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Representations of Canonical Anticommutation Relations*)

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Abstract. Cyclic Representations of canonical anticommutation relations (CARs) with charge conservation are studied. The algebra of zero charge polynomials of canonical variables (called the Q-algebra) is algebraically characterized by simple properties. It is proved that any cyclic representation of the Q-algebra is uniquely extendable to a cyclic representation of CARs with charge conservation. A cyclic representation of the Q-algebra is characterized by a certain functional E, satisfying a positivity condition and a condition related to the Fermi statistics. The functional E for the grand canonical ensemble of the free Fermi gas in an infinite volume is computed and the corresponding representation of CARs is analyzed.

§ 1. Introduction

Representations of canonical commutation relations (CCRs) have been studied by many authors and proved to be useful in the study of von Neumann algebras for the infinite free Bose gas[1]). Representations of the canonical anticommutation relations (CARs) for a finite system are completely analyzed by Jordan and Wigner[2]). Those for an infinite system have been studied by some authors[3]) but their analyses are not necessarily convenient for the discussion of the infinite Fermi gas. The purpose of the present paper is to develop a formulation for representations of canonical anticommutation relations, which can easily be applied to the infinite free Fermi gas.

We mainly consider the representations of CARs, where a total charge N can be defined as a selfadjoint operator having integer eigenvalues, such that the canonical field operators either increase or decrease N by 1 and such that there is a cyclic vector belonging to the eigenvalue 0 of N. Such representations of CARs will be called representations of CARs with charge conservation.

The central role in our formulation will be played by the Q-algebra, defined as the algebra of those polynomials of the canonical field variables which commute with N. This may be considered as the algebra of observables. We can characterize the Q-algebra algebraically by the commutation relations of the Lie algebra of finite rank

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operators K on a certain Hilbert space (the test function space) and by an additional relation directly related to the Fermi statistics. The most important property of the Q-algebra is that any of its cyclic representations is uniquely extendable to a cyclic representation of CARs with charge conservation.

A cyclic representation of the Q-algebra is characterized by a functional $E(e^K)$ satisfying a positivity condition and a condition related to the Fermi statistics. We compute the functional $E(e^K)$ for the grand cononical ensemble of the free Fermi gas in an infinite volume. The representation of CARs for the infinite free Fermi gas allows a particle-hole interpretation and has a great similarity to the representation of CCRs for the infinite free Bose gas.

In section 2, we define the representations of CARs and prove that they are always represented by bounded operators. In section 3, we define abstractly the algebra generated by the canonical variables and call it the CAR-algebra. In section 4, we prove properties of the zero charge part of the CAR-algebra and, using these properties, we define the Q-algebra abstractly in section 5. In section 6, we prove that the Q-algebra defined in section 5 is isomorphic to the zero charge part of the CAR-algebra. In section 7, we obtain all representations of the Q-algebra for a finite system and show that they are all given by the restriction of representations of the CAR-algebra. In section 8, using results in previous sections, we prove the main theorem that any cyclic representation of the Q-algebra is uniquely extendable to a cyclic representation of the CAR-algebra with charge conservation. In section 9, we introduce an auxiliary operator $U(e^K)$ and in section 10, we consider the functional $E(e^K)$ which is the expectation value of $U(e^K)$ for a cyclic vector. We prove that a few simple properties of $E(e^K)$ are equivalent to the existence of the corresponding unique cyclic representation of the Q-algebra. In section 11, we study a few simple examples of $E(e^K)$. Finally in section 12, we compute the $E(e^K)$ for the grand canonical ensemble of free Fermi particles without spin in an infinite volume. This turns out to be the example analyzed in section 11.

§ 2. Canonical Anticommutation Relations

We first give the definition of canonical anticommutation relations in a form, which is in line with Wightman's axioms for quantum field theory.

Definition 1: Let $\mathfrak{K}$ be a (not necessarily complete) complex vector space with a positive definite inner product. A representation of CARs over $\mathfrak{K}$ is the set of a pair of linear operators $(f\,\psi)$ and $(\psi^+ f)$, f in $\mathfrak{K}$, satisfying the following:

(1) $(f\,\psi)$ and $(\psi^+ f)$ are defined on a dense domain D in a Hilbert space $\mathfrak{H}$ and $(f\,\psi)\, D \subset D$ and $(\psi^+ f)\, D \subset D$.

(2) $(f\,\psi)$ is antilinear in f and $(\psi^+ f)$ is linear in f.

(3) $(f\,\psi)^* \supset (\psi^+ f)$ *).

(4) For any $\Phi \in D$, $f \in \mathfrak{K}$, $g \in \mathfrak{K}$,

$$\{(f\,\psi), (g\,\psi)\}_+ \Phi = \{(\psi^+ f), (\psi^+ g)\}_+ \Phi = 0 \tag{2.1}$$

*) A^* is the adjoint of A. $A \supset B$ means $D(A) \supset D(B)$ and $A\,\Phi = B\,\Phi$ for $\Phi \in D(B)$ where $D(A)$ and $D(B)$ are domains of A and B.

$$\{(f\,\psi),\,(\psi^+\,g)\}_+\,\Phi = (f,\,g)\,\Phi \tag{2.2}$$

where $\{A,\,B\}_+ = AB + BA$ and $(f,\,g)$ is the inner product*) of f and g.

The following theorem enables us to always take $D = \mathfrak{H}$ and the equality in (3).

Theorem 1. In any representation of CARs, $(f\,\psi)$ and $(\psi^+ f)$, for any f in $\mathfrak{K}$, are bounded with the norm

$$||\,(f\,\psi)\,|| = ||\,(\psi^+\,f)\,|| = ||\,f\,|| \tag{2.3}$$

and the mapping $f \to (f\,\psi)$ and $f \to (\psi^+ f)$ can be uniquely extended to norm preserving antilinear and linear mappings from the completion $\overline{\mathfrak{K}}$ of $\mathfrak{K}$ into $B(\mathfrak{H})$ (the set of all bounded linear operators on $\mathfrak{H}$).

Proof. For any $f \in \mathfrak{K}$, $||\,f\,|| \neq 0$, we define

$$n(f) = (f,\,f)^{-1}\,(\psi^+\,f)\,(f\,\psi) \tag{2.4}$$

By (1), $n(f)$ is defined on D and by (3) and (4)

$$n(f)^2 = n(f),\quad n(f)^* \supset n(f) \tag{2.5}$$

Hence, for any $\Phi \in D$, we have

$$||\,n(f)\,\Phi\,||^2 = (\Phi,\,n(f)\,\Phi) \leqq ||\,\Phi\,||\;||\,n(f)\,\Phi\,||$$

$$\text{i.e. } ||\,n(f)\,\Phi\,|| \leqq ||\,\Phi\,||$$

From this we have

$$||\,(f\,\psi)\,\Phi\,||^2 = ||\,f\,||^2\,(\Phi,\,n(f)\,\Phi) \leq ||\,f\,||^2\,||\,\Phi\,||^2$$

Therefore $(f\,\psi)$ is bounded on D and can be extended to a bounded operator on the whole $\mathfrak{H}$. By (3), $(\psi^+ f)$ is also bounded with the same bound as $(f\,\psi)$. From (2.5), $||\,n(f)\,|| = 1$ unless $n(f) = 0$. Hence, $||\,(f\,\psi)\,|| = ||\,f\,||$ unless $(f\,\psi) = 0$. However, $(f\,\psi) = 0$ contradicts (2.2) if $f \neq 0$. Hence, we have (2.3) for $f \neq 0$. On the other hand, (2.2) implies $(f\psi) = 0$ if $f = 0$. Therefore we have (2.3) for any f. The rest of the theorem is obvious.

Remark. In an application, $\mathfrak{K} = D \oplus \ldots \oplus D$ or $\overline{\mathfrak{K}} = L_2 \oplus \ldots \oplus L_2$, where $D = D\,(R^3)$ is the set of all infinitely continuously differentiable complex-valued functions of three real variables with compact supports, equipped with the inner product of L_2. The number of D in the direct sum may be taken to be 1 for the non-relativistic Fermi particle without spin, and 4 for the Dirac particle. Such structure of $\mathfrak{K}$ is relevant when one introduces a unitary representation of the Euclidian or the Lorentz group.

§ 3. CAR-algebra

The representation of CAR can be viewed as a *-representation of a certain *-algebra, which is constructed in the following way.

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*) We use the physicist's convention that $(f,\,g)$ is linear in g and antilinear in f.

The notation for the dual. If $\mathfrak{K}$ is as in the definition 1, $\bar{\mathfrak{K}}$ is a Hilbert space. For any $f \in \bar{\mathfrak{K}}$, we define f^* as the mapping from any $g \in \bar{\mathfrak{K}}$ to the complex number (f, g). (It is the adjoint of the mapping from a complex number c to the element c.f of $\bar{\mathfrak{K}}$.) We define $\mathfrak{K}^*$ as the set of all f^* with $f \in \mathfrak{K}$.

Definition 2. The free (non-commutative) *-algebra $P(\mathfrak{K}, \mathfrak{K}^*)$ over $\mathfrak{K}$ is a free complex vector space over monomials $[h_1] \ldots [h_n]$, $h_j \in \mathfrak{K} \cup \mathfrak{K}^*$ and $\mathbf{1}$, which is equipped with the vector space addition, the multiplication defined by

$$a^1 a^2 = \sum_{\lambda\mu} c^1_\lambda c^2_\mu [h^1_{1\lambda}] \ldots [h^1_{n_1(\lambda)\lambda}] [h^2_{1\mu}] \ldots [h^2_{n_2(\mu)\mu}] \tag{3.1}$$

and the involution defined by

$$(a^1)^* = \sum_\lambda c^{1*}_\lambda [h^{1\;\;*}_{n_1(\lambda)\lambda}] \ldots [h^{1*}_{1\lambda}] \tag{3.2}$$

where

$$a^l = \sum_\lambda c^l_\lambda [h^l_{1\lambda}] \ldots [h^l_{n_l(\lambda)\lambda}],\ h^l_{j\lambda} \in \mathfrak{K} \cup \mathfrak{K}^*,\ h^* = f^* \text{ if } h = f \in \mathfrak{K} \text{ and } h^* = f \text{ if } h = f^* \in \mathfrak{K}^*.$$

In our notation, $[h_1] \ldots [h_n]$ with $n = 0$ will always mean the element $\mathbf{1}$. The monomial $[h_1] \ldots [h_n]$ will also be written as $[h_1, \ldots, h_n]$. ($\mathbf{1}^* = \mathbf{1}$.)

Definition 3. Let I_C be the two-sided *-ideal of $P(\mathfrak{K}, \mathfrak{K}^*)$ generated by

$$c[f] + d[g] - [cf + dg],\ c[f^*] + d[g^*] - [cf^* + dg^*] \tag{3.3}$$

$$\{[f], [g]\}_+,\ \{[f^*], [g^*]\}_+,\ \{[f], [g^*]\}_+ - (g, f)\ \mathbf{1} \tag{3.4}$$

where f and g are arbitrary elements of $\mathfrak{K}$ and $c f^* = (c^* f)^*$. The CAR-algebra $\mathfrak{A}_C(\mathfrak{K})$ over $\mathfrak{K}$ is defined by

$$\mathfrak{A}_C(\mathfrak{K}) = \mathrm{P}(\mathfrak{K}, \mathfrak{K}^*)/\mathrm{I}_C \tag{3.5}$$

Lemma 1. For any representation $(f\psi)$ and $(\psi^+ f)$ of CARs, there exists a *-representation ψ of the CAR-algebra in $B(\mathfrak{H})$ defined by

$$\psi\Big(\sum_\lambda c_\lambda [h_{1\lambda}] \ldots [h_{n(\lambda)\lambda}]\Big) = \sum_\lambda c_\lambda \prod_{j=1}^{n(\lambda)} \psi([h_{j\lambda}]) \tag{3.6}$$

where $\psi([h]) = (\psi^+ f)$ if $h = f \in \mathfrak{K}$, $\psi([h]) = (f\psi)$ if $h = f^* \in \mathfrak{K}$. Conversely, any *-representation ψ of the CAR-algebra gives a representation of CARs through $(f\psi) = \psi([f^*])$ and $(\psi^+ f) = \psi([f])$.

Proof. Let ψ be defined by (3.6) for all elements of $P(\mathfrak{K}, \mathfrak{K}^*)$. Partly due to (3) of the definition 1, ψ is obviously a *-representation of $P(\mathfrak{K}, \mathfrak{K}^*)$. Because of (2) and (4) of the definition 1, the generating elements (3.3) and (3.4) of I_C are mapped to 0. Hence ψ gives a *-representation of $\mathfrak{A}_C(\mathfrak{K})$. The converse is also obvious, because the relations given by (3.3) and (3.4) are just (2) and (4) of the definition 1.

Definition 4. If $\Phi \in \mathfrak{H}$ has the property that $\{\psi(a)\,\Phi;\ a \in \mathfrak{A}_C(\mathfrak{K})\}$ is dense in $\mathfrak{H}$, Φ is called a cyclic vector of $\psi(\mathfrak{A}_C(\mathfrak{K}))$ and the representation ψ is called a cyclic representation.

For a cyclic representation, we can apply GELFAND's method to a cyclic vector Φ and the expectation functional

$$E(a) = (\Phi, \psi(a)\,\Phi), \quad a \in \mathfrak{A}_C(\mathfrak{K}) \tag{3.7}$$

Lemma 2. The functional $E(a)$ of (3.7) is linear in a and $E(a^*\,a) \geqq 0$ for any $a \in \mathfrak{A}_C(\mathfrak{K})$. Conversely, any functional $E(a)$, having these properties (linearity and positivity), can be written in the form (3.7), where the cyclic $*$-representation ψ of the CAR-algebra is unique up to a unitary equivalence. ($E(a^*) = E(a)^*$ follows from the requirement that $E(a^*\,a)$ is real.)

Remark. If $\mathfrak{K}$ is finite dimensional, it is known[2]) that any $*$-representation of CARs is fully reducible and the irreducible representation of CARs is unique up to a unitary equivalence. It is easy to prove (by a similar argument as the proof of the theorem 5) that $\mathfrak{A}_C(\mathfrak{K})$ is faithfully represented in the unique irreducible representation of CARs for the finite dimensional $\mathfrak{K}$ (and hence is isomorphic to the algebra of all $2^n \times 2^n$ matrices where $n = \dim \mathfrak{K}$). Since any element a of $\mathfrak{A}_C(\mathfrak{K})$ for an infinite dimensional $\mathfrak{K}$ can be considered as an element of $\mathfrak{A}_C(\mathfrak{K}_1)$ for some finite dimensional subspace $\mathfrak{K}_1$ of $\mathfrak{K}$, any $*$-representation ψ of $\mathfrak{A}_C(\mathfrak{K})$ is faithful and $\psi(a)$ has a unique operator norm. We can equip $\mathfrak{A}_C(\mathfrak{K})$ with this norm and obtain a C^*-algebra $\overline{\mathfrak{A}_C(\mathfrak{K})}$ by the completion of $\mathfrak{A}_C(\mathfrak{K})$. $\overline{\mathfrak{A}_C(\mathfrak{K})}$ is obviously simple, because any of its representation is faithful. We will, however, not use the concept of C^*-algebra in the following.

§ 4. The operator Q_ψ (K)

Definition 5. Let $F(\mathfrak{K}) = \mathfrak{K} \otimes \mathfrak{K}^*$ be the set of all finite rank operators of the form $K = \sum_{i=1}^{n} f_i \otimes g_i^*$ where $f_i, g_i \in \mathfrak{K}$ and $K f = \sum_{i=1}^{n} f_i\,(g_i, f)$ for any $f \in \bar{\mathfrak{K}}$. We define for any such K the following operator.

$$Q_\psi(K) = \psi\,(\sum_{i=1}^{n} [f_i]\,[g_i^*]) = \sum_{i=1}^{n} (\psi^+\,f_i)\,(g_i\,\psi) \tag{4.1}$$

where ψ is any representation of the CAR-algebra.

Since (3.3) is 0 in $\mathfrak{A}_C(\mathfrak{K})$, $Q_\psi(K)$ does not depend on any particular representation of K as $\Sigma f_i \otimes g_i^*$.

Theorem 2. For $Q = Q_\psi$, the following holds:

$$Q\,(c_1\,K_1 + c_2\,K_2) = c_1\,Q(K_1) + c_2\,Q(K_2) \tag{4.2}$$

$$Q(K^*) = Q(K)^* \tag{4.3}$$

$$[Q(K_1), Q(K_2)]_- = Q([K_1, K_2]_-) \tag{4.4}$$

$$Q\,(f \otimes f^*)^2 = (f, f)\;Q(f \otimes f^*) \tag{4.5}$$

where $[A, B]_- = AB - BA$.

The proof is obvious from (2.1), (2.2), the linearity of ψ and the identity $[ab, cd]_- = a\,\{b, c\}_+\,d - \{a, c\}_+\,b\,d + c\,a\,\{b, d\}_+ - c\,\{a, d\}_+\,b$.

Remark. As we shall see later, the properties (4.2) $\sim$ (4.5) completely characterize the mapping Q_ψ. On the other hand, a mapping Q satisfying (4.2) $\sim$ (4.4) without (4.5) can be constructed in the following way. Let ϕ and π be a representation of CCRs. Define

$$\psi(f) = (\phi(f) + i\,\pi(f))\,/\sqrt{2} \text{ and } Q_\psi(K) = \sum_{i=1}^{n} \psi(\bar{f}_i)^*\,\psi(\bar{g}_i) \text{ for } K = \sum_{i=1}^{n} f_i \otimes g_i^* .$$

Then Q_ψ so defined satisfies (4.2) $\sim$ (4.4) but not (4.5).

If $\mathfrak{K}$ is finite dimensional, any Q satisfying (4.2) $\sim$ (4.4) gives a representation of the Lie algebra of all $n \times n$ matrices where $n = \dim \mathfrak{K}$. Such $Q(K)$ can be obtained as the infinitesimal generator of *a* *-representation of the full linear group, or, if we restrict K to antihermitian matrix, as the infinitesimal generator of a unitary representation of unitary group. These representations are fully known for a finite dimensional $\mathfrak{K}$. We will see in section 7 that (4.5) restricts the representation to totally antisymmetric ones in the sense of Young tableau. The Q_ψ constructed from CCRs gives totally symmetric ones.

Theorem 3. The mapping Q_ψ of the definition 5 has the property

$$\| Q_\psi(K) \| \leqq \operatorname{tr}(K^* K)^{1/2} \tag{4.6}$$

and can be extended uniquely to a continuous mapping from the Banach space of trace class operators on $\bar{\mathfrak{K}}$ with the trace class norm ($\| K \|_{tr} = \operatorname{tr}(K^* K)^{1/2}$) into the Banach space $B(\mathfrak{H})$ with the operator norm.

Proof. Any $K \in F(\mathfrak{K})$ can be written as

$$K = \sum_{i,j=1}^{n} \hat{K}_{ij}\, f_i \otimes f_j^*$$

with a suitable orthonormal set $\{f_i\}$.

$$\text{Let } \Phi,\ \Psi \in \mathfrak{H},\ \| \Phi \| = \| \Psi \| = 1 \text{ and } M_{ij} = (\Phi,\ Q_\psi(f_j \otimes f_i^*)\ \Psi)\,.$$

Considering M as a $n \times n$ matrix, we have

$$(c, Md) = (\Phi,\ Q_\psi\,(g \otimes h^*)\ \Psi) \text{ with } h = \sum_{i=1}^{n} c_i f_i \text{ and } g = \sum_{i=1}^{n} d_i f_i\,.$$

By the theorem 1,

$$\| Q_\psi\,(g \otimes h^*) \| \leqq \| g \| \cdot \| h \| = \| d \| \cdot \| c \|\,.$$

Hence, $\| M \| = \sup \mid (c, Md) \mid \cdot \| c \|^{-1}\, \| d \|^{-1} \leqq 1$.

$$\text{Therefore } \| Q_\psi\,(K) \| \leqq \sup_{\| M \| \leqq 1} \mid \operatorname{tr}(\hat{K}, M) \mid = \operatorname{tr}(K^* K)^{1/2} = \| K \|_{tr}\,.$$

Because the set of finite rank operators is dense in the set of trace class operators relative to the trace class norm, the rest of the theorem is then obvious.

Remark. If K is hermitian, i.e. if

$$K = \sum_{i=1}^{n} \lambda_i f_i \otimes f_i^*$$

with some orthonormal set $\{f_i\}$ and real eigenvalues λ_i, then

$$|| Q_\psi(K) || = \sup_{0 \leqq M \leqq 1} | \mathrm{tr}(K\, M) | = \max \Big(\sum_{\lambda_i > 0} \lambda_i, -\sum_{\lambda_j < 0} \lambda_j\Big) . \tag{4.7}$$

§ 5. Q-algebra

In this section we consider the algebra of operators $Q(K)$ satisfying (4.2) $\sim$ (4.5).

Lemma 3. Let Q be a mapping from $F(\mathfrak{K})$ into the set of linear operators on $\mathfrak{H}$ such that the domains of $Q(K)$ and $Q(K)^*$, for any K, contain a common dense set D with the property $Q(K)\, D \subset D$, $Q(K)^*\, D \subset D$ and such that (4.2) $\sim$ (4.5) are satistied on D. Then $Q(K)$ is a bounded operator.

Proof. Due to the proof of the theorem 1, (4.5) and (4.3) imply that $Q(f \otimes f^*)$ is bounded. Since $f \otimes g^*$ can be expressed as a linear combination of $(f + \lambda\, g) \otimes (f + \lambda\, g)^*$ with $\lambda = \pm 1, \pm i$, $Q(f \otimes g^*)$ is also bounded due to (4.2). Hence any $Q(K)$ is bounded again due to (4.2).

The algebra of operators generated by $Q(K)$ satisfying (4.2) $\sim$ (4.5) can be viewed as a representation of a certain *-algebra, which is constructed in the following way.

Definition 6. The free (non-commutative) algebra $P(F(\mathfrak{K}))$ over $F(\mathfrak{K})$ is the free complex vector space over monomials $[K_1] \ldots [K_n]$, $K_j \in F(\mathfrak{K})$ and $\mathbf{1}$ ($[K_1] \ldots [K_n]$ with $n = 0$ will mean $\mathbf{1}$), equipped with the vector space addition, the multiplication defined by (3.1) and the involution defined by (3.2) where $h^l_{j\lambda}$ is now taken to be an element of $F(\mathfrak{K})$ and * on them is taken to be the hermitian conjugation of $F(\mathfrak{K})$. We sometimes denote the monomial $[K_1] \ldots [K_n]$ as $[K_1, \ldots, K_n]$.

Definition 7. Let I_Q be the two sided *-ideal of $P(F(\mathfrak{K}))$ generated by

$$c_1[K_1] + c_2[K_2] - [c_1\, K_1 + C_2\, K_2] \tag{5.1}$$

$$[[K_1], [K_2]]_- - [[K_1, K_2]_-] \tag{5.2}$$

$$[(f \otimes f^*)]^2 - (f, f)\, [f \otimes f^*] \tag{5.3}$$

where $K_1, K_2 \in F(\mathfrak{K})$ and $f \in \mathfrak{K}$. The Q-algebra $\mathfrak{A}_Q(\mathfrak{K})$ over $\mathfrak{K}$ is defined by

$$\mathfrak{A}_Q(\mathfrak{K}) = \mathrm{P}\,(\mathrm{F}(\mathfrak{K}))/\mathrm{I}_Q \tag{5.4}$$

Lemma 4. If Q is a mapping from $F(\mathfrak{K})$ into $B(\mathfrak{H})$ satisfying (4.2) $\sim$ (4.5), then there exists a *-representation of the Q-algebra defined by

$$Q\Big(\sum_{\lambda=1}^{m} c_\lambda [K_{1\lambda}] \ldots [K_{n(\lambda)\lambda}]\Big) = \sum_{\lambda=1}^{m} c_\lambda\, Q(K_{1\lambda}) \ldots Q(K_{n(\lambda)\lambda}) .$$

(If $n = 0$, $Q(K_1) \ldots Q(K_n)$ is to be understood as $\mathbf{1}$.) Conversely, any *-representation of the Q-algebra on a Hilbert space $\mathfrak{H}$ gives a mapping from $F(\mathfrak{K})$ into $B(\mathfrak{H})$ satisfying (4.2) $\sim$ (4.5).

The proof is similar to that of the lemma 1.

Lemma 5. If Q gives a *-representation of the Q-algebra and Ψ is a cyclic vector of $Q(\mathfrak{A}_Q(\mathfrak{K}))$, then the expectation functional

$$E(a) = (\Psi, Q(a)\, \Psi), \quad a \in \mathfrak{A}_Q(\mathfrak{K}) \tag{5.5}$$

is linear in a and $E(a^* a) \geqq 0$ for any $a \in \mathfrak{A}_Q(\mathfrak{K})$. Conversely, any linear non-negative functional $E(a)$, $a \in \mathfrak{A}_Q(\mathfrak{K})$ can be written in the form (5.5) where the cyclic $*$-representation Q of $\mathfrak{A}_Q(\mathfrak{K})$ is unique up to a unitary equivalence.

For the proof, see the ref. 4. The boundedness of the operator $Q(a)$ follows from the lemma 3.

We now prove a few formulas for the Q-algebra, which will be used in the next section.

Theorem 4. In the Q-algebra the following holds.

$$[f \otimes g^*]^2 = [f \otimes g^*]\,(g, f) \tag{5.6}$$

$$[f_1 \otimes g_1^*]\,[f_2 \otimes g_2^*] + [f_1 \otimes g_2^*]\,[f_2 \otimes g_1^*] = [f_1 \otimes g_1^*]\,(g_2, f_2) + [f_1 \otimes g_2^*]\,(g_1, f_2) \tag{5.7}$$

Proof. From the relation that (5.3) is zero, we obtain for arbitrary λ,

$$[(f + \lambda g) \otimes (f + \lambda g)^*]^2 = (f + \lambda g, f + \lambda g)\,[(f + \lambda g) \otimes (f + \lambda g)^*]$$

Using the relation that (5.1) is zero, we develop both sides of this equation into a polynomial of λ and λ^*. Comparing the coefficients of $(\lambda^*)^2$, we obtain (5.6). From (5.6), we have, for arbitrary complex λ and μ,

$$[(f_1 + \lambda f_2) \otimes (g_1 + \mu g_2)^*]^2 = (g_1 + \mu g_2, f_1 + \lambda f_2)\,[(f_1 + \lambda f_2) \otimes (g_1 + \mu g_2)^*]\,.$$

Comparing the coefficients of $\lambda \mu^*$, we have

$$\{[f_1 \otimes g_1^*], [f_2 \otimes g_2^*]\}_+ + \{[f_1 \otimes g_2^*], [f_2 \otimes g_1^*]\}_+$$
$$= [\{f_1 \otimes g_1^*, f_2 \otimes g_2^*\}_+] + [\{f_1 \otimes g_2^*, f_2 \otimes g_1^*\}_+]$$

Using the relation that (5.2) is zero, we obtain (5.7).

Corollary

$$[f_1 \otimes g^*]\,[f_2 \otimes g^*] = [f_1 \otimes g^*]\,(g, f_2) \tag{5.8}$$

$$[f \otimes g_1^*]\,[f \otimes g_2^*] = [f \otimes g_2^*]\,(g_1, f) \tag{5.9}$$

§ 6. The isomorphism of the Q-algebra and the zero charge part of the CAR-algebra

Charge quantum number. The algebra $P(\mathfrak{K}, \mathfrak{K}^*)$ can be split into a direct sum (as a vector space) of subsets according to the 'charge quantum number'. Namely,

$$\mathrm{P}(\mathfrak{K}, \mathfrak{K}^*) = \sum_{n=-\infty}^{+\infty} \mathrm{P}(\mathfrak{K}, \mathfrak{K}^*)_n \tag{6.1}$$

where $P(\mathfrak{K}, \mathfrak{K}^*)_n$ is the linear subset of $P(\mathfrak{K}, \mathfrak{K}^*)$, generated by all $[h_1, \dots, h_N]$ such that r of h_i are in $\mathfrak{K}$, s of h_i are in $\mathfrak{K}^*$ and $r + s = N$, $r - s = n$. We also have the following formulas,

$$\mathrm{P}(\mathfrak{K}, \mathfrak{K}^*)_n\, \mathrm{P}(\mathfrak{K}, \mathfrak{K}^*)_m = \mathrm{P}(\mathfrak{K}, \mathfrak{K}^*)_{m+n} \tag{6.2}$$

$$\mathrm{P}(\mathfrak{K}, \mathfrak{K}^*)_n^* = \mathrm{P}(\mathfrak{K}, \mathfrak{K}^*)_{-n} \tag{6.3}$$

Since each of the generating elements (3.3) and (3.4) of I_C is in one of the subsets $P(\mathfrak{K}, \mathfrak{K}^*)_n$, $n = 0, \pm 1, \pm 2$, the two sided ideal I_C has the direct sum decomposition

$$\mathrm{I}_C = \sum_{n=-\infty}^{+\infty} \mathrm{I}_{Cn} \tag{6.4}$$

where $I_{Cn} \subset P(\mathfrak{K}, \mathfrak{K}^*)_n$. Therefore, we have

$$\mathfrak{A}_C(\mathfrak{K}) = \sum_{n=-\infty}^{+\infty} \mathfrak{A}_C(\mathfrak{K})_n, \quad \mathfrak{A}_C(\mathfrak{K})_n = \mathrm{P}(\mathfrak{K}, \mathfrak{K}^*)_n / \mathrm{I}_{Cn} \tag{6.5}$$

From (6.2), we have

$$\mathfrak{A}_C(\mathfrak{K})_n \cdot \mathfrak{A}_C(\mathfrak{K})_m = \mathfrak{A}_C(\mathfrak{K})_{m+n} \tag{6.6}$$

$$\mathfrak{A}_C(\mathfrak{K})_n{}^* = \mathfrak{A}_C(\mathfrak{K})_{-n} \tag{6.7}$$

Therefore $\mathfrak{A}_C(\mathfrak{K})_0$ is a *-subalgebra of $\mathfrak{A}_C(\mathfrak{K})$ and will be called the zero charge part of $\mathfrak{A}_C(\mathfrak{K})$.

We now consider the mapping j from $P(F(\mathfrak{K}))$ into $\mathfrak{A}_C(\mathfrak{K})$, defined by

$$j\left(\sum_{\lambda=1}^{N} c_\lambda \left[K_{1\lambda}, \ldots, K_{n(\lambda)\lambda}\right]\right) = \sum_{\lambda=1}^{N} \sum_{j_l \ldots j_{n(\lambda)}} c_\lambda \left[f_{1\lambda}^{j_1}, g_{1\lambda}^{j_1 *}, \ldots, f_{n(\lambda)\lambda}^{j_{n(\lambda)}}, g_{n(\lambda)\lambda}^{j_{n(\lambda)} *}\right] \tag{6.8}$$

where $K_{l\lambda} = \sum_{j_l} f_{l\lambda}^{j_l} \otimes g_{l\lambda}^{j_l} {}^*$. Because of the relation that (3.3) is zero, j is well defined (i.e. does not depend on the various ways of expressing $K_{l\lambda}$ as a sum of tensor products of f's and g's). By the same calculation as in the theorem 2, we see that $j\, I_Q \subset I_C$. Hence j induces a *-homomorphism of $\mathfrak{A}_Q(\mathfrak{K})$ into $\mathfrak{A}_C(\mathfrak{K})$, which will be denoted by the same letter j. We now prove

Theorem 5. The mapping j defined by (6.8) is a *-isomorphism of $\mathfrak{A}_Q(\mathfrak{K})$ into $\mathfrak{A}_C(\mathfrak{K})_0$ (the zero charge part of $\mathfrak{A}_C(\mathfrak{K})$).

Proof. We already know that j is a *-homomorphism of $\mathfrak{A}_Q(\mathfrak{K})$ into $\mathfrak{A}_C(\mathfrak{K})_0$. We have to prove that it is onto $\mathfrak{A}_C(\mathfrak{K})_0$ and that $j\, a = 0$ implies $a = 0$.

Any element of $\mathfrak{A}_C(\mathfrak{K})$ or $\mathfrak{A}_Q(\mathfrak{K})$ can be considered as an element of $\mathfrak{A}_C(\mathfrak{K}_1)$ or $\mathfrak{A}_Q(\mathfrak{K}_1)$ for a finite dimensional subspace $\mathfrak{K}_1$ of $\mathfrak{K}$. Hence, if we prove for a finite dimensional $\mathfrak{K}$ that j is onto $\mathfrak{A}_C(\mathfrak{K})_0$ and that $j\, a = 0$ implies $a = 0$, we have the same for an arbitrary $\mathfrak{K}$.

Let $f_1 \ldots f_n$ be an orthonormal basis of a finite dimensional $\mathfrak{K}$ and let $a_i = [f_i]$ and $q_{ij} = [f_i \otimes f_j^*]$. Due to the anticommutation relation that (3.4) is zero, any monomial of $\mathfrak{A}_C(\mathfrak{K})$ is equivalent to a monomial with at most one a_i and one a_i^* for each i. Again using the anticommutation relation, we see that $\mathfrak{A}_C(\mathfrak{K})_0$ is linearly spanned by elements

$$a_{i_1} \cdot a_{j_1}^* \ldots a_{i_m} \cdot a_{j_m}^* \cdot a_{k_1} \cdot a_{k_1}^* \ldots a_{k_n} \cdot a_{k_n}^* \tag{6.9}$$

where all i, j, k are distinct and we impose the restriction

$$i_1 < \ldots < i_m, \; j_1 < \ldots < j_m, \; k_1 < \ldots < k_n \, . \tag{6.10}$$

Hence j is onto $\mathfrak{A}_C(\mathfrak{K})_0$.

The element (6.9) can be characterized by two index sets

$$I = \{i_1 \dots i_m \, k_1 \dots k_n\}, \quad J = \{j_1 \dots j_m \, k_1 \dots k_n \}$$

because I and J determines

$$\{k_1 \dots k_n\} = I \cap J, \ \{i_1 \dots i_m\} = I - (I \cap J), \ \{j_1 \dots j_m\} = J - (I \cap J)$$

and (6.10) will determine individual i_ν, j_ν and k_ν. We denote (6.9) by $a(I, J)$. We now prove that $a(I, J)$ are linearly independent. Let

$$\sum c(I, J) \, a(I, J) = 0 \tag{6.11}$$

Let us consider the partial ordering of the pair (I, J) defined by $(I_1, J_1) \subset (I_2, J_2)$ if $I_1 \subset I_2$ and $J_1 \subset J_2$. ($\subset$ and $\supset$ together implies $=$.) We prove that (6.11) implies $c(I, J) = 0$ for those (I, J) which are maximal in this ordering among (I, J) appearing in (6.11). This is, of course, sufficient to prove $c(I, J) = 0$ for all (I, J). If we define an operator $L_\pm(b)$ acting on $\mathfrak{A}_C(\mathfrak{K})$ for any element b of $\mathfrak{A}_C(\mathfrak{K})$ by

$$L_-(b) \, a = [b, a]_- \quad L_+(b) \, a = \{b, a\}_+$$

then we have, for a maximal (I, J)

$$\prod_\mu L_+(a_{k_\mu}) \, L_-(a^*_{k_\mu}) \prod_\nu L_+(a_{j_\nu}) \, L_-(a^*_{i\nu}) \, a = c(I, J) \, \mathbf{1}$$

for $a = \Sigma \, c(I, J) \, a(I, J)$. Hence, $a(I, J)$ are linearly independent.

Finally we prove that $\mathfrak{A}_Q(\mathfrak{K})$ is linearly spanned by

$$q(I, J) = q_{i_1 j_1}, \dots q_{i_m j_m}, q_{k_1 k_1}, \dots, q_{k_n k_n} \tag{6.12}$$

where i, j, k are restricted by (6.10) and I, J are defined as before. Since $j \, q(I, J) = a(I, J)$ and since $a(I, J)$ are linearly independent, this will guarantee that $j \, a = 0$ always implies $a = 0$.

Consider a monomial $q(\{\mu\}, \{\nu\}) = q_{\mu_1 \nu_1} \dots q_{\mu_p \nu_p}$. If some of μ's or ν's are the same, then we permute q's, using the commutation relations that (4.2) is zero, so that the q's with the same μ's or ν's come next to each other and subsequently use (5.8) or (5.9) so that we obtain a polynomial of lower order. Furthermore, if r and s are permutation of $1 \dots p$, with signatures $\varepsilon(r)$ and $\varepsilon(s)$, then, due to the theorem 4, $q(\{\mu\}, \{\nu\})$ and $\varepsilon(r) \cdot \varepsilon(s) \, q(\{\mu'\}, \{\nu'\})$ with $\mu'_i = \mu_{r(i)}$, $\nu'_i = \nu_{s(i)}$ differs by a polynomial of lower degree. Hence, by mathematical induction on the degree of polynomials, we see that $q(I, J)$ linearly spans $\mathfrak{A}_Q(\mathfrak{K})$. This completes the proof of the theorem 5.

§ 7. Representation of the Q-algebra for a finite dimensional K

In this section we obtain all the *-representation of the Q-algebra for a finite dimensional $\mathfrak{K}$. A corollary of the results in this section will be used in the next section.

Lemma 6. If $\mathfrak{K}$ is finite dimensional, $\mathfrak{A}_Q(\mathfrak{K})$ is finite dimensional.

Proof. We have already seen that (6.12) with the restriction (6.10) span the whole algebra, which proves the lemma.

Lemma 7. Let $\mathfrak{K}$ be finite dimensional and let Q be an irreducible *-representation of $\mathfrak{A}_Q(\mathfrak{K})$ on $\mathfrak{H}$. Then the operator

$$P = \sum_{i=1}^{n} Q(q_{ii}) \tag{7.1}$$

must be a constant p on $\mathfrak{H}$ and any two such Q with the same p are unitarily equivalent. Here $\{f_i;\ i = 1 \ldots n\}$ is an orthonormal basis of $\mathfrak{K}$ and $q_{ij} = [f_i \otimes f_j^*]$.

Proof. Let $Q_{ij} = Q(q_{ij})$. Since P commute with all Q_{ij} due to

$$\sum_l [Q_{ii}, Q_{lk}] = \sum_i (\delta_{il}\, Q_{ik} - \delta_{ki}\, Q_{li}) = 0\,,$$

P must be a constant in an irreducible representation. The uniqueness for the case $p = 0$ is trivial. Namely, $\Sigma\, Q_{ii} = 0$ implies $Q_{ii} = 0$ and, due to (5.8), $Q_{ij} = Q_{ij}\, Q_{jj} = 0$. That is $Q(a) = 0$ for any $a \in \mathfrak{A}_Q(\mathfrak{K})$.

Now we consider the case $p \neq 0$. Since $\{Q_{ii};\ i = 1 \ldots n\}$ is a commuting set of projections, they have a simultaneous eigenvector, say Ψ_0, such that

$$Q_{ii}\,\Psi_0 = \begin{cases} \Psi_0 & \text{if } i \in I_0 \\ o & \text{if } i \notin I_0 \end{cases} \tag{7.2}$$

for some subset I_0 of $\{1 \ldots n\}$ consisting of p indices. Let $I_0 = \{k_1 \ldots k_p\}$, $k_1 < \ldots < k_r \leqq p < k_{r+1} < \ldots < k_p$ and $\{1 \ldots p\} = \{k_1 \ldots k_r\, k'_1 \ldots k'_{p-r}\}$. Define

$$\Psi_0' = \prod_{\mu=1}^{p-r} Q'_{k_\mu k_{r+\mu}}\, \Psi_0\,.$$

Due to (4.3) and (5.7)

$$Q_{ji}^*\, Q_{ji} = Q_{ij}\, Q_{ji} = Q_{ii}\,(1 - Q_{jj})\,, \quad \text{if } i \neq j \tag{7.3}$$

By using (7.2), (7.3) and (4.4), we have

$$(\Psi_0', \Psi_0') = (\Psi_0, \Psi_0) \neq 0 \tag{7.4}$$

Due to (5.8) and (5.9)

$$Q_{ii}\, Q_{ki} = 0 \quad \text{if } k \neq i \tag{7.5}$$

$$Q_{ii}\, Q_{ik} = Q_{ik} \tag{7.6}$$

Hence

$$Q_{ii}\,\Psi_0' = \begin{cases} \Psi_0' & \text{if } i \leq p \\ o & \text{if } i > p \end{cases} \tag{7.7}$$

Now let $I = \{i_1 \ldots i_m\}$, $J = \{j_1 \ldots j_m\}$, $i_1 < \ldots < i_m \leqslant p < j_1 \ldots < j_m$ and define

$$\Psi(I, J) = \prod_{\mu=1}^{m} Q_{j_\mu i_\mu}\, \Psi_0' \tag{7.8}$$

We shall show that the inner product of vectors $\Psi(I, J)$ as well as the behavior of Q_{ij} on $\Psi(I, J)$ is uniquely determined by (4.2) $\sim$ (4.5).

Due to (7.5) $\sim$ (7.8),

$$Q_{ii}\,\Psi(I, J) = \begin{cases} \Psi(I, J) & \text{if } i \in I' \cup J \\ o & \text{if } i \in I \cup J' \end{cases} \tag{7.9}$$

where

$$I' = \{1 \ldots p\} - I \qquad J' = \{p+1 \ldots n\} - J \tag{7.10}$$

From (7.9), it obviously follows that

$$\big(\Psi(I_1, J_1), \Psi(I_2, J_2)\big) = 0 \qquad \text{unless } I_1 = I_2,\ J_1 = J_2 \tag{7.11}$$

Due to (7.3),

$$\|\,\Psi(I, J)\,\|^2 = (\Psi_0', \prod_{\mu=1}^{m} (1 - Q_{j_\mu j_\mu})\, Q_{i_\mu i_\mu}\,\Psi_0') = (\Psi_0', \Psi_0') \tag{7.12}$$

Since $Q_{ji} = Q_{ji}\,Q_{ii} = Q_{ji}\,(1 - Q_{jj})$, we have from (7.9)

$$Q_{ji}\,\Psi\,(I, J) = 0 \qquad \text{unless } j \in I \cup J' \text{ and } i \in I' \cup J\,. \tag{7.13}$$

Now if $i \in I'$, $j \in J'$, then $i_k < i < i_{k+1}$, $j_l < j < j_{l+1}$ for some k and l. (If $k = m$, $< i_{k+1}$ should be omitted and if $k = 0$, $i_k <$ should be omitted. The same for j.)

Because

$$Q_{j_1 i_1}\,Q_{j_2 i_2} = -\,Q_{j_1 i_2}\,Q_{j_2 i_1} = -\,Q_{j_2 i_1}\,Q_{j_1 i_2}$$

if all j_1, j_2, i_1, i_2 are different, we have

$$Q_{ji}\,\Psi(I, J) = (-1)^{k-l}\,\Psi(I \cup \{i\}, J \cup \{j\}) \tag{7.14}$$

If $i = j_k \in J$, $j \in J'$ and $j_l < j < j_{l+1}$, then we use

$$Q_{jj_k}\,Q_{j_k i_k} = Q_{ji_k}\,(1 - Q_{j_k j_k})$$

as well as the equality used for (7.14) and we have

$$Q_{ji}\,\Psi(I, J) = (-1)^{k-l+\theta(k-l)}\,\Psi(I, J \cup \{j\} - \{i\}) \tag{7.15}$$

where $\theta(x) = 1$, if $x > 0$ and $= 0$, if $x \leqq 0$.

If $j = i_k \in I$, $i \in I'$ and $i_l < i < i_{l+1}$, we use

$$Q_{i_k i}\,Q_{j_k i_k} = -\,Q_{j_k i}\,Q_{i_k i_k}$$

as well as the equality used for (7.14) and we have

$$Q_{ji}\,\Psi(I, J) = (-1)^{k-l+\theta(k-l)+1}\,\Psi(I \cup \{i\} - \{j\}, J) \tag{7.16}$$

Finally, if $j = i_k \in I$, $i = j_l \in J$, we use all equalities used above and we have

$$Q_{ji}\,\Psi(I, J) = (-1)^{k-l}\,\Psi(I - \{j\}, J - \{i\}) \tag{7.17}$$

Since $\{Q_{ij}\}$ is assumed to be irreducible, any vector is cyclic and hence $\Psi(I, J)$ span the whole space. Therefore (7.9) $\sim$ (7.17) shows the required uniqueness.

Theorem 6. Any representation Q of the Q-algebra over a finite dimensional $\mathfrak{K}$ is fully reducible. Its irreducible representation is uniquely characterized by the value of the operator P defined by (7.1).

Proof. Let $\mathfrak{A}$ be any *-algebra of finite dimension. By the transfinite induction, the Hilbert space is a direct sum of subspaces, in each of which $\mathfrak{A}$ is cyclic*). These cyclic spaces are finite dimensional if $\mathfrak{A}$ is finite dimensional. Furthermore the orthogonal complement of any invariant subspace of *-algebra is also invariant. Therefore any *-algebra of finite dimension is fully reducible and, due to the lemma 6, $Q(\mathfrak{A}_Q(\mathfrak{K}))$ is fully reducible. The rest of the theorem is due to the lemma 7.

Remark. If we consider the Q_ψ defined through the definition 5 with the unique irreducible representation[2]) ψ of the CARs for a finite dimensional $\mathfrak{K}$, and if we restrict the Hilbert space to the eigenspace of P belonging to the eigenvalue p, then we obtain a representation of the Q-algebra with $P = p$. This representation is irreducible as will be shown below. The above theorem asserts that this exhausts all possible representations of the Q-algebra.

The proof of the irreducibility: Take any vector Ψ in the eigenspace of P. Ψ is a cyclic vector of $\psi(\mathfrak{A}_C(\mathfrak{K}))$. Moreover, any monomials of $\psi(f)$ and $\psi(g^*)$ except those in $\mathfrak{A}_C(\mathfrak{K})_0$ brings Ψ into a different eigenspace of P and the monomials in $\mathfrak{A}_C(\mathfrak{K})_0$ are mapped by ψ into polynomials of $Q_\psi(K)$ due to the theorem 5. Hence restricted to one eigenspace of P, any Ψ is a cyclic vector of $Q_\psi(\mathfrak{A}_Q(\mathfrak{K}))$, which proves the desired irreducibility.

Corollary 1. Let Q be a representation of the Q-algebra over a finite dimensional $\mathfrak{K}$. There always exists a larger Hilbert space $\mathfrak{H}' \supset \mathfrak{H}$ and a representation ψ of CARs defined on $\mathfrak{H}'$ such that Q_ψ defined through the definition 5 coincides with the given Q on $\mathfrak{H}$.

Proof. Any representation of the Q-algebra is fully reducible and each irreducible representation can be embedded in an irreducible representation of CARs according to the above remark. Hence the entire representation can be embedded in the direct sum of the irreducible representations of CARs.

Definition 8. An element a of a *-algebra $\mathfrak{A}$ which can be written as $a = b^* \cdot b$ for some $b \in \mathfrak{A}$ will be called positive. If $a \in \mathfrak{A}_C(\mathfrak{K})_0$ is a positive element of $\mathfrak{A}_C(\mathfrak{K})$, then $j^{-1}\, a$ will be called a ψ-positive element of $\mathfrak{A}_Q(\mathfrak{K})$.

Corollary 2. Let Q be a representation of the Q-algebra over $\mathfrak{K}$, where $\mathfrak{K}$ need not be finite dimensional. Any ψ-positive element of $\mathfrak{A}_Q(\mathfrak{K})$ is represented by a positive semidefinite operator.

Proof. Any element of $\mathfrak{A}_C(\mathfrak{K})$ can be considered as an element of $\mathfrak{A}_C(\mathfrak{K}_1)$ for a suitable finite dimensional subspace $\mathfrak{K}_1$ of $\mathfrak{K}$ and hence any ψ-positive element of $\mathfrak{A}_Q(\mathfrak{K})$ can be considered as that of $\mathfrak{A}_Q(\mathfrak{K}_1)$ for some finite dimensional subspace $\mathfrak{K}_1$ of $\mathfrak{K}$. Then we can apply corollary 1 and corollary 2 follows.

§ 8. Cyclic representations of CARs with charge conservation

In this section, we introduce the definition of a cyclic representation of CARs with charge conservation and prove that they are determined already by a cyclic representation of the Q-algebra.

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*) See the ref. 4 page 253.

Definition 9. A cyclic representation of the CAR-algebra with charge conservation is a representation ψ of $\mathfrak{A}_C(\mathfrak{K})$ with a cyclic vector Ψ, such that

$$\mathfrak{H}_n = \overline{\psi(\mathfrak{A}_C(\mathfrak{K})_n)\,\Psi}$$

are orthogonal to each other for different n. (The total Hilbert space $\mathfrak{H}$ is decomposed as
$$\mathfrak{H} = \sum_{n=-\infty}^{\infty} \mathfrak{H}_n).$$

Theorem 7. If Q is any representation of the Q-algebra on a Hilbert space $\mathfrak{H}_Q$ with a cyclic vector Ψ_Q, then there exists a cyclic representation ψ of the CAR-algebra with charge conservation on a Hilbert space $\mathfrak{H}$ such that
(1) there exists a unitary operator S from $\mathfrak{H}_0$ onto $\mathfrak{H}_Q$, satisfying $S\,Q_\psi(a)\,S^{-1} = Q(a)$ for any $a \in \mathfrak{A}_Q(\mathfrak{K})$.
(2) and $S^{-1}\Psi_Q = \Psi$ is a cyclic vector of $\mathfrak{A}_C(\mathfrak{K})$ in $\mathfrak{H}$.
Such ψ is unique up to a unitary equivalence.

Proof. The uniqueness of ψ is seen in the following way. Since Ψ is cyclic, the expectation functional $E(a) = (\Psi, \psi(a)\,\Psi)$, $a \in \mathfrak{A}_C(\mathfrak{K})$ will determine ψ up to a unitary equivalence. However, by (6.5) any a can be written as

$$a = \sum_{n=-\infty}^{+\infty} a_n\,, \quad a_n \in \mathfrak{A}_C(\mathfrak{K})_n \tag{8.1}$$

and by the orthogonality of $\mathfrak{H}_n$ for different n, we have

$$E(a) = (\Psi_Q, Q(j^{-1}\,a_0)\,\Psi_Q) \tag{8.2}$$

Hence ψ is unique up to a unitary equivalence.

The existence of ψ is proved just as easily. We define $E(a)$ by (8.1) and (8.2). Note that, if $a \in \mathfrak{A}_C(\mathfrak{K})_n$, $n \neq 0$, then $E(a) = 0$. Since Q, j^{-1} and $a \to a_0$ are all linear, $E(a)$ is linear in a. Furthermore, due to corollary 2 to the theorem 6 for ψ-positive elements, we have

$$E(a^*\,a) = \sum_{n=-\infty}^{\infty} (\Psi_Q, Q(j^{-1}\,(a_n^*\,a_n))\,\Psi_Q) \geqq 0$$

Hence, due to the lemma 2, we have a Hilbert space $\mathfrak{H}$, a representation ψ of $\mathfrak{A}_C(\mathfrak{K})$ and a cyclic vector Ψ in $\mathfrak{H}$, satisfying $E(a) = (\Psi, \psi(a)\,\Psi)$. By definition, $\mathfrak{H}_n \perp \mathfrak{H}_m$ for $n \neq m$. Furthermore, by construction, $(\Psi, Q_\psi(a)\,\Psi) = (\Psi_Q, Q(a)\,\Psi_Q)$ for any $a \in \mathfrak{A}_Q(\mathfrak{K})$, which implies the existence of the unitary operator S satisfying (2) due to the lemma 5. This completes the proof of the theorem.

§ 9. The operator U(K)

By the theorem 7 and the lemma 4, the mapping Q from $F(\mathfrak{K})$ into $B(\mathfrak{H}_Q)$ with a cyclic vector Ψ_Q will uniquely define a cyclic representation of the CAR-algebra with charge conservation. We now consider another mapping U into $B(\mathfrak{H}_Q)$, which can replace Q.

We first consider the element

$$\exp [K] = \sum_{n=0}^{\infty} n!^{-1} [K]^n \tag{9.1}$$

for any $K \in F(\mathfrak{K})$. When one discusses an equation in $\mathfrak{A}_Q(\mathfrak{K})$ involving a finite number of $K_i \in F(\mathfrak{K})$, there always exists a finite dimensional subspace $\mathfrak{K}_1$ of $\mathfrak{K}$ such that all K_i are in $\mathfrak{K}_1 \otimes K_1^*$, and one can consider the equation in the finite dimensional subalgebra $\mathfrak{A}_Q(\mathfrak{K}_1)$ of $\mathfrak{A}_Q(\mathfrak{K})$. In particular, (9.1) converges in the unique topology of the finite dimensional vector space.

We now prove

Lemma 8. (1) $\exp [K]$, $K \in F(\mathfrak{K})$ linearly span $\mathfrak{A}_Q(\mathfrak{K})$.

(2) $\exp [K]$ depends only on $\exp K$.

(3) $(\exp [K])^* = \exp [K^*]$.

(4) $\exp [K_1] \exp [K_2] = \exp [K_1 \circ K_2]$ where $K_1 \circ K_2$ is any operator in $F(\mathfrak{K})$ satisfying $\exp K_1 \exp K_2 = \exp K_1 \circ K_2$.

(5) $\exp [f \otimes g^*] = 1 + \varphi(g, f) [f \otimes g^*]$ where $\varphi(x) = x^{-1} (e^x - 1)$.

Proof. It is sufficient to prove (1) $\sim$ (5) for a finite dimensional $\mathfrak{K}$. As is shown before, the infinite dimensional case follows from this.

(1) Clearly $[f \otimes g^*]$ algebraically generates $\mathfrak{A}_Q(\mathfrak{K})$. Hence, by (5), $\exp [K]$ algebraically generates $\mathfrak{A}_Q(\mathfrak{K})$. By (4), (1) follows.

(3) and (5) are obvious from the definition (9.1) and the relation (5.6).

(4) for commuting K's follows from the definition (9.1) where $K_1 \circ K_2$ is taken as $K_1 + K_2$. (4) for commuting K's with $K_1 \circ K_2$ different from $K_1 + K_2$ follows from (2). (4) for noncommuting K's is proved at the end.

(2) Any K_l can be written in the Jordan normal form

$$K_l = \sum_i E_i^l (\lambda_i^l + N_i^l)$$

where $\mathrm{E}_i^l \mathrm{E}_i^l = \delta_{ij} \mathrm{E}_i^l$, $\sum \mathrm{E}_i^l = 1$, $\lambda_i^l \neq \lambda_j^l$ for i $\neq$ j, $\mathrm{E}_i^l \mathrm{N}_j^l = \mathrm{N}_i^l \mathrm{E}_j^l = \delta_{ij} \mathrm{N}_i^l$, $(\mathrm{N}_i^l)^{d_i^l} = 0$ for a finite d_i^l. E_i^l, λ_i^l and N_i^l are uniquely defined by this equation and the stated conditions, apart from their order. If $\exp K_1 = \exp K_2$, then we have

$$\sum E_i^1 (\exp \lambda_i^1 + M_i^1) = \sum E_j^2 (\exp \lambda_j^2 + M_j^2)$$

where $M_i^l = (\exp \lambda_i^l) \sum_{k=1} (k!)^{-1} (N_j)^k$. Since a sufficiently high power of M_i^l vanishes, we have $E_i^1 = E_i^2$, $\exp \lambda_i^1 = \exp \lambda_i^2$ and $M_i^1 = M_i^2$ after a suitable rearrangement of suffices j. Now $\log (1 + (\exp -\lambda_i^l) M_i^l) = N_i^l$ where log is defined by a power series which terminates at finite term in this case. Hence $N_i^1 = N_i^2$. On the other hand, $\exp x = \exp y$ implies $x = y + 2 m i \pi$. Thus we see that $K_2 = K_1 + 2 \pi i L$ where $L = \Sigma m_i E_i$, the m_i are integers, $E_i E_j = \delta_{ij} E_i$ and E_i commutes with K_1. E_i can be written as $E_i = \sum_K f_{ik} \otimes g_{ik}^*$ where $(g_{ik}, f_{jl}) = \delta_{ij} \delta_{kl}$. Hence, using (4) for commuting K's with $K_1 \circ K_2 = K_1 + K_2$ repeatedly and using (5), we have (2).

(4) We use the formula

$$1 + z\,A = \exp \int_0^z (1 + \lambda\,A)^{-1}\,A\,d\lambda \tag{9.3}$$

where the integration is over any path Γ from 0 to z in the complex λ-plane, on which $(1 + \lambda\,A)$ is non-singular, and is well defined (for a fixed Γ) if $1 + z\,A$ is non-singular. Both sides of (9.3) are (operator-valued) analytic function of z and they coincide for small z as can be seen by the power series expansion. If $A = \Sigma\,E_i\,(\lambda_i + N_i)$ is the Jorgan normal form of A, $1 + z\,A$ becomes singular only at finite number of points $z = -\,\lambda_i^{-1}$. Hence we have (9.3).

We define

$$(\sigma\,K_1) \circ (\tau\,K_2) = \int_0^1 (1 + \lambda\,A\,(\sigma\,\tau))^{-1}\,A\,(\sigma\,\tau)\,d\lambda \tag{9.4}$$

where $A(\sigma\,\tau) = e^{\sigma K_1}\,e^{\tau K_2} - 1$ and the integration is over a path Γ from 0 to 1 in the complex λ-plane, on which $1 + \lambda\,A$ is non-singular. Since $1 + A(\sigma\,\tau)$ is non-singular, (9.4) is well defined for each Γ. Since singular points $\lambda = -\,\lambda_i^{-1}$ of $(1 + \lambda\,A(\sigma\tau))^{-1}$ depend on σ and τ continuously, (9.4) is locally analytic in σ and τ for a suitable choice of Γ. Furthermore, by (9.3),

$$e^{\sigma K_1}\,e^{\tau K_2} = e^{(\sigma K_1)\circ(\tau K_2)} \tag{9.5}$$

We now investigate the equation

$$e^{\sigma[K_1]}\,e^{\tau[K_2]} = e^{[(\sigma K_1)\circ(\tau K_2)]} \tag{9.6}$$

By the Baker-Hausdorf formula [5]), C, satisfying $e^A\,e^B = e^C$, can for sufficiently small A and B be given by a converging series, in which each term is a multiple commutator of A and B. Because of the relations that (5.1) and (5.2) are zero, we see that (9.6) follows from (9.5) for small σ and τ. Since the right hand side of (9.6) does not depend on Γ due to (2), it is analytic for any σ, τ and so is the left hand side. Therefore, (9.6) holds for any σ and τ.

We now want to prove that the properties (1) $\sim$ (5) of $\exp\,[K]$ is essentially equivalent to the properties of $[K]$.

Theorem 8. Let $\varepsilon(\mathfrak{K})$ be the multiplicative group of operators e^K, $K \in F(\mathfrak{K})$, equipped with an involution $(e^K)^* = e^{K^*}$. Let U be the mapping from $\varepsilon(\mathfrak{K})$ into linear operators on Hilbert space $\mathfrak{H}$, satisfying

(1) The $U(L)$, $L \in \varepsilon(\mathfrak{K})$ are defined on a common dense domain D such that $U(L)\,D \subset D$.

(2) U is a $*$-homomorphism, i.e.

$$U(L)^* \supset U(L^*) \tag{9.7}$$

$$U(L_1\,L_2) = U(L_1)\,U(L_2) \tag{9.8}$$

(3) $(\exp \lambda\,(f, f) - 1)^{-1}\,(U(\exp \lambda\,f \otimes f^*) - 1)$ is independent of λ.

Then $U(L)$, $L \in \varepsilon(\mathfrak{K})$ is bounded and there exists a unique representation Q of the Q-algebra on $\mathfrak{H}$ such that

$$U(e^K) = Q(\exp [K]) = e^{Q[K]} \tag{9.9}$$

Conversely, if Q is a representation of the Q-algebra on $\mathfrak{H}$, U defined by (9.9) is a *-representation of $\varepsilon(\mathfrak{K})$ satisfying (1) $\sim$ (3).

Proof. The converse part is obvious from lemma 8. Now let U be given. We define

$$n(f) = (e^{\lambda(f,f)} - 1)^{-1} \left(U(\exp \lambda f \otimes f^*) - 1\right)$$

which is independent of λ. Solving for $U(\exp \lambda f \otimes f^*)$ and substituting it for L's in (9.7) and (9.8) we see that $n(f)^* \supset n(f)$ and $n(f)^2 = n(f)$. By the proof of the theorem 1, $n(f)$ is bounded and so is $U(\exp \lambda f \otimes f^*)$. $U(\exp \lambda f \otimes f^*)$ is obviously continuous in λ.

For a moment, let us consider a finite dimensional subspace $\mathfrak{K}_1$ of $\mathfrak{K}$. $\varepsilon(\mathfrak{K}_1)$ is a Lie group and a finite number of $f \otimes f^*$ and $i\, f \otimes f^*$ linearly span its Lie algebra $F(\mathfrak{K}_1)$. Hence the product of $e^{\lambda f \otimes f^*}$ covers at least a certain neighbourhood of 1. Furthermore, since $\varepsilon(\mathfrak{K}_1)$ is connected, it is multiplicatively generated by a neighbourhood of 1. Hence $U(L)$ for any $L \in \varepsilon(\mathfrak{K})$ is bounded and it is a continuous representation.

According to GÅRDING[6]) the Lie algebra $F(\mathfrak{K})$ is then represented by linear operators on a common dense domain. Namely, $Q(K) = \lim_{\lambda \to 0} (U(e^{\lambda K}) - 1)/\lambda$ satisfies (4.2) $\sim$ (4.4) on a dense domain. In addition, due to (3), it satisfies (4.5). Hence all $Q(K)$ are bounded due to the lemma 3 and we have a *-representation Q of the Q-algebra $\mathfrak{A}_Q(\mathfrak{K}_1)$ due to the lemma 4. Furthermore, by definition, (9.9) is satisfied. The uniqueness of Q is obvious from the equation (9.9).

Since $\varepsilon(\mathfrak{K})$ and $\mathfrak{A}_Q(\mathfrak{K})$ are the unions of $\varepsilon(\mathfrak{K}_1)$ and of $\mathfrak{A}_Q(\mathfrak{K}_1)$, respectively, for all possible $\mathfrak{K}_1$, we have the theorem for any $\mathfrak{K}$.

§ 10. The functional E(L)

We again follow Gelfand and consider the functional

$$E(L) = (\Psi, U(L)\, \Psi) \tag{10.1}$$

Theorem 9. If U is as in the theorem 8, $E(L)$ defined by (10.1) satisfies

$$\sum_{i,j=1}^{N} c_i^* c_j\, E(L_i^* L_j) \geqq 0, \tag{10.2}$$

$$(e^{\lambda(f,f)} - 1)^{-1} \left(E(L_1\, e^{\lambda f \otimes f^*} L_2) - E(L_1 L_2)\right) = \text{constant of } \lambda, \tag{10.3}$$

where L, L_1, $L_2 \in \varepsilon(\mathfrak{K})$, $f \in \mathfrak{K}$ and c_i and λ are arbitrary complex numbers. Conversely, if $E(L)$ is a functional of $L \in \varepsilon(\mathfrak{K})$ and satisfies (10.2) and (10.3), there exists a Hilbert space $\mathfrak{H}$, a *-representation U of $\varepsilon(\mathfrak{K})$, satisfying (3) of the theorem 8, and a vector Ψ in $\mathfrak{H}$ which is cyclic with respect to $\{U(L);\, L \in \varepsilon(\mathfrak{K})\}$.

The proof is similar to those of lemma 2 and lemma 5. The boundedness of $U(L)$ comes from the theorem 8 in this case. ($E(L^*) = E(L)^*$ follows from the reality of the left hand side of (10.2).)

Remark 1. We can prove theorems similar to the theorem 8 and 9 even if we restrict L to unitary operators. In this case $U(L)$ will also be unitary. We can also prove similar theorems for the set of $L = e^K$ where K is an operator of the trace class on $\bar{\mathfrak{K}}$, due to the theorems 3 and 7.

Remark 2. In the example of CCRs in the remark after the theorem 2, we can construct*) a unitary operator $U(e^{iK}) = e^{iQ(K)}$ for a hermitian K and the functional $E(e^{iK})$. The operator $U(L)$ satisfies (9.7) and (9.8) but it does not satisfy (3) of the theorem 8. Although $U(L)$ for a unitary L is bounded, $Q(K)$ is not bounded, in this case.

§ 11. Examples

A simple and yet nontrivial (though well-known) example is given by

$$E_{JW}(L) = 1 \tag{11.1}$$

The representations of $\mathfrak{A}_C(\mathfrak{K})$ and $\mathfrak{A}_Q(\mathfrak{K})$ associated with this functional will be denoted by ψ_{JW} and Q_{JW}. The subspaces $\mathfrak{H}_n$ for ψ_{JW} will be 0 for negative n. $\mathfrak{H}_0$ is 1-dimensional and a vector in $\mathfrak{H}_0$ will be denoted by $\Psi_{JW}(\mathfrak{K})$. $\psi_{JW}(g^*)$ will annihilate $\Psi_{JW}(\mathfrak{K})$ for any $g \in \mathfrak{K}$. The $E(a)$ for $a \in \mathfrak{A}c(\mathfrak{K})$ can be calculated by the formula

$$(\Psi_{JW}(\mathfrak{K}), \psi_{JW}[f^*_1, \ldots, f^*_n, g_m, \ldots, g_1] \Psi_{JW}(\mathfrak{K})) = \delta_{mn} \det ((f_i, g_j)) \tag{11.2}$$

It is easy to see that $\Psi_{JW}(\mathfrak{K})$ is a cyclic vector of the subalgebra generated by $\psi_{JW}(f)$, $f \in \mathfrak{K}$. Furthermore, we have

Lemma 9. The representation ψ_{JW} is irreducible.

Proof. We know that $\Psi_{JW}(\mathfrak{K})$ is a cyclic vector. Hence it is enough to prove that the projector on $\Psi_{JW}(\mathfrak{K})$ is in the algebra. Let $\{f_\alpha\}$ be an orthonormal basis of $\bar{\mathfrak{K}}$ and let P_α be the spectral projector of $\psi_{JW}(f_\alpha \otimes f_\alpha^*)$ belonging to zero. They commute with each other we can define $P_0 = \prod_\alpha P_\alpha$. It is easy to see that $\Psi_{JW}(\mathfrak{K})$ is the unique simultanious eigenvector of $\psi_{JW}(f_\alpha^*)$ belonging to zero. Hence P_0 is the projector on $\Psi_{JW}(\mathfrak{K})$ and ψ_{JW} is irreducible.

We denote by θ_{JW} the operator which is $(-1)^n$ on $\mathfrak{H}_{JW}(\mathfrak{K})_n$ where $\mathfrak{H}_{JW}(\mathfrak{K})$ is the representation space of ψ_{JW}.

Another example is easily obtained from the above example by

$$\psi^T_{JW}(f) = \psi_{JW}((T f)^*) \tag{11.3}$$

$$\psi^T_{JW}(f^*) = \psi_{JW}(T f) \tag{11.4}$$

where T is any conjugation in $\mathfrak{K}$, i.e., $T^2 = 1$, $T i = -i T$ and $(T f, T g) = (g, f)$. If we write the Q_ψ for ψ^T_{JW} by Q^T_{JW}, we have $Q^T_{JW}(f \otimes g^*) = \mathrm{tr}\,(f \otimes g^*) - Q_{JW}(T g \otimes (T f)^*)$ and

$$E^T_{JW}(e^K) = e^{tr\,K} \tag{11.5}$$

*) See the footnote 23 of the ref. 1.

Contrary to ψ_{JW}, $\psi^T_{JW}(f)$ annihilate $\Psi_{JW}(\mathfrak{K})$.

Next we consider the space $\mathfrak{H} = \mathfrak{H}_{JW}(\mathfrak{K}) \otimes \mathfrak{H}_{JW}(\overline{\mathfrak{K}})$, the vector $\Psi_0 = \Psi_{JW}(\mathfrak{K}) \otimes \Psi_{JW}(\overline{\mathfrak{K}})$ and

$$\psi_{\varrho T}(h) = \psi_{JW}((1-\varrho)^{\frac{1}{2}} h) \otimes \mathbf{1} + \theta_{JW} \otimes \psi^T_{JW}(\varrho^{\frac{1}{2}} h) \tag{11.6}$$

where ϱ is an operator on $\overline{\mathfrak{K}}$, satisfying $\varrho > 0, 1 - \varrho > 0$, $h \in \mathfrak{K}$ or $\mathfrak{K}^*$, $\varrho^{1/2} f^* = (\varrho^{1/2} f)^*$ and $(1-\varrho)^{1/2} f^* = ((1-\varrho)^{1/2} f)^*$. Since θ_{JW} anticommute with all $\psi_{JW}(h)$, $\psi^{\varrho T}$ defined by (11.6) satisfy the Definition 1. Furthermore, since ϱ and $1-\varrho$ is assumed to be positive definite, $\varrho^{1/2}\,\mathfrak{K}$ and $(1-)\varrho^{1/2}\,\mathfrak{K}$ will be dense in $\overline{\mathfrak{K}}$. Hence the repeated application of $\psi_{\varrho T}(f), f \in \mathfrak{K}$ on Ψ_0 will give a dense set in $\mathfrak{H}_{JW}(\mathfrak{K}) \otimes \Psi_{JW}(\overline{\mathfrak{K}})$. Then, as can be seen by mathematical induction on n of $\mathfrak{H} = \sum_{n=0}^{\infty} \mathfrak{H}_{JW}(\mathfrak{K}) \otimes \mathfrak{H}_{JW}(\overline{\mathfrak{K}})_n$, the repeated application of $\psi_{\varrho T}(f^*)$, $f \in \mathfrak{K}$, on vectors in $\mathfrak{H}_{JW}(\mathfrak{K}) \otimes \Psi_{JW}(\overline{\mathfrak{K}})$ will give a total set in $\mathfrak{H}_{JW}(\mathfrak{K}) \otimes \mathfrak{H}_{JW}(\overline{\mathfrak{K}})$. Hence Ψ_0 is a cyclic vector of $\psi_{\varrho T}(\mathfrak{A}_C(\mathfrak{K}))$. Thus $\psi_{\varrho T}$ gives a cyclic representation of the CAR-algebra with charge conservation, where $\mathfrak{H}_n$ is given by

$$\mathfrak{H}_n = \sum_r \mathfrak{H}_{JW}(\mathfrak{K})_{n+r} \otimes \mathfrak{H}_{JW}(\overline{\mathfrak{K}})_r \tag{11.7}$$

If we define another representation of the CAR-algebra by

$$\psi'_{\varrho T}(f) = \theta_{JW}\, \psi_{JW}(\varrho^{\frac{1}{2}} f) \otimes \theta_{JW} + \mathbf{1} \otimes \psi^T_{JW}\left((1-\varrho)^{\frac{1}{2}} f\right) \theta_{JW} \tag{11.8}$$

where $f \in \mathfrak{K}$ and $\psi'_{\varrho T}(f^*) = \psi'_{\varrho T}(f)^*$, then $\psi'_{\varrho T}(h)$ commutes with any $\psi_{\varrho T}(h')$ and hence $\psi_{\varrho T}$ is not irreducible.

The functional $E(L)$ for $\psi_{\varrho T}$ is calculated in the appendix A and is given by

$$E(e^K) = \left(\Psi_0, Q_{\varrho T}(e^{[K]})\, \Psi_0\right) = \exp \operatorname{tr} \log\left(1 + (e^K - 1)\,\varrho\right) \tag{11.9}$$

The case where ϱ becomes a projection operator can be constructed in the following way. Let $\mathfrak{K} = \mathfrak{K}_1 + \mathfrak{K}_2$, $\mathfrak{K}_1 \perp \mathfrak{K}_2$ and let P_i be the projection operator on the subspace $\mathfrak{K}_i$, in $\mathfrak{K}$. We consider*) the decomposition $\mathfrak{H}_{JW}(\mathfrak{K}) = \mathfrak{H}_{JW}(\mathfrak{K}_1) \otimes \mathfrak{H}_{JW}(\mathfrak{K}_2)$ discussed in the appendix B and we define

$$\psi_{P_2 T}(h) = \psi_{JW}(P_1 h) \otimes \mathbf{1} + \theta_{JW} \otimes \psi^T_{JW}(P_2 h)\,. \tag{11.10}$$

It is obious, from the lemma 9, that $\psi_{P_2 T}$ is an irreducible representation of the CAR-algebra with charge conservation. The calculation in the appendix A shows that $E(L)$ for $\psi_{P_2 T}$ is given by

$$E(e^K) = (\Psi_{JW}(\mathfrak{K}), Q_{P_2 T}(e^{[K]})\, \Psi_{JW}(\mathfrak{K})) = \exp \operatorname{tr} \log\,(1 + (e^K - 1)\, P_2) \tag{11.11}$$

where $\Psi_{JW}(\mathfrak{K}) = \Psi_{JW}(\mathfrak{K}_1) \otimes \Psi_{JW}(\mathfrak{K}_2)$.

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*) In this decomposition, ψ_{JW} for $\mathfrak{K}$ can be defined as

$$\psi_{JW}(h) = \psi_{JW}(h_1) \otimes \mathbf{1} + \theta_{JW} \otimes \psi_{JW}(h_2)$$

where $h = h_1 + h_2$, $h_i \in K_i$ or K_i^*.

§ 12. Infinite free Fermi gas

The gas of non-relativistic Fermi particles without spin in a finite cubic volume V can be described by ψ_{JW} where $\mathfrak{K}$ is taken to be the Hilbert space $\mathfrak{K}_V$ of L_2-functions over V. Let $\{f_j\}$ be the complete orthonormal set of periodic functions $V^{-1/2} e^{ik_j x}$ where k_j takes discrete values. We define the operator H_V acting on $\mathfrak{K}_V$ by

$$H_V = \sum_j (k_j^2/2\,m)\, f_j \otimes f_j^* \tag{12.1}$$

By the appendix B, $U(\exp \beta\,(\mu - H_V))$ is a traceable operator and the grand canonical ensemble of free Fermi particles without spin in a volume V is defined by the state (i.e. linear continuous functional over observables $\mathfrak{A}_Q(\mathfrak{K}_V)$) characterized, according to the theorem 9, by

$$E(L) = \mathrm{tr}\,\big(U(L)\; U(e^{\beta\mu - \beta H_V})\big)/\mathrm{tr}\,\big(U(e^{\beta\mu - \beta H_V})\big) \tag{12.2}$$

$$= \exp \mathrm{tr} \log \big(1 + (L-1)\,\varrho\,(\beta, \mu)\big) \tag{12.3}$$

where (12.3) is calculated from (12.2) in the appendix C and

$$\varrho\,(\beta, \mu) = e^{\beta\mu - \beta H_V} \,/\, (1 + e^{\beta\mu - \beta H_V}) \tag{12.4}$$

By taking the limit $V \to \infty$, we obtain the grand canonical ensemble for the infinite free Fermi gas. It is given by (12.3) where $\varrho\,(\beta, \mu)$ is given by the same equation as (12.4) except that H_V is replaced by the non-relativistic free Hamiltonian $H = k^2/2\,m$. ($\mathfrak{K}$ is taken to be $L_2(R^3)$ and k is the multiplication by k of the Fourier transform of the function $f(x) \in L_2(R^3)$.)

Obviously, $\varrho > 0$, $1 - \varrho > 0$ for $\beta \neq \infty$. Such representation has already been discussed in the section 11. The formula (11.6) is very similar to the case of the infinite free Bose gas[1]). The formula (11.6) allows a particle-hole interpretation in an obvious way.

For the limit $\beta \to \infty$ of zero temperature, we obtain

$$E(L) = \exp \mathrm{tr} \log \big(1 + (L-1)\; P(\mu)\big) \tag{12.5}$$

where the projection operator $P(\mu)$ is for the subspace $k^2/2m \leqslant \mu$. This representation is also obtained in the section 11. In particular $\psi_{P(\mu)\,T}\,(\mathfrak{A}_C(\mathfrak{K}))$ is irreducible in the space $\mathfrak{H}_0$.

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Appendix A

We calculate*) $E(e^K)$ for $\psi_{\varrho T}$ given by (11.6). For this calculation we only assume $\varrho \geqslant 0$, $1 - \varrho \geqslant 0$ and hence the case for $\psi_{P_2 T}$ is included.

From (11.2) and $\psi_{JW}(f^*) \Psi_{JW}(\mathfrak{R}) = 0$, we have

$$(\Psi_0, \psi_{\varrho T}[f_1, \dots, f_n, g_n^*, \dots, g_1^*] \Psi_0) = \det\big((g_i, \varrho f_j)\big). \tag{A.1}$$

If $\{f_i\}$ is an orthonormal set, then we have

$$(\Psi_0, Q_{\varrho T}[f_1 \otimes f_1^*, \dots, f_n \otimes f_n^*] \Psi_0) = \det\big((f_i, \varrho f_j)\big).$$

Let K be hermitian and sufficiently small. We can write $K = \sum_{i=1}^{N} \lambda_i f_i \otimes f_i^*$ where $\{f_i\}$ is an orthonormal set and λ_i is small. Hence we have

$e^{[K]} = \prod_{i=1}^{N} (1 + \sigma(i) [f_i \otimes f_i^*])$, where $\sigma(i) = e^{\lambda_i} - 1$ is small and therefore

$$\begin{aligned}(\Psi_0, Q_{\varrho T}(e^{[K]}) \Psi_0) &= \sum_{n=0}^{N} \sum_{\{k_1 \cdots k_n\}} \Big(\prod_{i=1}^{n} \sigma(k_i)\Big) \det\big((f_{k_i}, f_{k_j})\big)\\ &= \det(1 + A) = \exp \operatorname{tr} \log(1 + A)\end{aligned}$$

where**) $A = (\sigma(i)(f_i, \varrho f_j))_{i,j=1 \dots N}$ and $\log(1 + A)$ can be defined by a power series expansion for small K. Since $\operatorname{tr} A^n = \operatorname{tr}((e^K - 1)\varrho)^n$, we have

$$E(e^K) = \exp \operatorname{tr} \log\big(1 + (e^K - 1)\varrho\big) = \det\big(1 + (e^K - 1)\varrho\big) \tag{A.2}$$

*) It is also possible to prove (11.9) by ordering creation and annihilation operators in $e^{Q(K)}$.

**) The formula $\det B = \exp \operatorname{tr} \log B$ can easily be proved for a hermitian B by the spectral decomposition of B. Considering $B = B_1 + z B_2$ with hermitian B's and using the analytic continuation in z from the real axis to i we obtain the formula for the general case. Here $\log B$ is defined as in the section 9 and if B is singular i.e. $\det B = 0$ then $\operatorname{tr} \log B$ should be taken to be $-\infty$.

for a sufficiently small and hermitian K. We now set $K = z_1 K_1 + z_2 K_2$ where K_1 and K_2 are hermitian. The two extreme sides of (A.2) are clearly analytic for all value of z_1 and z_2 and hence (A.2) holds for all z_1 and z_2. Thus we have (11.9). (Note that $(e^K-1)\,\varrho$ can be considered as a finite matrix on the finite dimensional space spanned by f_i and ϱf_i, $i = 1 \ldots N$.)

Appendix B

In this appendix, we introduce two different ways of obtaining ψ_{JW} by using the tensor product of Hilbert spaces. As an application we generalize the definition of $U_{JW}(L)$ for a wider class of operators L and calculate tr $U_{JW}(L)$ for a certain class of operators L.

Let $\{\alpha\}$ be an index set of ordinary numbers, $\mathfrak{K}_\alpha$ be subspaces of $\overline{\mathfrak{K}}$, $\mathfrak{K}_\alpha \perp \mathfrak{K}_\beta$ if $\alpha \neq \beta$ and $\overline{\mathfrak{K}} = \sum_\alpha \mathfrak{K}_\alpha$. We consider the incomplete infinite (or finite) direct product[7] $\mathfrak{H} = \prod_\alpha^\otimes \mathfrak{H}_{JW}(\mathfrak{K}_\alpha)$ containing $\prod_\alpha^\otimes \Psi_{JW}(\mathfrak{K}_\alpha)$. As is easily seen, there exists a unitary operator S mapping $\mathfrak{H}$ onto $\mathfrak{H}_{JW}(\mathfrak{K})$ satisfying

$$\Psi_{JW}(\mathfrak{K}) = S \prod_\alpha^\otimes \Psi_{JW}(\mathfrak{K}_\alpha) \tag{B.1}$$

$$\psi_{JW}(h) = S \left(\sum_\alpha \prod_{\beta<\alpha}^\otimes \theta_{JW\beta} \otimes \psi_{JW}(h_\alpha) \otimes \prod_{\beta>\alpha}^\otimes \mathbf{1}_\beta\right) S^{-1} \tag{B.2}$$

where $h = \Sigma h_\alpha$, $h_\alpha \in \mathfrak{K}_\alpha$ or $\mathfrak{K}_\alpha^*$, $\theta_{JW\beta}$ is the θ_{JW} for $\mathfrak{H}_{JW}(\mathfrak{K}_\beta)$, defined in the section 11. (The inside of the parenthesis in (B.2) clearly gives an irreducible representation of the CARs with charge conservation and the expectation functional $E(L)$ in the vector $\prod_\alpha^\otimes \Psi_{JW}(\mathfrak{K}_\alpha)$ is 1. Hence S exists.) Since ψ_{JW} is irreducible, S is unique (up to a number). In particular, if $\{f_\alpha\}$ is a complete orthonormal set and $\mathfrak{K}_\alpha = \{c f_\alpha\}$, then $\mathfrak{H}_{JW}(\mathfrak{K}_\alpha)$ is 2-dimensional, $\Psi_{JW}(\mathfrak{K}_\alpha)$ is, for example, $\begin{pmatrix}1\\0\end{pmatrix}$

$$\psi_{JW}(f_\alpha) \text{ and } \psi_{JW}(f_\alpha^*) \text{ are } \begin{pmatrix}00\\10\end{pmatrix} \text{ and } \begin{pmatrix}01\\00\end{pmatrix}$$

respectively, and (B.1) and (B.2) are familiar formulas.

Another method to construct $\mathfrak{H}_{JW}(\mathfrak{K})$ is the one by FOCK[8]). Let $\overline{\mathfrak{K}}^{\otimes n}$ be the tensor product of n copies of $\overline{\mathfrak{K}}$ and Asym $\overline{\mathfrak{K}}^{\otimes n}$ be its totally anti-symmetric part[9]). Consider

$$\mathfrak{H}_{JW}(\mathfrak{K}) = \sum_{n=0}^{\infty}{}^\oplus \text{ Asym } \overline{\mathfrak{K}}^{\otimes n} \tag{B.3}$$

where $\overline{\mathfrak{K}}^{\otimes 0}$ is one-dimensional and a vector in $\overline{\mathfrak{K}}^{\otimes 0}$ is to be identified with $\Psi_{JW}(\mathfrak{K})$. If B maps $\overline{\mathfrak{K}}^{\otimes m}$ into $\overline{\mathfrak{K}}^{\otimes n}$ then $\mathbf{1}_t \otimes B$ maps $\overline{\mathfrak{K}}^{\otimes (m+t)}$ into $\overline{\mathfrak{K}}^{\otimes (n+t)}$. Let E_n^{As} be the projection on Asym $\overline{\mathfrak{K}}^{\otimes n}$ and let As $(\mathbf{1}_t \otimes B)$ be the operator on $\mathfrak{H}_{JW}(\mathfrak{K})$, being 0 on all Asym

$\bar{\mathfrak{K}}^{\otimes l}$, $l \neq m + t$ and being $E^{As}_{n+t}\,(\mathbf{1}_t \otimes B)\,E^{As}_{m+t}$ on Asym $\bar{\mathfrak{K}}^{\otimes (m+t)}$. The generalized creation and annihilation operator is then defined

$$(b^{+m}\,B\,b^n) = \sum_{t=0}^{\infty} \mathrm{As}\,(\mathbf{1}_t \otimes B)\,\big((n+t)!\,(m+t)!/t!^2\big)^{1/2} \tag{B.4}$$

In particular, if $B = f \in \bar{\mathfrak{K}}$ or $f^* \in \bar{\mathfrak{K}}^*$ (which should be considered as a mapping from $\bar{\mathfrak{K}}^{\otimes 0}$ to $\bar{\mathfrak{K}}^{\otimes 1}$ and from $\bar{\mathfrak{K}}^{\otimes 1}$ to $\bar{\mathfrak{K}}^{\otimes 0}$ respectively), then we identify $(b^{+1} f\, b^0)$ as $\psi_{JW}(f)$ and $(b^{+0} f^*\, b^1)$ as $\psi_{JW}(f^*)$. It is easy to prove that ψ_{JW} so defined is a representation of CARs with charge conservation, $\psi_{JW}(f^*)$ annihilate $\Psi_{JW}(\mathfrak{K})$ (defined as any fixed vector in $\bar{\mathfrak{K}}^{\otimes 0}$) and $\Psi_{JW}(\mathfrak{K})$ is a cyclic vector with the charge 0. Hence $\mathfrak{H}_{JW}(\mathfrak{K})$, $\Psi_{JW}(\mathfrak{K})$ and ψ_{JW} defined here are unitarily equivalent to our earlier definitions. In this formulation, $Q_{JW}(K) = (b^{+1}\,K\,b^1)$.

We now define $U(L)$ for a wider class of L. Let L be an operator in $\bar{\mathfrak{K}}$ with $\| L \| \leqq 1$. We define

$$U(L) = \sum_{n=0}^{\otimes} L^{\otimes n} \tag{B.5}$$

where $L^{\otimes n}$ acts on Asym $\bar{\mathfrak{K}}^{\otimes n}$. Since $\| L^{\otimes n} \| \leqq L \|^n$, $\| U(L) \| \leqq 1$. If $L_j \to L$ strongly with $\| L_j \| \leqq 1$, then $U(L_j) \to U(L)$ on each Asym $\bar{\mathfrak{K}}^{\otimes n}$ strongly. Since $\| U(L_j) \|$ is bounded by 1 independent of j, this implies $U(L_j) \to U(L)$ strongly. Namely $U(L)$ is strongly continuous in L. Obviously $U(L_1)\,U(L_2) = U(L_1\,L_2)$. If K is positive semidefinite and $\lambda \geqq 0$, we may consider one-parameter semigroups $U(e^{-\lambda K})$ and $U(e^{i\lambda K})$ and we easily can prove that their infinitesimal generators are $Q_{JW}(-K)$ and $Q_{JW}(i\,K)$. Since K and $i\,K$ for $K \in F(\mathfrak{K})$, $K > 0$ linearly generate the Lie algebra $F(\mathfrak{K})$, we see that $U(L)$ coincides with the previously defined $U(L)$ for $L \in \varepsilon(\mathfrak{K})$. If $L = L_a\,L_b$, $L_a \in \varepsilon(\mathfrak{K})$, $\| L_b \| \leqq 1$, then we define $U(L) = U(L_a)\,U(L_b)$. As is easily seen, (B.5) and other properties hold also for such $U(L)$.

Finally, let us assume that L is a positive definite hermitian operator of the trace class. We have $L = e^{-K}$, $K = \sum_{i=1}^{\infty} \lambda_i f_i \otimes f_i^*$, $\sum e^{-\lambda i} < \infty$ and $\{f_i\}$ is a complete orthonormal set in $\bar{\mathfrak{K}}$. Let $\lambda_i \geqq 0$ for $i > N$. Then $U = U_a\,U_b$, $U_a = \prod_{i=1}^{N\,\otimes} U_i \otimes \prod_{i>N} \mathbf{1}_i$, $U_b = \prod_{i=1}^{N\,\otimes} \mathbf{1}_i \otimes \prod_{i>N}^{\otimes} U_i$, $U_i = e^{-\lambda_i Q(f_i \otimes f_i^*)}$ where the tensor product is in the tensor product decomposition $\mathfrak{H}_{JW}(\mathfrak{K}) = \prod_j^{\otimes} \mathfrak{H}_{JW}(\mathfrak{K}_j)$ with $\mathfrak{K}_j = \{c\,f_j\}$. Since $\mathrm{tr}\,U_i = 1 + \exp - \lambda_i$, we have

$$\mathrm{tr}\,U(L) = \prod_i \mathrm{tr}\,U(L_j) = \prod_j (1 + \exp - \lambda_j) = \exp \mathrm{tr} \log (1 + L) \tag{B.6}$$

where the product $\prod_j (1 + \exp - \lambda_j)$ is absolutely convergent due to $\Sigma\, e^{-\lambda_j} < \alpha$.

Appendix C. The calculation of (12.3)

Since $e^{\beta\mu - \beta H_V}$ is positive definite operator in the trace class, $U(e^{\beta\mu - \beta H_V})$ is also an operator in the trace class. First we consider a Hermitian K. We have

$$\operatorname{tr} U(e^K)\, U(e^{\beta\mu - \beta H_V}) = \operatorname{tr} U(L_K)$$

where $K \in F(\mathfrak{K})$ and $L_K = e^{K/2}\,(e^{\beta\mu - \beta H_V})\, e^{K/2}$ is a positive definite operator of the trace class. By (B.6) we have

$$\operatorname{tr} U(e^{\beta\mu - \beta H_V}) = \exp \operatorname{tr} \log\,(1 + e^{\beta\mu - \beta H_V})$$

$$\operatorname{tr} U(L_K) = \exp \operatorname{tr} \log\,(1 + L_K)$$

From the integral expression for log, we easily see

$$\operatorname{tr} \log\,(1 + L_K) = \operatorname{tr} e^{K/2}\,(\log\,(1 + L_K))\, e^{-K/2} = \operatorname{tr} \log\,(1 + e^{K/2}\, L_K\, e^{-K/2})$$

$$= \operatorname{tr} \log\,(1 + e^K\, e^{\beta\mu - \beta H_V})$$

For any operators A and B of finite rank, we have

$$\exp \operatorname{tr} \big(\log\,(1 + B) - \log\,(1 + A)\big) = \det\,(1 + B)\,(1 + A)^{-1}$$

$$= \exp \operatorname{tr} \log \big(1 + (B - A)\,(1 + A)^{-1}\big)$$

By continuity, the same equation is true for any operators A and B in the trace class. Hence

$$\langle e^K \rangle = \operatorname{tr} \{U(e^K)\, U(e^{\beta\mu - \beta H_V})\} \,/\, \operatorname{tr} U(e^{\beta\mu - \beta H_V})$$

$$= \exp \operatorname{tr} \log \big(1 + (e^K - 1)\,\varrho\big)$$

where

$$\varrho = e^{\beta\mu - \beta H_V} \,/\, (1 + e^{\beta\mu - \beta H_V}).$$

Finally, we set $K = K_1 + z\, K_2$ where K_1 and K_2 are Hermitian. By analytic continuation from a real z to $z = i$, we obtain the same formula for a general K.