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Integral Operators in the Theory of Scattering

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Summary. VON NEUMANN's work on the integral representation of self-adjoint operators is extended. It is shown that the S-operator is the sum of the identity transformation and a Carleman integral operator on the energy shell if and only if the total scattering cross section is finite.

1. Introduction

The theory of the scattering operator S can be considered from two different points of view, the abstract and the concrete. From the abstract point of view one is interested in the intrinsic properties of the scattering operator S. These are the properties of S which are invariant under arbitrary unitary transformations of the underlying Hilbert space.

In the concrete point of view one asks for extrinsic *properties* of S, that is, properties which manifest themselves only in a particular representation.

An example of the former point of view is the unitary property of the S-operator. Analytical properties of scattering amplitudes on the other hand are typical examples of the latter.

A general abstract theory of the scattering operator in Hilbert space was given by JAUCH in two papers [1][2] and it was clarified in a number of further publications [3][4][5][6]. One might say that the abstract formulation of the scattering theory has thereby reached a certain conclusion. Yet from the physical point of view there is a great deal more that can be said about the scattering operator.

The reason for this is that a physical experiment will generally be directly connected with some extrinsic properties of the scattering operator. Thus the cross-section of a scattering experiment is indeed merely proportional to the square of matrix elements of the scattering operator in a particular representation. In any *specific* scattering problem the extrinsic properties play an essential rôle in the actual theoretical interpretation of scattering experiments.

In this paper we shall be concerned with one of this properties, the property namely that the operator $R \equiv S - I$ is an integral operator on the energy shell. If this is the case in the momentum representation, the differential cross section exists.

In particular we shall be concerned with special classes of integral operators, namely the so called Carleman integral operators (CI-operator) and the even more restricted class of Hilbert-Schmidt operators (H.-S. operators). Both classes enjoy various interesting and useful mathematical properties, and their spectrum can be characterized in a simple way (cf. below). The paper consists of two parts.

The first mathematical part deals with properties of integral operators of the Carleman type (CIT operators) and is based on a paper by VON NEUMANN[11]). VON NEUMANN investigated a certain class of self-adjoint operators, which have *exactly 0 for a 'Weyl limit point' of their spectrum* (cf. below). In order to apply VON NEUMANN's results to the scattering operator it was necessary to generalize them. The next larger class for which this generalization is quite easy are the normal operators and this turns out to be sufficiently wide for the problem.

Since we are interested in the operator $R \equiv S - I$ on the energy shell, it is necessary to pursue VON NEUMANN's investigations in three directions.

1) The sufficient and necessary condition which VON NEUMANN gave for self-adjoint operators has to be generalized for non hermitian operators. The operator R is normal (it commutes with its adjoint) and for this class it is shown with help of a lemma on Carleman kernels («right multiplication») that VON NEUMANN's *condition* remains valid. (This lemma in fact applies to a wider class. This will be discussed in the appendix.)

2) The property of a CIT operator being an integral-kernel is not an invariant one; if K is a CI kernel $U^+ K U$ is not necessarily one. The first question therefore which arises is: which operators are integral kernels in every frame? VON NEUMANN gave as *sufficient* condition (for hermitian operators) that the kernel be of the so called Hilbert-Schmidt type. We show here that this condition is also *necessary* for operators which are not necessarily hermitian. We call here such an operator *strong*. The class of R operators of this type has a simple physical significance: R is strong if and only if the total scattering cross-section is finite.

3) Suppose the CIT operator is not strong, how can one decide in which frame the operator can be represented as an integral kernel? In particular if R does not belong to the Hilbert-Schmidt class, how can one decide whether or not R is an integral kernel on the energy shell? But for a few allusions we do not treat this problem here and hope to treat it in a further publication.

In the second part we show some applications of the results just mentioned to physics. Although these results are not bound to non relativistic physics, we restrict ourselves to (elastic) potential scattering. We discuss the relations between the existence of an S-matrix and the total cross section.

It can be shown that in many cases the operator $R \equiv S - I$ (on the energy shell) is indeed, in the momentum representation, a Carleman integral kernel. One would wish to have a proof of this statement, which follows from the mere existence of the S operator alone. In fact we conjecture much more, namely that (in the non-relativistic case) the operator R is even completely continuous. Neither statement however can be proved in general. Therefore in order to show the relevance of our assertions, we are forced to 'raise a loan' from the theory of potential scattering. We shall show that the 1st Born approximation indeed suggests that R is completely continous and therefore a fortiori, unitarily equivalent to a Carleman kernel. Furthermore we show that the existence of a finite total cross-section insures that R is an integral kernel in every frame. By sharpening slightly an argument due to MARTIN[20]) we show that this is the case if $V(r)$ falls off stronger than $1/r^2$.

2. Mathematical Preliminaries

We shall be interested in the Hilbert space $L^2(s, \mu)$ defined as the family of square summable functions over a continuous measure space S with the measure μ, and a certain class of linear operators K defined on a domain $D \subseteq L^2(s, \mu)$ by

$$(K f)(x) = \int_s K(x, y) f(y) \, d\mu(y)$$

and with range $R \subseteq L^2(s, \mu)$.

We call such an operator an *integral operator* (or I-operator). It is obvious that such an integral operator is linear, but conversely not all linear operator are I-operators; indeed some of the simplest operators are not, for instance the identity operator.

We want to find conditions under which a linear operator can be written as an I-operator. The work of VON NEUMANN quoted above has shown that such conditions can be given for selfadjoint operators which fulfill a condition formulated first by CARLEMAN.

Definition 1: A linear operator K in the space $L^2(s, \mu)$ is called a Carleman integral operator (CI-operator) if it can be written in the form (1) and if in addition

$$\int_s |K(x, y)|^2 \, d\mu(y) < \infty$$

for a.e. *) $x \in s$.

Condition (2) is called the Carleman condition in the following. It means that $K(x, y) \in L^2(s, \mu)$ for a.e. x S, as a function of y.

From the triangle inequality follows that sum and difference of two CI-operators are CI-operators too.

The property of an operator to be an I-operator (condition (1)) is in general not invariant under arbitrary unitary transformations. For instance if the hermitian operator H is represented by the kernel $K_H(x, y)$, such a representation of the operator $H' \equiv U^+ H U$ does not exist in general, cf. below sect. 4. Therefore a natural notion is that of an operator of *Carleman integral type*.

Definition 2: A linear operator K in $L^2(s, \mu)$ is of *Carleman integral type* (CIT) if it is unitarily equivalent to a CI-operator.

This means that there exists at least one unitary operator U and a kernel $K(x, y)$ such that

$$(U^* K U f)(x) = \int_s K(x, y) f(y) \, d\mu(y) \, .$$

The property of an operator to be CIT is an intrinsic property; for selfadjoint operators it can be expressed in terms of the spectrum alone.

It is this spectral condition which contains the essence of VON NEUMANN's result. In order to formulate it we use the notion of 'Weyl limit point' of the spectrum.

Definition 3: A point Z_0 of the spectrum of a normal operator is a *Weyl limit point* (W.l.p.) of the spectrum if it satisfies one or several of the following conditions

*) We use the abbreviation a.e. for 'almost every'.

(a) It is an eigenvalue of multiplicity ∞
(b) It is limit point of a sequence of eigenvalues
(c) It belongs to the continuous part of the spectrum.

In the case of self-adjoint operators these conditions can be expressed in terms of their spectral projections. Indeed the point λ_0 is Weyl limit of a self-adjoint operator if and only if

$$E(\Delta) \equiv \int_\Delta dE_\lambda$$

is infinite dimensional for every interval which contains the point λ_0 in its interior. An equivalent way of characterizing a Weyl limit is the following: λ_0 is a Weyl limit point of the spectrum of the operator A if there exists an infinite linearly independent sequence of elements f_n all of norm 1, such that $\| H f_n \| \to \lambda_0$ for $n \to \infty$. WEYL has recognized that zero is always a W.l.p. of the eigenvalues of a Carleman-kernel. VON NEUMANN then demonstrated that this necessary condition is also sufficient[11]):

A self-adjoint operator H is of CIT if and only if zero is Weyl limit point of its spectrum.

But the theorem says nothing about the question whether a given CIT operator H in $L^2(s, \mu)$ is indeed an integral operator or not.

We shall characterize below the CI-operators for which the group of unitary transformations which transform a given CI-kernel again into a CI-kernel is the group of *all* unitary transformations. For these operators then the property of being a CI-operator is invariant under all unitary transformations.

That there are operators of CI-type for which this is not the case is easily seen. Consider for instance a self-adjoint operator A with a simple and purely continuous spectrum which includes the point zero. According to VON NEUMANN's theorem this is a CIT-operator. According to the theorem on the spectral representation*) there exists a representation of the operator A acting on functions

$$g(\lambda) \in L^2 (\Lambda, \mu) \quad \text{such that} \quad (A\, g)\, (\lambda) = \lambda\, g(\lambda)\,.$$

Barring δ-functions, such an operator can not be represented as an I-operator[10]). Thus we see that an operator with continuous spectrum is not a CI-kernel in every frame.

On the other hand it is known[11]) that a certain class of operators, the so-called Hilbert-Schmidt operators, are CI-kernels in every representation (cf. sect. 4).

Definition 4: An operator is called completely continuous ($c\,c$) if it is bounded and if it transforms every bounded set of vectors into a compact set, i.e. into a set every infinite subset of which has at least one limit vector. For normal operators this second condition means that zero is the only l.p. of its spectrum**).

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*) We call spectral representation the space of L^2 functions $f(\lambda)$ over the spectrum Λ of a self-adjoint operator such that the operator A is represented by $(A\, f)\, (\lambda) = \lambda\, f\, (\lambda)$.

**) We recall that an operator N is called normal if it admits a spectral resolution. A necessary and sufficient condition for this is that it commutes with its adjoint: $[N, N^*] = 0$.

Definition 5: An integral kernel belongs to the Hilbert-Schmidt class if

$$\int | k(x, y) |^2 \, dx \, dy < \infty .$$

It is therefore a fortiori a CI-operator.

This definition implies that

$$\sum_{r,s} | (f_r, K f_s) |^2$$

converges in every orthonormal frame, or as we might say: an operator is of the H.-S. type if the 'double-norm' $T r\, 1/2\, (K K^* + K^* K)$ is finite. For normal operators this definition implies that the eigenvalues are a square summable sequence:

$$\sum_n | \lambda_n |^2 < \infty .$$

Note that of course not every cc. operator belongs to the H.-S. class: e.g. the (hermitian) operator with eigenvalues $\lambda_n = 1/\sqrt{n}$ does not.

We add the following *Lemma*:

If a Carleman-integral operator on a Hilbert space of functions is defined on a compact interval, and its kernel depends on the difference of the variables only, then it is even of Hilbert-Schmidt type.

By definition we have

$$\int_a^b \int_a^b | k(x, y) |^2 \, dx \, dy = \int_a^b \left[\int_{.a}^b | k\,(x - y)\, 2\, dx \right] dy .$$

The substitution $x - y = t$ in the inner bracket leads to

$$\int_{a-y}^{b-y} | k(t) |^2 \, dt$$

and since $k(t)$ can be taken as continued periodically beyond (a, b) the result is a number independent of y, since $\int_{a-y}^{b-y} = \int_a^b$ the second integration morely multiplies this number by $b - a$.

The proof shows also that the theorem is not true if a or b or both are infinite.

3. Some non-Hermitian Operators of Carleman Type

We begin this section with a lemma which permits various extensions of VON NEUMANN's theorem needed for the applications to scattering theory.

Lemma 1: (Main lemma) Let K be any CI-operator and B bounded then $K B$ is a CI-operator too. Moreover if K is represented by the kernel $k(x, y)$, then $K B$ by the kernel $(J B^* J k_x)\,(y)$ *) where $k_x = k(x, y)$ is the vector in $L^2(s, \mu)$ as a function of y

*) J denotes complex conjugation; $J\, k(x, y) = \overline{k(x, y)}$.

for fixed x (Carleman condition!) and $J K_x$ is the vector with complex conjugate components.

Proof: $(K B g)(x) = \int K(x, y) (B g)(y) \, d\mu(y) = (J k_x(y), (B g)(y))$

$$= (B^* J k_x(y), g(y)) = \int (J B^* J k(x, y)) g(y) \, d\mu(y) .$$

This shows that $K B$ is an I-operator.

Furthermore

$$\int |(J B^* J k_x)(y)|^2 \, d\mu(y) = \| J B^* J k_x \|^2 \leq \| B \|^2 \cdot \| k_x \| < \infty .$$

Thus the I-operator $K B$ satisfies the Carleman condition.

With respect to bounded CI-operators this lemma can also be expressed in the following way: the set of (bounded) CIT-operators which are CI-operators *in the same frame* are a right sided ideal in the algebra of all bounded operators in $\mathfrak{H}$.

All further results presented here will depend upon this lemma. We note that the condition that B be bounded, is essential in the sense that the lemma is not *always* true for unbounded operators. For instance if A is a CI-operator and A^{-1} exists $A A^{-1} = I$ is *not* a CI-operator. Although this lemma seems quite obvious the analogous statement for left multiplication with a bounded operator is, as we shall show, in general false. An immediate consequence of the main lemma and better suited for applications is the

Lemma 2: If K is a CIT-operator and B bounded, then $K B$ is also a CIT-operator.

Proof: By assumption there exists a U such that

$$\tilde{K} = U^+ K U$$

may be realized as a CI-kernel, then:

$$U^+ K B U = (U^+ K U)(U^+ B U) = \tilde{K} \tilde{B}$$

too can be realized as a CI-renal by the main lemma, and $K B$ is unitarily equivalent to it.

Note, however, that the sum of two CIT-operators is in general *not* again a CIT-operator. A straightforward counter-example is provided by the following 2 (even commuting!) hermitian operators:

Let in $\mathfrak{H} = \mathfrak{H}^1 \oplus \mathfrak{H}^2$ be given an orthonormal system $f_m \in \mathfrak{H}^1$, $g_n \in \mathfrak{H}^2$; $m, n = 1, 2, \ldots \infty$; $f_m \perp g_n$ for every m, n.

Consider now the 2 hermitian operators A and B, both diagonal in this frame and defined by their spectra

A: $$\alpha_m^1 = \frac{1}{m} , \quad \alpha_n^2 = 1 - \frac{1}{n} ,$$

B: $$\beta_m^1 = 1 - \frac{1}{m} , \quad \alpha_n^2 = \frac{1}{n} .$$

Evidently both A and B are CIT-operators, but $A + B = I$! Therefore in spite of lemma 2 the bounded CIT-operators are *not* an ideal.

The following statements are immediate consequence of VON NEUMANN's theorem and the preceding lemma.

Corollary: No unitary operator is a CIT-operator.

Proof: A unitary operator U is bounded, so is U^*. Thus if U were a CIT-operator so would $U\,U^* = I$, by lemma 2. But this is false, hence U cannot be a CIT-operator.

The same reasoning applies to every operator A, whose right inverse exists and is bounded:

$$A\,A^{-1} = I\,.$$

There exist, of course, unitary integral operators, for instance the Fourier transform, defined as an operator in $L^2\,(-\infty, \infty)$

$$(F\,g)\,(x) = \frac{1}{\sqrt{2\,\pi}} \int\limits_{-\infty}^{\infty} e^{i\,x\,y}\, f(y)\, dy\,.$$

But this operator is not of Carleman type since

$$\int\limits_{-\infty}^{\infty} |\,e^{i\,x\,y}\,|^2\, dy = \infty\,.$$

On the other hand if a finite interval (a, b) is chosen, then the transformation

$$(F\,g)\,(x) = \frac{1}{\sqrt{2\,\pi}} \int\limits_{a}^{b} e^{i\,x\,y}\, f(y)\, dy$$

belongs to the Hilbert-Schmidt class, hence it is not unitary, but *it is* a CI-operator.

In order to prove the main result of this section we use the polar decomposition of normal operators (cf. (12) no. 110):

Every normal operator can be written as the product of a non-negative hermitian operator R and a unitary operator U; this decomposition is unique and R and U commute:

$$N = R\,U = U\,R\,, \quad R = N\,U^* = U^*\,N\,.$$

Therefore we have as an obvious consequence of lemma 2 the

Theorem 1: A normal operator is a CIT-operator if and only if the hermitian factor in its polar decomposition is a CIT-operator. Therefore: a normal operator is a CIT-operator if and only if zero is a W.l.p. of its spectrum.

For generalisations of this theorem to non-normal operators cf. App. 3. Note, however, that no restriction had to be imposed on the spectrum.

Corollary: If the CI-operator K is normal, then its hermitian adjoint is also a CI-operator, for

$$K = U\,R = R\,U\,; \quad K^* = R\,U^* = R\,U\,U^*\,U^* = K\,U^*\,U^*\,.$$

For non-normal operators this statement is in general not true. Ref. (13) pp. 230–232 contains several other sufficient conditions such that K^* be a CI-operator together with K.

We conclude this section with additional corollaries of these results.

We have already seen that a unitary operator U is never a CIT-operator. On the other hand, the operator $U - I$ can very well be one, namely precisely then, if 1 is W.l.p. of its spectrum. More generally we see:

The operator $e^{-i\gamma} U - I$ (γ real), is a CIT-operator if and only if $e^{i\gamma}$ is W.l.p. of the spectrum of U.

For any given U there always exists a real γ such that the condition of the preceding statement is satisfied. This follows from the spectral theorem for unitary operators, and the Bolzano-Weierstrass theorem which states that an infinite set on a compact domain (here the unit circle on which the eigenvalues of U lie) has at least one l.p.

A similar statement can be proved for bounded normal operators:

Theorem 2: Every bounded normal operator is unitarily equivalent to the sum of a CIT-operator and a multiple of the identity.

The first part of the statement follows from the fact that every bounded normal operator must contain at least one Weyl limit point, say z_0 in its spectrum. For then $N - z_0 I$ is a normal operator and zero is W.l.p. of its spectrum.

4. Strong C-integral Operators

So far we distinguished carefully between a CI-operator and a CIT-operator. The former is a Carleman integral operator in a definite representation. The prefix CI makes therefore only sense with reference to a specific representation; it is, in the terminology of the first section an extrinsic property. In contrast to this CIT designates an intrinsic property, namely the existence of a unitary transformation which transforms a given operator into a CI-operator.

In this section we shall be concerned with the question under which conditions a given operator is a CI-operator in *every representation*. If this is the case the CI-property becomes an intrinsic property too. These conditions must therefore be expressible in terms of invariant characteristics of the spectrum.

It is useful to introduce here the following notion:

Definition 6: A CI-operator is called *strong* if every unitary transformation of it is also a CI-operator.

VON NEUMANN showed that all hermitian operators of the Hilbert-Schmidt class are strong, but he left the question open, whether this sufficient condition is also necessary.

To answer this question we prove first that a strong operator must certainly be completely continuous (compact), i.e. has no other l.p. besides the one at 0.

Suppose that the normal operator K is strong and that its spectrum contains a second finite W.l.p. say λ_0. Since K is strong and $K' \equiv K - \lambda_0 I$ a CIT-operator, a unitary operator U exists such that $\tilde{K}' \equiv U^+ K' U$ as well as $\tilde{K} \equiv U^+ K U$ are both

CI-kernels; whence follows that $\tilde{K}' - \tilde{K} = {}_0I$ is also a CI-kernel. But this is impossible, as we saw; therefore K cannot be strong. It also follows that if a normal operator N has two finite W.l.p. in the spectrum z_1 and z_2 say, $N - z_1 I$ and $N - z_2 I$ cannot be CI-kernels in the same frame.

We note here that except the existence of a second finite W.l.p. no further assumption about K is needed for this proof: K may be unbounded, one may even drop the condition that K is normal (cf. App. III) etc. There remains however one case not covered by this proof: the unbounded operators the only finite l.p. of which is zero.

Here one sees that if a left multiplication would be permitted in general then *all* bounded CIT-operators would be strong, for from K is a CI-operator would follow $U^+ K U$ is CI too.

We now want to show that the class of all bounded strong CI-operators coincides with the class of all Hilbert-Schmidt operators. For this purpose we proceed in the following way. First we show that for strong operators also a «left multiplication» analogous to the 'right multiplication' of sect. 2 (lemma 1) is valid. (The converse statement of the foregoing remark.) Thus the set of strong operators is a two-sided ideal which is closed under hermitian conjugation. Therefore together with K (whether normal or not) also K^* will be a strong CI-operator. The decomposition $K \equiv H_1 + i H_2$ with H_1 and H_2 Hermitian then allows to carry through the proof for hermitian operators alone.

Lemma 3: If K is a strong CI-operator and B bounded, then $B K$ is also strong CI-operator.

Proof: 1. Let K and L be two strong operators. We saw that together with K and L, $K + L$ is a CI-operator too, and since $U^* (K + L) U = U^* K U + U^* L U$; $K + L$ is also strong.

2. Next we note that every bounded operator B can be expressed as sum of four unitary operators $B = \sum_{k=1}^{4} U_k$ *). On account of remark 1. it is sufficient to prove the lemma for unitary operators.

3. This, however is straightforward: by our assumption $U K U^*$ is a CI-operator, whence by lemma 1 also $(U K U^*) U = U K (U^* U) = U K$.

4. From this statement combined with lemma 1 follows that $U K$ is also strong:

$$V^* (U K) V = \big(V^* (U K)\big) V = (V^* U K) V$$

for *every* unitary V.

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*) Essentially:

$$B = H_1 + H_2 = \frac{n_1}{2} \left[\frac{H_1}{n_1} + i \left(1 - \frac{1}{n_1^2} H_1^2\right)^{1/2}\right] + \text{compl. conj.},$$

$$\frac{n_2}{2} \left[\frac{H_2}{n_2} + i \left(1 - \frac{1}{n_2^2} H_2^2\right)^{1/2}\right] + \text{compl. conj.},$$

where n_1, n_2 are the norms of H_1 and H_2 resp. Cf. [7]), p. 4, proposition 3.

This concludes the proof of

Theorem 3: The class of all strong CI-operators is a two-sided ideal in the algebra of all bounded operators in $\mathfrak{H}$.

But every two-sided ideal in the algebra of all bounded operators is closed under hermitian conjugation (cf. [7]), chapter 1 and 6).

Whence $K \in S$ implies $K^* \in S$ and further more since

$$K = \frac{K + K^*}{2} + i \frac{K - K^*}{2} = H_1 + i H_2 , \quad H_1 \in S , \quad H_2 \in S .$$

The result may be summed up in

Lemma 4: A CI-operator is strong if and only if its hermitian and its antihermitian part are both strong.

Thus we can concentrate on hermitian operators.

Let us now consider a Hermitian completely continuous strong C-integral operator K in $L^2(a, b)$ with a and b finite. There exists a complete orthonormal system of eigenfunctions $\{g_n(x)\}$ of K with corresponding eigenvalues λ_n:

$$K g_n = \lambda_n g_n .$$

For the corresponding kernel we have

$$k(x, y) \sim \sum_{n=0} \lambda_n g_n(x) \overline{g_n(y)}$$

in the following sense. For almost every fixed value of x, k considered as a function of y has a Fourier expansion which converges in the mean to the value given by the right-hand side; $\lambda_n g_n(x)$ then is a constant, a Fourier coefficient, and the Carleman condition (2) now involves:

$$\sum_{n=0}^{\infty} | \lambda_n |^2 \, | g_n(x) |^2 < \infty$$

for *a e x*. If K is a strong CI-operator, this inequality must hold for all orthonormal systems $\{g_n(x)\}$.

Now a and b being finite

$$g_n(x) = e^{2 \pi i n x / b - a} \cdot \frac{1}{\sqrt{b - a}}$$

is a complete orthonormal set, and since $| g_n(x) | \equiv 1/b - a$ for every x, we conclude:

$$\sum_{n=0}^{\infty} | \lambda_n |^2 < \infty$$

as a consequence of K being strong. This show that a strong CI-operator on $L^2(a, b)$ belongs to the Hilbert-Schmidt class (see definition 4).

In the above notation, λ_n figures as often as its multiplicity.

We can free ourselves from the restriction that a and b are finite by using a suitable unitary transformation which maps $L^2(a, b)$ onto $L^2(-\infty, \infty)$ and is such that C-integral operators are mapped into C-integral operators only (see Appendix II where such transformations are explicitly exhibited). As mentioned previously the condition of hermiticity can be disposed of by means of the decomposition. Thus we have proved

Theorem 4: A bounded C-integral operator is strong if and only if it belongs to the Hilbert-Schmidt class.

5. Illustrations with Scattering Theory

As an illustration of our results we consider the elastic scattering of a spinless particle by a spherically symmetric potential $V(r)$ for which the S-operator shall exist. S commutes with the free Hamiltonian H_0, with L^2 and L_z. These operators are a complete set of commuting observables and therefore S a function of them. In the spectral representation of this set the effect of S will appear as multiplication by a function:

$$(S\,g)\,(E, l, m) = s\,(E, l, m)\,g\,(E, l, m)\,,$$

where E, l, m are the eigenvalues. Moreover S is independant of m and has the form

$$S(E, l) = e^{2i\delta_l(E)}\,.$$

According to the results of section 3 $R \equiv S - I$ is a CIT-operator if and only if there exists an infinite subsequence of phases which tend to zero:

$$\lim_{j\to\infty} \delta_{l_j} \to 0\,.$$

Clearly, one would wish to have a general theorem which proves from the mere existence of the S-operator that there is indeed such a subsequence. Physically this seems plausible enough. In fact we conjecture (at least for potential scattering) much more, namely that R is completely continuous (i.e. there is no other l.p. besides zero). The 1st Born approximation, as we shall see, supports this.

Here we will not pursue the question in which frame R actually is a CI-operator. Instead we shall state the condition under which R is *strong*.

The necessary and sufficient condition given in the last section was

$$T\,r\,R^*\,R < \infty$$

or:

$$\sum_l (2\,l + 1)\,(e^{2i\delta_l} - 1)\,(e^{-2i\delta_l} - 1) = 4\sum_l (2\,l + 1)\,\sin^2\delta_l \sim \sigma_{tot} < \infty\,. \quad (1)$$

Thus R is a strong CI-operator if and only if the total cross-section exists.

Let us illustrate the range of this result in the case of elastic potential scattering. We want to show that the total cross-section exists if roughly speaking the potential falls off at infinity faster than $1/r^2$, to be exact is:

$$|\,r^{2+k}\,V(r)\,| \leq N = \text{const}\,, \quad k > 0\,^{*})\,. \quad (2)$$

This condition excludes strong singularities at the origin etc., but here we are interested only in the asymptotic behaviour of the potential.

*) This condition, however, does not include potentials like $1/r^2\,l\,g\,r$.

The potential is assumed also to fulfil the following well known condition *):

$$\int_0^\infty | r\, V(r) |\, dr < \infty . \tag{3}$$

A particularly simple proof for the existence of the total cross-section then follows from an interesting relation due to MARTIN[20]). He showed for potentials which fulfil (3) that if l is sufficiently large, the exact phases δ_l are smaller than the phases calculated in the 1st Born-approximation δ_l^B multiplied with a certain factor **):

$$\delta_l < \frac{\delta_l^B}{1 - \sqrt{\frac{\pi}{2\,l+1}} \int_0^\infty r\, | V(r) |\, dr} .$$

The first Born-approximation yields with the help of (2):

$$\delta_l^B = \int_0^\infty r\, | V |\, J^2_{l+1/2}\, dr = \int_0^\infty r^{2+k}\, | V |\, \frac{J^2_{l+1/2}}{r^{1+k}}\, dr \leq N \int_0^\infty \frac{J^2_{l+1/2}}{r^{1+k}}\, dr .$$

The value of the last integral is:

$$\int_0^\infty \frac{J^2_{l+1/2}}{r^{1+k}}\, dr = \frac{\Gamma\,(1+k)}{2^{1+k}\, \Gamma^2 \left(\frac{1+k}{2}\right)}\; \frac{\Gamma\left(l+1-\frac{1+k}{2}\right)}{\Gamma\left(l+1+\frac{1+k}{2}\right)} .$$

Stirling's formula shows that for $l \to \infty$ this quotient goes $\sim 1/l^{1+k}$.

Thus σ_{tot} exists if (2) is fulfilled. On the other hand it is known[8]) that no total cross-section exists for the potential $V = 1/r^2$.

What can be said about R if the potential falls off slower than $1/r^2$? JAUCH loc. cit. has proved the existence of an S-operator for potentials which fall off faster than $1/r$. If the potential falls off $\sim 1/r^{1+k}$ the first Born phases δ_l^B will go $\sim 1/l^k$ for $l \to \infty$, thus suggesting that R be completely continuous. But of course MARTIN's relation cannot be used for confirming this guess.

The result of this section then may be summed up:

If the potential $V(r)$ fulfils condition (2) R is a H.-S. operator.

The potential $V = 1/r^2$, on the other hand, offers an example of a curious mathematical possibility: in this case R is a completely-continuous integral operator in the momentum representation, hence of CI-type.

But since the potential is spherically symmetric the lemma at the end of sect. 2 applies. Therefore since σ_{tot} is not finite, *R cannot be a CI operator in this representation*, although it is an I-operator and of CI-type!

Nothing seems to be known if $V(r)$ falls off slower than $1/r^2$.

*) For the importance of this condition cf.[18]).

**) Here δ_l^B denotes the phase-shift in first Born approximation' of the potential $| V |$.

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Appendix I

Integral representation of the operator $e^{-iHt} - I$

Let $H = H^*$ operate on $L^2(a, b)$; let moreover H belong to the Hilbert-Schmidt class; thus H is a strong C-integral operator, represented by the kernel $h(x, y)$. We shall show that the normal operator

$$U_t - I = e^{-iHt} - I = \sum_{n=1}^{\infty} \frac{(-i\,t)^n}{n!} H^n, \tag{A.1.1}$$

then also belongs to the (generalized) Hilbert-Schmidt class. As well known, H has a square-convergent pure point spectrum; for any eigenfunction φ_n of H,

$$\left.\begin{aligned}(e^{-iHt} - I)\,\varphi_n = \sum_{m=1}^{\infty} \frac{(-i\,t)^m}{m!} H^m\,\varphi_n &= \sum_{m=1}^{\infty} \frac{(-i\,t)^m}{m!} \lambda_n^m\,\varphi_n \\ &= (e^{-it\lambda_n} - 1)\,\varphi_n = \mu_n\,\varphi_n\,.\end{aligned}\right\} \tag{A.1.2}$$

Thus $U_t - I$ possesses a (complex) pure point spectrum since φ_n is a complete system.

We find

$$\sum |\mu_n|^2 = 4 \sum \left|\sin \lambda_n \frac{t}{2}\right|^2 \le t^2 \sum |\lambda_n|^2 < \infty\,.$$

Thus $U_t - I$ belongs to the (generalized) Hilbert-Schmidt class, thus it is a strong C-integral operator, represented by a kernel

$$\omega(x, y; t)\,.$$

The kernel representing the product of two Hilbert-Schmidt operators is given by

$$K_{A \cdot B}(x, y) = \int_a^b K_A(x, \xi)\, K_B(\xi, y)\, d\xi = K_A * K_B\,, \tag{A.1.3}$$

thus (A.1.1) suggests for ω the form

$$\omega(x, y; t) = \sum_{n=1}^{\infty} \frac{(-i\,t)^n}{n!} h_n(x, y) \quad \text{with} \tag{A.1.4}$$

$$h_n(x, y) = h(x, y) * \ldots * h(x, y) \qquad (n \text{ times})\,. \tag{A.1.5}$$

We shall see that our assumption is in fact correct. We need the notion of the double norm

$$|||\Omega||| = \left[\iint |\omega|^2\, dx\, dy\right]^{1/2} < \infty$$

with the well-known properties ([12]), pp. 242–243)

$$|||\Omega_1 + \Omega_2||| \leq |||\Omega_1||| + |||\Omega_2|||, \tag{A.1.6}$$

$$|||\Omega^n||| \leq |||\Omega|||^n . \tag{A.1.7}$$

From (A.1.4) and (A.1.5), (A.1.6) and (A.1.7) lead to

$$|||\omega(x, y_i t)||| \leq \sum_{n=1}^{\infty} \frac{|t|^n}{n!} |||h|||^n = e^{|t| \cdot |||h|||} - 1 < \infty ,$$

thus ω belongs to the Hilbert-Schmidt class. We still have to show that it represents $U_t - I$.

For any finite N

$$s_N = \sum_{n=1}^{N} \frac{(-i\,t)^n}{n!} h_n(x, y) \tag{A.1.8}$$

certainly represents

$$S_N = \sum_{n=1}^{N} \frac{(-i\,t)^n}{n!} H^n . \tag{A.1.9}$$

For $N \to \infty$, both (A.1.8) and (A.1.9) converge in their respective norms, for every finite value of t; thus

$$|||\omega(x, y; t) - s_n(x, y; t)||| \leq \varepsilon , \tag{A.1.10}$$

$$\| U_t - I - S_N \| \leq \varepsilon . \tag{A.1.11}$$

We do not yet know that ω represents $U_t - I$; it certainly represents a Hilbert-Schmidt operator Ω and we know (see [12]), No. 66) that

$$|||\Omega||| = |||\omega||| \geq \|\Omega\|,$$

thus from (A.1.10)

$$\| \Omega - S_N \| \leq \varepsilon . \tag{A.1.12}$$

Addition of (A.1.12) and (A.1.10) leads to

$$2\,\varepsilon \| S_N - \Omega \| + \| U_t - I - S_N \| \geq \| U_t - I - \Omega \|$$

for every ε, thus

$$U_t - I = \Omega ,$$

thus $U_t - I$ is represented by $\omega(x, y; t)$.

The explicit calculation of the iterated kernel (A.1.5) is not always simple; it becomes very easy, however, if H is the multiple of a projection operator*); then

$$\sum_{n=1}^{\infty} \frac{(-i\,\mu\,t)^n}{n!} p^n = p \sum_{n=1}^{\infty} \frac{(-i\,\mu\,t)^n}{n!} = (e^{-i\mu t} - 1)\, p .$$

If, in particular, H is represented by a $\varphi(x)\,\bar{\varphi}(y)$ $U_t - I$ is represented by

$$\omega(x, y; t) = (e^{-i a t} - 1)\, \varphi(x)\, \bar{\varphi}(y) .$$

*) In that case the kernel representing P is, in some representation at least, 'separable', i.e. $h(x, y) = \alpha\, \varphi(x)\, \bar{\varphi}(y)$.

Appendix II

The VON NEUMANN *transforms*

We show that

$$(\mathfrak{N}_1 \varphi)(x) = \frac{1}{\sqrt{x}} \varphi(\ln x) \tag{A.2.1}$$

and

$$(\mathfrak{N}_2 \varphi)(x) = \frac{1}{1-x} \varphi\left(\frac{x}{1-x}\right) \tag{A.2.2}$$

are unitary transformations, $\mathfrak{N}_1$ mapping $L^2(-\infty, \infty)$ into $L^2(0, \infty)$ and $\mathfrak{N}_2$ mapping $L^2(0, \infty)$ into $L^2(0, 1)$. Moreover, both transforms leave the Carleman integral character of a transformation invariant. By simple substitution one finds

$$\int_0^\infty |(\mathfrak{N}_1 \varphi)(x)|^2 \, dx = \int_{-\infty}^\infty |\varphi(x)|^2 \, dx , \tag{A.2.3}$$

$$\int_0^1 |(\mathfrak{N}_2 \varphi)(x)|^2 \, dx = \int_0^\infty |\varphi(x)|^2 \, dx , \tag{A.2.4}$$

thus $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are unitary.

It is somewhat more lengthy to show that the integral character of any transformation is preserved under $\mathfrak{N}_1$ and $\mathfrak{N}_2$. We first note

$$(\mathfrak{N}_1^{-1} \varphi)(x) = (\mathfrak{N}_1^* \varphi)(x) = \exp\frac{x}{2} \, \varphi(\exp x) , \tag{A.2.5}$$

$$(\mathfrak{N}_2^{-1} \varphi)(x) = (\mathfrak{N}_2^* \varphi)(x) = \frac{1}{1+x} \varphi\left(\frac{x}{1+x}\right). \tag{A.2.6}$$

We consider now

$$(\Omega \varphi)(x) = \int_{-\infty}^\infty \varphi(y) \, \omega(x, y) \, dy = \tilde{\varphi}(x) . \tag{A.2.7}$$

Writing

$$\mathfrak{N}_1 \varphi = \Phi ; \quad \mathfrak{N}_1 \tilde{\varphi} = \tilde{\Phi}$$

(A.2.7) becomes

$$\int_{-\infty}^\infty \mathfrak{N}_1^* \Phi(y) \, \omega(x, y) \, dy = \mathfrak{N}_1^* \tilde{\Phi}(x) \tag{A.2.8}$$

multiplying both sides in (A.2.8) with $\mathfrak{N}_1$ and applying (A.2.5) and (A.2.1)

$$\int_{-\infty}^\infty e^{y/2} \Phi(e^y) \frac{1}{\sqrt{x}} \omega(\ln x, y) \, dy = \tilde{\Phi}(x) , \tag{A.2.9}$$

substitution of $y = \ln s$ leads from (A.2.9) to

$$\int_0^\infty \frac{1}{\sqrt{s\,x}} \Phi(s) \, \omega(\ln x, \ln s) \, ds = \tilde{\Phi}(x) . \tag{A.2.10}$$

Comparing (A.2.7) with (A.2.10) we find the following situation: in $L^2(-\infty, \infty)$, the elements φ and $\tilde{\varphi}$ are linked by the kernel $\omega(x, y)$, which satisfies the Carleman condition

$$\int_{-\infty}^{\infty} |\omega(x, y)|^2 \, dy < \infty$$

for almost all values of x.

In $L^2(0, \infty)$ the corresponding elements Φ, $\tilde{\Phi}$ are linked by the kernel

$$\Omega(x, s) = (x\, s)^{-1/2} \, \omega(\ln x, \ln s) \, .$$

We have to show that Ω satisfies the Carleman condition

$$\int_{0}^{\infty} |\Omega(x, s)|^2 \, ds < \infty \tag{A.2.11}$$

for almost all values of x.

Substituting $y = \ln s$ we find

$$\int_{0}^{\infty} \left| \frac{\omega(\ln x, \ln s)}{\sqrt{x\, s}} \right|^2 ds = \int_{-\infty}^{\infty} \frac{|\omega(\ln x, y)|^2}{|x|} \, dy \, . \tag{A.2.12}$$

We know that

$$\int_{-\infty}^{\infty} |\omega(x, y)|^2 \, dy < \infty$$

for almost every x; then $e^{-x} \int_{-\infty}^{\infty} |\omega(x, y)|^2 \, dy < \infty$ for almost every x; and writing $x = \ln \xi$

$$\frac{1}{\xi} \int_{-\infty}^{\infty} |\omega(\ln \xi, y)|^2 \, dy < \infty$$

for almost every ξ since the mapping $x = \ln \xi$ is strictly monotonous. From (A.2.12) we thus find that (A.2.11) is satisfied. Ω remains an integral operator in $L^2(0, \infty)$. The proof for $\mathfrak{N}_2$ is exactly the same; we merely give the outlines.

If the kernel in $L^2(0, \infty)$ is $\omega(x, y)$ the corresponding kernel in $L^2(0, 1)$ is found to be

$$\Omega(x, s) = \frac{\omega\left(\dfrac{x}{1-x}, \dfrac{s}{1-s}\right)}{(1-x)\,(1-s)} \, .$$

On the same lines as in (A.2.12) we find that is a Carleman kernel.

Ultimately, we can easily see that the transformation

$$(U\, \varphi)\,(x) = \frac{1}{\sqrt{b-a}} \, \varphi\left(\frac{x-a}{b-a}\right)$$

maps $L^2(0, 1)$ unitarily into $L^2(a, b)$; one can also see at once that any integral operator in $L^2(0, 1)$ is also an integral operator in $L^2(a, b)$.

The VON NEUMANN proof is thus given first for $L^2(-\infty, +\infty)$ from there extended to $L^2(0, \infty)$ using $\mathfrak{N}_1$; from there to $L^2(0, 1)$ using $\mathfrak{N}_2$; and from there to $L^2(a, b)$ using U.

Appendix III

We shall generalize in this appendix the results found in sect. 3 to bounded non-normal operators. This seems to have some interest for multichannel scattering where R if reduced to the energy shell is not necessarily normal.

VON NEUMANN's proof shows that the essential property of a CIT-operator is that it makes an infinite subset of an orthonormal system $\{g_n\}$ arbitrarily small:

$$\lim_{i\to\infty} \| H g_{n_i} \| = 0 .$$

If this is the case, then VON NEUMANN's construction goes through.

In order to find a sufficient condition that an arbitrary bounded operator X be of CI-type we use the following general theorem: Every closed operator may be decomposed into the product of a non-negative hermitian operator R and a partial isometry V:

$$X = R V$$

and this decomposition is unique.

Theorem: X is a CIT-operator if there exists an infinite orthonormal set $\{g_n\}$ such that $\| X g_{n_i} \| \to 0$ $\{g_{n_i}\}$ being a subset of $\{g_n\}$

Proof: Consider first the case where V is an isometry $V^* V = I$, $V V^* = F$, F a projection.

Take an infinite set of orthonormal vectors $\{g_n, \| g_n \| = 1\}$ such that

$$\| X g_n \| = \| R V g_n \| = \| R h_n \| \to 0 ,$$

where $h_n = V g_n$ and $\| h_n \| = 1$ since V is an isometry. VON NEUMANN's theorem then says that R is a CIT-operator and so is $X = R V$ because of lemma 2.

If V is only a partial isometry: $V^* V = G$ is a projection too, the conclusion is not changed since $\| V g_n \| \leqq \| g_n \| = 1$.

Thus we have found a sufficientcondition. But is it also necessary?

Suppose $X = R V$ is a CIT-operator. By lemma 2 then $X V^* = R V V^* = R F$ is also a CIT-operator.

If now V is unitary $V V^+ = F = I$, R has 0 as W.l.p. for an infinite set g_n. Therefore

$$\| R g_n \| = \| X V^* g_n \| = \| X g_n' \| \to 0 ,$$

where

$$\| g_n' \| = \| V^* g_n \| = 1 .$$

Thus if $V V^+ = I$ the condition $\| X g_n \| \to 0$ is also necessary. This comprises in particular the case of normal operators $N = H U = U H$ where the operator U commutes with H. However, the statement is slightly more general since no use of commutativity was made.

The same argument holds if $F \neq I$, but such that the dimension of space M annihilated by $F (F M = 0)$ is finite.

For $\| R F g_n \| \to 0$ for an infinite orthonormal set, the same must be true for $R \equiv F (R F) F = F R F$ since $\| F R F \| \leq \| F \| \cdot \| R F g_n \|$ and $F V F \equiv V$ is a unitary operator on an *infinite* dimensional space.

The condition, however, seems not to be necessary in the case where F annihilates a space of infinite dimension.

Thus we may state the following

Theorem: In order that a closed operator X be a CIT-operator it is sufficient that there exists an infinite orthonormal set of unit vectors g_n such that $\| X g_n \| \to 0$ for $n \to \infty$.

If in the polar decomposition of X, $X = R V$ the partial isometry V is such that $V V^+ = F$ annihilates only a space of finite dimension, this condition is also necessary.

Under the same restriction we might state as criterion the following

Corollary: In order that X is a CIT-operator it is sufficient and necessary that the non-negative hermitian operator in the polar decomposition $X = R V$ of X is a CIT-operator. This is an immediate consequence of lemma 2.

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