \neq j).$$

$$(b) \quad \overline{B} | \hat{g}_1 \dots \hat{g}_n^{ex} \rangle = s - \lim_{t \rightarrow \pm \infty} B \prod_{i=1}^n A^*(g_i, t) \Omega \in \overline{D} \quad *$$

$$\overline{B_1} (\overline{B_2} | \hat{g}_1 \dots \hat{g}_n^{ex} \rangle) = \overline{B_1 B_2} | \hat{g}_1 \dots \hat{g}_n^{ex} \rangle \text{ for } B, B_1, B_2 \in \mathfrak{P}.$$

$$(c) \quad s - \lim_{t \rightarrow \pm \infty} A^{(*)}(f, t) | \hat{g}_1 \dots \hat{g}_n^{in} \rangle = A_{in}^{(*)}(f) | \hat{g}_1 \dots \hat{g}_n^{out} \rangle \text{ for } \tilde{f} \in \mathfrak{S}(D_M) \text{ (and for } \tilde{f} \in \mathfrak{S}(R^4) \text{ in the weak topology in } \mathfrak{H}_{ex})$$

For  $n = 0, 1$  the proof follows immediately from theorem 1.1 and corollary 1.1. All the reduction formulae necessary for the proof of the analyticity properties of the 2-particle scattering amplitude can be derived using only this information. For  $n > 1$  and  $B \in \mathfrak{P}$  one uses the SCHWARZ inequality:

$$|| B \frac{d}{dt} \prod_{i=1}^n A^*(g_i, t) \Omega ||^2 \leq || \frac{d}{dt} \prod_{i=1}^n A^*(g_i, t) \Omega || || B^* B \frac{d}{dt} \prod_{i=1}^n A^*(g_i, t) \Omega ||. \quad (1.15)$$

\*) One can choose e.g.  $\tilde{g}_i(p) = \hat{g}_i(p) \hat{\alpha}_i(\sqrt{p^2 + m^2} - p^0)$  with  $\hat{\alpha}_i \in \mathfrak{D}(\langle -\varepsilon, +\varepsilon \rangle)$ ,  $\varepsilon > 0$  sufficiently small,  $\sqrt{2\pi} \hat{\alpha}(0) = 1$ .

For non-overlapping wave-packets  $\{\hat{g}_i\}$  the first factor decreases stronger than any power of  $(1 + |t|)^{-1}$  for  $t \rightarrow \pm \infty$  (see <sup>9)</sup>), whilst the second factor increases only polynomially in  $|t|$ , due to the temperedness of the WIGHTMAN distributions. This proves the convergence of (b) and similarly of (c). One can see that on the smooth non-overlapping states  $|\hat{g}_1 \dots \hat{g}_n^{ex}\rangle$  the mapping  $\varphi \rightarrow \overline{A(\varphi)} |\hat{g}_1 \dots \hat{g}_n^{ex}\rangle$  is a vector-valued tempered distribution. Finally corollary 1.2 can be extended to the general Haag-Ruelle scattering theory \*).

Under the assumptions (A), (B), (C) the Fock spaces  $\mathfrak{H}_{in}$  and  $\mathfrak{H}_{out}$  are related by a unitary  $S$ -matrix, which is defined in terms of its matrix elements:

$$S_{mn}(\tilde{f}_1^*, \dots, \tilde{f}_m^*, \tilde{g}_1, \dots, \tilde{g}_n) \equiv (\hat{f}_1, \dots, \hat{f}_m^{out} | \hat{g}_1, \dots, \hat{g}_n^{in}) . \quad (1.16)$$

The tempered distributions

$$S_{mn}(p_1, \dots, p_m, -p_{m+1}, \dots, -p_{m+n}) = \prod_{i=1}^{m+n} \theta_0(p_i) \delta(p_i^2 - m^2) \langle \hat{a}_{out}(p_1) \dots \hat{a}_{in}^*(p_{m+n}) \rangle_0 \quad (1.17)$$

are  $L_+^\dagger$ - and TCP-invariant, symmetric in the  $\{p_1, \dots, p_m\}$  and  $\{p_{m+1}, \dots, p_{m+n}\}$  and have their supports on the mass shell  $\{p_i^2 = m^2, p_i^0 > 0, 1 \leq i \leq m+n, \sum_{i=1}^m p_i = \sum_{j=m+1}^{m+n} p_j\}$  as well as the scattering amplitudes  $T_{mn}$ , defined in (1.16) by  $S = 1 + iT$ .

The aim of our investigation is to derive analyticity properties of the 2-particle scattering amplitude  $T_{22}$ . For that purpose we express  $T_{22}$  and its 'imaginary part' by certain matrix elements of retarded or time-ordered products:

$$R(x, x_1, \dots, x_n) \equiv (i)^n \sum_{p \in \gamma^n} \theta(x - x_{p(1)}) \dots \theta(x_{p(n-1)} - x_{p(n)}) [[A(x), A(x_{p(1)})], \dots, A(x_{p(n)})] \\ T(x_1, \dots, x_n) \equiv \sum_{p \in \gamma^n} \theta(x_{p(1)} - x_{p(2)}) \dots \theta(x_{p(n-1)} - x_{p(n)}) A(x_{p(1)}) \dots A(x_{p(n)}) . \quad (1.18)$$

$R$  and  $T$  are well-defined for  $C^\infty$ -functions  $\theta$  with  $\text{supp}(\theta - \theta_0)$  compact ( $\theta_0$ : Heaviside step function). Let  $\delta_{ab}^m(p)$  be the  $O_M(R^4)$ -function:

$$\delta_{ab}^m(p) \equiv [2\pi i(p_0 - \omega_p) 2\omega_p]^{-1} \{e^{i(p_0 - \omega_p)a} - e^{-i(p_0 - \omega_p)b}\}, \quad (1.19)$$

which converges to  $\theta_0(p) \delta(p^2 - m^2)$  for  $a, b$  tending independently to  $\infty$ . In this notation one can prove the following 'reduction formula'<sup>15)</sup>:

**Theorem 1.2:** Under the assumptions (A) and (B) one has in the strong topology for  $p_1^2, q_1^2 < M^2$ :

$$T_{22}(p_1, p_2, -q_1, -q_2) = \\ 2\pi \lim_{a, b \rightarrow \infty} \lim_{c, d \rightarrow \infty} \delta_{ab}^m(p_1)^* \delta_{cd}^m(q_1) [(p_1^2 - m^2)(q_1^2 - m^2) \langle p_2 | \tilde{R}(p_1, -q_1) | q_2 \rangle] , \quad (1.20)$$

independently of the order of  $\lim_{a, b \rightarrow \infty}$  and  $\lim_{c, d \rightarrow \infty}$ . If (C) holds in addition, then (1.20) is true for all  $p_1, q_1$ .

\*) Professor R. HAAG has kindly informed me that results similar to those in corollary 1.2 have been obtained by him and D. W. ROBINSON.

*Proof:* We choose  $\tilde{f}_1, \tilde{g}_1 \in \mathfrak{S}(D_M)$  ( $\in \mathfrak{S}(R^4)$ , if (C) holds),  $\hat{f}_2, \hat{g}_2 \in \mathfrak{S}(R^3)$ , and a  $C^\infty$ -function  $\theta$  with  $\text{supp } (\theta - \theta_0)$  compact. Then one proves that for fixed  $s$  the limit

$$\lim_{t \rightarrow -\infty} \int dx dy f_1^*(x, s) g_1(y, t) \theta(x - y) \langle \hat{f}_2 | A(x) A(y) | \hat{g}_2 \rangle \quad (1.21)$$

is equal to  $(\hat{g}_1(\mathbf{p}) = \tilde{g}_1(\omega_p, \mathbf{p}))$  and using theorem 1.1 or corollary 1.1):

$$\begin{aligned} &= \lim_{t \rightarrow -\infty} \int dx dy f_1^*(x, s) g_1(y, t) \langle \hat{f}_2 | A(x) A(y) | \hat{g}_2 \rangle = \\ &= \lim_{t \rightarrow -\infty} (A^*(f_1, s) \Phi_{\hat{f}_2}, A^*(g_1, t) \Phi_{\hat{g}_2}) = (A^*(f_1, s) \Phi_{\hat{f}_2}, \Phi_{\hat{g}_1 \hat{g}_2}^{\text{in}}). \end{aligned} \quad (1.22)$$

This follows from the fact that for  $\text{supp } (\theta - \theta_0)$  compact there exists an  $R > 0$ , such that the tempered distribution  $(1 - \theta(x - y)) \langle \hat{f}_2 | A(x) A(y) | \hat{g}_2 \rangle$  has support in  $G_R \equiv \{(x, y) \in R^8, y_0 > x_0 - R\}$ . Therefore (1.21) is equal to (1.22) in the limit  $t \rightarrow -\infty$ , if for fixed  $s$

$$\lim_{t \rightarrow -\infty} f_1^*(s, x) g_1(t, y) = 0 \quad (1.23)$$

in the topology of  $\mathfrak{S}(G_R)$ . For any monomial  $\mathfrak{P}_1(y)$  in the  $y_i$  and  $\mathfrak{P}_2(\partial/\partial y)$  in the  $\partial/\partial y_i$  one has

$$\begin{aligned} &\mathfrak{P}_1(y) \mathfrak{P}_2\left(\frac{\partial}{\partial y}\right) g_1(y, t) = \\ &= (2\pi)^{-5/2} \int dp e^{-i(p, y)} \left[ \mathfrak{P}_1\left(-i \frac{\partial}{\partial p}\right) \mathfrak{P}_2(-i p) \tilde{g}_1(p) \left(\frac{\omega_p + p_0}{2\omega_p}\right) e^{i(p_0 - \omega_p)t} \right] \\ &= \sum_{\mu=0}^M t^\mu \int dp e^{-i(p, y) + i(p_0 - \omega_p)t} \tilde{g}_{1, \mu}(p), \end{aligned} \quad (1.24)$$

with  $\tilde{g}_{1, \mu} \in \mathfrak{S}(R^4)$  and  $M = M(\mathfrak{P}_1)$ . Therefore:

$$\begin{aligned} &|y_0 - t|^L |\mathfrak{P}_1(y) \mathfrak{P}_2\left(\frac{\partial}{\partial y}\right) g_1(y, t)| \leq \\ &\leq \sum_{\mu=0}^M |t|^\mu \left| \int dp e^{-i(p, y) + i(p_0 - \omega_p)t} \left(\frac{\partial}{\partial p_0}\right)^L \tilde{g}_{1, \mu}(p) \right| \leq \sum_{\mu=0}^M c_\mu^L |t|^\mu, \end{aligned} \quad (1.25)$$

for all  $L$  and  $t$ . Finally one has for fixed  $s$  and  $K, L > M$  and for sufficiently large  $|t|$ :

$$\begin{aligned} &\sup_{y_0 > x_0 - R} |\mathfrak{P}_1(x, y) \mathfrak{P}_2\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) f_1^*(s, x) g_1(t, y)| \leq \\ &\leq \sup_{y_0 > x_0 - R} c |t|^M (|x_0 - s|^K + 1)^{-1} (|y_0 - t|^L + 1)^{-1} \rightarrow 0, \end{aligned} \quad (1.26)$$

for  $t \rightarrow -\infty$ . This proves (1.23).

Applying theorem 1.1 or corollary 1.1 again to (1.21) one obtains:

$$\begin{aligned} &\lim_{s \rightarrow \pm \infty} \lim_{t \rightarrow -\infty} \int dx dy f_1^*(x, s) g_1(y, t) \theta(x - y) \langle \hat{f}_2 | A(x) A(y) | \hat{g}_2 \rangle \\ &= \langle \hat{f}_1 \hat{f}_2^{\text{out}} | \hat{g}_1 \hat{g}_2^{\text{in}} \rangle. \end{aligned} \quad (1.27)$$

By the same argument one can show that:

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \int dx dy f_1^*(s, x) g_1(y, t) \theta(x - y) \langle \hat{f}_2 | A(x) A(y) | \hat{g}_2 \rangle = 0 \\ & = \left( \lim_{s \rightarrow +\infty} - \lim_{s \rightarrow -\infty} \right) \lim_{t \rightarrow \pm\infty} \int dx dy f_1^*(x, s) g_1(y, t) \theta(x - y) | \langle \hat{f}_2 | A(y) A(x) | \hat{g}_2 \rangle . \end{aligned} \quad (1.28)$$

Collecting these terms one obtains in momentum space:

$$\begin{aligned} & \langle \hat{f}_1 \hat{f}_2^{out} | \hat{g}_1 \hat{g}_2^{in} \rangle - \langle \hat{f}_1 \hat{f}_2^{out} | \hat{g}_1 \hat{g}_2^{out} \rangle = 2\pi \lim_{a, b \rightarrow \infty} \lim_{c, d \rightarrow \infty} \\ & \times \int dp dq \tilde{f}_1^*(p) \tilde{g}_1(q) \delta_{ab}^m(p)^* \delta_{cd}^m(q) [(p^2 - m^2)(q^2 - m^2) \langle \hat{f}_2 | \tilde{R}(p, -q) | \hat{g}_2 \rangle] . \end{aligned} \quad (1.29)$$

In (1.29)  $a, b$  and  $c, d$ , respectively, tend independently to  $\infty$  and the order of  $\lim_{a, b \rightarrow \infty}$  and  $\lim_{c, d \rightarrow \infty}$  is immaterial. The majorizations leading to (1.27) and (1.28) can be carried

through uniformly in  $\tilde{f}_1 \in \tilde{B}_1, \tilde{g}_1 \in \tilde{B}'_1, \hat{f}_2 \in \hat{B}_2, \hat{g}_2 \in \hat{B}'_2$  for bounded sets  $\tilde{B}_1, \tilde{B}'_1 \subset \mathfrak{S}(D_M), \hat{B}_2, \hat{B}'_2 \subset \mathfrak{S}(R^3)$ . As a consequence of the nuclear theorem<sup>7)</sup> the limit (1.29) equally holds for any  $\Psi \in \mathfrak{S}(R^6 \times D_M \times D_M)$ . Since weak and strong sequential limits are identical in the topology of  $\mathfrak{S}'^{17)}$ , theorem 1.2 is proved.

*Remark:* One sees that the reduction formula (1.20) can be derived for  $p_1^2, q_1^2 < M^2$  without using the postulate (C) of asymptotic completeness. In such a framework  $T_{22}$  is again defined by (1.16) but might lose its physical significance as scattering amplitude. For the proof of dispersion relations for  $T_{22}$  one has then to make assumptions on the mass spectrum of the theory (see postulate (D) in section 2). Another reduction formula will be necessary for the proof of the analyticity properties of  $T_{22}$ :

*Theorem 1.3:* Under (A), (B) one has (independently of the order of  $\lim_{c, d \rightarrow \infty}$  and  $\lim_{a, b \rightarrow \infty}$ ) for  $p_1^2, p_2^2 < M^2$ :

$$\begin{aligned} & T_{22}(p_1, p_2, -q_1, -q_2) \\ & = 2\pi \lim_{a, b \rightarrow \infty} \lim_{c, d \rightarrow \infty} \delta_{ab}^m(p_1)^* \delta_{cd}^m(p_2)^* [(p_1^2 - m^2)(p_2^2 - m^2) \langle \tilde{R}(p_1, p_2) | q_1 q_2^{in} \rangle] \end{aligned} \quad (1.30)$$

and for  $p_1^2, p_2^2, q_1^2, q_2^2 < M^2$  (with  $a, b, c, d \rightarrow \infty$  independently):

$$\begin{aligned} & (p_1^2 - m^2)(q_1^2 - m^2) \langle p_2 | \tilde{A}(p_1) \tilde{A}(-q_1) | q_2 \rangle \\ & = 2\pi \lim_{a, b, c, d \rightarrow \infty} \delta_{ab}^m(p_2)^* \delta_{cd}^m(q_2) [(p_1^2 - m^2)(p_2^2 - m^2)(q_1^2 - m^2)(q_2^2 - m^2) \\ & \times \langle \tilde{R}(p_1, p_2) | \tilde{R}(-q_1, -q_2) \rangle] . \end{aligned} \quad (1.31)$$

*Proof:* One chooses  $\tilde{f}_1, \tilde{f}_2 \in \mathfrak{S}(D_M)$  with  $\tilde{f}_i(\omega_p, \mathbf{p}) = \hat{f}_i(\mathbf{p})$ . Then theorem 1.1 gives for all  $t$ :

$$| \hat{f}_1 \hat{f}_2^{out} \rangle = \lim_{s \rightarrow \pm\infty} A^*(f_1, s) A^*(f_2, t) \Omega . \quad (1.32)$$

Furthermore one has as in theorem 1.2 for fixed  $s$ :

$$A^*(f_1, s) A^*(f_2, t) \Omega = \lim_{t \rightarrow -\infty} \int dx dy f_1^*(x, s) f_2^*(y, t) \theta(x - y) A(x) A(y) \Omega \quad (1.33)$$

for any  $C^\infty$ -function  $\theta$  with  $\text{supp } (\theta - \theta_0)$  compact. Since the other terms in (1.30) and (1.31) can be treated in the same way, theorem 1.3 is proved in the weak and therefore in the strong topology.

We finally remark that  $T_{22}$  can be expressed (essentially as in <sup>15</sup>) by the VEV of a 4-fold retarded or time-ordered product (1.18) of  $A(x)$ .

## § 2. Integral Representations

The reduction formulae in section 1 gave a large class of 'off-shell' extrapolations for the 2-particle scattering amplitude  $T_{22}$  and for the 1-particle matrix element  $\langle p_2 | \tilde{j}(p_1) \tilde{j}(-q_1) | q_2 \rangle$  of the currents  $\tilde{j}(p) = (p^2 - m^2) \tilde{A}(p)$ . All these retarded expressions derived from the 4-point function  $\langle A(x_1) \dots A(x_4) \rangle_0$  turned out to be equivalent on the mass shell.

In this section we shall construct a class of 'sharp' admissible extrapolations, which are tempered  $iL_+^\uparrow$ -invariant distributions with the necessary support properties in  $x$ - and  $p$ -space, for the proof of the analyticity properties of  $T_{22}$ . The construction is based on two lemmas on invariant distributions.

Let  $\mu > 0$  and  $\bar{V}_+^\mu \equiv \{p: p_0 \geq \sqrt{p^2 + \mu^2}\}$ . Let  $\mathring{\mathfrak{S}}'(\bar{V}_+^\mu \times R^{4n}; L_+^\uparrow)$  be the subspace of  $L_+^\uparrow$ -invariant tempered distributions  $T \in \mathring{\mathfrak{S}}'(R^{4(n+1)})$  with  $\text{supp } T \subset \bar{V}_+^\mu \times R^{4n}$ . Let  $[\mu^2, \infty)$  be the interval  $\mu^2 \leq t < \infty$ , let  $O_+$  be the proper real rotation group in 3 dimensions and  $\mathring{\mathfrak{S}}'([\mu^2, \infty) \times R^{4n}; O_+)$  be the subspace of tempered distributions  $\tilde{T}(t, q_1^0, \mathbf{q}_1, \dots, q_n^0, \mathbf{q}_n)$  with  $\text{supp } \tilde{T} \subset [\mu^2, \infty) \times R^{4n}$ , which are  $O_+$ -invariant in  $\mathbf{q}_1, \dots, \mathbf{q}_n$ . Then one has:

**Lemma 2.1:**  $\mathring{\mathfrak{S}}'(\bar{V}_+^\mu \times R^{4n}; L_+^\uparrow)$  and  $\mathring{\mathfrak{S}}'([\mu^2, \infty) \times R^{4n}; O_+)$  are (topologically) isomorphic.

*Proof:* For any  $p \in V_+$  let  $L(p)$  be the pure Lorentz transformation into the standard rest-frame of  $p$  corresponding to the  $A(p) \in SL(2, C)$  ( $\sigma_i$ : Pauli matrices):

$$A(p) = [2\sqrt{(p, p)} (\sqrt{(p, p)} + p_0)]^{-1} \{ (\sqrt{(p, p)} + p_0) \sigma_0 - \mathbf{p} \cdot \boldsymbol{\sigma} \}. \quad (2.1)$$

Then for any  $\mu > \bar{\mu} > 0$  and  $\varphi \in \mathring{\mathfrak{S}}(V_+^{\bar{\mu}} \times R^{4n})$  the function

$$(M\varphi)(t, p_1, \dots, p_n) \equiv \int dp \delta(p, p) - t \varphi(p, L^{-1}(p) p_1, \dots, L^{-1}(p) p_n) \quad (2.2)$$

lies in  $\mathring{\mathfrak{S}}([\mu^2, \infty) \times R^{4n})$  (see e.g. <sup>7</sup>). One can further prove that the linear mapping

$$M: \mathring{\mathfrak{S}}(V_+^{\bar{\mu}} \times R^{4n}) \rightarrow \mathring{\mathfrak{S}}([\mu^2, \infty) \times R^{4n}) \quad (2.3)$$

is onto and continuous. The homomorphism  $M$  defines then an isomorphism  $M'$  between  $\mathring{\mathfrak{S}}'(\bar{V}_+^\mu \times R^{4n}; L_+^\uparrow)$  and  $\mathring{\mathfrak{S}}'([\mu^2, \infty) \times R^{4n}; O_+)$  by

$$\langle T, \varphi \rangle = \langle M' T, M\varphi \rangle. \quad (2.4)$$

In this sense  $T \in \mathring{\mathfrak{S}}'(\bar{V}_+^\mu \times R^{4n}; L_+^\uparrow)$  is given by its 'value'  $\tau \in \mathring{\mathfrak{S}}'([\mu^2, \infty) \times R^{4n}; O_+)$  in the standard rest-frame of  $p$ :

$$T(p, p_1, \dots, p_n) = \tau((p, p), L(p) p_1, \dots, L(p) p_n) \theta_0(p), \quad (2.5)$$

a formal notation, which will be convenient in the sequel.

Let  $\mathfrak{S}'(R^6; O_+)$  be the subspace of tempered distributions  $F(\mathbf{q}_1, \mathbf{q}_2)$ , which are  $O_+$ -invariant in  $\mathbf{q}_1, \mathbf{q}_2$ . Let  $D$  be the closed convex set

$$\{(a, b, c) \in R^3: a, c \geq 0, a c \geq b^2\}.$$

Then the following lemma holds:

*Lemma 2.2:* The spaces  $\mathfrak{S}'(R^6; O_+)$  and  $\mathring{\mathfrak{S}}'(D)$  are (topologically) isomorphic.

*Proof:* Let  $dR$  be the invariant Haar measure of  $O_+$  normalized to  $\int_{O_+} dR = 1$ . Then the mapping

$$N_1: \varphi(\mathbf{x}, \mathbf{y}) \rightarrow \int_{O_+} dR \varphi(R^{-1} \mathbf{x}, R^{-1} \mathbf{y}) \quad (2.6)$$

is a continuous projection from  $\mathfrak{S}(R^6)$  onto the subspace  $\mathfrak{S}(R^6; O_+)$  of the  $O_+$ -invariant testing functions ( $\varphi(\mathbf{x}, \mathbf{y}) = \varphi(S^{-1} \mathbf{x}, S^{-1} \mathbf{y})$  for all  $S \in O_+$ ). As in <sup>18)</sup> one can see that the mapping

$$N_2: \psi(a, b, c) \rightarrow \psi(\mathbf{x}^2, \mathbf{x} \mathbf{y}, \mathbf{y}^2) \quad (2.7)$$

gives a topological isomorphism between  $\mathfrak{S}(R^6; O_+)$  and  $\mathfrak{S}(D)$ . For  $N_2$  is evidently one-to-one, linear and continuous. Furthermore to every  $\varphi \in \mathfrak{S}(R^6; O_+)$  there exists an in  $D$  continuous function  $\psi$  such that  $\psi(\mathbf{x}^2, \mathbf{x} \mathbf{y}, \mathbf{y}^2) = \varphi(\mathbf{x}, \mathbf{y})$ . By evaluation of the derivatives of  $\psi$  in special  $O_+$ -frames for  $\varphi$  one can show by complete induction that  $\psi \in \mathfrak{S}(D)$  and that  $N_2^{-1}$  is continuous. Setting  $N \equiv N_2^{-1} \circ N_1$  the isomorphism  $N'$  between  $\mathfrak{S}'(R^6; O_+)$  and  $\mathring{\mathfrak{S}}'(D)$  is given by duality:

$$\langle T, \varphi \rangle = \langle N' T, N \varphi \rangle. \quad (2.8)$$

If  $T(\mathbf{x}, \mathbf{y})$  is a  $O_+$ -invariant tempered continuous function, then there exists in  $D$  a tempered continuous function  $\tilde{T}$  with  $\tilde{T}(\mathbf{x}^2, \mathbf{x} \mathbf{y}, \mathbf{y}^2) = T(\mathbf{x}, \mathbf{y})$ . For  $\varphi \in \mathfrak{S}(R^6)$  and  $\tilde{\varphi} = N \varphi \in \mathfrak{S}(D)$  one has

$$\langle T, \varphi \rangle = 2 \pi^2 \int \tilde{T}(\mathbf{x}^2, \mathbf{x} \mathbf{y}, \mathbf{y}^2) \tilde{\varphi}(\mathbf{x}^2, \mathbf{x} \mathbf{y}, \mathbf{y}^2) d\mathbf{x}^2 d\mathbf{x} \mathbf{y} d\mathbf{y}^2 \quad (2.9)$$

and therefore  $N' T = 2 \pi^2 \tilde{T} \in \mathring{\mathfrak{S}}'(D)$ . In the sense of (2.8) we shall sometimes use the functional notation (2.9) in section 3.

Let  $I: R^{12} \rightarrow R^6$  be the mapping into the  $L_+^\uparrow$ -invariants:

$$I: (p_1, p_2, p_3) \in R^{12} \rightarrow ((p_1, p_1), \dots, (p_3, p_3)) \in R^6. \quad (2.10)$$

Let  $G \subset R^6$  be the closed convex  $I$ -image of  $\bar{V}_+^\mu \times R^8$ . Then by combining lemma 2.1 and 2.2 and using the nuclear theorem of L. SCHWARTZ<sup>7)</sup> one obtains:

*Lemma 2.3:* The spaces  $\mathring{\mathfrak{S}}'(\bar{V}_+^\mu \times R^8; L_+^\uparrow)$  and  $\mathring{\mathfrak{S}}'(G)$  are (topologically) isomorphic.

In trying to generalize lemma 2.3 to more than 3 four-vectors one encounters the difficulty that the image of  $\bar{V}_+^\mu \times R^{4n}$ ,  $n \geq 3$ , in the space of the  $L_+^\uparrow$ -invariants is an algebraic variety  $\hat{V} \subset R^m$  ( $m = (n+1)(n+2)/2 + \binom{n+1}{4}$ ) with singularities, on which no reasonable spaces of testing functions have yet been defined. For  $L_+^\uparrow$ -invariant



continuous functions  $T(p_0, \dots, p_n)$  with  $\text{supp } T \subset \bar{V}_+^\mu \times R^{4n}$ , however, there exists a continuous function  $\hat{T}$  on  $\hat{V}$  with  $T(p_0, \dots, p_1) = \hat{T}((p_0, p_0), \dots, (p_n, p_n))$ , a frequently-used result in relativistic scattering theory.

Lemma 2.1 justifies the study of Lorentz invariant distributions in special Lorentz frames, if certain support properties are satisfied. Furthermore the construction of a sharp extrapolation for  $T_{22}$  or for  $\langle p_2 | \hat{j}(p_1) \hat{j}(-q_1) | q_2 \rangle$  in the Breit- or centre-of-mass system can be  $iL_\perp^\dagger$ -invariantly extended into an arbitrary Lorentz frame. This we shall illustrate by constructing and discussing the sharp 2-fold retarded commutator  $\langle \tilde{R}_0(p_1, p_2) \tilde{R}_0(-q_1, -q_2) \rangle_0$ .

Let  $E_0$  be the projection on the vacuum and  $E_1$  the projector on the 1-particle space  $\mathfrak{H}_{[m,0]}$ . Then in order to derive analyticity properties of the 2-particle scattering amplitude we have to make some assumptions on the mass spectrum of the theory. For convenience we postulate for the spectral measure  $E(\Delta) = \int_\Delta dE(p)$  corresponding to  $U(a, 1) = \int e^{i(p,a)} dE(p)$  <sup>11)</sup>:

$$(D) \quad (1 - E_0 - E_1) E(\Delta) = 0 \text{ for } \Delta \cap \{p_0 \geq \sqrt{4m^2 + \mathbf{p}^2}\} = \emptyset,$$

(D) follows from (C) for an asymptotic complete theory of one kind of  $[m, 0]$ -particles. More general mass-spectra are treated in the literature <sup>1)</sup>.

The 2-fold commutator  $\langle [\tilde{A}(p_1) \tilde{A}(p_2)] E_0^\perp [\tilde{A}(-q_1) \tilde{A}(-q_2)] \rangle_0$  has support in  $\{p_1 + p_2 \in \bar{V}_+^m\}$  and is of the form:

$$\delta(p_1 + p_2 - q_1 - q_2) \theta_0(p_1 + p_2) \tau_{00} \left( (p_1 + p_2)^2, L(p_1 + p_2) \frac{p_1 - p_2}{2}, L(p_1 + p_2) \frac{q_1 - q_2}{2} \right), \quad (2.11)$$

as a consequence of translation invariance and of Lemma 2.1. The tempered distribution  $\tau_{00}(s, k_2, k_3)$  is  $O_+$ -invariant in  $\mathbf{k}_2, \mathbf{k}_3$  and has its support in  $\{s \geq m^2, |k_i^0| \geq \sqrt{m^2 + \mathbf{k}_i^2} - 1/2 \sqrt{s}, i = 1, 2\}$ . Since the partial Fourier transform  $\tilde{\tau}_{00}(s, \xi_2, \xi_3)$  vanishes for  $(\xi_2, \xi_2) < 0$  or  $(\xi_3, \xi_3) < 0$ ,  $\tau_{00}(s, k_2, k_3)$  can be extended <sup>5)22)</sup> to a rotation-symmetric tempered solution  $\hat{\tau}_{00}(s, K_2, K_3)$  of the 6-dimensional wave equation in  $K_2, K_3$  ( $K_i = (k_i^0, \mathbf{k}_i, k_i^4, k_i^5)$ ).

With the same methods as for a simple 6-dimensional wave-equation (see e.g. <sup>20)</sup>) it can be shown that the Cauchy problem has a unique solution in terms of the Cauchy data on the surface  $\{k_2^0 = k_3^0 = 0\}$ . This gives a 2-fold JOST-LEHMANN-DYSON (JLD) representation <sup>12)5)22)</sup> for  $\tau_{00}(s, k_2, k_3)$ , which is (because of the symmetry of  $\text{supp } \tau_{00}$ ) of the form:

$$\tau_{00}(s, k_2, k_3) = \int d\mathbf{u}_2 d\mathbf{u}_3 d\kappa_2^2 d\kappa_3^2 \varepsilon(k_2^0) \varepsilon(k_3^0) \delta((k_2^0)^2 - (\mathbf{k}_2 - \mathbf{u}_2)^2 - \kappa_2^2) \times \delta((k_3^0)^2 - (\mathbf{k}_3 - \mathbf{u}_3)^2 - \kappa_3^2) [\Phi_{11} + k_2^0 \Phi_{21} + k_3^0 \Phi_{12} + k_2^0 k_3^0 \Phi_{22}] \quad (2.12)$$

The integral (2.12) is to be understood in the sense that for  $\varphi(k_2, k_3) \in \mathfrak{S}(R^8)$

$$\int dk_2 dk_3 \varphi(k_2, k_3) \varepsilon(k_2^0) \varepsilon(k_3^0) \delta((k_2^0)^2 - (\mathbf{k}_2 - \mathbf{u}_2)^2 - \kappa_2^2) \delta((k_3^0)^2 - (\mathbf{k}_3 - \mathbf{u}_3)^2 - \kappa_3^2)$$

is a testing function in the variables  $\mathbf{u}_2, \mathbf{u}_3, \kappa_2^2, \kappa_3^2$ . The  $\Phi_{ij} = \Phi_{ij}(\mathbf{u}_2, \mathbf{u}_3, \kappa_2^2, \kappa_3^2, s)$  (being derivatives of the initial values of  $\hat{\tau}_{00}, \hat{\tau}_{00, \kappa_2^0}, \hat{\tau}_{00, \kappa_3^0}, \hat{\tau}_{00, \kappa_2^0, \kappa_3^0}$  <sup>22)</sup>) are



tempered distributions, in  $\mathbf{u}_2, \mathbf{u}_3$   $O_+$ -invariant, which have for fixed  $s \geq m^2$  their support in

$$G(s) = \left\{ \mathbf{u}_i^2 \leq \frac{s}{4}, \kappa_j \geq \max \left\{ 0, m - \left( \frac{s}{4} - \mathbf{u}_j^2 \right)^{1/2} \right\}, i, j = 1, 2 \right\}. \quad (2.13)$$

In the standard construction<sup>1)</sup> of a sharp retarded commutator  $\langle \tilde{R}(p_1, p_2) E_0^\perp \tilde{R}(-q_1, -q_2) \rangle_0$  one studies the integral

$$\tau(s, k_2, k_3) = \int d\mathbf{u}_2 d\mathbf{u}_3 d\kappa_2^2 d\kappa_3^2 \chi [\Phi_{11} + k_2^0 \Phi_{21} + k_3^0 \Phi_{12} + k_2^0 k_3^0 \Phi_{22}] , \quad (2.14)$$

with the kernel  $\chi$  defined by:

$$\chi = \left( \frac{i}{2\pi} \right)^2 \frac{((k_2^0)^2 - (\mathbf{k}_2 - \mathbf{u}_2)^2 + \kappa_{20}^2)^{N_2} ((k_3^0)^2 - (\mathbf{k}_3 - \mathbf{u}_3)^2 + \kappa_{30}^2)^{N_3}}{((k_2^0)^2 - (\mathbf{k}_2 - \mathbf{u}_2)^2 - \kappa_2^2) ((k_3^0)^2 - (\mathbf{k}_3 - \mathbf{u}_3)^2 - \kappa_3^2) (\kappa_2^2 + \kappa_{20}^2)^{N_2} (\kappa_3^2 + \kappa_{30}^2)^{N_3}} \quad (2.15)$$

(2.14) converges in the weak sense, for  $\kappa_{20}^2, \kappa_{30}^2 > 0$  and sufficiently large integers  $N_2, N_3 \geq 0$ , uniformly for all  $(k_2, k_3) \in D(s)$ :

$$D(s) = \{(k_i^0)^2 \neq (\mathbf{k}_i - \mathbf{u}_i)^2 + \kappa_i^2, i = 2, 3 \quad \forall (\mathbf{u}_2, \mathbf{u}_3, \kappa_2^2, \kappa_3^2) \in G(s)\}, \quad (2.16)$$

$\tau(s, k_2, k_3)$  is  $O_+$ -invariant and holomorphic in  $(k_2, k_3) \in D(s)$  and a tempered measure in  $s \geq m^2$ <sup>11)</sup>.  $D(s)$  contains for all  $s \geq m^2$  the direct product of the forward and backward tubes  $(\mathfrak{T}_+ \cup \mathfrak{T}_-) \times (\mathfrak{T}_+ \cup \mathfrak{T}_-)$ .  $\tau(s, k_2, k_3)$  fulfills due to the temperedness of the  $\Phi_{ij}$  the growth condition in  $(\mathfrak{T}_+ \cup \mathfrak{T}_-) \times (\mathfrak{T}_+ \cup \mathfrak{T}_-)$  (see <sup>7)</sup>20)), which guarantees the existence of the boundary values:

$$\tau_{rr}(s, k_2, k_3) = \lim_{\substack{aa \\ \text{Im } k_2, \text{Im } k_3 \in V_\pm \rightarrow 0}} \tau(s, k_2, k_3), \quad (2.17)$$

as  $O_+$ -invariant tempered distributions. From (2.14) one proves the following support properties of the partial Fourier transform  $\tilde{\tau}_{rr}(s, \xi_2, \xi_3)$ :

$$\begin{aligned} \tilde{\tau}_{rr} - \tilde{\tau}_{ra} - \tilde{\tau}_{ar} - \tilde{\tau}_{aa} &= \tilde{\tau}_{00}, \\ \tilde{\tau}_{rr}(s, \xi_2, \xi_3) &= 0 \text{ for } (\xi_2, \xi_2) < 0 \text{ or } (\xi_3, \xi_3) < 0 \end{aligned} \quad (2.18)$$

and  $\tilde{\tau}_{rr}(s, \xi_2, \xi_3) = 0$ , if  $\xi_2^0$  or  $\xi_3^0 < 0$  for  $(r, r)$ ;  $-\xi_2^0$  or  $\xi_3^0 < 0$  for  $(a, r)$ ;  $\xi_2^0$  or  $-\xi_3^0 < 0$  for  $(r, a)$ ;  $\xi_2^0$  or  $\xi_3^0 > 0$  for  $(a, a)$ .

Therefore the  $i L_+^\dagger$ -invariant tempered distribution

$$\begin{aligned} \langle \tilde{R}_0(p_1, p_2) E_0^\perp \tilde{R}_0(-q_1, -q_2) \rangle_0 &\equiv i^2 \delta(p_1 + p_2 - q_1 - q_2) \theta_0(p_1 + p_2) \\ &\times \tau_{rr} \left( (p_1 + p_2)^2, L(p_1 + p_2) \frac{p_1 - p_2}{2}, L(p_1 + p_2) \frac{q_1 - q_2}{2} \right) \end{aligned} \quad (2.19)$$

is a sharp 2-fold retarded commutator, which can be used for the off-shell extrapolation of  $\langle p_2 | \tilde{j}(p_1) \tilde{j}(-q_1) | q_2 \rangle = \langle p_2 | \tilde{j}(p_1) E_0^\perp \tilde{j}(-q_1) | q_2 \rangle$  in (1.31).

$\tau_{rr}(s, k_2, k_3)$  is unique up to  $O_+$ -invariant tempered distributions  $\theta_i(s, k_2, k_3)$ ,  $i = 2, 3$ , which are polynomials in  $k_i$ ,  $i = 2, 3$ , and which result e.g. from a different

choice of the subtractions in (2.15). In the reduction formula (1.31) this ambiguity does not contribute to

$$(p_1^2 - m^2) (p_2^2 - m^2) (q_1^2 - m^2) (q_2^2 - m^2) \theta_0(p_1 + p_2) \\ \times \tau_{rr} \left( (p_1 + p_2)^2, L(p_1 + p_2) \frac{p_1 - p_2}{2}, L(p_1 + p_2) \frac{q_1 - q_2}{2} \right).$$

By (2.18) the latter distribution differs from any sharp retarded commutator

$$\tau'_{rr} \left( (p_1 + p_2)^2, L(p_1 + p_2) \frac{p_1 - p_2}{2}, L(p_1 + p_2) \frac{q_1 - q_2}{2} \right) \theta_0(p_1 + p_2)$$

constructed as above from

$$\langle [\tilde{j}(p_1) \tilde{j}(p_2)] E_0^\perp [\tilde{j}(-q_1) \tilde{j}(-q_2)] \rangle_0$$

only by trivial terms of similar structure. For the discussion of the analyticity properties of the sharp off-shell extrapolations one has therefore only to study  $\tau'_{rr}$ , where the support of the spectral distributions in (2.14) is due to (D) determined by the mass  $2m$  of the lowest 2-particle state.

Sometimes it is convenient to treat the 'pole term' in  $\langle \tilde{R}_0(p_1, p_2) E_0^\perp \tilde{R}_0(-q_1, -q_2) \rangle_0$  separately. Since  $E_1$  is of the form:

$$E_1 = \int \frac{d^2 \mathbf{p}}{2\omega_p} | a_{ex}^*(\mathbf{p}) \Omega \rangle \langle a_{ex}^*(\mathbf{p}) \Omega | \quad (2.20)$$

$\langle \tilde{R}_0(p_1, p_2) E_1 \tilde{R}_0(-q_1, -q_2) \rangle_0$  leads to a product of vertex functions

$$\int d\mathbf{p} \theta_0(p) \delta(p^2 - m^2) \langle 0 | \tilde{R}_0(p_1, p_2) | \mathbf{p} \rangle \langle \mathbf{p} | \tilde{R}_0(-q_1, -q_2) | 0 \rangle.$$

Then the continuum contribution

$$\langle \tilde{R}_0(p_1, p_2) E_2 \tilde{R}_0(-q_1, -q_2) \rangle_0 \quad (E_2 \equiv \mathbf{1} - E_0 - E_1)$$

starts from  $s \geq 4m^2$ .

Executing the same standard construction for the other cases in the Breit- or centre-of-mass system, one obtains

*Theorem 2.1:* Sharp retarded commutators (RC) can be defined in the form\*):

$$\langle p_2 | \tilde{R}_0(p_1, -q_1) | q_2 \rangle = i \delta(p_1 + p_2 - q_1 - q_2) \theta_0(p_2) \delta(p_2^2 - m^2) \theta_0(q_2) \delta(q_2^2 - m^2) \\ \times \tau_{1r} \left( L(p_2 + q_2) \frac{p_2 - q_2}{2}, L(p_2 + q_2) \frac{p_1 + q_1}{2} \right), \quad (2.21a)$$

$$\langle \tilde{R}_0(p_1, p_2) | q_1 q_2^{in} \rangle = i \delta(p_1 + p_2 - q_1 - q_2) \theta_0(q_1) \delta(q_1^2 - m^2) \theta_0(q_2) \delta(q_2^2 - m^2) \\ \times \tau_{2r} \left( L(q_1 + q_2) \frac{p_1 - p_2}{2}, L(q_1 + q_2) \frac{q_1 - q_2}{2} \right), \quad (2.21b)$$

\*) The space-part of  $q \in R^4$  is denoted here by  $\underline{q}$ .

$$\langle \tilde{R}_0(p_1, p_2) E_1 \tilde{R}_0(-q_1, -q_2) \rangle_0 = i^2 \delta(p_1 + p_2 - q_1 - q_2) \theta_0(p_1 + p_2) \\ \times \delta\left((p_1 + p_2)^2 - m^2\right) \tau_{3r}(L(p_1 + p_2) \frac{p_1 - p_2}{2}) \tau_{4r}\left(L(p_1 + p_2) \frac{q_1 - q_2}{2}\right), \quad (2.21c)$$

$$\langle \tilde{R}_0(p_1, p_2) E_2 \tilde{R}_0(-q_1, -q_2) \rangle_0 = i^2 \delta(p_1 + p_2 - q_1 - q_2) \theta_0(p_1 + p_2) \\ \times \tau_{rr}^2\left((p_1 + p_2)^2, L(p_1 + p_2) \frac{p_1 - p_2}{2}, L(p_1 + p_2) \frac{q_1 - q_2}{2}\right). \quad (2.21d)$$

Here  $\tau_{1r}(\Delta, \omega)$ ,  $\tau_{2r}(k_2, k_3)$ ,  $\tau_{3r}(k_2)$ ,  $\tau_{4r}(k_3)$  and  $\tau_{rr}^2(s, k_2, k_3)$  are  $O_+$ -invariant tempered distributions, boundary values from  $\mathfrak{T}_+$  or  $\mathfrak{T}_+ \times \mathfrak{T}_+$  of weakly convergent integral representations of the type (2.14), which are derived from the JLD-representations of the corresponding commutator matrix-elements.

Although the 'sharp' RC (2.21) are not of the class considered in section 1, it is easy to see that theorem 1.2 and 1.3 also hold for them, the difference between a 'sharp' and a 'smooth' RC being a tempered distribution, which does not contribute in (1.20), (1.30), (1.31), by an argument like (1.23).

We conclude this section with a remark on a doubt expressed recently as to the validity of the proof of dispersion relations<sup>6</sup>). It follows directly from the temperedness of the  $\Phi'_{ij}$  in (2.14) that for a sufficiently large  $M \geq 0$  the integral

$$\int_{4m^2}^{\infty} \frac{ds' \tau'_{22}(s', k_2, k_3)}{(s' - s)^{M+1}} \quad (2.22)$$

converges uniformly for all complex  $s \notin [4m^2, \infty)$  and all  $(k_2, k_3)$  mapped by (3.9) into the compact set (3.10), with  $M$  independent of  $s$  and  $\varepsilon$ . For, it is seen by a tedious, but straightforward majorization that for any choice of  $N_2, N_3 \geq 0$  and  $\kappa_{20}^2, \kappa_{30}^2 > 0$  in (2.15) the function:

$$\chi_{N_2 N_3} = \left(\frac{i}{2\pi}\right)^2 \frac{((k_2^0)^2 - (\mathbf{k}_2 - \mathbf{u}_2)^2 + \kappa_{20}^2)^{N_2} ((k_3^0)^2 - (\mathbf{k}_3 - \mathbf{u}_3)^2 + \kappa_{30}^2)^{N_3}}{((k_2^0)^2 - (\mathbf{k}_2 - \mathbf{u}_2)^2 - \kappa_2^2) ((k_3^0)^2 - (\mathbf{k}_3 - \mathbf{u}_3)^2 - \kappa_3^2)} \quad (2.23)$$

is for all  $(k_2, k_3) \in (3.10)$  a multiplier  $\in O_M$  in a small neighbourhood of the support of the tempered distributions  $\Phi'_{ij}$ . Since in the (formal) integral:

$$\int_{4m^2}^{\infty} \frac{ds' \tau'_{22}(s', k_2, k_3)}{(s' - s)^{M+1}} = \int \frac{ds' d\kappa_2^2 d\kappa_3^2}{(s' - s)^{M+1} (\kappa_2^2 + \kappa_{20}^2)^{N_2} (\kappa_3^2 + \kappa_{30}^2)^{N_3}} \times \\ \times \int d\mathbf{u}_2 d\mathbf{u}_3 \chi_{N_2 N_3} \{\Phi'_{11} + k_2^0 \Phi'_{21} + k_3^0 \Phi'_{12} + k_2^0 k_3^0 \Phi'_{22}\}, \quad (2.24)$$

one has  $\mathbf{u}_2^2, \mathbf{u}_3^2 \leq s/4$  in  $\text{supp } \Phi'_{ij}$ , only the polynomial growth in  $s$  of the inner integral will be affected by changing the powers  $N_2, N_3 \geq 0$ . Since the  $\Phi'_{ij} \chi_{N_2 N_3}$  increase only polynomially in  $\kappa_2^2, \kappa_3^2$ , the left-hand side of (2.24) will converge uniformly for sufficiently large  $N_2, N_3 \geq 0$  and  $M = M(N_2, N_3) \geq 0$ .

### § 3. Analyticity on the Mass Shell

In this section the off-shell extrapolations of the 2-particle scattering amplitude  $T_{22}$  by sharp retarded commutators will be used to derive analyticity properties of  $T_{22}$

on the mass shell. The investigation is scarcely original and will be centered around the question as to how the classical results of dispersion theory<sup>1)</sup> can be rigorously proved from the general postulates (A), (B), and (D) without making additional 'technical assumptions'.

The behaviour of  $T_{22}$  in the momentum transfer  $t = (p_1 - q_1)^2$  can be easily obtained by the well-known argument of LEHMANN<sup>13)</sup>. Combining the results of theorems 1.3 and 2.1 one gets for  $T_{22}$  the distribution identity (for  $p_1^2, p_2^2 < 4m^2$ ):

$$\begin{aligned} T_{22}(p_1, p_2, -q_1, -q_2) &= 2\pi i \delta(p_1 + p_2 - q_1 - q_2) \theta_0(q_1) \delta(q_1^2 - m^2) \theta_0(q_2) \delta(q_2^2 - m^2) \\ &\quad \times \lim_{a, b \rightarrow \infty} \lim_{c, d \rightarrow \infty} \delta_{ab}^m(p_1)^* \delta_{cd}^m(p_2)^* \\ &\quad \times \left[ (p_1^2 - m^2) (p_2^2 - m^2) \tau_{2r} \left( L(q_1 + q_2) \frac{p_1 - p_2}{2}, \underline{L(q_1 + q_2) \frac{q_1 - q_2}{2}} \right) \right]. \end{aligned} \quad (3.1)$$

The distribution

$$(p_1^2 - m^2) (p_2^2 - m^2) \tau_{2r} \left( L(q_1 + q_2) \frac{p_1 - p_2}{2}, \underline{L(q_1 + q_2) \frac{q_1 - q_2}{2}} \right)$$

can be replaced (neglecting an  $O_+$ -invariant distribution, which is a polynomial in  $L(q_1 + q_2) ((p_1 - p_2)/2)$  by a retarded commutator

$$\tau'_{2r} \left( L(q_1 + q_2) \frac{p_1 - p_2}{2}, \underline{L(q_1 + q_2) \frac{q_1 - q_2}{2}} \right)$$

of the currents, defined by an integral representation  $\tau'_{2r}(k_2, \mathbf{k}_3) = \lim_{\text{Im } k_2 \in V_+ \rightarrow 0} \tau_2(k_2, \mathbf{k}_3)$ :

$$\tau'_2(k_2, \mathbf{k}_3) = \left( \frac{i}{2\pi} \right) \int \frac{d^4u \, d\kappa^2 \Phi'(u, \kappa^2, \mathbf{k}_3) ((u - k_2)^2 + \kappa_0^2)^N}{((u - k_2)^2 - \kappa^2) (\kappa^2 + \kappa_0^2)^N}. \quad (3.2)$$

Now, the support of the JLD-spectral distribution  $\Phi'$  of the commutator  $(\Omega, [\tilde{j}(p_1) \tilde{j}(p_2)] | q_1 q_2^{in})$  is such that  $\tau'_2(k_2, \mathbf{k}_3)$  (and therefore  $\tau'_{2r}(k_2, \mathbf{k}_3)$ ) is for fixed  $\mathbf{k}_3$  analytic in  $k_2$  in a neighbourhood of the mass shell  $\{k_2^0 = 0, \mathbf{k}_2^2 = \mathbf{k}_3^2\}^{13)}$ . Thus in the limit  $a, b \rightarrow \infty, c, d \rightarrow \infty$  (3.1) is just the product of  $\theta_0(p_1) \delta(p_1^2 - m^2) \theta_0(p_2) \delta(p_2^2 - m^2)$  with  $(p_1^2 - m^2) (p_2^2 - m^2) \tau_{2r}$ , which is a real-analytic function in the critical variables essentially given by (3.2).

$\tau'_2(k_2, \mathbf{k}_3)$  is for real  $k_2, \mathbf{k}_3$  a tempered  $O_+$ -invariant distribution and thus by lemma 2.2 a distribution in the invariants  $k_2^0, \mathbf{k}_2^2, \mathbf{k}_2 \mathbf{k}_3, \mathbf{k}_3^2$ . The well-defined restriction to the mass shell is then a tempered distribution in  $s = (p_1 + p_2)^2 = 4m^2 + 4\mathbf{k}_3^2$  and  $t = (p_1 - q_1)^2 = 2(\mathbf{k}_2 \mathbf{k}_3 - \mathbf{k}_3^2)$ .

For  $\mathbf{k}_3^2 \neq 0$  (to be understood in the weak sense, i.e. integrated over a testing function  $\chi(\mathbf{k}_3)$  with sufficiently small support) the only dependence of the  $O_+$ -invariant distribution  $\tau'_2(k_2, \mathbf{k}_3)$  on  $\mathbf{k}_2 \mathbf{k}_3$  can be brought into the kernel of (3.2) by taking the mean over the rotation group around  $\mathbf{k}_3$ . In this way it can be shown<sup>13)</sup> that on the mass shell  $\tau'_2(k_2, \mathbf{k}_3)$  is, for fixed  $s > 4m^2$ , holomorphic in  $t$  in the small Lehmann ellipse:

$$E_1(s) = \left\{ |t| + |t + s - 4m^2| < \left[ (s - 4m^2)^2 + \frac{36m^4}{s} (s - 4m^2) \right]^{1/2} \right\}. \quad (3.3)$$

The same holds for the reduced 2-particle scattering amplitude, which is defined by  $\lim [(k_2^0)^2 - \mathbf{k}_2^2 + \mathbf{k}_3^2)^2 - 4 (k_2^0)^2 (m^2 + \mathbf{k}_3^2)] \tau_{2r}(k_2, \mathbf{k}_3)$ :

*Theorem 3.1:* Under the assumptions (A), (B), (D) the 2-particle scattering amplitude is of the form:

$$T_{22}(p_1, p_2, -q_1, -q_2) = \delta(p_1 + p_2 - q_1 - q_2) \theta_0(p_1) \delta(p_1^2 - m^2) \theta_0(p_2) \delta(p_2^2 - m^2) \\ \times \theta_0(q_1) \delta(q_1^2 - m^2) \theta_0(q_2) \delta(q_2^2 - m^2) T_{22}((p_1 + p_2)^2, (p_1 - q_1)^2). \quad (3.4)$$

The reduced scattering amplitude  $T_{22}(s, t)$  is a tempered distribution in the physical domain  $\{(s, t) : s \geq 4m^2, 2m^2 - s/2 \leq t \leq 0\}$ , which has for fixed  $s > 4m^2$  an analytic continuation in  $t \in E_1(s)$ .

Next we discuss the behaviour of  $T_{22}(s, t)$  for fixed momentum transfer  $t$  in the centre-of-mass energy  $s$ , following closely the classical idea of BOGOLIUBOV<sup>4)</sup>. By theorem 1.2 and 2.1 one can use the following sharp off-shell extrapolation for  $T_{22}$ :

$$T_{22}(p_1, p_2, -q_1, -q_2) = 2\pi i \delta(p_1 + p_2 - q_1 - q_2) \theta_0(p_2) \delta(p_2^2 - m^2) \theta_0(q_2) \delta(q_2^2 - m^2) \\ \times \lim_{a, b \rightarrow \infty} \lim_{c, d \rightarrow \infty} \delta_{ab}^m(p_1)^* \delta_{cd}^m(q_1) \\ \times \left[ (p_1^2 - m^2) (q_1^2 - m^2) \tau_{1r} \left( L(q_2 + p_2) \frac{p_2 - q_2}{2}, L(q_2 + p_2) \frac{p_1 + q_1}{2} \right) \right], \quad (3.5)$$

$\tau_{1r}(\Delta, \omega)$  is defined as the limit,  $\text{Im } \omega \in V_+ \rightarrow 0$ , of

$$\tau_1(\Delta, \omega) = \left( \frac{i}{2\pi} \right) \int \frac{d\mathbf{u} d\kappa^2 \{ \varphi_1(\mathbf{u}, \kappa^2, \Delta) + \omega_0 \varphi_2 \} (\omega_0^2 - (\omega - \mathbf{u})^2 + \kappa_0^2)^N}{(\omega_0^2 - (\omega - \mathbf{u})^2 - \kappa^2) (\kappa^2 + \kappa_0^2)^N}. \quad (3.6)$$

The distribution  $[(\omega^2 - m^2 - \Delta^2)^2 - 4(\omega \Delta)^2] \tau_{1r}(\Delta, \omega)$  is, up to a polynomial in  $\omega$ , identical with the boundary value of a  $\tau_1'(\Delta, \omega)$ , defined as in (3.6) with spectral distributions  $\varphi_i'(\mathbf{u}, \kappa^2, \Delta)$  corresponding to the commutator of the currents:

$$\langle p_2 | [\tilde{j}(p_1) \tilde{j}(-q_1)] | q_2 \rangle = \delta(p_1 + p_2 - q_1 - q_2) \theta_0(p_2) \delta(p_2^2 - m^2) \theta_0(q_2) \delta(q_2^2 - m^2) \\ \times \tau_{10}' \left( L(p_2 + q_2) \frac{p_2 - q_2}{2}, L(p_2 + q_2) \frac{p_1 + q_1}{2} \right), \quad (3.7)$$

where

$$\tau_{10}'(\Delta, \omega) = \int d\mathbf{u} d\kappa^2 \varepsilon(\omega_0) \delta(\omega_0^2 - (\omega - \mathbf{u})^2 - \kappa^2) \{ \varphi_1' + \omega_0 \varphi_2' \}. \quad (3.8)$$

$\tau_1(\Delta, \omega)$  is, for fixed  $\Delta$ , holomorphic in the domain  $D_1(\Delta) = \{\omega : \omega_0^2 \neq (\omega - \mathbf{u})^2 + \kappa^2 \forall (\mathbf{u}, \kappa^2) \in G_1(\Delta)\}$  with  $G_1(\Delta) = \{(\mathbf{u}, \kappa^2) : \mathbf{u}^2 \leq m^2 + \Delta^2, \kappa \geq \max\{0, 2m - \sqrt{m^2 + \Delta^2 - \mathbf{u}^2}\}\}$ .

$D_1(\Delta)$  contains  $\mathfrak{T}_+ \cup \mathfrak{T}_-$  and real  $\omega$  with  $|\omega_0| < \sqrt{4m^2 + \omega^2} - \sqrt{m^2 + \Delta^2}$ , except for 'pole terms' singular at  $|\omega_0| = \sqrt{m^2 + \omega^2} - \sqrt{m^2 + \Delta^2}$ . The domain of holomorphy  $D_1(\Delta)$  is well known<sup>6)</sup>, and one can see that no conclusive analyticity properties of  $T_{22}(s, t)$  can be proved using only the information contained in (3.5).

Following the argument of BOGOLIUBOV et al.<sup>4)</sup> we shall therefore construct an analytic continuation of  $[(\omega^2 - m^2 - \Delta^2)^2 - 4(\omega \Delta)^2] \tau_1(\Delta, \omega)$  by a Cauchy integral representation for 'virtual masses'. The commutator (3.7) will appear in the integrand,

and we shall use for (3.7) the far-reaching information contained in the reduction formula (1.31) with a sharp retarded commutator (2.19).

$$(p_1^2 - m^2) (p_2^2 - m^2) (q_1^2 - m^2) (q_2^2 - m^2) \langle \tilde{R}_0(p_1, p_2) E_0^\perp \tilde{R}_0(-q_1, -q_2) \rangle_0$$

is the boundary value of an analytic function  $\tau'(s, k_2, k_3)$ , which is, due to its  $O_+$ -invariance, a holomorphic function  $\bar{\tau}'^{10)}$  in the invariants  $k_2^0, k_3^0, k_2^2, k_3^2, k_2 \cdot k_3, k_2^2 \cdot k_3^2$ . These variables are for  $s > 0$  biholomorphically related to the Lorentz invariants:

$$\begin{aligned} z_1 &= p_1^2 = \left( \frac{\sqrt{s}}{2} + k_2^0 \right)^2 - k_2^2 & z_2 &= p_2^2 = \left( \frac{\sqrt{s}}{2} - k_2^0 \right)^2 - k_2^2, \\ z_3 &= q_1^2 = \left( \frac{\sqrt{s}}{2} + k_3^0 \right)^2 - k_3^2 & z_4 &= q_2^2 = \left( \frac{\sqrt{s}}{2} - k_3^0 \right)^2 - k_3^2, \\ z_5 &= t = (k_2^0 - k_3^0)^2 - (k_2 - k_3)^2. \end{aligned} \quad (3.9)$$

The analysis of the domain of analyticity  $\bigcap_{s=4m^2}^\infty D(s) \cap D(m^2)$  of  $\bar{\tau}' = \bar{\tau}'(s, z_1, \dots, z_5)$  shows <sup>4), 13), 1)</sup>, that  $\bar{\tau}'$  is for  $s = m^2, s \geq 4m^2$  holomorphic in  $z_1, \dots, z_5$  from:

$$\begin{aligned} |z_1 - \zeta^2| &\leq \delta, \quad |z_2 - m^2| \leq \delta, \quad |z_3 - \zeta^2| \leq \delta, \\ |z_4 - m^2| &\leq \delta, \quad -8m^2 + \varepsilon \leq \operatorname{Re} z_5 \leq 0, \quad |\operatorname{Im} z_5| \leq \frac{\delta}{s}, \end{aligned} \quad (3.10)$$

with  $\delta = \delta(\varepsilon) > 0$  for all  $\varepsilon > 0$  and  $3m^2 \leq \zeta^2 \leq m^2$ .

The physical values of  $p_1^2, p_2^2, q_1^2, q_2^2, t$  are for  $-8m^2 < t \leq 0$  contained in (3.10). The same applies to

$$(p_1^2 - m^2) (p_2^2 - m^2) (q_1^2 - m^2) (q_2^2 - m^2) \langle \tilde{R}_0(-q_1, p_2) E_0^\perp \tilde{R}_0(p_1, -q_2) \rangle_0,$$

which is given by  $\bar{\tau}'(u, z_1, \dots, z_5)$ , where  $u \equiv s + t - (p_1^2 + p_2^2 + q_1^2 + q_2^2)$ . Using this one sees that in the reduction formula (1.31) the limit  $\lim_{a, b \rightarrow \infty} \delta_{ab}^m(p_2)^* \lim_{c, d \rightarrow \infty} \delta_{cd}^m(q_2)$  leads to the product of the  $\delta$ -distributions  $\theta_0(p_2) \delta(p_2^2 - m^2) \theta_0(q_2) \delta(q_2^2 - m^2)$  with the functions  $\bar{\tau}'$ , which are again regular in the critical variables.

In order to insert the reduction formula (1.31) in (3.7), we compute the invariants (3.9) in the variables  $\Delta = \frac{L(q_2 + p_2)(p_2 - q_2)}{2}$  and  $\omega = \frac{L(p_2 + q_2)(p_1 + q_1)}{2}$  of the Breit-system:

$$p_1^2, q_1^2 = \omega^2 \pm 2\omega\Delta - \Delta^2; \quad t = -4\Delta^2; \quad s, u = m^2 + \Delta^2 + \omega^2 \pm 2\omega_0 \sqrt{m^2 + \Delta^2}. \quad (3.11)$$

Combining (1.31), (2.19) and (3.11) one obtains the important identity:

$$\tau'_{10}(\Delta, \omega) = 2\pi \{ \bar{\tau}'(s, p_1^2, m^2, q_1^2, m^2, t) - \bar{\tau}'(u, p_1^2, m^2, q_1^2, m^2, t) \} \quad (3.12)$$

for real  $\Delta, \omega$  satisfying  $\Delta^2 < 2m^2, \omega^2 \pm \omega\Delta - \Delta^2 \leq m^2 + \delta$ . It follows from the regularity properties of  $\bar{\tau}'$  in (3.10) that  $\tau'_{10}(\Delta, \omega)$  is a measure in  $\omega_0$  and real-analytic in  $\omega^2, \omega\Delta, \Delta^2$  in the above domain.

For sufficiently large  $M$  and for real  $\omega^2, \omega\Delta, \Delta^2$  with  $\Delta^2 \leq 2m - \varepsilon, \omega^2 \pm 2\omega\Delta - \Delta^2 \leq m^2 + \delta(\varepsilon)$ , the  $\omega'_0$ -integral of the measure  $\bar{\tau}'_{10}(\omega'_0, \omega^2, \omega\Delta, \Delta^2)$  over the



testing function  $(\omega'_0 - \omega_0)^{-(M+1)}$  exists for all  $\text{Im } \omega_0 \neq 0$  and  $\varepsilon > 0$ . Using (3.12) and changing variables one obtains:

$$\begin{aligned} \tau^M(\omega_0, \omega^2, \omega \Delta, \Delta^2) &\equiv \frac{M!}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\omega'_0 \bar{\tau}'_{10}(\omega'_0, \omega^2, \omega \Delta, \Delta^2)}{(\omega'_0 - \omega_0)^{M+1}} \\ &= -i M! \int_{m^2}^{\infty} ds (2\sqrt{m^2 + \Delta^2})^M \bar{\tau}'(s, \omega^2 + 2\omega \Delta - \Delta^2, m^2 \omega^2 - 2\omega \Delta - \Delta^2, m^2, -4\Delta^2) \\ &\quad \times \left\{ \left( s - \omega^2 - m^2 - \Delta^2 - 2\omega_0 \sqrt{m^2 + \Delta^2} \right)^{-(M+1)} \right. \\ &\quad \left. + (-1)^M \left( s - \omega^2 - m^2 - \Delta^2 + 2\omega_0 \sqrt{m^2 + \Delta^2} \right)^{-(M+1)} \right\}. \end{aligned} \quad (3.13)$$

According to section 2 the right-hand side of (3.13) converges for sufficiently large  $M$  also for complex values of  $\omega^2, \omega \Delta$ . Therefore  $\tau^M(\omega_0, \omega^2, \omega \Delta, \Delta^2)$  is for  $0 \leq \Delta^2 < 2m^2$  a  $C^\infty$ -function in  $\Delta^2$ , which is for fixed  $\Delta^2$  holomorphic in  $\omega_0, \omega^2, \omega \Delta$  in a domain, which contains the points:

$$\begin{aligned} |\text{Im } \omega^2| &< 2\sqrt{m^2 + \Delta^2} |\text{Im } \omega_0| \\ |\omega^2 \pm 2\omega \Delta - \Delta^2 - \zeta^2| &< \delta(\Delta) \text{ for } -3m^2 \leq \zeta^2 \leq m^2. \end{aligned} \quad (3.14)$$

In order to relate  $\tau^M(\omega_0, \omega^2, \omega \Delta, \Delta^2)$  to an analytic continuation  $[(\omega^2 - m^2 - \Delta^2)^2 - 4(\omega \Delta)^2] \tau_1(\Delta, \omega)$  of the 2-particle scattering amplitude, we prove:

*Lemma 3.1:* Let  $f_\chi^N(\omega; \Delta, \mathbf{u}, \kappa^2)$  be defined as

$$f_\chi^N(\omega; \Delta, \mathbf{u}, \kappa^2) \equiv \frac{i}{2\pi} \int_{O_+} dR \frac{\chi(R^{-1} \Delta) (\omega_0^2 - (\omega - R^{-1} \mathbf{u})^2 + \kappa_0^2)^N}{(\kappa^2 + \kappa_0^2)^N (\omega_0^2 - (\omega - R^{-1} \mathbf{u})^2 - \kappa^2)}, \quad (3.15)$$

for

$$\chi \in \mathfrak{D}(R^3), X = \text{supp } \chi \subset \{\Delta^2 \leq 2m^2 - \varepsilon, \varepsilon > 0\},$$

$$(\mathbf{u}, \kappa^2) \in \bigcup_X G_1(\Delta), \omega \in \bigcap_X D_1(\Delta), \kappa_0^2 > 0.$$

Then one has for integer  $M \geq M(N) \geq 0$ :

$$\begin{aligned} &\left( \frac{\partial}{\partial \omega_0} \right)^M f_\chi^N(\omega; \Delta, \mathbf{u}, \kappa^2) \Big|_{\omega^2 = \omega_0^2 - \kappa^2 \text{ fixed}} \\ &= \frac{M!}{2\pi i} \int_{-\infty}^{+\infty} d\omega'_0 \int_{O_+} dR \frac{\varepsilon(\omega'_0) \delta((\omega'_0)^2 - (\omega - R^{-1} \mathbf{u})^2 - \kappa^2) \chi(R^{-1} \Delta)}{(\omega'_0 - \omega_0)^{M+1}}, \end{aligned} \quad (3.16)$$

for all  $\text{Im } \omega_0 \neq 0, \omega^2 < 0$  and sufficiently small real  $\omega \Delta$ .

For the proof one remarks that, for  $(\mathbf{u}, \kappa^2) \in \bigcup_X G_1(\Delta)$  and  $\omega \in \bigcap_X D_1(\Delta)$ ,  $f_\chi^N(\omega; \Delta, \mathbf{u}, \kappa^2)$  is  $O_+$ -invariant in  $\Delta, \mathbf{u}$  and holomorphic in  $\omega$  with  $f_\chi^N(\omega_1; \Delta, \mathbf{u}, \kappa^2) = f_\chi^N(\omega_2; \Delta, \mathbf{u}, \kappa^2)$  for  $\omega_1^2 = \omega_2^2, \omega_{1,0} = \omega_{2,0}, \omega_1 \Delta = \omega_2 \Delta$ . Therefore  $f_\chi^N$  is a holomorphic function  $\bar{f}_\chi^N(\omega_0, \omega^2, \omega \Delta; \Delta, \mathbf{u}, \kappa^2)$  in the variables  $\omega_0, \omega^2, \omega \Delta$  in the image  $\bigcap_X \bar{D}_1(\Delta)$  of the saturated domain  $\bigcap_X D_1(\Delta)^{10}$ . For fixed  $\omega^2 < 0$  and real  $\omega \Delta$  sufficiently small,  $\bar{f}_\chi^N$  can be seen to be holomorphic in the complex  $\omega_0$ -plane, except possibly for singulari-



ties on the real axis. After having replaced  $(\omega_0^2 - (\omega - \mathbf{u})^2 - \kappa^2)^{-1}$  in the definition (3.15) of  $f_\chi^0$  by the following expression (under the  $R$ -integration equivalent):

$$\frac{\omega^2 - \mathbf{u}^2 - \kappa^2 + 2(\omega \mathbf{\Delta}) (\mathbf{u} \mathbf{\Delta}) (\mathbf{\Delta} \mathbf{\Delta})^{-1}}{(\omega^2 - \mathbf{u}^2 - \kappa^2 + 2(\omega \mathbf{\Delta}) (\mathbf{u} \mathbf{\Delta}) (\mathbf{\Delta} \mathbf{\Delta})^{-1})^2 - 4(\omega_0^2 - \omega^2 - (\omega \mathbf{\Delta}) \mathbf{\Delta}^{-2}) (\mathbf{u}^2 - (\mathbf{u} \mathbf{\Delta})^2 \mathbf{\Delta}^{-2})} \quad \text{for } \mathbf{\Delta}^2 \neq 0$$

$$\frac{\omega^2 - \mathbf{u}^2 - \kappa^2}{\omega^2 - \mathbf{u}^2 - \kappa^2)^2 - 4(\omega_0^2 - \omega^2) \mathbf{u}^2} \quad \text{for } \mathbf{\Delta}^2 = 0 \quad (3.17)$$

one obtains (3.16) for  $N = 0$  by CAUCHY's theorem. The general case follows from the fact that for sufficiently large  $M \geq M(N)$  one has:

$$\left( \frac{\partial}{\partial \omega_0} \right)^M \bar{f}_\chi^N(\omega_0, \omega^2, \omega \mathbf{\Delta}; \mathbf{u}, \kappa^2, \mathbf{\Delta}) = \left( \frac{\partial}{\partial \omega_0} \right)^M \bar{f}_\chi^0(\omega_0, \omega^2, \omega \mathbf{\Delta}; \mathbf{u}, \kappa^2, \mathbf{\Delta}). \quad (3.18)$$

We apply Lemma 3.1 to

$$\int d\mathbf{\Delta} \tau'_1(\mathbf{\Delta}, \omega) \chi(\mathbf{\Delta}) = \int d\mathbf{\Delta} \bar{\tau}'_1(\mathbf{\Delta}, \omega^2, \omega \mathbf{\Delta}, \omega_0) \chi(\mathbf{\Delta})$$

$$= \int d\mathbf{u} d\kappa^2 d\mathbf{\Delta} \{ \varphi'_1(\mathbf{u}, \kappa^2, \mathbf{\Delta}) + \omega_0 \varphi'_2(\mathbf{u}, \kappa^2, \mathbf{\Delta}) \} \bar{f}_\chi^N(\omega_0, \omega^2, \omega \mathbf{\Delta}; \mathbf{u}, \kappa^2, \mathbf{\Delta}) \quad (3.19)$$

and obtain for a sufficiently large  $M$ :

$$\left( \frac{\partial}{\partial \omega_0} \right)^M \int \bar{\tau}'_1(\mathbf{\Delta}, \omega^2, \omega \mathbf{\Delta}, \omega_0) \chi(\mathbf{\Delta}) d\mathbf{\Delta} =$$

$$= \frac{M!}{2\pi i} \int d\mathbf{u} d\kappa^2 d\mathbf{\Delta} \int_{-\infty}^{+\infty} d\omega'_0 \int_{O_+} dR \frac{\varepsilon(\omega'_0) \delta((\omega'_0)^2 - (\omega - R^{-1} \mathbf{u})^2 - \kappa^2) \chi(R^{-1} \mathbf{\Delta}) \{ \varphi'_1 + \omega_0 \varphi'_2 \}}{(\omega'_0 - \omega_0)^{M+1}}$$

$$= \int d\mathbf{\Delta} \chi(\mathbf{\Delta}) \frac{M!}{2\pi i} \int_{-\infty}^{+\infty} d\omega'_0 \frac{\bar{\tau}'_{10}(\omega_0, \omega^2, \omega \mathbf{\Delta}, \mathbf{\Delta}^2)}{(\omega'_0 - \omega_0)^{M+1}}. \quad (3.20)$$

In (3.20) the  $R$ -integration can be omitted, as the  $\varphi_1(\mathbf{u}, \kappa^2, \mathbf{\Delta})$  are  $O_+$ -invariant, and the interchange of the  $\omega'_0$ - and  $(\mathbf{u}, \kappa^2)$ -integration is justified by FUBINI's theorem<sup>17)</sup>. Since  $\tau'_1(\mathbf{\Delta}, \omega)$  and  $[(\omega^2 - m^2 - \mathbf{\Delta}^2)^2 - 4(\omega \mathbf{\Delta})^2] \tau_1(\mathbf{\Delta}, \omega)$  only differ by a polynomial in  $\omega$ , one finally obtains for sufficiently large  $M$  the distribution identity:

$$\left( \frac{\partial}{\partial \omega_0} \right)^M [(\omega^2 - m^2 - \mathbf{\Delta}^2)^2 - 4(\omega \mathbf{\Delta})^2] \tau_1(\mathbf{\Delta}, \omega) \Big|_{\omega^2 = \omega_0^2 - \mathbf{u}^2 \text{ fixed}}$$

$$= \frac{M!}{2\pi i} \int d\omega'_0 \frac{\bar{\tau}'_{10}(\omega'_0, \omega^2, \omega \mathbf{\Delta}, \mathbf{\Delta}^2)}{(\omega'_0 - \omega_0)^{M+1}} = \tau^M(\omega_0, \omega^2, \omega \mathbf{\Delta}, \mathbf{\Delta}^2), \quad (3.21)$$

for  $\mathbf{\Delta}^2 \leq 2m^2 - \varepsilon$ ,  $\omega^2 < 0$ , sufficiently small real  $\omega \mathbf{\Delta}$ ,  $\text{Im } \omega_0 \neq 0$  and  $\varepsilon > 0$ . Since the left-hand side is analytic in  $\overline{D_1(\mathbf{\Delta})}$  and the right-hand side in the set  $\overline{D_2(\mathbf{\Delta})}$  defined by (3.14), (3.21) holds identically in  $\overline{D_1(\mathbf{\Delta})} \cap \overline{D_2(\mathbf{\Delta})}$ . This shows that the  $M$ -fold  $\omega_0$ -integral of  $\tau^M$  is an analytic continuation of  $[(\omega^2 - m^2 - \mathbf{\Delta}^2)^2 - 4(\omega \mathbf{\Delta})^2] \tau_1(\mathbf{\Delta}, \omega)$  except for an  $O_+$ -invariant distribution  $P(\mathbf{\Delta}, \omega)$ , which as a distribution in the invariants  $\omega_0, \omega^2, \omega \mathbf{\Delta}, \mathbf{\Delta}^2$  is a polynomial of  $M^{\text{th}}$  degree in  $\omega_0$ .

For all fixed real  $\Delta$  the physical values of  $\omega_0, \omega^2, \omega \Delta$  (corresponding to real 4-vectors  $\omega$ ) lie on the boundary of  $\overline{\mathfrak{T}_{\pm}(\Delta)}$  and also on the boundary of the domains  $\overline{D_{\pm}(\Delta)} \equiv \overline{D_2(\Delta)} \cap \{\text{Im } \omega_0 \leq 0\}$  for  $0 \leq \Delta^2 < 2m^2$ , as well as on the boundary of  $\overline{D_{\pm}(\Delta)} \cap \overline{\mathfrak{T}_{\pm}(\Delta)^1}$ . The boundary value of  $\tau^M(\omega_0 - i\varepsilon, \omega^2, \omega \Delta, \Delta^2)$  for real  $\omega_0, \omega^2, \omega \Delta, \Delta^2$  with  $\Delta^2 < 2m^2, \omega^2 \pm 2\omega \Delta - \Delta^2 < m^2 + \delta(\Delta)$  is for  $\varepsilon \downarrow 0$  the tempered distribution in  $\omega_0$ :

$$\lim_{\varepsilon \downarrow 0} \tau^M(\omega_0 + i\varepsilon, \omega^2, \omega \Delta, \Delta^2) = \frac{1}{2} \left( \frac{\partial}{\partial \omega_0} \right)^M \bar{\tau}'_{10}(\omega_0, \omega^2, \omega \Delta, \Delta^2) + \frac{M!}{2\pi i} P \int_{-\infty}^{+\infty} d\omega'_0 \frac{\bar{\tau}'_{10}(\omega'_0, \omega^2, \omega \Delta, \Delta^2)}{(\omega'_0 - \omega_0)^{M+1}}, \quad (3.22)$$

and the same limit distribution is obtained taking a sequence from  $\overline{\mathfrak{T}_{+}(\Delta)} \cap \overline{D_{+}(\Delta)}$  converging to  $\omega_0, \omega^2, \omega \Delta, \Delta^2$ . This shows that  $\tau^M$  is for these points an analytic continuation of the  $M^{\text{th}}$   $\omega_0$ -derivative (for  $\omega^2 = \omega_0^2 - \omega^2$  fixed) of the off-shell extrapolation  $[(\omega^2 - m^2 - \Delta^2)^2 - 4(\omega \Delta)^2] \tau_{1r}(\Delta, \omega)$  of the 2-particle scattering amplitude.

The mass-shell is characterized by  $\omega^2 = m^2 + \Delta^2, \omega \Delta = 0$ . It follows from (3.22) that  $\tau^M(\omega_0, \omega^2, \omega \Delta, \Delta^2)$  is, for physical points with  $\Delta^2 \leq 2m^2 - \varepsilon, \varepsilon > 0, \omega^2 \pm 2\omega \Delta - \Delta^2 \leq m^2 + \delta(\varepsilon)$ , regular in  $\omega^2$  and  $\omega \Delta$  in a neighbourhood of the mass shell. Therefore the limit of  $\delta_{ab}^m(p_1) * \delta_{cd}^m(q_1) \tau^M$  again exists as the product of  $\tau^M$  with  $\theta_0(p_1) \delta(p_1^2 - m^2) \theta_0(q_1) \delta(q_1^2 - m^2)$ .

From (3.21) and (3.4)  $\tau^M(\omega_0, m^2 + \Delta^2, 0, \Delta^2)$  is up to a factor  $(2\sqrt{m^2 + \Delta^2})^M$  identical with  $(\partial/\partial s)^M T_{22}(s, t)$ . Therefore  $(\partial/\partial s)^M T_{22}(s, t)$  is for  $-8m^2 < t \leq 0$  infinitely often differentiable in  $t$  and for fixed  $t$  a boundary distribution of an analytic function for  $\text{Im } s \neq 0$ , which fulfills the integral representation (3.13). Integrating  $M$  times with respect to  $s$  one obtains the same result for  $T_{22}(s, t)$ , except for a polynomial  $\sum_{\mu=0}^{M-1} c_{\mu}(t) s^{\mu}$  in  $s$  with coefficient distributions in  $t$ . But  $T_{22}(s, t)$  is for fixed  $s > 4m^2$  holomorphic in  $t \in E_1(s)$  and the  $M$ -fold  $s$ -integral of  $\tau^M$  is again  $C^{\infty}$  in  $t$  for  $0 \geq t > -8m^2$ . Therefore the  $c_{\mu}(t)$  must also be  $C^{\infty}$ -functions in  $t$  for  $0 \geq t > -8m^2$ , and we obtain the

**Theorem 3.2:** Under (A), (B), (D) the reduced 2-particle scattering amplitude  $T_{22}(s, t)$  is the boundary value from  $\text{Im } s > 0$  of a holomorphic function in  $s$  for  $\text{Im } s \neq 0$ , which is  $C^{\infty}$  in  $t$  and fulfills in  $s$  an  $M$ -fold subtracted dispersion relation. Furthermore  $\text{Im } T_{22}(s, t)$  is a measure in  $s$  and for fixed  $s$  holomorphic in  $t$  in the 'large Lehmann ellipse'

$$E_2(s) = \left\{ t : |t| + |t + s - 4m^2| < (s - 4m^2) \left( 2 \left[ 1 + \frac{36m^4}{s(s - 4m^2)} \right]^{1/2} - 1 \right) \right\}. \quad (3.23)$$

To prove the last statement we remark that, for physical points,  $\text{Im } T_{22}(s, t)$  is essentially given by the measure

$$\bar{\tau}'(s, m^2, m^2, m^2, m^2, t) - \bar{\tau}'(u(s, t), m^2, m^2, m^2, m^2, t).$$

Then the analyticity properties of  $\text{Im } T_{22}(s, t)$  follow by studying the domain of analyticity  $D'(s)$  of  $\bar{\tau}'(s, z_1, \dots, z_5)$  (see <sup>13)</sup> <sup>1)</sup> <sup>6)</sup>).

The theorems 3.1 and 3.2 contain the necessary analyticity properties for the proof of MANDELSTAM and LEHMANN<sup>14)</sup> that  $T_{22}(s, t)$  is the boundary value of a function simultaneously holomorphic in  $s$  and  $t$ .

### Bibliography

- <sup>1)</sup> For an introduction into the extensive literature on dispersion relations see e.g. R. OMNÈS in *Dispersion Relations and Elementary Particles* (Paris 1960).
- <sup>2)</sup> H. ARAKI, *Einführung in die axiomatische Quantenfeldtheorie II* (ETH, Zürich 1961/62).
- <sup>3)</sup> H. ARAKI, K. HEPP and D. RUELE, *Helv. Phys. Acta* **35**, 164 (1962).
- <sup>4)</sup> N. N. BOGOLIUBOV and D. V. SHIRKOV, *Introduction to the Theory of Quantized Fields* (Interscience, New York 1959); N. N. BOGOLIUBOV, B. V. MEDVEDEV, and M. K. POLIVANOV, *Voprossy teorii dispersionnykh sootnoshenii* (Moscow 1958).
- <sup>5)</sup> F. J. DYSON, *Phys. Rev.* **110**, 1460 (1958).
- <sup>6)</sup> M. FROISSART, *XXIXth Course of the International School of Physics 'Enrico Fermi'* (Varenna 1963).
- <sup>7)</sup> L. GÅRDING and J. L. LIONS, *Suppl. Nuovo Cim.* **14**, 9 (1959).
- <sup>8)</sup> R. HAAG, *Phys. Rev.* **112**, 669 (1958).
- <sup>9)</sup> K. HEPP, *Acta Physica Austriaca* **17**, 85 (1963), *Helv. Phys. Acta* **37**, 659 (1964).
- <sup>10)</sup> K. HEPP, *Math. Ann.* **152**, 149 (1963).
- <sup>11)</sup> R. JOST, *General Theory of Quantized Fields*, American Math. Soc. Publications (1964).
- <sup>12)</sup> R. JOST, H. LEHMANN, *Nuovo Cimento* **5**, 1598 (1957).
- <sup>13)</sup> H. LEHMANN, *Nuovo Cim.* **10**, 579 (1958).
- <sup>14)</sup> H. LEHMANN, *XXIXth Course of the International School of Physics 'Enrico Fermi'* (Varenna 1963), S. MANDELSTAM, *Nuovo Cim.* **15**, 658 (1960).
- <sup>15)</sup> H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN, *Nuovo Cimento* **1**, 205 (1955); **6**, 319 (1957); V. GLASER, H. LEHMANN and W. ZIMMERMANN, *Nuovo Cimento* **6**, 1122 (1957).
- <sup>16)</sup> D. RUELE, *Helv. Phys. Acta* **35**, 147 (1962).
- <sup>17)</sup> L. SCHWARTZ, *Théorie des distributions*, I, II (Paris 1957/59).
- <sup>18)</sup> L. SCHWARTZ, *Séminaire 1954/55*, Exposé no. 7.
- <sup>19)</sup> O. STEINMANN, *Helv. Phys. Acta* **36**, 90 (1963).
- <sup>20)</sup> V. S. VLADIMIROV, 'Methods of Function Theory of Several Complex Variables' (Moscow 1964).
- <sup>21)</sup> A. S. WIGHTMAN, *Phys. Rev.* **101**, 860 (1956).
- <sup>22)</sup> A. S. WIGHTMAN in *Dispersion Relations and Elementary Particles* (Paris 1960).