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Lorentz Invariant Analytic S-Matrix Amplitudes

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Abstract. A canonical decomposition of analytic relativistic S-matrix elements (for processes between non-massless particles with arbitrary spins) into a finite sum of Lorentz invariant amplitudes multiplied with covariant polynomials is investigated. Necessary and sufficient conditions are given for the existence of a set of amplitudes, which are analytic functions in the Lorentz invariants. The restrictions by Π-, T- and C-invariance and by the Pauli principle are discussed, and the results are exemplified for the spin $^1/_2$ – spin $^1/_2$ scattering. In an appendix the theory of 'multivalued' holomorphic tensor fields over the complex mass shell and their analytic continuation is considered.

§ 1. Introduction and Statement of the Problem

The S-matrix is of central importance in every relativistic theory of elementary processes. In a certain basis the S-matrix elements for scattering and production processes of particles with arbitrary spins are tensor fields transforming covariantly under the real Lorentz group.

Considering the complexity of the mathematical structure of relativistic quantum mechanics, it is useful to overcome as trivially as possible the complications due to the spin degrees of freedom of the interacting particles, in order to concentrate on the true dynamical problems of the theory. Therefore we investigate the possibility of decomposing the scattering amplitudes

$$T_{(\alpha)}^{(\varkappa)} = \sum_{\lambda=1}^{L} T_{\lambda}^{(\varkappa)} \, Q_{(\alpha)}^{\lambda} \tag{1.1}$$

into a simple set of covariants $Q_{(\alpha)}^{\lambda}$, which absorb the constraints of relativistic kinematics, multiplied by a finite number of invariant amplitudes $T_{\lambda}^{(\varkappa)}$.

Such a covariant decomposition without kinematical singularities is certainly only possible under sufficiently strong assumptions for the S-matrix elements. Now, it is widely believed and for the 2-particle scattering amplitude supported by general principles, that the scattering amplitudes are boundary values of 'multivalued' holomorphic tensor fields over the complex mass shell.

The Mandelstam representation [31]) for the Lorentz invariant 2-particle spin-zero scattering amplitudes, which one assumes to be analytic in the invariants $s = (p_1 + p_2)^2$

and $t = (p_1 - p_4)^2$, has been the starting point for many theoretical and numerical investigations. For realistic processes, as pion-nucleon[8]) or nucleon-nucleon scattering[1])[14]), one knows from perturbation theory[22]) or from algebraic considerations[4]) a finite set of covariants, for which a decomposition (1.1) should most plausibly lead to holomorphic amplitudes in the $L(C)$-invariants. Yet only recently[23]) a general investigation of Lorentz covariant analytic functions proved the absence of kinematical singularities in these cases (see also [48]) for an independent treatment).

Starting from the representation theory of the inhomogeneous Lorentz group, we shall discuss, under which analyticity assumptions the general scattering and production amplitudes allow a covariant decomposition of the form (1.1) with holomorphic or meromorphic amplitudes in the Lorentz invariants.

The necessary and sufficient conditions for a (local) holomorphic decomposition are given in theorems 1 and 2. For production processes this decomposition is highly non-unique and (if globally possible) only of little physical interest. On the other hand, there always exists an independent set of meromorphic amplitudes in the Lorentz invariants, for which the restrictions due to possible discrete symmetries can be fully discussed.

As stated above, a global holomorphic decomposition always exists for the 2-particle scattering amplitudes (in saturated domains). Here we give a complete classification for the restrictions due to the discrete symmetries with an explicit construction for the spin $^1/_2$–spin $^1/_2$ scattering. This will justity the usual approach, e.g. in the framework of the Mandelstam representation, for non-massless particles with arbitrary spins.

In an appendix the analytic structure of 'multivalued' holomorphic tensor fields over the complex mass shell is investigated. Some results on the analytic completion with respect to covariant analytic functions are also useful for the Wightman and Green functions in general quantum field theory.

In conclusion I wish to thank both Professor M. Fierz and Professor R. Jost for continuous helpful criticism and Dr. M. Kummer and Dr. D. N. Williams for some clarifying duscussions. The guidance of Dr. H. Araki and Dr. D. Ruelle at an earlier of this work is gratefully acknowledged.

§ 2. Covariant Decomposition of Scattering Amplitudes

In this chapter we shall prove under sufficiently strong assumptions the existence of a covariant analytic decomposition for relativistic scattering amplitudes.

The kinematic properties of a relativistic scattering system can be naturally characterized by the representation theory of the quantum mechanical (restricted) Poincaré group $\tilde{P}_+^\uparrow$, the universal covering group of the inhomogeneous restricted Lorentz group $P_+^\uparrow$. In this framework we shall first define the basic quantities of our investigation.

We consider at most countably many different asymptotically free particles. These particles are distinguished by a real index $\varkappa = \bar{\varkappa} = 1, 2, \ldots$ for a Majorana particle and by a purely imaginary index $\varkappa \neq \bar{\varkappa} = \pm i, \pm 2i, \ldots$ for a particle–antiparticle pair, when required by a gauge group. All particles are assumed to have a nonvanishing rest mass $m_\varkappa = m_{\bar{\varkappa}} > 0$.

In the Hilbert space of a relativistic scattering theory there are 2 sets of basic vectors to asymptotic incoming and outgoing states. From general principles[19]) both form a Fock basis over the Hilbert spaces $\mathfrak{H}_{\underset{\varkappa}{(-)}}$ of the one $\overset{(-)}{\varkappa}$-particle states. The $\mathfrak{H}_{\underset{\varkappa}{(-)}}$ carry a unitary irreducible representation $[m_\varkappa, s_\varkappa]$ of $\tilde{P}_+^\uparrow$ with the spin $s_\varkappa = s_{\bar{\varkappa}} = 0, {}^1/_2, 1, \ldots$[47]). All information about the asymptotic configurations is contained in a set of creation and annihilation operators $\underset{\varkappa}{(-)}a^*_{ex}(p)_{\dot\beta}$, $\underset{\varkappa}{(-)}a_{ex}(p)_\alpha$ (ex = in, out). They obey the following transformation law under $\tilde{P}_+^\uparrow$ *)

$$U_{ex}(a, A)\ {}_\varkappa a_{ex}(p)_\alpha\ U_{ex}^{-1}(a, A) = e^{-iAp\cdot a}\ D^{s_\varkappa}_{\alpha\beta}(A^{-1})\ {}_\varkappa a_{ex}(A p)_\beta \tag{2.1}$$

and the commutation relations (assuming the normal connection of spin and statistics[28])[42]):

$$\left[\underset{\varkappa}{(-)}a_{ex}(p)_\alpha\,,\ \underset{\lambda}{(-)}a^*_{ex}(q)_{\dot\beta}\right]_{\eta_{\varkappa\lambda}} = \delta_{\underset{\varkappa}{(-)}\underset{\lambda}{(-)}}(p)_{\alpha\dot\beta}\,2\,p^0\,\delta\,(\boldsymbol{p}-\boldsymbol{q}) \tag{2.2}$$

$$\left[\underset{\varkappa}{(-)}a_{ex}(p)_\alpha\,,\ \underset{\lambda}{(-)}a_{ex}(q)_\beta\right]_{\eta_{\varkappa\lambda}} = 0 \tag{2.3}$$

with $\eta_{\varkappa\lambda} = +$ for $s_\varkappa$ and s_λ half-integer and $\eta_{\varkappa\lambda} = -$ otherwise.

Here we shall briefly explain our notation: For $A \in SL\,(2, C)$ and $\Lambda\,(A) \in L_+^\uparrow$ we set $(A\,p)^\mu = \Lambda\,(A)^\mu_\nu\,p^\nu$. Furthermore $A \to D^s\,(A)$ denotes the $(2\,s+1)$-dimensional irreducible tensor representation $[2\,s, 0]$ of $S\,L\,(2, C)$[43]). The transformation law (2.1) corresponds to a choice of a spinor basis[26]) in $\mathfrak{H}_{\underset{\varkappa}{(-)}}$, in which the Lorentz covariance is most simply expressed. In this connection we shall use the fact, that on the elements φ_α, $\chi_{\dot\beta}$ of the representation spaces $\mathfrak{S}^{[2s,o]}$, $\mathfrak{S}^{[o,2s]}$ of D^s and the complex conjugate representation $\overline{D^s}$ a (generalized) spinor calculus is defined. Using the numerical spinors

$$\varepsilon^{\alpha\beta} = \varepsilon^{\dot\alpha\dot\beta} = (-1)^{2s}\,\varepsilon_{\alpha\,\beta} = (-1)^{2s}\,\varepsilon_{\dot\alpha\dot\beta} = (-1)^{s+\alpha}\,\delta_{\alpha,-\beta} \tag{2.4}$$

one can contract and raise and lower spinor indices covariantly. Furthermore the matrix

$$(p)_{\alpha\dot\beta} = D^s_{\alpha\beta}\left(\frac{1}{m}\,(p^0 + \boldsymbol{p}\cdot\boldsymbol{\sigma})\right) \tag{2.5}$$

is defined and non-singular for $p^0 = \sqrt{m^2 + \boldsymbol{p}^2}$, $m > 0$, as $(p)_{\alpha\dot\beta}\,(p)^{\dot\beta\gamma} = \delta^\gamma_\alpha$. Because of $(A\,p)_{\alpha\dot\beta} = \overset{s}{D}_{\alpha\alpha'}\,(A)\,\overset{s}{\overline{D}}_{\dot\beta\dot\beta'}\,(A)\,(p)_{\alpha'\dot\beta'}$, the new creation operators

$$a^*_{ex}(p)_\alpha = \varepsilon_{\alpha\,\beta}\,(p)^{\dot\gamma\beta}\,a^*_{ex}(p)_{\dot\gamma} \tag{2.6}$$

are equally $\tilde{P}_+^\uparrow$-covariant. Hence the type of spinor index (lower-upper, dotted—undotted) is irrelevant for the representation of the free field operators $a_{ex}\,(p)$ in a spinor basis, as well as for the scattering amplitudes defined by (2.8).

The S-matrix is the unitary mapping between the asymptotic outgoing and incoming states, defined by the following vacuum expectation values:

*) With summation over repeated indices.

$$\langle {}_{\varkappa_1}a_{out}\,(p_1)_{\alpha_1} \ldots S \ldots {}_{\varkappa_n}a^*_{out}\,(p_n)_{\alpha_n} \rangle_0 \equiv \langle {}_{\varkappa_1}a_{out}\,(p_1)_{\alpha_1} \ldots {}_{\varkappa_n}a^*_{in}\,(p_n)_{\alpha_n} \rangle_0 \,. \qquad (2.7)$$

For a Lorentz invariant scattering theory ($[S, U_{out}(a, A)] = 0$) the scattering amplitudes in the spinor basis (the M-functions of Stapp[39][4])

$$T^{\varkappa_1 \ldots \varkappa_n}_{\alpha_1 \ldots \alpha_n}\,(p_1 \ldots | \ldots p_n)\,\delta\,(p_1 + \ldots - p_n) \equiv \langle {}_{\varkappa_1}a_{out}\,(p_1)_{\alpha_1} \ldots (S-1) \ldots {}_{\varkappa_n}a^*_{out}\,(p_n)_{\alpha_n} \rangle_0 \qquad (2.8)$$

transform as tensor fields under the restricted homogeneous Lorentz group $L^\uparrow_+$:

$$T^{\varkappa_1 \ldots \varkappa_n}_{\alpha_1 \ldots \alpha_n}\,(p_1 \ldots | \ldots p_n) = \prod_{i=1}^{n} D^{s_{\varkappa_i}}_{\alpha_i \beta_i}\,(A^{-1})\, T^{\varkappa_1 \ldots \varkappa_n}_{\beta_1 \ldots \beta_n}\,(A\,p_1, \ldots | \ldots A\,p_n) \qquad (2.9)$$

The central problem is now to find a canonical representation for these covariant tensor fields $T^{(\varkappa)}_{(\alpha)}$, which satisfy the constraints of relativistic kinematics in a more transparent way. This is an unsolved problem in the interesting case, where $T^{(\varkappa)}_{(\alpha)}$ is a tensorvalued tempered distribution[3]). We shall have to make considerably stronger regularity assumptions in order to prove satisfactory results.

In recent years there has been much interest in demonstrating and applying the analyticity properties of relativistic S-matrix elements. In general quantum field theory the rigorous results of MANDELSTAM[32]), ZIMMERMANN[49]) and LEHMANN[30]) show that the 2-particle scattering amplitude for certain processes has an analytic continuation into an $L\,(C)$-invariant domain over the complex mass shell

$$M^{(\varkappa)} = \{(p_1, \ldots | \ldots p_n) : \sum_{i=1}^{n} p_i = 0,\; p_i^2 = m^2_{\varkappa_i},\; 1 \le i \le n\}\,, \qquad (2.10)$$

which contains physical points.

The assumption that the scattering amplitude $T^{(\varkappa)}_{(\alpha)}$ is holomorphic in an $L_+\,(C)$-invariant neighbourhood $U\,(\tilde{p})$ of one physical point $\tilde{p}$ on $M^{(\varkappa)}$ is further supported by the singularity structure of Feynman amplitudes in perturbation theory (see e.g. [10])). Then due to the $L^\uparrow_+$-covariance of $T^{(\varkappa)}_{(\alpha)}$ in a real neighbourhood of $\tilde{p}$ in $U\,(\tilde{p})$ the anylytic continuation (which we again denote by $T^{(\varkappa)}_{(\alpha)}$) transforms $L_+\,(C)$-covariantly in $U\,(\tilde{p})$:

$$T^{(\varkappa)}_{(\alpha)}\,(p) = D^{(\beta)}_{(\alpha)}\,(\Lambda^{-1})\; T^{(\varkappa)}_{(\beta)}\,(\Lambda\,p) \qquad (2.11)$$

for all $\Lambda \in L_+\,(C)$ and all $p \in U\,(\tilde{p})$. By maximal analytic continuation along $M^{(\varkappa)}$, $T^{(\varkappa)}_{(\alpha)}$ becomes an $L_+\,(C)$-covariant holomorphic tensor field on an $L_+\,(C)$-invariant unramified (in general non-schlicht) domain $(R, \pi, M^{(\varkappa)})$ over $M^{(\varkappa)}$ (see appendix A_1).

Therefore the following two theorems about the existence of a covariant singularity-free decomposition of covariant holomorphic tensor fields over the complex mass shell are applicable at least in the following cases: for certain 2-particle scattering amplitudes in general quantum field theory and without restrictions in analytic S-matrix theories[9][39][18]), where the existence of an $L_+\,(C)$-covariant analytic continuation of $T^{(\varkappa)}_{(\alpha)}$ is postulated.

The first theorem is a generalization of a theorem of BARGMANN, HALL and WIGHTMAN[20]). It gives necessary and sufficient conditions for an $L_+\,(C)$-invariant 'multivalued' holomorphic function on M to be a 'multivalued' holomorphic function of certain L_+ (C)-invariants. In more precise mathematical terms (for definitions see appendix A_1) we shall prove the ('$(+)$' refers to the full as well as to the proper group):

Theorem 1: (a) The complex mass shell M as well as its image $\hat{M}_{(+)}$ in the space $C^{r(+)}$ of the $L_{(+)}$ (C)-invariants is a normal algebraic set.

(b) To every $L_{(+)}$ (C)-invariant function T, holomorphic on an $L_{(+)}$ (C)-invariant domain (R, π, M), corresponds uniquely a holomorphic function $\hat{T}_{(+)}$ on $(\hat{R}_{(+)}, \hat{\pi}_{(+)}, \hat{M}_{(+)})$, such that on the $I_{(+)}$-saturated kernel $(R^{s(+)}, \pi, M)$ of (R, π, M) T is given by

$$T = \hat{T}_{(+)} \circ \hat{I}_{(+)} \tag{2.12}$$

Remark: On algebraic sets, such as M and $\hat{M}_{(+)}$, there exist in general different inequivalent notions of holomorphy or 'complex structures' (see e.g.[16]). On a normal algebraic set M all these structures coincide: every local holomorphic function on M has a convergent power series in the coordinates of the imbedding space in a neighbourhood of every point of holomorphy. The first part of theorem 1 therefore implies, that T and $\hat{T}_{(+)}$ have local (not uniquely determined) analytic continuations off the mass shell M and $\hat{M}_{(+)}$.

Proof: The normality of $M \subset C^{4n}$ has been proved in [23]). We remark that for $n \leq 2$ and for $n > 2$, when the masses $m_{\varkappa_i}$ cannot fulfil the condition $\sum_{i=1}^{n} m_{\varkappa_i} \sigma_i = 0$ for a choice of signs $\sigma_i = \pm 1$, $M^{(\varkappa)}$ is a complex manifold.

Let $\{V^i_{(+)}\}$ be the canonical covering of the $I_{(+)}$-saturated kernel of (R, π, M) defined by $(A, 4)$ in appendix A_1. Given a holomorphic tensor field T_α on (R, π, M) the functions $T_\alpha \circ \pi^{-1}(p)$ are strongly holomorphic on $U^{(+)}_{\varepsilon i}(p_i) \cap M$, since M is normal. By the theorem B of H. CARTAN[7]) there exist holomorphic functions $T^i_\alpha(p)$ in the domain of holomorphy $U^{(+)}_{\varepsilon i}(p_i)$ with

$$T_\alpha \circ \pi^{-1}(p) = T^i_\alpha(p) \text{ on } M \cap U^{(+)}_{\varepsilon i}(p_i)\,. \tag{2.13}$$

We take the mean value

$$\tilde{T}^i_\alpha(p) = \Omega^{-1}_{(+)} \int_{K_{(+)}} d\Lambda\, D_{\alpha\alpha'}(\Lambda^{-1})\, T^{\alpha'}_i(\Lambda p) \tag{2.14}$$

over the compact subgroup $K_{(+)}$ of $L_{(+)}(C)$

$$K_{(+)} = \{\Lambda \in L_{(+)}(C) : \Lambda = HMH^{-1}\,,\ M \in O_{(+)}(4, R)\}\,. \tag{2.15}$$

Here H is a square root of the metric form $G = (g^{\mu\nu})$ in Minkowski space, $O_{(+)}(4, R)$ is the real (proper) 4-dimensional orthogonal group and $\Omega_{(+)}$ is the volume of $K_{(+)}$ with respect to its Haar measure [21]) $d\Lambda$. Then by the uniqueness theorem $T_\alpha \circ \pi^{-1}(p)$ is on $U^{(+)}_{\varepsilon i}(p_i) \cap M$ trace of the $L_{(+)}(C)$-covariant tensor field $T^i_\alpha(p)$, holomorphic in $U^{(+)}_{\varepsilon i}(p_i)$. Let now T be holomorphic and $L_{(+)}(C)$-invariant on (R, π, M). Then according to [24]) an $L_{(+)}(C)$-invariant analytic continuation $T^i(p)$ of $T \circ \pi^{-1}(p)$ into each $U^{(+)}_{\varepsilon i}(p_i)$ determines a strongly holomorphic function $\hat{T}^i_{(+)}$ in $I_{(+)}(U^{(+)}_{\varepsilon i}(p_i))$. T defines uniquely the strongly holomorphic function $\hat{T}^i_{(+)} \mid_{\hat{M}_{(+)}}$ on $\hat{U}^i_{(+)} \equiv I_{(+)} (U^{(+)}_{\varepsilon_i} (p_i)) \cap \hat{M}_{(+)}$ with $\hat{T}^i_{(+)} \mid_{\hat{M}_{(+)}} = \hat{T}^j_{(+)} \mid_{\hat{M}_{(+)}}$ in $U^i_{(+)} \cap U^j_{(+)}$ for $\varepsilon^{ij} = 1$ (see appendix A_1). Therefore [6]) there exists a global strongly holomorphic function $\hat{T}_{(+)}$ on $(\hat{R}_{(+)}, \hat{\pi}_{(+)}, \hat{M}_{(+)})$ with (2.12) on $(R^{s(+)}, \pi, M)$. Finally the normality of $\hat{M}_{(+)}$ follows as in [24]).

Theorem 1 can be applied to the $L_{(+)}(C)$-invariant 'Riemann surface' (R_T, π, M) of a $L_{(+)}(C)$-invariant holomorphic germ[40]) T_p on M. Then $\hat{T}_{(+)}$ is the maximal (unramified) analytic continuation of T_p in the space of the $L_{(+)}(C)$-invariants. It can happen[23]) that $(R^{s(+)}_T, \pi, M)$ is a proper subdomain of (R_T, π, M) and that a representation (2.12) of T as an analytic function $\hat{T}_{(+)}$ of the invariants is not everythere possible in (R_T, π, M).

In the same notation, let $T^{(\varkappa)}_{(\alpha)}$ be an $L_{(+)}(C)$-covariant holomorphic tensor field in an $L_{(+)}(C)$-invariant domain (R, π, M) over M (e.g. in the unramified analytic configuration of $T^{(\varkappa)}_{(\alpha)}$). Then one has the

Theorem 2: In every holomorphically-convex $L_{(+)}(C)$-invariant subdomain $(R^{r(+)}, \pi, M)$ of $(R^{s(+)}, \pi, M)$ a holomorphic tensor field $T^{(\varkappa)}_{(\alpha)}$ can be decomposed

$$T^{(\varkappa)}_{(\alpha)} = \sum_{\lambda=1}^{L} T^{(\varkappa)}_{\lambda} \, Q^{\lambda}_{(\alpha)} \tag{2.16}$$

into a fixed finite set of tensor polynomials $Q^{\lambda}_{(\alpha)}$ with $L_{(+)}(C)$-invariant amplitudes $T^{(\varkappa)}_{\lambda}$, $1 \leq \lambda \leq L$, holomorphic in $(R^{H(+)}, \pi, M)$.

For the proof, we first remark that we need only consider tensor fields of irreducible representations $[2\,s, 0]$ of $L_+(C)$. For, by application of $\varepsilon_{\alpha\beta} (p_i)^{\dot{\gamma}\beta}$ and repeated reduction with the Clebsch-Gordon series, any $L_+(C)$-covariant function over M can be represented as a sum of such tensor fields without introducing singularities.

As in the proof of theorem 1, a local $L_+(C)$-covariant analytic continuation $T^{(\varkappa)i}_{(\alpha)}$ into $U^{(+)}_{\varepsilon_i}(p_i)$ gives [24]) a local covariant decomposition

$$T^{(\varkappa)i}_{(\alpha)} (p) = \sum_{\lambda=1}^{L} T^{(\varkappa)}_{\lambda} (p) \, Q^{\lambda}_{(\alpha)} (p) \tag{2.17}$$

in $U^{(+)}_{\varepsilon_i}(p_i)$ and therefore in $V^i_{(+)}$ (by restriction of (2.17) to M and by lifting with π).

For an irreducible representation $[2\,s, 0]$ and for a fixed n we have the following standard covariants $Q^{\lambda}_{(\alpha)}$ [23]):

$$\begin{aligned} &\text{(a) for } s = 0: \; 1, \; \det \mid p_{\lambda_1}, p_{\lambda_2}, p_{\lambda_3}, p_{\lambda_4} \mid \\ &\text{(b) for } s > 0: \; \underset{\alpha}{\mathrm{S}} \, (p_{\lambda_1} \cup p_{\lambda_2})_{\alpha_1\alpha_2} \cdots (p_{\lambda_{2s-1}} \cup p_{\lambda_{2s}})_{\alpha_{2s-1}\,\alpha_{2s}} . \end{aligned} \tag{2.18}$$

Here $(p_i \cup p_r)_{\alpha\beta} = (p_i)_{\alpha\dot{\gamma}} (p_r)_{\beta\dot{\delta}} \, \varepsilon^{\dot{\gamma}\dot{\delta}}$, and $\underset{\alpha}{\mathrm{S}}$ stands for the total symmetrization of the 2-spinor indices $\alpha_1, \ldots \alpha_{2s}$ [43]). The $\lambda_i \in \{1, \ldots n\}$ are to be chosen to give all (up to a sign) different covariants (2.18). On $M^{\varkappa_1 \cdots \varkappa_n}$ one can eliminate e.g. p_n by momentum conservation. Further, because of $p_i^2 = m_{\varkappa_i}^2 > 0$, the only remaining relation between the covariants (2.18) for $n = 4$ and non-vanishing Gram determinant $G(p_1, p_2, p_3) \neq 0$ is

$$\sum_{i,j=1}^{3} (p_i, p_j) \underset{\alpha}{\mathrm{S}} M^i_{\alpha_1\alpha_2} M^j_{\alpha_3\alpha_4} = 0 \tag{2.19}$$

with $M^1_{\alpha\dot\beta} = (p_2 \cup p_3)_{\alpha\dot\beta}$, cycl. After having solved (2.19) e.g. for $\underset{\alpha}{\mathrm{S}} M^1_{\alpha_1\alpha_2} M^1_{\alpha_3\alpha_4}$ the remaining $(2s+1)$ covariants

$$\begin{aligned} &\underset{\alpha}{\mathrm{S}} M^2_{\alpha_1\alpha_2} \dots M^2_{\alpha_{2r-1}\alpha_{2r}} M^3_{\alpha_{2r+1}\alpha_{2r+2}} \dots M^3_{\alpha_{2s-1}\alpha_{2s}} \quad (1 \le r \le s) \\ &\underset{\alpha}{\mathrm{S}} M^1_{\alpha_1\alpha_2} M^2_{\alpha_3\alpha_4} \dots M^2_{\alpha_{2r-1}\alpha_{2r}} M^3_{\alpha_{2r+1}\alpha_{2r+2}} \dots M^3_{\alpha_{2s-1}\alpha_{2s}} \quad (2 \le r \le s) \end{aligned} \tag{2.20}$$

are linearly independent in $\mathfrak{S}^{[2s,0]}$ (for the proof one computes (2.20) for $(p_i)_{\alpha\dot\beta} = (\delta_i)_{\alpha\dot\beta}$, $1 \le i \le 3$, to which arbitrary p_1, p_2, p_3 with $G(p_1, p_2, p_3) \neq 0$ can be non-singularly transformed).

Therefore for $n \le 4$ the decomposition (2.17) is unique on M, hence on $R^{s(+)}$, and one obtains global $L_{(+)}(C)$-invariant amplitudes, holomorphic on $(R^{s(+)}, \pi, M)$.

For $n \ge 5$ this is no longer generally true. Here the existence of global holomorphic invariant amplitudes can only be guaranteed in every holomorphically-convex subdomain $(R^{H(+)}, \pi, M)$, using methods of analytic sheaf theory [24]).

The usefulness of a covariant holomorphic decomposition (2.16) of analytic S-matrix elements is, even if it exists globally, rather limited. Firstly, it is in general not convenient to describe a relativistic scattering process by functions of the Lorentz invariants, since their number $r(+)$ is in general much larger than the dimensionality $4n$ of the space of n 4-vectors. Secondly, for $n > 4$ the number of (local) singularity-free invariant amplitudes exceeds considerably the number $\prod_{i=1}^{n} (2 s_{\varkappa_i} + 1)$ of spinor amplitudes for the process $(\varkappa_1, \dots \varkappa_n)$. Thus the invariant amplitudes loose their physical significance, being no longer in one-to-one correspondence with the (sometimes) observable spin-state amplitudes.

The problems connected with the non-uniqueness of the decomposition (2.16) can be overcome, if one admits for $n > 4$ meromorphic invariant amplitudes. In the points P, where the Gramian is different from zero for certain $p_i, p_j, p_k \in \{\pi(P)_1, \dots \pi(P)_n\}$, a unique decomposition is possible with respect to the covariants (2.20) formed from these p_i, p_j, p_k. As the determinant

$$\det \left| Q^{\lambda_{ijk}}_{\alpha_1 \cdots \alpha_n}(p_i, p_j, p_k)\, Q^{\mu_{ijk}}_{\beta_1 \cdots \beta_n}(p_i, p_j, p_k) \prod_{i=1}^{n} \varepsilon^{\alpha_i\beta_i} \right| \tag{2.21}$$

is proportional to $G(p_i, p_j, p_k)^a (p_i, p_i)^b$, the amplitudes $T^{(\varkappa)}_{\lambda_{ijk}}(p_1, \dots p_n)$ can be computed by solving the system of linear equations:

$$\begin{aligned} &\prod_{i=1}^{n} \varepsilon^{\alpha_i\beta_i}\, T^{(\varkappa)}_{\alpha_1 \dots \alpha_n}(p_1, \dots p_n)\, Q^{\mu_{ijk}}_{\beta_1 \dots \beta_n}(p_i, p_j, p_k) = \\ &= \sum_{\lambda_{ijk}=1}^{L} \prod_{i=1}^{n} \varepsilon^{\alpha_i\beta_i}\, T^{(\varkappa)}_{\lambda_{ijk}}(p_1, \dots p_n)\, Q^{\lambda_{ijk}}_{\alpha_1 \dots \alpha_n}(p_i, p_j, p_k)\, Q^{\mu_{ijk}}_{\beta_1 \dots \beta_n}(p_i, p_j, p_k) \end{aligned} \tag{2.22}$$

A meromorphic decomposition certainly holds in any $L_+ (C)$-invariant domain (R, π, M). The additional information from theorems 1 and 2 is that in the saturated kernel $(R^{s(+)}, \pi, M)$ the meromorphic invariant amplitudes are strongly meromorphic functions of the $L_{(+)} (C)$-invariants, and that their singularities can be resolved by a local holomorphic decomposition (2.17).

The meromorphic decompositions for different choices of p_i, p_j, p_k for $n > 4$ will be important for the proof of lemma 1 in appendix A_2. They are also a convenient tool to discuss the restrictions due to discrete symmetries on the S-matrix. This may be done along the same line as in § 3. for the 2-particle scattering amplitudes. One finds, by choosing covariants $Q^{\lambda_{ijk}}$ with a definite parity under the space inversion (3.21), that certain invariant meromorphic amplitudes $T_{\lambda_{ijk}}$ will vanish identically for a Π-invariant scattering process. The other discrete symmetries will further restrict the functional behaviour of the $T_{\lambda_{ijk}}$.

§ 3. 2-Particle Scattering Amplitudes and Discrete Symmetries

The covariant holomorphic decomposition (2.16) of the physically important 2-particle scattering amplitude expresses in an optimal way the connection between $\tilde{P}_+^{\uparrow}$-covariance and the (here generally accepted) analyticity in $(p_1, \dots p_4)$. The well known applications in the framework of the Mandelstam representation exemplify the usefulness of this decomposition.

Essential for the success of this approach is the uniqueness of the representation (2.16) on $M^{\varkappa_1 \dots \varkappa_4}$. This guarantees the holomorphy of the invariant amplitudes $T_\lambda^{(\varkappa)}$ in the saturated kernel $(R^{s(+)}, \pi, M)$, and that the $T_\lambda^{(\varkappa)}$ can be globally obtained by solving the system of linear equations (2.22). Over $M^{\varkappa_1 \dots \varkappa_4}$ every I_+-saturated set is I-saturated, and the mass shell M is a 2-dimensional linear manifold. A highly symmetric choice of the independent invariants is given by [31]):

$$s = (p_1 + p_2)^2 , \quad t = (p_1 - p_4)^2 \tag{3.1}$$

Hence the canonical representation of every 2-particle scattering amplitude $T_{(\alpha)}^{(\varkappa)}$, holomorphic in (R, π, M), is given by $\prod_{i=1}^{4} (2 s_{\varkappa_i} + 1)$ $L (C)$-invariant amplitudes $T_\lambda^{(\varkappa)}$, which are holomorphic in the domain $(I (R^s), \hat{\pi})$ over the $C^2 (s, t)$. The Mandelstam representation for $T_{(\alpha)}^{(\varkappa)}$ and the analytic continuation in complex angular momentum (see e.g. [4])) rely strongly on these analyticity properties.

In addition to $\tilde{P}_+^{\uparrow}$-covariance general principles of relativistic quantum mechanics (see e.g. [28])[42])) further restrict the relativistic S-matrix elements: by the Pauli principle and the $T C \Pi$-invariance.

The Pauli principle is expressed by the normal connection between spin and statistics in the Fock bases of the incoming and outgoing states. Formulae (2.2) and (2.3) imply for the scattering amplitudes the symmetry

$$T_{\alpha_1 \dots \alpha_n}^{\varkappa_1 \dots \varkappa_n} (p_1, \dots | \dots p_n) = \sigma(0)\, T_{\alpha_{o(1)} \dots \alpha_{o(n)}}^{\varkappa_{o(1)} \dots \varkappa_{o(n)}} (p_{o(1)} \dots | \dots p_{o(n)}) \tag{3.2}$$

Here 0 is a permutation, which does not interchange incoming with outgoing particles, and σ (0) is the signature of the fermion permutation under 0.

For the formulation of $T\,C\,\Pi$-invariance we first define in the Fock basis a class of unitary and antiunitary representations of the factor group $P/P_+^\uparrow$. Here we restrict ourselves to such representations [3][17], which are compatible with a positive semidefinite energy spectrum and which are local with respect to the asymptotic free field operators [12].

$$
{}_\varkappa\varphi_{ex}(x)_\alpha = (2\,\pi)^{-3/2} \int\limits_{p^0=\omega_p} \frac{d^3p}{2\,\omega_p} \{{}_\varkappa a_{ex}(p)_\alpha\, e^{-ipx} + {}_{\bar\varkappa} a^*_{ex}(p)_\alpha\, e^{ipx}\} \tag{3.3}
$$

$$
{}_\varkappa\chi_{ex}(x)^{\dot\beta} = D^s_{\beta\alpha}\left(\frac{i}{m}(\partial_0 - \boldsymbol{\partial}\cdot\boldsymbol{\sigma})\right) {}_\varkappa\varphi_{ex}(x)_\alpha \tag{3.4}
$$

The unitary representations (2.1) can be extended to a local representation of the full group $\tilde P$ by the unitary representation of the space inversion

$$
\Pi_{ex}\,{}_\varkappa\varphi_{ex}(x)_\alpha\,\Pi_{ex}^{-1} = {}_\varkappa\eta^\Pi_{ex}\,{}_\varkappa\chi_{ex}(\Pi p)^{\dot\alpha} \tag{3.5}
$$

and by the antiunitary representation T_{ex} of the time reflection

$$
T_{ex}\,{}_\varkappa\varphi_{ex}(x)_\alpha\,T_{ex}^{-1} = {}_\varkappa\eta^T_{ex}\,{}_\varkappa\varphi_{ex}(Tx)^\alpha\,. \tag{3.6}
$$

Together with the unitary representation C_{ex} of the charge conjugation

$$
C_{ex}\,{}_\varkappa\varphi_{ex}(x)_\alpha\,C_{ex}^{-1} = {}_\varkappa\eta^C_{ex}\,{}_\varkappa\chi^*_{ex}(x)_\alpha \tag{3.7}
$$

and the gauge transformations

$$
U_{ex}(\lambda)\,{}_\varkappa\varphi_{ex}(x)_\alpha\,U_{ex}(\lambda)^{-1} = e^{i\lambda q_\varkappa}\,{}_\varkappa\varphi_{ex}(x)_\alpha \tag{3.8}
$$

(with λ, $q_\varkappa$ real) they generate all unitary local representations of $P/P_+^\uparrow$ to positive energy. With

$$
({}_\varkappa\eta^\Pi_{ex})^2 = |\,{}_\varkappa\eta^T_{ex}\,| = |\,{}_\varkappa\eta^C_{ex}\,| = 1
$$

one has

$$
\Pi^2 = \pm 1\,,\ C^2 = 1\,,\ T^2 = (\Pi\,T)^2 = (-1)^{2s}
$$

in the Fock space of the $\varkappa\,\bar\varkappa$-particle states.

General principles imply [27][39], that the antiunitary involution

$$
\theta_{ex}\,{}_\varkappa\varphi_{ex}(x)_\alpha\,\theta_{ex} = \begin{cases} {}_\varkappa\varphi^*_{ex}(-x)_{\dot\alpha} & \text{for } s_\varkappa \equiv 0\ (1) \\ i\,{}_\varkappa\varphi^*_{ex}(-x)_{\dot\alpha} & \text{for } s_\varkappa \equiv \frac{1}{2}\ (1) \end{cases} \tag{3.9}
$$

(coinciding with the product $T_{ex}\,C_{ex}\,\Pi_{ex}$ for special choices of the phases η) is a symmetry of a relativistic scattering theory: $S\,\theta_{out} = \theta_{out}\,S^*$. This implies for the scattering amplitudes

$$
T^{\varkappa_1\ldots\varkappa_n}_{\alpha_1\ldots\alpha_n}(p_1,\ldots\,|\,\ldots p_n) = \sigma(X)\,T^{\varkappa_n\ldots\varkappa_1}_{\alpha_n\ldots\alpha_1}(p_n\ldots\,|\,\ldots p_1) \tag{3.10}
$$

with X: $(1, \ldots n) \to (n, \ldots 1)$ and $\sigma(X)$ defined as in (3.2). If in addition the theory is covariant under Π, T or C (i.e. $[\Pi_{out}, S] = 0$, $T_{out} S^* = S\, T_{out}$ or $[C_{out}, S] = 0$ for certain phases η), then the scattering amplitudes are further restricted by

$$T^{\varkappa_1 \ldots \varkappa_n}_{\alpha_1 \ldots \alpha_n} (p_1 \ldots | \ldots p_n) = {}_{(\varkappa)}\eta^{\Pi} (\Pi p_1)^{\dot{\alpha}_1 \beta_1} \cdots (\Pi p_n)^{\dot{\alpha}_n \beta_n}\ T^{\varkappa_1 \ldots \varkappa_n}_{\beta_1 \ldots \beta_n} (\Pi p_1, \ldots | \ldots \Pi p_n)$$

$$(\Pi\text{-invariance}, {}_{(\beta)}\eta^{\Pi} = \pm 1) \tag{3.11}$$

$$T^{\varkappa_1 \ldots \varkappa_n}_{\alpha_1 \ldots \alpha_n} (p_1 \ldots | \ldots p_n) = {}_{(\varkappa)}\eta^{T} (\Pi p_n)^{\dot{\alpha}_n \beta_n} (\Pi p_1)^{\dot{\alpha}_1 \beta_1}\ T^{\bar{\varkappa}_n \ldots \bar{\varkappa}_1}_{\beta_n \ldots \beta_1} (\Pi p_n \ldots | \ldots \Pi p_1)$$

$$(T\text{-invariance}, | {}_{(\varkappa)}\eta^{T} | = 1) \tag{3.12}$$

$$T^{\varkappa_1 \ldots \varkappa_n}_{\alpha_1 \ldots \alpha_n} (p_1 \ldots | \ldots p_n) = {}_{(\varkappa)}\eta^{c}\ T^{\bar{\varkappa}_1 \ldots \bar{\varkappa}_n}_{\alpha_1 \ldots \alpha_n} (p_1 \ldots | \ldots p_n)$$

$$(C\text{-invariance}, | {}_{(\varkappa)}\eta^{c} | = 1) \tag{3.13}$$

We now discuss the restrictions on the invariant amplitudes $T^{(\varkappa)}_{\lambda}$ by (3.2), (3.10) and possibly by (3.11), (3.12), (3.13). For that purpose we remark that the Mandelstam invariants (3.1) remain unchanged under a transformation $(p_1, \ldots p_4) \to (p_{0(1)}, \ldots p_{0(4)})$ with 0 equal to $X = (14)(23)$, $Y = (12)(34)$ or $X\,Y = (13)(24)$. Let $\{Q^{\lambda}_{(\varkappa)}(p)$: $1 \leq \lambda \leq L = \prod_{i=1}^{4} (2\, s_{\varkappa_i} + 1)\}$ be a covariant basis of the representation $\bigotimes_{i=1}^{4} [2\, s_{\varkappa_i}, 0]$ on $M^{\varkappa_1 \ldots \varkappa_4}$. If one decomposes for $0 = X, Y$ $\; Q^{\lambda}_{\alpha_{o(1)} \ldots \alpha_{o(4)}} (p_{o(1)} \ldots p_{o(4)})$ and $(\Pi p_1)^{\dot{\alpha}_1 \beta_1} \ldots (\Pi p_4)^{\dot{\alpha}_4 \beta_4} Q^{\lambda}_{\beta_1 \ldots \beta_4} (\Pi p_1, \ldots \Pi p_4)$ with respect to this basis, one obtains

$$Q^{\lambda}_{\alpha_4 \alpha_3 \alpha_2 \alpha_1} (p_4, p_3, p_2, p_1) = \sum_{\mu=1}^{L} X(s, t)_{\lambda\mu} Q^{\mu}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} (p_1, p_2, p_3, p_4)$$

$$Q^{\lambda}_{\alpha_2 \alpha_1 \alpha_4 \alpha_3} (p_2, p_1, p_4, p_3) = \sum_{\mu=1}^{L} Y(s, t)_{\lambda\mu} Q^{\mu}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} (p_1, p_2, p_3, p_4)$$

$$(\Pi p_1)^{\dot{\alpha}_1 \beta_1} \cdots (\Pi p_4)^{\dot{\alpha}_4 \beta_4}\ Q^{\lambda}_{\beta_1 \ldots \beta_4} (\Pi p_1, \ldots \Pi p_4) = \sum_{\mu=1}^{L} Z(s, t)_{\lambda\mu} Q^{\mu}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} (p_1, p_2, p_3, p_4)\,. \tag{3.14}$$

Evidently the 3 operations (3.14) commute and are involutions. Because of the uniqueness of the decomposition (3.14), the matrices $X(s, t)$, $Y(s, t)$ and $Z(s, t)$ are $L(C)$-invariant commuting involutions, holomorphic on $M^{\varkappa_1 \ldots \varkappa_4}$. They can be simultaneously diagonalized on $M^{\varkappa_1 \ldots \varkappa_4}$ by a nonsingular linear transformation

$$\tilde{Q}^{\lambda}_{(\alpha)} (p) = \sum_{\mu=1}^{L} A(s, t)_{\lambda\mu}\, Q^{\mu}_{(\alpha)} (p) \tag{3.15}$$

The existence of globally holomorphic matrix elements $A(s, t)_{\gamma\mu}$ follows from the triviality of the vector space bundles generated by the projections $1/2\,(1 \pm X(s, t))$,

$1/2\,(1 \pm Y(s,t))$, $1/2\,(1 \pm Z(s,t))$ over the C^2 using a theorem by GRAUERT (Math. Ann. *135*, 263 (1958), Theorem 6). In terms of the covariants $\tilde{Q}^{\lambda}$, which have now definite parities $\sigma_X^{\lambda}, \sigma_Y^{\lambda}, \sigma_Z^{\lambda} = \pm 1$ under (3.14), the restrictions due to the discrete symmetries can be easily discussed.

(1.) Pauli principle and $T C \Pi$-invariance: In a scattering process between identical initial or final particles (and sometimes between identical ingoing particles and outgoing antiparticles) these symmetries together entail useful restrictions on the invariant amplitudes $T_{\lambda}^{(\varkappa)}$. Suppose that $(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4) = (\varkappa_{o(1)}, \varkappa_{o(2)}, \varkappa_{o(3)}, \varkappa_{o(4)})$ for $0 \in \{X, Y, X\,Y\}$. Then (3.2) and (3.10) induce the relation

$$T^{\varkappa_1 \cdots \varkappa_4}_{\alpha_1 \cdots \alpha_4}(p_1 \cdots | \cdots p_4) = \sigma\,(0)\; T^{\varkappa_1 \cdots \varkappa_4}_{\alpha_{o(1)} \;\alpha_{o(4)}}(p_{o(1)} \cdots | \cdots p_{o(1)}) \tag{3.16}$$

with the signature $\sigma\,(0) = \pm 1$ of the fermion permutations in 0. Let now both sides of (3.16) be decomposed with respect to the covariants (3.15). Then the invariant amplitudes $T_{\lambda}^{(\varkappa)}(s,t)$ with $\sigma_0^{\lambda} \neq \sigma\,(0)$ $(\sigma_{XY}^{\lambda} \equiv \sigma_X^{\lambda}\,\sigma_Y^{\lambda})$ vanish identically, because of the invariance of s and t under 0.

The Pauli principle also restricts the amplitudes to processes of the type $(\lambda\,\lambda \mid \mu\,\nu)$ or $(\lambda\,\mu \mid \nu\,\nu)$. Here it is convenient to choose covariants $\approx\!\!\!\!Q^{\lambda}$ with definite parities $\sigma^{\lambda} = \pm 1$ under the transformation

$$Q^{\lambda}_{\alpha_1\alpha_2\alpha_3\alpha_4}(p_1, p_2, p_3, p_4) \to Q^{\lambda}_{\alpha_2\alpha_1\alpha_3\alpha_4}(p_2, p_1, p_3, p_4)$$

$$= \sum_{\mu=1}^{L} W_{\lambda\mu}(s,t)\; Q^{\mu}_{\alpha_1\alpha_2\alpha_3\alpha_4}(p_1, p_2, p_3, p_4) \tag{3.17}$$

and under $Q^{\lambda}_{\alpha_1\alpha_2\alpha_3\alpha_4}(p_1, p_2, p_3, p_4) \to Q^{\lambda}_{\alpha_1\alpha_2\alpha_4\alpha_3}(p_1, p_2, p_4, p_3)$.
Then one obtains with $u = (p_1 - p_3)^2$:

$$T_{\lambda}^{(\varkappa)}(s,t) = \sigma_{\lambda}\, T_{\lambda}^{(\varkappa)}(s,u) \tag{3.18}$$

(2.) T- and C-invariance relate the scattering amplitudes belonging to the in general different processes $(\varkappa_1\,\varkappa_2 \mid \varkappa_3\,\varkappa_4)$ and $(\bar{\varkappa}_1\,\bar{\varkappa}_2 \mid \bar{\varkappa}_3\,\bar{\varkappa}_4)$. Again, if $(\varkappa_1\,\varkappa_2 \mid \varkappa_3\,\varkappa_4) = (\bar{\varkappa}_{o(1)}\,\bar{\varkappa}_{o(2)} \mid \bar{\varkappa}_{o(3)}\,\bar{\varkappa}_{o(4)})$ holds for $0 \in \{X, Y, X\,Y\}$, then one has in case of T-invariance

$$T^{\varkappa_1 \cdots \varkappa_4}_{\alpha_1 \cdots \alpha_4}(p_1, \cdots | \cdots p_4) = \tag{3.19}$$

$$= {}_{(\varkappa)}\eta^{T} \cdot \sigma(0)\,\sigma(X)\,(\Pi\,p_{o(1)})^{\dot{\alpha}_{o(1)}\beta_1} \cdot (\Pi\,p_{o(4)})^{\dot{\alpha}_{o(4)}\beta_4}\; T^{\varkappa_1 \cdots \varkappa_4}_{\beta_1 \cdots \beta_4}(\Pi\,p_{o(1)} \cdots | \cdots \Pi\,p_{o(4)})$$

and for C-invariance

$$T^{\varkappa_1 \cdots \varkappa_4}_{\alpha_1 \cdots \alpha_4}(p_1, \cdots | \cdots p_4) = {}_{(\varkappa)}\eta^{C} \cdot \sigma(0)\; T^{\varkappa_1 \cdots \varkappa_4}_{\alpha_{o(1)} \cdots \alpha_{o(4)}}(p_{o(1)} \cdots | \cdots p_{o(4)}) \tag{3.20}$$

with phase factors ${}_{(\varkappa)}\eta^{T} = {}_{(\varkappa)}\eta^{C} = \pm 1$. Therefore the invariant amplitudes $T_{\lambda}^{(\varkappa)}$ in the basis (3.15) vanish identically for $\sigma_Z^{\lambda}\,\sigma_0^{\lambda} \neq {}_{(\varkappa)}\eta^{T}\,\sigma(0)\,\sigma(X)$ or for $\sigma_0^{\lambda} \neq {}_{(\varkappa)}\eta^{C}\,\sigma(0)$. For 2-body processes of the type $(\lambda\,\bar{\lambda} \mid \mu\,\nu)$ or $(\mu\,\nu \mid \lambda\,\bar{\lambda})$ with Majorana particles μ, ν, T- and C-invariance entail relations of the kind (3.18) between the invariant amplitudes at different points.

(3.) Π-invariance always relates the scattering amplitudes of the same process at the points $(p_1, \dots p_n)$ and $(\Pi p_1, \dots \Pi p_n)$, which have the same $L(C)$-invariants. Therefore the $T^{(\varkappa)}_{\lambda}$ for a Π-invariant 2-body scattering process vanish for all λ with $\sigma_Z^{\lambda} \neq {}_{(\varkappa)}\eta^{\Pi}$. An analogous statement holds in a Π-invariant production process for all sets of unique meromorphic amplitudes $\{T^{(\varkappa)}_{\lambda ijk}\}$ (2.16) corresponding to covariants

$$\tilde{Q}^{\lambda ijk}_{\alpha_1 \dots \alpha_n}(p_1, \dots p_n) = \pm (\Pi p_1)^{\alpha_1 \dot{\beta}_1} (\Pi p_n)^{\alpha_n \dot{\beta}_n} \; Q^{\lambda ijk}_{\beta_1 \dots \beta_n}(\Pi p_1, \dots \Pi p_n) \quad (3.21)$$

with well-defined Π-parities for $G(p_i, p_j, p_k) \neq 0$.

It is easy to establish the connection of the invariant amplitudes with the spin-state amplitudes $T^{\varkappa_1 \dots \varkappa_n}_{\mu_1 \dots \mu_n}(p_1 \dots | \dots p_n)$ describing processes between particles $\varkappa_i$ of momentum p_i and the spin component μ_i in a characteristic direction. For each 4-vector $p = (\sqrt{m^2 + \boldsymbol{p}^2}, \boldsymbol{p})$ we choose a fixed $L(p) \in SL(2, C)$ with $L(p)\, p = (m, \boldsymbol{o})$. Then the creation operators

$$a^*(p)_{\mu} = \overline{D^s_{\mu\alpha}(L(p))}\, a^*(p)_{\dot{\alpha}} \quad (3.22)$$

create from the vacuum 1-particle states $| p, \mu \rangle$ transforming as

$$U(a, A) \,| p, \mu \rangle = e^{iAp \cdot a} \,| A p, \mu' \rangle\, D^s_{\mu' \mu}(R(A, p)) \quad (3.23)$$

with $R(A, p) = L(Ap)\, A\, L^{-1}(p) \in SU(2, C)$. The $| p, \mu \rangle$ are eigenstates of the 3-component of the spin-operator

$$S_r(p) = (2m)^{-1} \sum_{\alpha,\beta,\gamma,\delta = 0}^{3} \varepsilon_{\alpha\beta\gamma\delta}\, L(P)^{\alpha}_{r}\, P^{\beta}\, M_{\gamma\delta} \quad (3.24)$$

corresponding to the eigenvalue μ. The scattering amplitudes (2.5) expressed by the operators ${}_{\varkappa}a^{(*)}_{ex}(p)_{\mu}$ define the spin-state amplitudes corresponding to $S_3(p)$. For instance, the helicity amplitudes [25] for a 2-particle scattering process are obtained by choosing $(p_1, \dots | \dots p_4)$ in the centre-of-mass system and by the choice (with the polar angles ϑ, φ of $\boldsymbol{p}$):

$$L(p) = m^{-1/2} \begin{pmatrix} \sqrt{p_0 - |\boldsymbol{p}|}\, e^{i\varphi/2} \cos\frac{\vartheta}{2}, & \sqrt{p_0 - |\boldsymbol{p}|}\, e^{-i\varphi/2} \sin\frac{\vartheta}{2} \\ -\sqrt{p_0 + |\boldsymbol{p}|}\, e^{i\varphi/2} \sin\frac{\vartheta}{2}, & \sqrt{p_0 + |\boldsymbol{p}|}\, e^{-i\varphi/2} \cos\frac{\vartheta}{2} \end{pmatrix} \quad (3.25)$$

The spinor amplitudes and the spin-state amplitudes are via (3.22) in a 1-to-1 correspondence for physical p. Equally, the spinor amplitudes and the invariant amplitudes are biholomorphically connected for $G(p_1, p_2, p_3) \neq 0$. Therefore one has for the discrete symmetries generated by X, Y and Z just as many nonvanishing invariant amplitudes as there are independent helicity amplitudes.

As an illustration we shall discuss in appendix B the invariants and symmetries for all spin $^1/_2$–spin $^1/_2$ scattering processes.

Appendix A: On the Complex Structure of Analytic Scattering Amplitudes

From the general standpoint of analytic function theory it is unnatural to restrict the discussion of the 'multivalued' scattering amplitudes $T^{(\varkappa)}_{(\alpha)}$ to a so-called' physical sheet' [39]): Firstly, it is not at all clear that such a 'physical sheet' can be defined. Secondly, a reasonable behaviour of the invariant amplitudes (2.12), (2.16) can only be proved in the interior of the domain of analyticity of $T^{(\varkappa)}_{(\alpha)}$. Since the properties of the scattering amplitudes 'across the cuts' are of importance e.g. for the Mandelstam representation, any restriction of the analyticity domain of $T^{(\varkappa)}_{(\alpha)}$ should only be made at later stages of the calculation. Lastly, we shall discuss in this appendix the simultaneous analytic continuation of holomorphic tensor fields, where one has to consider (in general) non-schlicht hulls of holomorphy over M and $\hat{M}_{(+)}$. But it will turn out that apart from topological complications all results are essentially the same as in the schlicht case.

A_1: *Domains over the complex mass shell.*

The analytic configuration of a 'multivalued' holomorphic function on M is – except for ramification points – a domain (R, π, M), defined by a Hausdorff space R and a locally topological mapping π from R into M (for an introduction into the theory of functions of several complex variables see e.g. [5])[46])). If π is a homeomorphism, then (R, π, M) is called schlicht.

A complex-valued continuous function f on a domain $D \subset R$ is holomorphic, if for every $P \in D$ there exists a schlicht neighbourhood $U(P)$, such that $f \circ \pi^{-1}$ is holomorphic on $\pi\,(U(P) \cap D) \subset M$. Therefore (R, π, M) has locally the structure of the normal algebraic set $M \subset C^{4n}$ and $f \circ \pi^{-1}$ is locally strongly holomorphic.

The maximal analytic continuation f along M of a locally convergent power series generates a set R_f of holomorphic germs $f_p, p \in M$. By the usual topologization one obtains a domain (R_f, π, M) over M with $\pi\,(f_p) = p$. On (R_f, π, M) f is unique and holomorphic and separates points $P, P' \in R_f$ over the same ground-point $\pi\,(P) = \pi\,(P')$. A domain (R, π, M) is called $L_{(+)}(C)$-invariant, if there exists a schlicht open $L_{(+)}(C)$-invariant covering $\mathfrak{U} = \{U_\alpha\}$ of R (i.e. with $L_{(+)}(C)\,\pi\,(U_\alpha) = \pi\,(U_\alpha)$). Then $L_{(+)}(C)$ operates as a complex Lie group on (R, π, M):

$$\Lambda\,(P) = (\pi \mid_{U_\alpha})^{-1} \circ \Lambda \circ \pi\,(P)\,,\quad U_\alpha \in \mathfrak{U}\,,\ \Lambda \in L_{(+)}\,(C), \mathrm{P} \in \mathrm{U}_\alpha \tag{A.1}$$

A holomorphic vector field F^α on an $L_{(+)}(C)$-invariant domain (R, π, M) is called $L_{(+)}(C)$-covariant, if in every $\pi\,(U_\beta)$, $U_\beta \in \mathfrak{U}$, $F^\alpha \circ \pi^{-1}$ has a tensor transformation law under $L_{(+)}(C)$.

An $L_{(+)}(C)$-invariant domain is generated by the maximal analytic continuation along M of a $L_{(+)}(C)$-covariant holomorphic function $T^\alpha\,(p)$ due to the permanence of the functional equation*):

$$T^\alpha\,(p) = \sum_{\alpha'} D^\alpha{}_{\alpha'}\,(\Lambda^{-1})\;T^{\alpha'}\,(\Lambda\,p) \tag{A.2}$$

*) This question has also been treated by H. P. Stapp[40]). I thank Dr. Stapp for correspondence.

Let $I_{(+)}$ be the mapping, which maps $p = (p_1, \dots p_n) \in C^{4n}$ on the $r\,(+) = (n+1)\,n/2$ $(+ \max\{0, (n/4)\})$ essentially different typical invariants [14] of $L_{(+)}(C)$ (viz. the scalar products (p_i, p_j), $1 \leq i \leq j \leq n$ and possibly the determinants det $| p_i, p_j, p_r, p_s |$, $1 \leq i < j < r < s \leq n$). Then an open set $D \subset C^{4n}$ (resp. in $M^{(\varkappa)}$) is called $I_{(+)}$-saturated, if $D = I_{(+)}^{-1} \circ I_{(+)}(D)$ holds or, equivalently, if D contains for every $p \in D$ the closed $L_{(+)}(C)$-orbit

$$\overline{B_{(+)}(p)} = \overline{\{\Lambda\, p : \Lambda \in L_{(+)}(C)\}} \tag{A.3}$$

or, equivalently, a regular point [24] $\check{p}$ with $I_{(+)}(\check{p}) = I_{(+)}(p)$. Examples of I-saturated open sets are the extended tubes $\mathfrak{T}_n' \subset C^{4n}$ [45] and the Mandelstam domain [31] on $M^{\varkappa_1 \dots \varkappa_4}$. If one omits from a $L_{(+)}(C)$-invariant domain D (in C^{4n} or an $M^{(\varkappa)}$) the points p with $\overline{B_{(+)}(p)} \not\subset D$, then one obtains the $I_{(+)}$-saturated kernel $D^{s(+)}$ of D. It follows from [20][24] that $D^{s(+)}$ is open and connected for connected D.

A domain (R, π, M) is called $I_{(+)}$-saturated, if there exists a $I_{(+)}$-saturated schlicht covering $\mathfrak{U} = \{U_\alpha\}$ of R (i.e. with $I_{(+)}^{-1} \circ I_{(+)}\, (\pi\,(U_\alpha)) = \pi\,(U_\alpha)$). The $I_{(+)}$-saturated kernel $(R^{s(+)}, \pi, M)$ of a $L_{(+)}\,(C)$-invariant domain (R, π, M) is canonically related to a domain $(\hat{R}_{(+)}, \hat{\pi}_{(+)}, \hat{M}_{(+)})$ over the complex mass shell $\hat{M}_{(+)}$ in the space $C^{r(+)}$ of the $L_{(+)}(C)$-invariants. For the proof, we remark that $(R^{s(+)}, \pi, M)$ has a countable schlicht covering $\{V_{(+)}^i, 1 \leq i < \infty\}$, where $\pi\,(V_{(+)}^i) = U_{(+)}^i$ have the form $U_{\varepsilon_i}^{(+)}(p_i) \cap M$ with

$$U_{\varepsilon_i}^{+}\,(p_i) = \{p : | I_{(+)}^{\varrho}\,(p) - I_{(+)}^{\varrho}\,(p_i) | < \varepsilon_i\,, 1 \leq \varrho \leq r\,(+)\} \tag{A.4}$$

and where furthermore, if we set for $U_{(+)}^i$, $U_{(+)}^j$

$$\varepsilon^{ij} = \begin{cases} 0 \text{ if } V_{(+)}^i \cap V_{(+)}^j = \phi \\ 1\,, \text{ otherwise} \end{cases}$$

$\{V_{(+)}^i\}$ has the property that

(a) to each $U_{(+)}^i$, $i > 1$, there exists a $U_{(+)}^j$, $j < i$, with $U_{(+)}^i \cap U_{(+)}^j \neq \phi$ and $\varepsilon^{ij} = 1$ (connectedness)

(b) for $U_{(+)}^i \cap U_{(+)}^j \cap U_{(+)}^k \neq \phi$ and $\varepsilon^{ij} = 1$ one has $\varepsilon^{ik} = \varepsilon^{jk}$ (transitivity of the equality of points in $R^{s(+)}$).

Inversely (a) and (b) are sufficient conditions [6] for a set $\{U_{(+)}^j \subset M, \varepsilon^{kl}\}$ to define a domain over M. Therefore the set $\{I_{(+)}\,(U_{(+)}^i) \equiv \hat{U}_{(+)}^i, \varepsilon^{kl}\}$ defines a domain $(\hat{R}_{(+)}, \hat{\pi}_{(+)}, \hat{M}_{(+)})$ over $\hat{M}_{(+)}$, where every point $\hat{P}_{(+)} \in \hat{R}_{(+)}$ is an equivalence class of points $\hat{p}_{(+)}^i \in \hat{U}_{(+)}^i$ having the same coordinates on $\hat{M}_{(+)}$ and $\varepsilon^{ij} = 1$, and where $\hat{\pi}_{(+)}$ maps $\hat{P}_{(+)} \in \hat{R}_{(+)}$ on its coordinates in $\hat{M}_{(+)}$.

If one sets for $P \in R^{s(+)}$ $\hat{I}_{(+)}\,(P) \equiv \hat{P}_{(+)}$, then one obtains a natural mapping $\hat{I}_{(+)}$ from $R^{s(+)}$ onto $\hat{R}_{(+)}$ with the commutative diagram:

$$\begin{tikzcd} R \supset R^{s(+)} \arrow[r, "\hat{I}_{(+)}"] \arrow[d, "\pi"] & \hat{R}_{(+)} \arrow[d, "\hat{\pi}_{(+)}"] \\ M \quad M \arrow[r, "I_{(+)}"'] & \hat{M}_{(+)} \end{tikzcd} \tag{A.5}$$

$\hat{I}_{(+)}$ is holomorphic and pointwise open [20] [24], and every $\hat{I}_{(+)}$-saturated domain over the C^{4n} and M is the $\hat{I}_{(+)}$-preimage of a domain over $\hat{C}_{(+)}$ and $\hat{M}_{(+)}$.

A_2: Holomorphy envelopes for covariant analytic funktions.

In general quantum field theory as well as in an analytic S-matrix theory it is useful to know, into which domain all $L_+(C)$-covariant functions, holomorphic in a domain over C^{4n} or $M^{(\varkappa)}$, can be simultaneously continued. In this connection we first remark that for any domain (U, π, V) over a normal analytic set $V \subset C_k$ the unramified envelope of holomorphy $(H(U), \pi', V)$ is again a domain over V. Then we can prove the following:

Lemma 1: In a $I_{(+)}$-saturated domain $\mathfrak{R} = (R, \pi, C^{4n})$ every $L_+(C)$-covariant holomorphic function F is holomorphic in the envelope of holomorphy $\tilde{H}_{(+)}(\mathfrak{R}) \equiv (I_{(+)}^{-1}(H(I_{(+)}(R))), \pi', C^{4n})$ of $\mathfrak{R}$ in the space of the Lorentz invariants.

Proof: For $n \leq 2$ and for $n = 3$ and representations $[r, s]$ with $|r - s| \leq 2$, a global holomorphic decomposition of F as in (2.17) gives the desired analytic continuation into $\tilde{H}_{(+)}(\mathfrak{R})$, since the invariant coefficient functions are holomorphic in $(\hat{I}_{(+)}(R), \hat{\pi}_{(+)}, \hat{C}_{(+)})$ and hence in $(H(\hat{I}_{(+)}(R)), \hat{\pi}'_{(+)}, \hat{C}_{(+)})$. For $n = 3$ and $|r - s| \geq 4$ and for $n \geq 4$ one obtains different meromorphic continuations of F into $\tilde{H}_{(+)}(\mathfrak{R})$, according to the different possible meromorphic eliminations of redundant covariants $Q^{\lambda}_{(\alpha)(\dot{\beta})}$. Except for an at least 2-codimensional set of exceptional points (which lie e.g. for $n = 3$, $|r - s| \geq 4$, only over the (p_1, p_2, p_3) with all $(p_i, p_j) = 0$, $1 \leq i \leq j \leq 3$) at least one meromorphic continuation of F is holomorphic in $\tilde{H}_{(+)}(\mathfrak{R})$. A holomorphic continuation of F into $\tilde{H}_{(+)}(\mathfrak{R})$ is obtained by applying the 2. Riemann theorem to the removable set of exceptional points.

Since the 2. Riemann theorem remains valid on a normal analytic set [16], an analogous statement is true for saturated domains (R, π, M) over the complex mass shell M.

If $\mathfrak{R}$ is separated by its family of $L_+(C)$-covariant holomorphic fonctions, then also by the $L_{(+)}(C)$-invariant holomorphic functions in $\mathfrak{R}$. Hence $(\hat{I}_{(+)}(R), \hat{\pi}_{(+)}, \hat{C}_{(+)})$ is holomorphically separable and the continuous ground-point true mapping of $(\hat{I}_{(+)}(R), \hat{\pi}_{(+)}, \hat{C}_{(+)})$ in its holomorphy envelope is one-to-one. Then $\mathfrak{R}$ can be considered as a subdomain of $\tilde{H}_{(+)}(\mathfrak{R})$.

In the following lemma we shall prove for an I-saturated domain $\mathfrak{R}$ in the case $n \leq 4$ that $\tilde{H}(\mathfrak{R})$ is the maximal domain, into which all $L(C)$-covariant functions, holomorphic in $\mathfrak{R}$, can be analytically continued.

Lemma 2: If $\mathfrak{R}$ is I-saturated and $n \leq 4$, then $\tilde{H}(\mathfrak{R})$ is convex with respect to the class of $L(C)$-covariant functions holomorphic in $\tilde{H}(\mathfrak{R})$.

For the proof we have to show that for every sequence $\{P_i\} \subset \tilde{H}(\mathfrak{R})$ without an accumulation point in $\tilde{H}(\mathfrak{R})$ there exists a $L(C)$-covariant function, which is holomorphic in $\tilde{H}(\mathfrak{R})$ and unbounded on $\{P_i\}$. Now, on such a sequence either the ($L(C)$-covariant) coordinate functions are unbounded or the sequence $\{\hat{I}(P_i)\}$ has no accumulation point in $(H(\hat{I}(R), \hat{\pi}', \hat{C})$. Since $\hat{C} = I(C^{4n}) = C^{(n+1)n/2}$, the unramified holomorphy envelope $(H(\hat{I}(R)), \hat{\pi}', \hat{C})$ is holomorphically convex [34]. Therefore there exists an $\hat{F}$, which is holomorphic on $(H(\hat{I}(R)), \hat{\pi}', \hat{C})$ and unbounded on $\{\hat{I}(P_i)\}$. Then $\hat{F} \circ \hat{I}$ has all required properties in $\tilde{H}(\mathfrak{R})$.

Remark: $\hat{M} = I(M)$ is for $n \leq 4$ a linear manifold in the $C^{(n+1)n/2}$. Therefore lemma 2 applies alos for I-saturated domains (R, π, M) over the complex mass shell. Since, under the assumptions of lemma 2, the theorem B of Cartan [7] is valid on the holomorphically convex set $\tilde{H}(\mathfrak{R})$, one can prove as in [24]:

Lemma 3: For an I-saturated domain $\mathfrak{R} = (R, \pi, C^{4n})$, $n \leq 4$, every $L_+(C)$-covariant analytic function F_α can be covariantly decomposed

$$F_\alpha = \sum_{\lambda=1}^{L} F_\lambda \, Q_\alpha^\lambda \tag{A.6}$$

with global $L(C)$-invariant coefficient functions, holomorphic in $\tilde{H}(\mathfrak{R})$.

Lemma 3 guarantees for $n \leq 4$ a covariant decomposition of the Wightman [45] and Green functions [41][36][2]. It follows that for the $(n+1)$-point functions, $n \leq 4$, the analytic completion (e.g. of $\underset{P}{U} \mathfrak{T}_p'$ [29][37]) is possible in the space of the invariants and leads there to the maximal domain of holomorphy for the class of $L_{(+)}(C)$-covariant analytic functions.

For $n > 4$ (and for I_+-saturated domains with $n = 4$) a new difficulty arises: $\hat{C}_{(+)} = I_{(+)}(C^{4n})$ is then an algebraic set in the $C^{r(+)}$, and therefore $(H(\hat{I}_{(+)}(R), \hat{\pi}'_{(+)}, \hat{C}_{(+)})$ is not necessarily holomorphically convex [15]. Yet at theorem of D. Ruelle [38] states that for $n \geq 4$ every $L_{(+)}(C)$-invariant domain of holomorphy (as e.g. $(H(R), \pi', C^{4n})$ is also the exact domain of regularity of a $L_{(+)}(C)$-invariant analytic function.

Appendix B: Spin $\frac{1}{2}$ – Spin $\frac{1}{2}$ Scattering

We shall illustrate the general theory by constructing a set of covariants for an arbitrary spin $^1/_2$–spin $^1/_2$ scattering process with a classification of all possible discrete symmetries. In the most symmetrical case the relevant covariants will turn out to be essentially the Fermi covariants [14][1].

According to (2.20) the spin $^1/_2$–spin $^1/_2$ scattering amplitude $T_{\alpha_1 \ldots \alpha_4}^{\varkappa_1 \ldots \varkappa_4}(p_1, p_2, p_3, p_4)$ can be decomposed in any saturated domain into 16 invariant amplitudes $T_\lambda^{(\varkappa)}(s, t)$.

The covariants of the representation $[1, 0]^{\otimes 4}$ are linear combinations of 2-spinors [42]) of the type $\varepsilon_{\alpha_1\alpha_2}\,\varepsilon_{\alpha_3\alpha_4}$, $\varepsilon_{\alpha_1\alpha_2}\,(p_i \cup p_j)_{\alpha_3\alpha_4}$, $(p_i \cup p_j)_{\alpha_1\alpha_2}\,\varepsilon_{\alpha_3\alpha_4}$ and $(p_i \cup p_j)_{\alpha_1\alpha_2}(p_r \cup p_s)_{\alpha_3\alpha_4}$. If we use on $M^{\varkappa_1 \dots \varkappa_4}$ the relations $p_i^2 = m_{\varkappa_i}^2 > 0$ and $p_1 + p_2 = p_3 + p_4$ and the identity

$$\varepsilon_{\alpha_1\alpha_2}\,\varepsilon_{\alpha_3\alpha_4} = \varepsilon_{\alpha_1\alpha_3}\,\varepsilon_{\alpha_2\alpha_4} - \varepsilon_{\alpha_1\alpha_4}\,\varepsilon_{\alpha_2\alpha_3}\,, \tag{B.1}$$

we can express all these covariants by the following (almost everythere) basis $\{Q^\lambda_{(\alpha)}\}$, which has a very simple transformation law under the symmetries (3.14):

$$\begin{aligned}
\hat{Q}^1_{(\alpha)}(p) &= \varepsilon_{\alpha_1\alpha_2}\,\varepsilon_{\alpha_3\alpha_4}\,, \quad Q^3_{(\alpha)}(p) = \varepsilon_{\alpha_1\alpha_2}\,(p_3 \cup p_4)_{\alpha_3\alpha_4}\,,\\
\hat{Q}^2_{(\alpha)}(p) &= \varepsilon_{\alpha_1\alpha_3}\,\varepsilon_{\alpha_2\alpha_4} + \varepsilon_{\alpha_1\alpha_4}\,\varepsilon_{\alpha_2\alpha_3}\,,\\
\hat{Q}^4_{(\alpha)}(p) &= \varepsilon_{\alpha_1\alpha_3}\,(p_2 \cup p_4)_{\alpha_2\alpha_4} + \varepsilon_{\alpha_1\alpha_4}\,(p_2 \cup p_3)_{\alpha_2\alpha_3}\,,\\
\hat{Q}^5_{(\alpha)}(p) &= \varepsilon_{\alpha_1\alpha_3}\,(p_2 \cup p_4)_{\alpha_2\alpha_4} - \varepsilon_{\alpha_1\alpha_4}\,(p_2 \cup p_3)_{\alpha_2\alpha_3}\,,\\
\hat{Q}^6_{(\alpha)}(p) &= \varepsilon_{\alpha_1\alpha_2}\,\{(p_3 \cup [p_1 - p_2])_{\alpha_3\alpha_4} + ([p_1 - p_2] \cup p_4)_{\alpha_3\alpha_4}\}\\
&\quad + \{(p_1 \cup [p_4 - p_3])_{\alpha_1\alpha_2} + ([p_4 - p_3] \cup p_2)_{\alpha_1\alpha_2}\}\,\varepsilon_{\alpha_3\alpha_4}\,,\\
\hat{Q}^7_{(\alpha)}(p) &= \varepsilon_{\alpha_1\alpha_2}\,\{(p_3 \cup [p_1 - p_2])_{\alpha_3\alpha_4} + ([p_1 - p_2] \cup p_4)_{\alpha_3\alpha_4}\}\\
&\quad - \{(p_1 \cup [p_4 - p_3])_{\alpha_1\alpha_2} + ([p_4 - p_3] \cup p_2)_{\alpha_1\alpha_2}\}\,\varepsilon_{\alpha_3\alpha_4}\,,\\
\hat{Q}^8_{(\alpha)}(p) &= \varepsilon_{\alpha_1\alpha_2}\,\{(p_3 \cup [p_1 - p_2])_{\alpha_3\alpha_4} - ([p_1 - p_2] \cup p_4)_{\alpha_3\alpha_4}\}\\
&\quad + \{(p_1 \cup [p_4 - p_3])_{\alpha_1\alpha_2} - ([p_4 - p_3] \cup p_2)_{\alpha_1\alpha_2}\}\,\varepsilon_{\alpha_3\alpha_4}\,,
\end{aligned} \tag{B.2}$$

and the Π-transformed covariants

$$\hat{Q}^{8+\lambda}_{\alpha_1 \cdot \alpha_4}(p_1, \dots p_4) = (\Pi\, p_1)^{\dot{\alpha}_1\beta_1} \dots (\Pi\, p_4)^{\dot{\alpha}_4\beta_4}\, \hat{Q}^\lambda_{\beta_1 \cdots \beta_4}(\Pi\, p_1, \dots \Pi\, p_4) \tag{B.3}$$

The reduction of all $[1, 0]^{\otimes 4}$-covariant polynomials to (B.2) and (B.3) over the ring of the $L(C)$-invariant polynomials is essentially trivial in the van der Waerden spinor calculus [43]). This is a considerable advantage over the conventional representation of the scattering amplitudes, where one has to operate repeatedly with the Fierz [11]) and Michel identities [33]) in the γ-algebra (see e.g. [1])) and to use the Dirac equation to eliminate redundant degrees of freedom (here from 256 to 16).

Then the standard covariants

$$Q^i_{(\alpha)}(p) = \hat{Q}^i_{(\alpha)}(p) + (\Pi\, p)^{(\dot\alpha)(\beta)}\, \hat{Q}^i_{(\beta)}(\Pi\, p)\,,\quad 1 \le i \le 8$$

$$Q^j_{(\alpha)}(p) = \hat{Q}^{j-8}_{(\alpha)}(p) - (\Pi\, p)^{(\dot\alpha)(\beta)}\, \hat{Q}^{j-8}_{(\beta)}(\Pi\, p)\,,\quad 9 \le j \le 16 \tag{B.4}$$

have a well defined parity under the commuting set of discrete symmetries X, Y and Z (see (3.14)). One can verify in particular that only $Q^1_{(\alpha)}, \ldots Q^5_{(\alpha)}$ have nonvanishing coefficient functions for a scattering process of the type $(\varkappa\,\varkappa \mid \bar\varkappa\,\bar\varkappa)$ (e.g. proton-proton scattering), which is Π- and C-invariant (with ${}_{(\varkappa)}\eta^{\Pi} = {}_{(\varkappa)}\eta^{C} = 1$). Less symmetrical processes can also easily be discussed with (B.4). Furthermore the covariants $Q^1_{(\alpha)}, \ldots Q^5_{(\alpha)}$ and $Q^9_{(\alpha)}, \ldots Q^{13}_{(\alpha)}$ have a well-defined parity under the symmetry W in (3.17). Therefore one has e.g. in the above case of proton-proton scattering the additional relation for the invariant amplitudes:

$$T^{(\varkappa)}_{\lambda}(s\,,t) = (-1)^{1+\lambda}\, T^{(\varkappa)}_{\lambda}(s\,,u)\,,\quad 1 \le \lambda \le 5\,. \tag{B.5}$$

The standard covariants $Q^{\lambda}_{(\alpha)}(p)$ are related to the spin-state covariants $Q^{\lambda}_{\mu_1\mu_2\mu_3\mu_4}$ $(p_1, \ldots p_4)$ by (3.22). After defining

$$u_{\mu}(p)_{\alpha} = D^{s}_{\alpha\mu}\left(L^{-1}(p)\right),\quad v_{\mu}(p)^{\dot\beta} = (p)^{\dot\beta\alpha}\, u_{\mu}(p)_{\alpha} \tag{B.6}$$

one has to contract $Q^{\lambda}_{(\alpha)}(p)$ with $v^*_{\mu_1}(p_1)^{\alpha_1}\, v^*_{\mu_2}(p_2)^{\alpha_2}$ for the outgoing and with $u_{\mu_3}(p_3)^{\alpha_3}$ $u_{\mu_4}(p_4)^{\alpha_4}$ for the incoming particles. Then one sees that the $Q^i_{\mu_1\mu_2\mu_3\mu_4}(p_1, p_2, p_3, p_4)$, $1 \le i \le 5$, are (up to a Fierz transformation) identical with the Fermi invariants for nucleon-nucleon scattering[1][14]. For instance one has

$$Q^1_{\mu_1\cdots\mu_4}(p_1\cdots p_4) = \frac{1}{2}\,\{(\bar\psi_1\,\psi^c_2)\,(\bar\psi^c_3\,\psi_4) + (\bar\psi_1\,\gamma^5\,\psi^c_2)\,(\bar\psi^c_3\,\gamma^5\,\psi_4)\} \tag{B.7}$$

with the Dirac spinors ψ_i and their charge conjugate ψ^c_i:

$$\psi_i \equiv \begin{pmatrix} u_{\mu_i}(p_i)_{\alpha_i} \\ v_{\mu_i}(p_i)^{\dot\beta_i} \end{pmatrix} \qquad \psi^c_i \equiv \begin{pmatrix} \overline{v_{\mu_i}(p_i)_{\alpha_i}} \\ \overline{u_{\mu_i}(p_i)^{\dot\beta_i}} \end{pmatrix} \tag{B.8}$$

in a representation of the γ-matrices with γ^5 diagonal.

Bibliography

1) D. Amati, E. Leader, B. Vitale, Nuovo Cimento *17*, 68 (1960).
2) H. Araki, J. Math. Phys. *2*, 163 (1961), Progr. Theor. Phys. Suppl. *18*, 83 (1961).
3) V. Bargmann, Gruppentheoretische Analyse der Lorentzinvarianz, ETH (1963).
4) A. O. Barut, I. Munzinich, D. N. Williams, Phys. Rev. *130*, 442 (1963).

[5]) H. BEHNKE, P. THULLEN, Erg. d. Math. *3*, 3 (1934); H. BEHNKE, H. GRAUERT, in Analytic Functions, Priceton (1960).
[6]) H. CARTAN, P. THULLEN, Math. Ann. *106*, 617 (1932).
[7]) H. CARTAN, Variétés analytiques complexes et cohomologie, C. B. R. M., Bruxelles (1953).
[8]) G. F. CHEW, M. L. GOLDBERGER, F. E. LOW, Y. NAMBU, Phys. Rev. *106*, 1337 (1957).
[9]) G. F. CHEW, *S*-Matrix Theory of Strong Interactions, New York (1961).
[10]) R. J. EDEN and J. C. POLKINGHORNE, in 1961 Brandeis Summer Institute in Theoretical Physics, New York (1962).
[11]) M. FIERZ, Z. Phys. *104*, 553 (1937); Helv. Phys. Acta *31*, 587 (1958).
[12]) M. FIERZ, Helv. Phys. Acta *12*, 3 (1939).
[13]) L. GÅRDING, J. L. LIONS, Suppl. Nuovo Cim. *14*, 9 (1959).
[14]) M. L. GOLDBERGER, M. T. GRISARU, S. W. MC DOWELL, D. Y. WONG, Phys. Rev. *120*, 2250 (1960).
[15]) H. GRAUERT, R. REMMERT, Math. Zeitschr. *67*, 103 (1957).
[16]) H. GRAUERT, R. REMMERT, Math. Ann. *136*, 245 (1958).
[17]) G. GRAWERT, G. LÜDERS, H. ROLLNIK, Fortschr. Physik *7*, 290 (1959).
[18]) J. GUNSON, preprint.
[19]) R. HAAG, Dan. Vid. Selsk. Mat.-Fys. Medd. *29*, No. 12 (1955).
[20]) D. HALL, A. S. WIGHTMAN, Dan. Vid. Selsk. Mat.-Fys. Medd. *31*, No. 5 (1957).
[21]) P. R. HALMOS, Measure Theory, New York (1950).
[22]) A. C. HEARN, Nuovo Cim. *21*, 333 (1961).
[23]) K. HEPP, Helv. Phys. Acta *36*, 355 (1963).
[24]) K. HEPP, Math. Ann. *152*, 149 (1963).
[25]) M. JACOB, G. C. WICK, Ann. Phys. *7*, 404 (1959).
[26]) H. JOOS, Fortschr. Physik *11*, 65 (1962).
[27]) R. JOST, Helv. Phys. Acta *30*, 409 (1957); ibid. *36*, 77 (1963).
[28]) R. JOST, Lectures on the General Theory of Quantized Fields, to appear.
[29]) G. KÄLLÉN, A. S. WIGHTMAN, Dan. Vid. Selsk. Mat.-Fys. Skrifter *1*, No. 6 (1958).
[30]) H. LEHMANN, preprint.
[31]) S. MANDELSTAM, Phys. Rev. *112*, 1344 (1958), ibid. *115*, 1743 (1959).
[32]) S. MANDELSTAM, Nuovo Cim. *15*, 658 (1960).
[33]) L. MICHEL, J. Phys. et Rad. *12*, 793 (1951).
[34]) K. OKA, J. Math. Soc. Japan *3* (1951).
[35]) K. OKA, Jap. J. Math. *27*, 97 (1953).
[36]) D. RUELLE, Nuovo Cim. *19*, 356 (1961).
[37]) D. RUELLE, Helv. Phys. Acta *34*, 587 (1961).
[38]) D. RUELLE, unpublished.
[39]) H. P. STAPP, Phys. Rev. *125*, 2139 (1962).
[40]) H. P. STAPP, UCRL–10843 preprint.
[41]) O. STEINMANN, Helv. Phys. Acta *33*, 257, 347 (1960).
[42]) R. F. STREATER and A. S. WIGHTMAN, PTC, Spin, Statistics and All That, New York (1964).
[43]) B. L. VAN DER WAERDEN, Die gruppentheoretische Methode in der Quantenmechanik, Berlin (1932).
[44]) H. WEYL, Classical Groups, Princeton (1946).
[45]) A. S. WIGHTMAN, Phys. Rev. *101*, 860 (1956); J. Indian Math. Soc. *24*, 625 (1960).
[46]) A. S. WIGHTMAN, in Relations de dispersion et particules élémentaires, Paris (1960).
[47]) E. P. WIGNER, Ann. Math. *40*, 149 (1939).
[48]) D. N. WILLIAMS, UCRL-11113 preprint.
[49]) W. ZIMMERMANN, Nuovo Cim. *21*, 249 (1961).