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Potential scattering at high energies

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Abstract. An asymptotic expansion of the scattering amplitude for Schrödinger- and Dirac-potential scattering in inverse powers of the wave number k is developed. In both cases the first few coefficients of this expansion are given explicitely in terms of the potential $V(\boldsymbol{x})$. In order to simplify the proofs it is assumed that $V(\boldsymbol{x})$ is infinitely often differentiable and vanishes with all its derivatives faster than any inverse power of x as $x \to \infty$.

The present work had its roots in a paper of N. N. KHURI and S. B. TREIMAN[1]) on dispersion relations for Dirac potential scattering, where it is claimed that for fixed momentum transfer the scattering amplitude diverges less rapidly than E^2 as $E \to \infty$ (E = energy). Searching for a proof, we found a method leading to an asymptotic expansion of the scattering amplitude in inverse powers of the wave number k. The main part of this paper is devoted to the development of this expansion both for the Schrödinger- and the Dirac case. Among the various possible applications we discuss two which are related to the original problem: the behaviour of the Dirac scattering amplitude for fixed real momentum transfer and large complex energies in the physical sheet $\operatorname{Im} k \geq 0$, and the determination of the potential from the Dirac scattering amplitude.

When this work was completed we learned that essentially the same high energy approximation for the Schrödinger case has been previously derived by others[2]). However, it seems that proofs are still desirable.

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1. Potential Scattering

The wave function $\psi(\boldsymbol{x})$ describing the scattering of a particle of initial momentum $\boldsymbol{k}$ under the influence of a potential $V(\boldsymbol{x})$ is a solution of the integral equation

$$\psi(\boldsymbol{x}) = \psi_0(\boldsymbol{x}) + \int d^3y \; G(\boldsymbol{x} - \boldsymbol{y}) \; U(\boldsymbol{y}) \; \psi(\boldsymbol{y}) \, , \tag{1}$$

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or, in operator notation,

$$\psi = \psi_0 + G\, U\, \psi\,,$$

where

$$G(\boldsymbol{x}) = -\frac{1}{4\pi}\,\frac{e^{ikx}}{x}$$

is the GREEN's function of $\Delta + k^2$ satisfying the outgoing wave boundary condition. The meaning of k, ψ_0 and U is the following:

a) Schrödinger theory (units $\hbar = 2\,m = 1$)

$\psi_0 = e^{i\boldsymbol{k}\boldsymbol{x}}$ (incident plane wave), $k = |\boldsymbol{k}| = E^{1/2}$ (E = energy), $U = V$.

b) Dirac theory (units $\hbar = c = 1$)

Here it is convenient to work with ψ-functions whose values are 4×4-matrices rather than 4-component spinors[1]).

$\psi_0 = 1\, e^{i\boldsymbol{k}\boldsymbol{x}}$ ($1 = 4 \times 4$-unit matrix), $k = |\boldsymbol{k}| = (E^2 - m^2)^{1/2}$ (m = rest mass),

$$U = 2\,E\,V - V^2 - i\,\alpha_k\,V_{,k}\,,$$

where the α_k are the usual Dirac matrices and $V_{,k} = \partial V/\partial x_k$. Let us assume that $\psi(\boldsymbol{x})$ is a continuous and bounded solution of (1). Then – under certain conditions on the potential which are far more general than our later assumptions (30) – the scattered wave $\varphi = G\,U\,\psi$ has the following asymptotic behaviour as $x \to \infty$:

$$\varphi(\boldsymbol{x}) = \frac{e^{ikx}}{x}\,T(\boldsymbol{k}', \boldsymbol{k}) + o(x^{-1})\,, \tag{2}$$

$$T(\boldsymbol{k}', \boldsymbol{k}) = -\frac{1}{4\pi}\int d^3y\, e^{-i\boldsymbol{k}'\boldsymbol{y}}\, U(\boldsymbol{y})\,\psi(\boldsymbol{y})\,, \tag{3}$$

where $\boldsymbol{k}' = k\,\boldsymbol{x}/x$. $T(\boldsymbol{k}', \boldsymbol{k})$ is the scattering amplitude. In the Dirac case it is a 4×4-matrix, which is sometimes called the T-matrix. Our aim is to discuss the behaviour of $T(\boldsymbol{k}', \boldsymbol{k})$ for $k \to \infty$.

2. Formal Developments

Let $F(\boldsymbol{z})$ be a given function of $\boldsymbol{z}$. We do not impose precise conditions on $F(\boldsymbol{z})$, since the following developments are only formal and preliminary. We define

$$G\,F\,G(\boldsymbol{x}, \boldsymbol{y}) = (4\pi)^{-2}\int d^3z\, \frac{e^{ik|\boldsymbol{x}-\boldsymbol{z}|}}{|\boldsymbol{x}-\boldsymbol{z}|}\, F(\boldsymbol{z})\, \frac{e^{ik|\boldsymbol{z}-\boldsymbol{y}|}}{|\boldsymbol{z}-\boldsymbol{y}|}\,,$$

and we shall now derive an expansion of $G\,F\,G(\boldsymbol{x}, \boldsymbol{y})$ in powers of k^{-1}. First we choose new cartesian coordinates $\boldsymbol{v}$ instead of $\boldsymbol{z}$, so that the origin $\boldsymbol{v} = 0$ is at the point $1/2\,(\boldsymbol{x} + \boldsymbol{y})$ and the positive v_3-axis goes through the point $\boldsymbol{x}$. Then we introduce elliptic coordinates φ, η, t with foci $\boldsymbol{x}$, $\boldsymbol{y}$ ($\boldsymbol{x} \neq \boldsymbol{y}$):

$$\left.\begin{aligned} v_1 &= a\,t\,(t^2+2)^{1/2}\,(1-\eta^2)^{1/2}\cos\varphi\,, && 0 \le t < \infty\,,\\ v_2 &= a\,t\,(t^2+2)^{1/2}\,(1-\eta^2)^{1/2}\sin\varphi\,, && -1 \le \eta \le +1\,,\\ v_3 &= a\,(t^2+1)\,\eta\,, && 0 \le \varphi < 2\pi\,, \end{aligned}\right\} \tag{4}$$

where $a = 1/2\,|\boldsymbol{x} - \boldsymbol{y}|$. We note that $\boldsymbol{v}$ is an analytic function of t in a region containing the positive real t-axis. This is the reason why we introduced t instead of the usual elliptic coordinate $\xi = a\,(t^2 + 1)$. It follows from (4) that

$$\left.\begin{aligned} &|\boldsymbol{x} - \boldsymbol{z}| = a\,(t^2 + 1 - \eta)\,, \quad |\boldsymbol{z} - \boldsymbol{y}| = a\,(t^2 + 1 + \eta)\,, \\ &d^3 z = d^3 v = |\boldsymbol{x} - \boldsymbol{z}|\,|\boldsymbol{z} - \boldsymbol{y}|\,2\,a\,t\,d\varphi\,d\eta\,dt\,, \\ &G\,F\,G(\boldsymbol{x}, \boldsymbol{y}) = (4\,\pi)^{-2}\,e^{2\,i\,k\,a} \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\eta \int_0^{\infty} dt\,2\,a\,t\,e^{2\,i\,k\,a\,t^2}\,F(\varphi, \eta, t)\,. \end{aligned}\right\} \tag{5}$$

We assume that $F(\varphi, \eta, t)$, and its derivatives with respect to t, vanish sufficiently fast for $t \to \infty$. Then we obtain by successive partial integrations

$$\int^{\infty} dt\,2\,a\,t\,e^{2\,i\,k\,a\,t^2}\,F(\varphi, \eta, t) = -\frac{1}{2\,i\,k}\left[\sum_{n=0}^{N} (4\,k\,a)^{-n/2}\,f_n(0)\,\frac{\partial^n F}{\partial t^n}\,(\varphi, \eta, 0) + (4\,k\,a)^{-N/2} \int_0^{\infty} dt\,f_N\,(2\,k^{1/2}\,a^{1/2}\,t)\,\frac{\partial^{N+1} F}{\partial t^{N+1}}\,(\varphi, \eta, t)\right],$$

where

$$f_0(z) = e^{i\,z^2/2}\,, \quad f_n(z) = \int_z^{\infty} f_{n-1}(z)\,dz\,, \qquad (n = 1, 2, 3 \ldots)\,. \tag{6}$$

The functions $f_n(z)$ are uniformly bounded in the quadrant $\mathrm{Re}\,z \geq 0$, $\mathrm{Im}\,z \geq 0$, and their values at $z = 0$ are determined in appendix *I*. In this way we arrive at the following formal expansion of $G\,F\,G(\boldsymbol{x}, \boldsymbol{y})$:

$$G\,F\,G(\boldsymbol{x}, \boldsymbol{y}) = \frac{G\,(\boldsymbol{x} - \boldsymbol{y})}{4\,\pi\,i\,k}\left[\sum_{n=0}^{N} k^{-n/2}\,D_{n/2} F(\boldsymbol{x}, \boldsymbol{y}) + k^{-N/2}\,I_{N/2}\,F(\boldsymbol{x}, \boldsymbol{y})\right], \tag{7}$$

where

$$D_{n/2} F(\boldsymbol{x}, \boldsymbol{y}) = f_n(0)\,2^{-n}\,a^{(1-(n/2))} \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\eta\,\frac{\partial^n F}{\partial t^n}\,(\varphi, \eta, 0)\,,$$

$$I_{n/2}\,F(\boldsymbol{x}, \boldsymbol{y}) = 2^{-n}\,a^{(1-(n/2))} \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\eta \int_0^{\infty} dt\,f_n\,(2\,k^{1/2}\,a^{1/2}\,t)\,\frac{\partial^{n+1} F}{\partial t^{n+1}}\,(\varphi, \eta, t)\,.$$

Next we show that, for odd n, $D_{n/2} F(\boldsymbol{x}, \boldsymbol{y}) = 0$, so that (7) is actually an expansion in powers of k^{-1}. It follows from (4) that $F(\varphi, \eta, -t) = F\,(\varphi + \pi, \eta, t)$. Therefore

$$D(t) \equiv \int_0^{2\pi} d\varphi\,F(\varphi, \eta, t)$$

is an even function of t and consequently, for odd n, $d^n D/dt^n(t=0)=0$. For even n the numbers $f_n(0)$ are given by

$$f_0(0)=1\,,\quad f_n(0)=\frac{i^{n/2}}{1\cdot 3\cdot 5\cdot\ldots(n-1)}\qquad (n=2,4,6\ldots)\,.$$

For later use we list here the first few coefficients $D_{n/2}F(\boldsymbol{x},\boldsymbol{y})$ expressed in terms of $F(\boldsymbol{z})$. It is to be noted that $t=0$ is nothing but the straight line joining $\boldsymbol{x}$ and $\boldsymbol{y}$, so that the coefficients $D_{n/2}F(\boldsymbol{x},\boldsymbol{y})$ are integrals over this line only:

$$\left.\begin{aligned}
D_0F(\boldsymbol{x},\boldsymbol{y}) &= a\int_{-1}^{+1} d\eta\, F(\boldsymbol{s})\,,\\
D_1F(\boldsymbol{x},\boldsymbol{y}) &= \frac{i\,a^2}{4}\int_{-1}^{+1} d\eta\,(1-\eta^2)\,(\Delta F)\,(\boldsymbol{s})\,,\\
D_2F(\boldsymbol{x},\boldsymbol{y}) &= -\frac{a^3}{32}\int_{-1}^{+1} d\eta\,(1-\eta^2)^2\,(\Delta\Delta F)\,(\boldsymbol{s})\,,\\
&\quad -\frac{a}{8}\int_{-1}^{+1} d\eta\,(1-\eta^2)\,(\Delta F)\,(\boldsymbol{s})\,,
\end{aligned}\right\}\tag{8}$$

where $\boldsymbol{s}=1/2\,[(\boldsymbol{x}+\boldsymbol{y})+\eta\,(\boldsymbol{x}-\boldsymbol{y})]$. This is derived in appendix II. Sometimes we will find it convenient to introduce the arc length $s=a\,(1+\eta)$ measured from the point $\boldsymbol{y}$, in place of η. So we write, for instance,

$$D_0F(\boldsymbol{x},\boldsymbol{y})=\int_{\boldsymbol{y}}^{\boldsymbol{x}} F(\boldsymbol{s})\,ds\,.$$

3. Formal Expansion of the Resolvent

a) Schrödinger theory

We define two operators A and R by

$$A=GV\,,\quad 1+R=(1-A)^{-1}\,,$$

and assume R to be an integral operator with a kernel $G\,(\boldsymbol{z}-\boldsymbol{y})\,H(\boldsymbol{z},\boldsymbol{y})\,V(\boldsymbol{y})$. For $H(\boldsymbol{z},\boldsymbol{y})$ we try an expansion in powers of k^{-1}:

$$H=H_0+k^{-1}H_1+k^{-2}H_2+\ldots\,.\tag{9}$$

R has to satisfy the equation $A\,R=R-A$. Applying the results of section 2, we obtain a formal expansion of $A\,R$ in powers of k^{-1}, which we equate to the corresponding expansion of $R - A$:

$$\left.\begin{aligned}(2\,i\,k)^{-1}\,\{D_0(V\,H_0) + k^{-1}\,[D_1(V\,H_0) + D_0(V\,H_1)]&\\ + k^{-2}\,[D_2(V\,H_0) + D_1(V\,H_1) + D_0(V\,H_2)] + \ldots\}&\\ = -1 + H_0 + k^{-1}\,H_1 + k^{-2}\,H_2 + \ldots\,.&\end{aligned}\right\} \tag{10}$$

Comparing coefficients yields

$$\left.\begin{aligned}H_0 &= 1\,,\\ H_1 &= (2\,i)^{-1}\,D_0(V\,H_0)\,,\\ H_2 &= (2\,i)^{-1}\,[D_1(V\,H_0) + D_0(V\,H_1)]\,,\\ H_3 &= (2\,i)^{-1}\,[D_2(V\,H_0) + D_1(V\,H_1) + D_0(V\,H_2)]\,,\\ &\ldots\end{aligned}\right\} \tag{11}$$

We see that each H_p is given in terms of H_n, $n < p$. Since H_0 is known, we can compute H_p recursively:

$$\left.\begin{aligned}H_0 &= 1\,,\\ H_1 &= \frac{1}{2\,i}\int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} V(\boldsymbol{s})\,ds\,,\\ H_2 &= \frac{1}{2}\left(\frac{1}{2\,i}\int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} V(\boldsymbol{s})\,ds\right)^2 + \frac{1}{4}\int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} ds\,s\left(1 - \frac{s}{|\boldsymbol{x}-\boldsymbol{y}|}\right)\Delta V(\boldsymbol{s})\,,\\ &\ldots\end{aligned}\right\} \tag{12}$$

It is worth mentioning that there is another way of obtaining from (10) a recurrence relation for H_p:

$$\left.\begin{aligned}H_0 &= 1 + (2\,i\,k)^{-1}\,D_0(V\,H_0)\,,\\ H_1 &= (2\,i\,k)^{-1}\,D_1(V\,H_0) + (2\,i\,k)^{-1}\,D_0(V\,H_1)\,,\\ H_2 &= (2\,i\,k)^{-1}\,[D_2(V\,H_0) + D_1(V\,H_1)] + (2\,i\,k)^{-1}\,D_0(V\,H_2)\,,\\ &\ldots\end{aligned}\right\} \tag{13}$$

Recalling the explicit form of $D_0F(\boldsymbol{x}, \boldsymbol{y})$, we see that in general

$$H_p(\boldsymbol{x}, \boldsymbol{y}) = G_p(\boldsymbol{x}, \boldsymbol{y}) + (2\,i\,k)^{-1}\int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} ds\,V(\boldsymbol{s})\,H_p(\boldsymbol{s}, \boldsymbol{y})\,, \tag{14}$$

where the functions $G_p(\boldsymbol{x}, \boldsymbol{y})$ are given explicitely in terms of H_n, $n < p$. (14) is a simple one-dimensional integral equation for $H_p(\boldsymbol{x}, \boldsymbol{y})$. Differentiation with respect to $\boldsymbol{x}$, keeping the direction of $\boldsymbol{x}-\boldsymbol{y}$ fixed, reduces it to an ordinary differential equation, which has to be solved for the initial value $H_p(\boldsymbol{y}, \boldsymbol{y}) = G_p(\boldsymbol{y}, \boldsymbol{y})$. The solution is

$$H_p(\boldsymbol{x}, \boldsymbol{y}) = G_p(\boldsymbol{x}, \boldsymbol{y}) + (2\, i\, k)^{-1} \int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} ds\, V(\boldsymbol{s})\, G_p(\boldsymbol{s}, \boldsymbol{y})\, e^{\frac{1}{2ik} \int\limits_{\boldsymbol{s}}^{\boldsymbol{x}} V(\boldsymbol{s}')\, ds'} . \tag{15}$$

By aid of this formula and (8), the functions $H_p(\boldsymbol{x}, \boldsymbol{y})$, for $p = 0, 1, 2$, can easily be calculated. We obtain for H_0 and H_1:

$$\left.\begin{aligned} H_0(\boldsymbol{x}, \boldsymbol{y}) &= e^{\frac{1}{2ik} \int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} V(\boldsymbol{s})\, ds} , \\ H_1(\boldsymbol{x}, \boldsymbol{y}) &= G_1(\boldsymbol{x}, \boldsymbol{y}) + 0(k^{-2}) \\ &= \frac{1}{4\,k} \int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} ds \left(1 - \frac{s}{|\boldsymbol{x}-\boldsymbol{y}|}\right) s \left[\Delta_{\boldsymbol{x}} \left(V(\boldsymbol{x})\, e^{\frac{1}{2ik} \int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} V(\boldsymbol{s})\, ds}\right)\right]_{\boldsymbol{x}=\boldsymbol{s}} + 0(k^{-2}) . \end{aligned}\right\} \tag{16}$$

It is clear that the $H_p(\boldsymbol{x}, \boldsymbol{y})$ defined by (13) differ from those defined by (11) and depend on k, so that (9) is no longer an expansion in powers of k^{-1} in the strict sense. However, the first expansion is easily reconstructed from the second, if

$$\exp\left[(2\, i\, k)^{-1} \int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} V(\boldsymbol{s})\, ds\right]$$

is replaced by its power series.

The expression (16) for H_0 clearly shows the connection with the WKB-approximation. In such an approximation, the resolvent kernel would be given by

$$G\,(\boldsymbol{x} - \boldsymbol{y})\, H_{WKB}(\boldsymbol{x}, \boldsymbol{y}) = -\frac{1}{4\,\pi} \frac{e^{i \int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} \sqrt{k^2 - V(\boldsymbol{s})}\, ds}}{|\boldsymbol{x}-\boldsymbol{y}|} , \tag{17}$$

where it is assumed that the energy is so high that the classical path from $\boldsymbol{y}$ to $\boldsymbol{x}$ can be replaced by a straight line. Expansion of the square root yields

$$H_{WKB}(\boldsymbol{x}, \boldsymbol{y}) = e^{\frac{1}{2ik} \int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} V(\boldsymbol{s})\, ds} + 0(k^{-3}) .$$

b) *Dirac theory*

We proceed in the same way as in the non-relativistic case. The main difference is that U depends now linearly on E. (This energy-dependence makes the Born series an inappropriate tool at high energies, contrary to the Schrödinger case). In order to make this explicit we write

$$U = 2\,k\,W\,, \quad W = V + k^{-1}\,Q\,, \quad Q = (E - k)\,V - \frac{1}{2}\,(V^2 + i\,\alpha_k\,V_{,k})\,. \tag{18}$$

$E - k$ is of the order of k^{-1} for $k \to \infty$, so that the energy dependence of Q is harmless. The operators A, R are defined by

$$A = G\,W\,, \quad 1 + 2\,k\,R = (1 - 2\,k\,A)^{-1}\,.$$

Again the kernel of R is written in the form $G\,(\boldsymbol{z} - \boldsymbol{y})\;H(\boldsymbol{z}, \boldsymbol{y})\;W(\boldsymbol{y})$, and for H we try the expansion

$$H = H_0 + k^{-1}\,H_1 + k^{-2}\,H_2 + \dots \tag{19}$$

From $2\,k\,A\;R = R - A$ it follows, as before:

$$\left.\begin{aligned} H_0 &= 1 - i\,D_0(V\,H_0)\,, \\ H_1 &= -i\,[D_1(V\,H_0) + D_0(Q\,H_0)] - i\,D_0(V\,H_1)\,, \\ H_2 &= -i\,[D_2(V\,H_0) + D_1(V\,H_1 + Q\,H_0) + D_0(Q\,H_1)] - i\,D_0(V\,H_2)\,, \\ &\dots \end{aligned}\right\} \tag{20}$$

or, in general,

$$H_p = G_p - i\,D_0(V\,H_p)\,,$$

where again G_p is determined by all H_n with $n < p$. This is the same kind of equation as (14). The solution is

$$H_p(\boldsymbol{x}, \boldsymbol{y}) = G_p(\boldsymbol{x}, \boldsymbol{y}) - i \int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} ds\; V(\boldsymbol{s})\;G_p(\boldsymbol{s}, \boldsymbol{y})\; e^{-i \int\limits_{\boldsymbol{s}}^{\boldsymbol{x}} V(\boldsymbol{s}')\,ds'}\,. \tag{21}$$

From this we obtain after some partial integrations:

$$\left.\begin{aligned} H_0 &= e^{-i \int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} V(\boldsymbol{s})\,ds}\,, \\ H_1 &= -i\,e^{-i \int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} V(s)\,ds} \int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} Q(\boldsymbol{s})\;ds \\ &\quad + \frac{1}{2} \int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} ds\; e^{-i \int\limits_{\boldsymbol{s}}^{\boldsymbol{x}} V(\boldsymbol{s}')\,ds'} \int\limits_{\boldsymbol{y}}^{\boldsymbol{s}} ds' \left(1 + \frac{s'}{s^2}\right) s' \left[\Delta_{\boldsymbol{x}} \left(V(x)\; e^{-i \int\limits_{\boldsymbol{y}}^{\boldsymbol{x}} V(\boldsymbol{s})\,ds} \right)\right]_{\boldsymbol{x} = \boldsymbol{s}'} \\ &\dots \end{aligned}\right\} \tag{22}$$

Again H_0 can be understood as the high energy limit of a WKB-approximation. If we neglect the term $k^{-1}\,Q$ in W for high energies, the WKB-resolvent kernel is given by

(17), provided V is replaced by $2\,k\,V$. Expansion of the square root yields $H_{WKB}(\boldsymbol{x},\boldsymbol{y}) = H_0(\boldsymbol{x},\boldsymbol{y}) + 0(k^{-1})$.

Furthermore, since $H_0 \neq 1$, the resolvent does not converge to the first Born approximation as $k \to \infty$, contrary to the Schrödinger case. In fact, a formal application of (7) to the n'th Born term shows that this term gives a non-vanishing contribution

$$\frac{1}{n!}\left(-i\int_{\boldsymbol{y}}^{\boldsymbol{x}} V(s)\,ds\right)^n$$

to $H(\boldsymbol{x},\boldsymbol{y})$ in the limit $k \to \infty$. Summing over these limits, we again obtain (22) for $H_0(\boldsymbol{x},\boldsymbol{y})$. However, nothing is known about the convergence of the Born series at high energies.

4. Proofs

a) Schrödinger theory

Two things are to be proved: first, that for sufficiently high energies the resolvent $(1 - A(k))^{-1} = 1 + R(k)$ exists, and secondly, that $R(k)$ admits the asymptotic expansion (9) (13) derived formally in section 3. (The proof of (11) is completely analogous.) We start by giving a precise meaning to the operators A, R. For this we introduce two function spaces C_0, C_1, whose elements are complex valued continuous functions $\psi(\boldsymbol{x})$ normalizable in the following sense:

$$\left.\begin{aligned} \psi \in C_0: &\quad \|\psi\|_0 \equiv \sup_{\boldsymbol{x}} |\psi(\boldsymbol{x})| < \infty\,, \\ \psi \in C_1: &\quad \|\psi\|_1 \equiv \sup_{\boldsymbol{x}} (1 + x)\, |\psi(\boldsymbol{x})| < \infty\,. \end{aligned}\right\} \tag{23}$$

Obviously $C_1 \subset C_0$ and $\|\psi\|_0 \leq \|\psi\|_1$ for all $\psi \in C_1$. Next we consider an operator A mapping C_0 into C_1 and define its norm in the usual way by

$$|A| = \sup_{\|\psi\|_0 = 1} \|A\,\psi\|_1\,. \tag{24}$$

Let A_1, A_2 be two operators of this kind. It follows from $\|\psi\|_0 \leq \|\psi\|_1$ that $|A_1\,A_2| \leq |A_1|\,|A_2|$. Together with the completeness of C_1 this implies

Lemma 1: Let A be an operator mapping C_0 into C_1 and $|A| < 1$. Then $(1 - A)^{-1} = 1 + R$ exists. R is an operator mapping C_0 into C_1 and

$$|R| < \frac{|A|}{1 - |A|}\,.$$

Now we can explain the idea of the proof. We define an integral operator $R^n(k)$ by the kernel

$$G\,(\boldsymbol{x} - \boldsymbol{y})\,H^{(n)}(\boldsymbol{x},\boldsymbol{y})\,V(\boldsymbol{y})\,,\quad H^{(n)}(\boldsymbol{x},\boldsymbol{y}) = \sum_{p=0}^{n} k^{-p}\,H_p(\boldsymbol{x},\boldsymbol{y})\,, \tag{25}$$

where the functions $H_p(\boldsymbol{x}, \boldsymbol{y})$ are explicitely defined by (8), (13), (15) in terms of the potential. If these functions turn out to be sufficiently regular, so that the formal calculations leading to (13) can be justified, it is clear that $R^{(n)}(k)$ is a solution of the equation $A\ R = R - A$, up to terms of the order of $k^{-n-3/2}$. More precisely, we expect

$$\varepsilon^{(n)}(k) \equiv A\ R^{(n)} - R^{(n)} + A \tag{26}$$

to be an operator mapping C_0 into C_1 and

$$|\,\varepsilon^{(n)}(k)\,| \leq \lambda\, k^{-n-3/2}\,, \tag{27}$$

where λ is some constant independent of k. (26) is equivalent to

$$[1 - A(k)]\ [1 + R^{(n)}(k)] = 1 - \varepsilon^{(n)}(k)\,. \tag{28}$$

For sufficiently large k, we have $|\,\varepsilon^{(n)}(k)\,| \leq 1/2$, so that by Lemma 1, $(1 - \varepsilon^{(n)}(k))^{-1} = 1 + \delta^{(n)}(k)$ exists and $|\,\delta^{(n)}(k)\,| \leq 2\,\lambda\, k^{-n-3/2}$. Then we conclude from (28) that $(1 - A(k))^{-1}$ exists and is given by

$$1 + R(k) = [1 + R^{(n)}(k)]\ [1 + \delta^{(n)}(k)]\,.$$

If $|\,R^{(n)}(k)\,|$ turns out to be uniformly bounded for large k, we finally arrive at the result we are looking for:

$$|\,R^{(n)}(k) - R(k)\,| \leq \text{const.}\ k^{-n-3/2} \qquad (k \to \infty)\,. \tag{29}$$

Leaving the question of the validity of this result for arbitrary n open, we content ourselves with the proof for $n = 2$. In order to confirm (27), we have to justify the formal expansion (7) of $G\,F\,G(\boldsymbol{x}, \boldsymbol{y})$, to the required order N, where

$$F(\boldsymbol{z}) = V(\boldsymbol{z})\ H_p(\boldsymbol{z}, \boldsymbol{y}) \qquad (p = 0, 1, 2)\,,$$

and to estimate all the coefficients $D_n F(\boldsymbol{x}, \boldsymbol{y})$, $I_n F(\boldsymbol{x}, \boldsymbol{y})$ which occur. In order to simplify these estimations, we assume

$$V \in \mathfrak{S}\,, \tag{30}$$

where $\mathfrak{S}$ denotes the class of infinitely often differentiable functions $f(\boldsymbol{x})$ which, together with all its derivatives, tend to zero faster than any negative power of x as $x \to \infty$. (It should be noticed, however, that this condition is far from being necessary. For instance, it would be sufficient to assume the existence of a finite number of derivatives vanishing faster than some finite power of x^{-1} and $x \to \infty$. Any more general condition would at least have to ensure the boundedness of the integral operators with kernels $G(\boldsymbol{x}, \boldsymbol{y})\ H_P(\boldsymbol{x}, \boldsymbol{y})\ V(\boldsymbol{y})$. Between this and (30), there is still a wide field for possible generalizations.)

Under the assumption (30) the functions $H_P(\boldsymbol{x}, \boldsymbol{y})$ $(p = 0, 1, 2)$ have a common property on which all further estimations will be based, namely:

For $p = 0, 1, 2$ $H_p(\boldsymbol{z}, \boldsymbol{y})$ may be written as $H_p(\boldsymbol{z}, \boldsymbol{y}) = W_p(\boldsymbol{z}, |\boldsymbol{z}-\boldsymbol{y}|, \boldsymbol{y})$, where $W(u_1 \dots u_4, \boldsymbol{y})$ is infinitely often differentiable with respect to $u_1 \dots u_4$ and

$$\left| \frac{\partial^k W}{\partial u_1^{k_1}, \dots \partial u_4^{k_4}} (\boldsymbol{z}, |\boldsymbol{z}-\boldsymbol{y}|, \boldsymbol{y}) \right| \leq C\,(1 + |\boldsymbol{z}-\boldsymbol{y}|)^n, \tag{31}$$

C and n depending only on p and $k_1 \dots k_4$. We shall refer to this property by writing

$$H_p \in \mathfrak{A}.$$

Proof: 1. If $H \in \mathfrak{A}$, then by (8) and (30) $D_n(V\,H) \in \mathfrak{A}$ for $n = 0, 1, 2$.
2. If $G_p \in \mathfrak{A}$, then by (15) $H_p \in \mathfrak{A}$.
3. $G_0 = 1 \in \mathfrak{A}$, therefore, by (13) and (15) $H_p \in \mathfrak{A}$ for $p = 0, 1, 2$.

We know from (4) and (5) that $\boldsymbol{z}$ and $|\boldsymbol{z}-\boldsymbol{y}|$ are analytic functions of t in a region containing the positive real t-axis, provided that $a \neq 0$. It is clear therefore that, for fixed $\boldsymbol{x} \neq \boldsymbol{y}$, the formal procedure of section 2 is justified to every order N; since according to (30) and (31) all the derivatives with respect to t exist and the convergence of the integrals is ensured by (30). We are left with the estimation of the coefficients $D_n F(\boldsymbol{x}, \boldsymbol{y})$, $I_n\, F(\boldsymbol{x}, \boldsymbol{y})$. The results are collected in the following Lemma:

Lemma 2: We assume $F(\boldsymbol{z}, \boldsymbol{y}) = S(\boldsymbol{z})\, A(\boldsymbol{z}, \boldsymbol{y})$, $S \in \mathfrak{S}$ and $A \in \mathfrak{A}$.

Then there exist constants C and m such that

$$|D_n F(\boldsymbol{x}, \boldsymbol{y})| \leq C\,(1+y)^m, \qquad n = 0, 1, 2,$$

$$|I_n\, F(\boldsymbol{x}, \boldsymbol{y})| \leq C\,(1+y)^m, \qquad n = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2,$$

$$|I_{5/2}\, F(\boldsymbol{x}, \boldsymbol{y})| \leq C\,(1+y)^m\,(1 + |\boldsymbol{x}-\boldsymbol{y}|^{-1/2}).$$

We omit the proof of this Lemma because there are too many integrals to be considered and the method is always the same. An example is given in appendix III. We are now ready to estimate the norm of the integral operator $\varepsilon^{(2)}(k)$, whose kernel is $G\,(\boldsymbol{x}-\boldsymbol{y})\, E(\boldsymbol{x}, \boldsymbol{y})\, V(\boldsymbol{y})$, where

$$E = \frac{k^{-7/2}}{2\,i} \left[I_{5/2}\,(V\,H_0) + I_{3/2}\,(V\,H_1) + I_{1/2}\,(V\,H_2) \right]. \tag{32}$$

$V \in \mathfrak{S}$ implies $|V(\boldsymbol{y})| \leq A\,(1+y)^{-p}$, p arbitrarily large and A depending on p only. Then it follows from (31) and Lemma 2 that

$$|\varepsilon^{(2)}(k)| \leq \text{const}\; k^{-7/2} \int_0^\infty dy\, y^2\,(1+y)^{m-p} \sup_{\boldsymbol{x}} \left[\frac{1+x}{4\,\pi} \int d\Omega\, [|\boldsymbol{x}-\boldsymbol{y}|^{-1} + |\boldsymbol{x}-\boldsymbol{y}|^{-3/2}] \right].$$

The supremum can be estimated using the formula

$$\frac{1}{4\,\pi} \int d\Omega\, f\,(|\boldsymbol{x}-\boldsymbol{y}|) = \frac{1}{2\,x\,y} \int_{|x-y|}^{x+y} r\, f(r)\, dr$$

and turns out to be bounded by const. $(1 + y^{-3/2})$. Therefore (27) is proved if we choose $p > m + 3$. In the same way it follows from (13) and Lemma 2 that $| R^{(2)}(k) |$ is uniformly bounded in k. This completes the proof of (29) for $n = 2$.

b) *Dirac theory*

After a few minor modifications, the foregoing proof also applies to this case. Since $\psi(\boldsymbol{x})$ is a matrix-valued function, $| \psi(\boldsymbol{x}) |$ has to be identified with some reasonable kind of matrix norm, for instance,

$$| \psi |^2 = \sup_{\chi} (\psi \chi, \psi \chi) ,$$

where χ varies over the unit sphere $(\chi, \chi) = 1$ in four-dimensional unitary spinor space. Continuity of $\psi(\boldsymbol{x})$ can be defined with respect to this norm. The definitions of C_0, C_1 and $| A |$ are taken over literally and Lemma 1 remains true as it stands. $R^{(n)}(k)$ is defined by its kernel

$$G (\boldsymbol{x} - \boldsymbol{y}) H^{(n)}(\boldsymbol{x}, \boldsymbol{y}) W(\boldsymbol{y}) , \quad H^{(n)}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{p=0}^{n} k^{-p} H_p(\boldsymbol{x}, \boldsymbol{y}) , \tag{33}$$

where the $H_p(\boldsymbol{x}, \boldsymbol{y})$ are the functions determined by (20) and (21). Then we expect

$$\varepsilon^{(n)}(k) \equiv 2 k A R^{(n)} - R^{(n)} + A \tag{34}$$

to be an operator mapping C_0 into C_1 and

$$| \varepsilon^{(n)}(k) | \leq \lambda k^{-n-1/2} . \tag{35}$$

(34) is equivalent to

$$(1 - 2 k A) (1 + 2 k R^{(n)}) = 1 - 2 k \varepsilon^{(n)}(k) , \tag{36}$$

therefore we have to take at least $n = 1$ in order to prove the existence of $(1 - A(k))^{-1}$ for sufficiently large k. Provided that $| R^{(n)}(k) |$ is uniformly bounded for large k, it follows from (35) and (36) that

$$| R^{(n)}(k) - R(k) | \leq \text{const } k^{-n+1/2} \qquad (k \to \infty) . \tag{37}$$

In order to prove (35) for $n = 2$, we note that (31) also applies to the Dirac case. The kernel of $\varepsilon^{(2)}(k)$ is $G (\boldsymbol{x} - \boldsymbol{y}) E(\boldsymbol{x}, \boldsymbol{y}) W(\boldsymbol{y})$, where now

$$E = - i k^{-5/2} [I_{5/2} (V H_0) + I_{3/2} (Q H_0 + V H_1) + I_{1/2} (Q H_1 + V H_2) \\ + k^{-1/2} D_0(Q H_2) + k^{-1/2} I_0 (Q H_2)] .$$

Using (31) and Lemma 2, the proof of (37) for the case $n = 2$ is completed as in the Schrödinger case.

5. The Scattering Amplitude

a) *Schrödinger theory*

It is clear that the scattering amplitude exists whenever $(1 - A)^{-1} = 1 + R$ exists as an operator on C_0, and is given by

$$T(\boldsymbol{k}', \boldsymbol{k}) = \lim_{\boldsymbol{x}\to\infty} x\, e^{-i k x}\, \varphi(\boldsymbol{x}) \,, \tag{38}$$

where $\varphi = R\,\psi_0$, and $x \to \infty$ is to be understood in the fixed direction of $\boldsymbol{k}'$. On the other hand, we define

$$T^{(2)}(\boldsymbol{k}', \boldsymbol{k}) = \lim_{\boldsymbol{x}\to\infty} x\, e^{-i k x}\, \varphi^{(2)}(\boldsymbol{x}) \,, \tag{39}$$

where $\varphi^{(2)} = R^{(2)}\,\psi_0$. The existence of this limitfollows from (13), (8), (30) and (31).

For the difference $T^{(2)} - T$ we obtain, by (29) and $\| \psi_0 \|_0 = 1$, for sufficiently large k:

$$\left.\begin{aligned} | T^{(2)}(\boldsymbol{k}', \boldsymbol{k}) - T(\boldsymbol{k}', \boldsymbol{k}) | &= | \lim_{\boldsymbol{x}\to\infty} x\, e^{-i k x}\, (\varphi^{(2)}(\boldsymbol{x}) - \varphi(\boldsymbol{x})) | \\ &\leq \| \varphi^{(2)} - \varphi \|_1 \leq | R^{(2)} - R | \leq \text{const}\, k^{-7/2} \,, \end{aligned}\right\} \tag{40}$$

uniformly for all directions of $\boldsymbol{k}'$ and $\boldsymbol{k}$. The explicit form of $T^{(2)}$ is too long to be reproduced here; we content ourselves with terms up to order k^{-2}:

$$\left.\begin{aligned} T(\boldsymbol{k}', \boldsymbol{k}) = &-\frac{1}{4\pi} \int d^3y\, e^{-i(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{y} + \frac{1}{2ik}\int_0^\infty V(\boldsymbol{s})\,ds}\, V(\boldsymbol{y}) \\ &- \frac{1}{16\pi k^2} \int d^3y\, e^{-i(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{y}}\, V(\boldsymbol{y}) \int_0^\infty ds\, s \left[\Delta_{\boldsymbol{x}} \left(V(\boldsymbol{x})\, e^{\frac{1}{2ik}\int_y^x V(\boldsymbol{s})\,ds} \right) \right]_{\boldsymbol{x}=\boldsymbol{s}} + 0(k^{-3}) \,, \end{aligned}\right\} \tag{41}$$

where $\boldsymbol{s} = \boldsymbol{y} + s\,\boldsymbol{k}'/k$ and $| 0(k^{-3}) | \leq C\, k^{-3}$, C being independent of $\boldsymbol{k}', \boldsymbol{k}$. This follows from (16) and Lemma 2. An equivalent expansion is obtained either from (12) or by replacing $\exp\,((2\,i\,k)^{-1} \int V\,ds)$ in (41) by its power series:

$$\begin{aligned} T(\boldsymbol{k}', \boldsymbol{k}) = &-\frac{1}{4\pi} \int d^3y\, e^{-i(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{y}}\, V(y) \\ &\cdot \left[1 + \frac{1}{2ik} \int_0^\infty V(\boldsymbol{s})\,ds + \frac{1}{2} \left(\frac{1}{2ik} \int_0^\infty V(\boldsymbol{s})\,ds \right)^2 + \frac{1}{4k^2} \int_0^\infty ds\, s\, \Delta V(\boldsymbol{s}) \right] + 0(k^{-3}) \,. \end{aligned}$$

It is seen that for $\boldsymbol{k} = \boldsymbol{k}'$, the coefficients of odd powers of k^{-1} are purely imaginaty, those of even powers real. By the optical theorem, the total cross section $\sigma(\boldsymbol{k})$ is given by

$$\sigma(\boldsymbol{k}) = \frac{4\pi}{k}\, \text{Im}\, T(\boldsymbol{k}, \boldsymbol{k}) \,.$$

We conclude that $\sigma(\boldsymbol{k})$ admits an asymptotic expansion in powers of E^{-1}. The leading term is found to be

$$\sigma(\boldsymbol{k}) = \frac{1}{4\,E} \iint dy_1\, dy_2 \left(\int dy_3\, V(\boldsymbol{y})\right)^2 + 0(E^{-2})\,,$$

where the y_3-axis has been taken along the direction of $\boldsymbol{k}$. For a spherically symmetric potential $V = V(r) \neq 0$, this leading term cannot vanish.

b) Dirac theory

In the same way as in the non-relativistic case, we obtain

$$|\, T^{(2)}(\boldsymbol{k}', \boldsymbol{k}) - T(\boldsymbol{k}', \boldsymbol{k})\,| \leq \text{const}\ k^{-1/2}\,. \tag{42}$$

The leading term of the T-matrix is isotropic, i.e. proportional to the 4×4-unit matrix:

$$T(\boldsymbol{k}', \boldsymbol{k}) = -1\, \frac{k}{2\,\pi} \int d^3y\; e^{-i(\boldsymbol{k}'-\boldsymbol{k}) - i \int_0^\infty V(\boldsymbol{s})\, ds}\; V(\boldsymbol{y}) + 0(k^0)\,. \tag{43}$$

The leading term of the non-isotropic part $T - 1/4$ trace T, which is responsible for the polarisation phenomena at high energies, is found to be

$$\left.\begin{aligned} &\left(T - \frac{1}{4}\ \text{trace}\ T\right) (\boldsymbol{k}', \boldsymbol{k}) \\ &= \frac{\alpha_k}{4\,\pi} \int d^3y\; e^{-i(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{y} - i\int_0^\infty V(\boldsymbol{s})\, ds} \left[i\, V_{,k}(\boldsymbol{y}) + V(\boldsymbol{y}) \int_0^\infty V_{,k}(\boldsymbol{s})\, ds \right] + 0(k^{-1/2})\,. \end{aligned}\right\} \tag{44}$$

In the derivation of the last result use is made of the fact that trace $\alpha_k = 0$ for all k.

6. Further Discussion of the Dirac Scattering Amplitude

a) Complex energies

In the following we discuss only the spherically symmetric case $V = V(r)$, so that the T-matrix can be considered as a function of energy E and momentum transfer $\Delta = 1/2\, |\boldsymbol{k} - \boldsymbol{k}'|$. In ref. [1]) it is claimed that $T(E, \Delta)/E^2 \to 0$ for $E \to \infty$, uniformly in the cut plane Im $(E^2 - m^2)^{1/2} \geq 0$, for fixed real Δ. For real E this is confirmed by our result (43) which implies

$$\lim_{E\to\infty} \frac{T(E, \Delta)}{E} = -\frac{1}{2\,\pi} \int d^3y\; e^{-2i\boldsymbol{\Delta}\boldsymbol{y} - i\int_0^\infty V(\boldsymbol{s})\, ds}\; V(\boldsymbol{y}) \equiv T_\infty(\Delta)\,, \tag{45}$$

where $\boldsymbol{s} = \boldsymbol{y} + s\,\boldsymbol{n}$, $\boldsymbol{n}$ being any unit vector perpendicular to $\boldsymbol{\Delta}$. We shall show now that this remains true for complex energies, if the potential satisfies the additional conditions

$$\int_0^\infty dr\, e^{\alpha r} \,|\, V(r) \,|< \infty\,, \quad \int_0^\infty dr\, e^{\alpha r} \left| \frac{dV}{dr}(r) \right| < \infty$$

for some $\alpha > 0$. The precise statement we want to prove is the following:

For any fixed real Δ, the analytic continuation of $T(E, \Delta)$ to complex E exists in some region $|E| > M(\Delta)$ of the cut plane $\mathrm{Im}\, k \geq 0$, and satisfies

$$\lim_{E\to\infty} \frac{T(E, \Delta)}{E} \equiv T_\infty(\Delta)\,, \tag{46}$$

uniformly in all directions.

Proof

1) We introduce the vectors

$$\boldsymbol{\Delta} = \frac{1}{2}(\boldsymbol{k}' - \boldsymbol{k})\,, \quad \boldsymbol{n}^2 = 1\,, \quad \boldsymbol{p} = \frac{1}{2}(\boldsymbol{k}' + \boldsymbol{k}) = p\,\boldsymbol{n}\,, \quad \boldsymbol{n}\,\boldsymbol{\Delta} = 0\,. \tag{47}$$

If E becomes complex, so that $\mathrm{Im}\, k \geq 0$, $\boldsymbol{\Delta}$ and $\boldsymbol{n}$ remain real and $p = (E^2 - m^2 - \Delta^2)^{1/2}$ is uniquely defined by the requirement $\mathrm{Im}\, p \geq 0$.

2) The incident wave $1\, e^{i\boldsymbol{k}\boldsymbol{x}}$ is no longer bounded for complex k. For this reason we consider the transformed scattering equation

$$\psi' = \psi'_0 + G'\, U\, \psi'\,, \tag{48}$$

where

$$\psi'(\boldsymbol{x}) = \psi(\boldsymbol{x})\, e^{-ik\boldsymbol{n}\boldsymbol{x}}\,, \quad \psi'_0(\boldsymbol{x}) = 1\, e^{-i\boldsymbol{\Delta}\boldsymbol{x} + i(p-k)\boldsymbol{n}\boldsymbol{x}}\,, \quad G'(\boldsymbol{x}, \boldsymbol{y}) = G(\boldsymbol{x}, \boldsymbol{y})\, e^{-ik\boldsymbol{n}(\boldsymbol{x}-\boldsymbol{y})}$$

and, instead of C_0, the larger space C_α of all continuous functions $\psi(\boldsymbol{x})$ normalizable in the following sense:

$$\|\psi\|_\alpha = \sup_{\boldsymbol{x}} e^{-\alpha x}\, |\psi(\boldsymbol{x})| < \infty\,.$$

For $k \to \infty$, we have $p - k \approx \Delta^2/2\,k$, so that $|\mathrm{Im}\,(p-k)| < \alpha$ for sufficiently large $|E|$, and consequently $\psi'_0 \in C_\alpha$.

3) The whole of sections 3b and 4b applies also to the transformed equation (48), if G is replaced everywhere by G' and C_0 by C_α. $R'(k)$ and $R'^{(n)}(k)$ are then defined as operators mapping C_α into C_1. The functions $H_P(\boldsymbol{x}, \boldsymbol{y})$ remain unchanged. For sufficiently large $|k|$ $R'(k)$ exists and

$$|R'^{(2)}(k) - R'(k)| \leq \text{const}\, |k|^{-3/2} \qquad (|k| \to \infty,\ \mathrm{Im}\, k \geq 0)\,. \tag{49}$$

4) The definition (38) of the scattering amplitude is no longer appropriate for complex k; instead we use the integral representation (3). It is then natural to split the T-matrix into the first Born approximation (which is a linear function of E and therefore can be trivially continued to complex E) and the contribution $T_1(E, \Delta)$ of the scattered wave. The continuation of $T_1(E, \Delta)$ is defined by

$$T_1(E, \Delta) = -\frac{1}{4\pi} \int d^3x\, e^{-i\boldsymbol{\Delta}\boldsymbol{x} - i(p-k)\boldsymbol{n}\boldsymbol{x}}\, U(\boldsymbol{x})\, \varphi'(\boldsymbol{x})\,, \tag{50}$$

where $\varphi' = 2\,k\,R'\,\psi_0'$. There exists a constant $M(\Delta)$, depending only on Δ, such that for $|E| > M(\Delta)$ and $\operatorname{Im} k \geq 0$ $R'(k)$ exists and $|\operatorname{Im}(p - k)| < \alpha$. This implies $\varphi_1' \in C_1$ and therefore the integral (50) is absolutely convergent. Moreover, the function $T_1(E, \Delta)$ defined by (50) is known to be analytic in E for $|E| > M(\Delta)$, $\operatorname{Im} k > 0$ and continuous for $|E| > M(\Delta)$, $\operatorname{Im} k \geq 0$[1])[3]). If we define $T_1^{(n)}(E, \Delta)$ in the same way, but with $\varphi'^{(n)} = 2\,k\,R'^{(n)}\,\psi_0'$ instead of φ', it follows from (49) that

$$|\,T_1^{(2)}(E, \Delta) - T_1(E, \Delta)\,| \leq \text{const}\,|\,k\,|^{1/2} \tag{51}$$

for $|E| \to \infty$, uniformly in the cut plane $\operatorname{Im} k \geq 0$.

To prove (46), it is therefore sufficient to deal with

$$\left.\begin{aligned} T_1^{(2)}(E, \Delta) = (4\,\pi)^{-2} \int d^3x\, d^3y\, \frac{e^{i k [|\boldsymbol{x}-\boldsymbol{y}| - \boldsymbol{n}(\boldsymbol{x}-\boldsymbol{y})]}}{|\boldsymbol{x}-\boldsymbol{y}|}\, H^{(2)}(\boldsymbol{x}, \boldsymbol{y}; k) \\ \cdot\, U(\boldsymbol{x})\, U(\boldsymbol{y})\, e^{-i \boldsymbol{\Delta}(\boldsymbol{x}+\boldsymbol{y}) - i (p-k) \boldsymbol{n}(\boldsymbol{x}-\boldsymbol{y})}\,, \end{aligned}\right\} \tag{52}$$

where $H^{(2)}(\boldsymbol{x}, \boldsymbol{y}; k)$ is the function defined by (33) and (20).

5) The high energy behaviour of $T_1^{(2)}(E, \Delta)$ is obtained by adapting the method of section 2. Let $F(\boldsymbol{z})$ be a given function of $\boldsymbol{z}$, which is sufficiently regular and vanishes sufficiently fast as $z \to \infty$. We define

$$T\,F(\boldsymbol{y}) = -\frac{1}{4\,\pi} \int d^3z\, \frac{e^{i k [|\boldsymbol{z}-\boldsymbol{y}| - \boldsymbol{n}(\boldsymbol{z}-\boldsymbol{y})]}}{|\boldsymbol{z}-\boldsymbol{y}|}\, F(\boldsymbol{z})\,.$$

In order to obtain an expansion of $T\,F(\boldsymbol{y})$ in inverse powers of k we first introduce new cartesian coordinates $\boldsymbol{v}$ instead of $\boldsymbol{z}$, with the origin $\boldsymbol{v} = 0$ at the point $\boldsymbol{y}$, and with the positive v_3-axis in the direction of $\boldsymbol{n}$. Then we define parabolic coordinates φ, η, t by

$$v_1 = t\left(\frac{\eta}{2}\right)^{1/2} \cos\varphi\,, \qquad 0 \leq t < \infty\,,$$

$$v_2 = t\left(\frac{\eta}{2}\right)^{1/2} \sin\varphi\,, \qquad 0 \leq \eta < \infty\,,$$

$$v_3 = \frac{1}{2}\left(\eta - \frac{t^2}{2}\right), \qquad 0 \leq \varphi < 2\,\pi\,.$$

It follows, as in section 2, that

$$T\,F(\boldsymbol{y}) = \frac{1}{4\,i\,k}\left[\sum_{n=0}^{N} k^{-n/2}\, D_{n/2} F(\boldsymbol{y}) + k^{-N/2}\, I_{N/2}\, F(\boldsymbol{y})\right],$$

with

$$D_{n/2} F(\boldsymbol{y}) = \frac{f_n(0)}{2\,\pi} \int_0^{2\pi} d\varphi \int_0^{\infty} d\eta\, \frac{\partial^n F}{\partial t^n}\,(\varphi, \eta, 0)\,,$$

$$I_{n/2}\, F(\boldsymbol{y}) = \frac{1}{2\,\pi} \int_0^{2\pi} d\varphi \int_0^{\infty} d\eta \int_0^{\infty} dt\, f_n\,(k^{1/2}\,t)\, \frac{\partial^{n+1} F}{\partial t^{n+1}}\,(\varphi, \eta, t)\,,$$

where the functions $f_n(z)$ are defined by (6) and where again $D_{n/2}\, F(\boldsymbol{y}) = 0$ for odd n. $t = 0$ is the ray with origin $\boldsymbol{y}$, in the direction of $\boldsymbol{n}$, so that

$$D_0 F(\boldsymbol{y}) = 2 \int_0^\infty ds\, F(\boldsymbol{s}) \,, \quad \boldsymbol{s} = \boldsymbol{y} + s\, \boldsymbol{n} \,. \tag{54}$$

We now apply this to $T_1^{(2)}(E, \Delta)$. (The justification, which is based on (31), is omitted here.) From (52) and (53) we find, noticing that $\boldsymbol{\Delta}\, \boldsymbol{n} = 0$,

$$T_1^{(2)}(E, \Delta) = -\frac{k}{2\pi i} \int d^3y\, e^{-2i\boldsymbol{\Delta y}}\, V(\boldsymbol{y}) \int_0^\infty ds\, V(\boldsymbol{s})\, e^{-i \int_y^s V(s')\, ds' - i(p-k)s} + 0(|\,k\,|^0) \,.$$

Because $p - k \approx \Delta^2/2\,k$ for $k \to \infty$, we can replace $\exp\left(-i\,(p-k)\,s\right)$ by 1 to the same order of accuracy. Then the integration over s can be carried out, and we obtain

$$T_1^{(2)}(E, \Delta) = -\frac{k}{2\pi} \int d^3y\, e^{-2i\boldsymbol{\Delta y}}\, V(\boldsymbol{y}) \left[e^{-i \int_0^\infty V(s)\, ds} - 1 \right] + 0(|\,k\,|^0) \,.$$

Adding the first Born approximation removes the term -1. Together with (51), this completes the proof of (46).

b) *Determination of the potential*

As an application of (45), we show that the potential is uniquely determined by the high energy limit of the T-matrix, i.e. by the function $T_\infty(\Delta)$. The corresponding result in the non-relativistic case is well known: $\lim_{E\to\infty} T(E, \Delta)$ is equal to the first Born approximation $T_0(\Delta)$, which is the Fourier transform of the potential. To compute $V(r)$ from $T_\infty(\Delta)$ we write

$$V(r) = W(r^2) \,, \quad T_\infty(\Delta) = g(\tau^2) \,, \tag{55}$$

with $\tau = 2\,\Delta$. Choosing the y_3-axis in the direction of $\boldsymbol{n}$, we get from (45):

$$\left.\begin{aligned} g(\tau^2) &= -\frac{1}{2\pi} \int dy_1\, dy_2\, e^{-i(\tau_1 y_1 + \tau_2 y_2)}\, h(\varrho^2) \,, \\ h(\varrho^2) &= \int dy_3\, e^{-i \int_{y_3}^\infty W(\varrho^2 + s^2)\, ds}\, W(\varrho^2 + y_3^2) \,, \end{aligned}\right\} \tag{56}$$

where $\tau^2 = \tau_1^2 + \tau_2^2$ and $\varrho^2 = y_1^2 + y_2^2$. $g(\tau^2)$ is the two-dimensional Fourier transform of $h(\varrho^2)$, therefore, by Fourier inversion:

$$h(u) = -\frac{1}{2} \int_0^\infty dv\, J_0\left(\sqrt{u\, v}\right) g(v) \,, \tag{57}$$

where $J_0(z)$ is the usual zero order Bessel function. Next we note that the integral (56) can be evaluated in closed form, yielding

$$h(u) = -i \left[1 - e^{-i \int_{-\infty}^{+\infty} W(u + s^2)\, ds} \right] .$$

Together with (57), this implies

$$f(u) \equiv \int_{-\infty}^{+\infty} W(u + s^2)\, ds = i \log \left[1 + \frac{i}{2} \int_0^\infty dv\, J_0(\sqrt{u\, v})\, g(v) \right], \tag{58}$$

where the logarithm is uniquely defined by the requirement that $f(u)$ be continuous and vanishes as $u \to \infty$. The calculation of W from f is easily carried out in terms of the Laplace transforms $\tilde{W}$ and $\tilde{f}$:

$$\tilde{W}(t) = \left(\frac{t}{\pi}\right)^{1/2} \tilde{f}(t)\,. \tag{59}$$

(55), (58) and (59) determine the potential $V(r)$ explicitely in terms of $T_\infty(\Delta)$.

Appendix I

We have to compute

$$f_n(0) = \int_0^\infty dz_1 \int_{z_1}^\infty dz_2 \ldots \int_{z_{n-1}}^\infty dz_n\, e^{i z_n^2/2} \qquad (n = 1, 2, 3 \ldots)\,,$$

where all the paths are to be taken along the real axis. We can displace these paths so that they become

$$z_k = x_k\, e^{i\pi/4}\,, \qquad (x_k \text{ real}, 0 \leq x_1 \leq x_2 \leq \ldots \leq x_n)\,.$$

Therefore

$$f_n(0) = e^{i n \pi/4} \int_0^\infty dx_1 \int_{x_1}^\infty dx_2 \ldots \int_{x_{n-1}}^\infty dx_n\, e^{-x_n^2/2}\,,$$

and after $n - 1$ successive partial integrations one arrives at

$$f_n(0) = \frac{e^{i n\pi/4}}{(n-1)!} \int_0^\infty dx\, x^{(n-1)}\, e^{-x^2/2} = e^{i n\pi/4}\, 2^{((n/2)-1)}\, \frac{\left(\frac{n}{2} - 1\right)!}{(n-1)!}\,.$$

For even n this is equal to the result claimed in section 2.

Appendix II

In order to express $D_{n/2} F(\boldsymbol{x}, \boldsymbol{y})$ in terms of $F(\boldsymbol{z})$ and its derivatives with respect to the cartesian coordinates $\boldsymbol{z}$, we put

$$\left.\begin{aligned} v_1 &= \varrho \cos\varphi\,, \quad v_2 = \varrho \sin\varphi\,, \quad v_3 = \sigma\,, \\ \partial_{12} &= \cos\varphi \frac{\partial}{\partial v_1} + \sin\varphi \frac{\partial}{\partial v_2}\,, \quad \partial_3 = \frac{\partial}{\partial v_3}\,, \\ D^{(n)} &= \frac{\partial^n \varrho}{\partial t^n}\, \partial_{12} + \frac{\partial^n \sigma}{\partial t^n}\, \partial_3\,. \end{aligned}\right\} \tag{60}$$

The functions $\varrho(\eta, t)$ and $\sigma(\eta, t)$ are given by (4). We add a subscript 0 to a function of φ, η, t to denote its values at $t = 0$. The following formulae will prove useful:

$$\left.\begin{aligned} &(D^{(1)}F)_0 = 2^{1/2}\, a\, (1-\eta^2)^{1/2}\, (\partial_{12}F)_0\,, \quad (D^{(2)}F)_0 = 2\, a\, \eta\, \partial_3 F_0\,, \\ &(D^{(3)}F)_0 = 3 \cdot 2^{-1/2}\, a\, (1-\eta^2)^{1/2}\, (\partial_{12}F)_0\,, \quad (D^{(4)}F)_0 = 0\,, \quad \ldots \end{aligned}\right\} \tag{61}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi\, (\partial_{12}^n F)_0 = \begin{cases} 2^{-n}\begin{pmatrix} n \\ \frac{n}{2} \end{pmatrix} (\Delta_{12}^{n/2} F)_0 & (n \text{ even})\,, \\ 0 \quad (n \text{ odd})\,, \end{cases} \tag{62}$$

where

$$\Delta_{12} = \frac{\partial^2}{\partial v_1^2} + \frac{\partial^2}{\partial v_2^2}\,.$$

Applying this to $D_1 F(\boldsymbol{x}, \boldsymbol{y})$, for example, we get

$$\frac{\partial^2 F}{\partial t^2} = (D^{(1)})^2\, F + D^{(2)} F\,,$$

and therefore, by (61) and (62),

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{-1}^{+1} d\eta \left(\frac{\partial^2 F}{\partial t^2}\right)_0 = a^2 \int_{-1}^{+1} d\eta\, (1-\eta^2)\, (\Delta_{12}F)_0 + 2\, a \int_{-1}^{+1} d\eta\, \eta\, \partial_3 F_0\,.$$

But since $v_3 = a\,\eta$ for $t = 0$, we can transform the second integral by partial Integration and then combine it with the first one. In this way, we obtain the result

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{-1}^{+1} d\eta \left(\frac{\partial^2 F}{\partial t^2}\right)_0 = a^2 \int_{-1}^{+1} d\eta\, (1-\eta^2)\, (\Delta F)_0\,,$$

which leads immediately to the expression (8) for $D_1\, F(\boldsymbol{x}, \boldsymbol{y})$. The analogous calculation of $D_2 F(\boldsymbol{x}, \boldsymbol{y})$ is easily carried out using (61) and (62).

Appendix III

We use the cartesian coordinates $\boldsymbol{v}$ introduced in section 2 and define $v_4 = |\boldsymbol{z} - \boldsymbol{y}|$. The connection with the elliptic coordinates is given by (4) and (5). The assumptions of Lemma 2 imply that

$$F(\boldsymbol{z}, \boldsymbol{y}) = W(v_1 \ldots v_4, \boldsymbol{x}, \boldsymbol{y})\,,$$

$$\left|\frac{\partial^k W}{\partial {v_1}^{k_1} \ldots \partial {v_4}^{k_4}}\, (\varphi, \eta, t, \boldsymbol{x}, \boldsymbol{y})\right| \leq C\, (1+z)^{-(P+n)}\, (1+v_4)^n \leq C\, (1+y)^{P+n}\, (1+v_4)^{-P}$$

for all φ, η, t, $\boldsymbol{x}$, $\boldsymbol{y}$, p arbitrarily large. The first inequality follows from (31) and $S \in \mathfrak{S}$, the second from $1 + |\boldsymbol{z} - \boldsymbol{y}| \leq 1 + z + y \leq (1+z)\,(1+y)$. Now we consider as an example the estimation of $I_{3/2}\, F(\boldsymbol{x}, \boldsymbol{y})$, where we must majorize

$$a^{-1/2} \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\eta \int_0^{\infty} dt \left|\frac{\partial^4 F}{\partial t^4}\, (\varphi, \eta, t)\right|.$$

$\partial^4 F/\partial t^4$ is a sum of products of the form

$$P(\varphi, \eta, t) \frac{\partial^k W}{\partial v_1^{k_1} \ldots \partial v_4^{k_4}},$$

where P is any product of derivatives of the v_k with respect to t of total order 4, e.g. $v_1'' v_4''$, v_2'''', $(v_1')^3 v_3'$. The last of these examples yields, for instance,

$$P(\varphi, \eta, t) = 16\, a^4 \cos^3\varphi\, \eta\, (1-\eta)^{3/2}\, t\, (t^2+1)^3\, (t^2+2)^{-3/2}.$$

In this case, we have to estimate

$$(1+y)^{P+n}\, a^{7/2} \int_{-1}^{+1} d\eta\, |\eta|\, (1-\eta)^{3/2} \int_0^\infty dt\, t\, (t^2+1)^3\, (t^2+2)^{-3/2} \cdot [1 + a\,(t^2+1+\eta)]^{-P} \equiv (1+y)^{P+n}\, I(a).$$

To verify the result claimed in Lemma 2, we have to show that $I(a)$ is uniformly bounded in $0 \leq a < \infty$. If we choose $p > 5/2$, it is clear that $I(a)$ exists and is continuous for $0 < a < \infty$, so that only the behaviour of $I(a)$ for $a \to 0$ and $a \to \infty$ needs further consideration:

$a \to 0$:

$$I(a) \leq \text{const}\; a^{7/2} \int_0^\infty dt\, t\, (t^2+1)^{3/2}\, [1 + a\, t^2]^{-P},$$

$$\leq \text{const}\; a \int_0^\infty ds\, (s+1)^{3/2-P} \qquad (a \leq 1).$$

Taking $p > 5/2$, we have

$$I(a) \leq \text{const}\; a \qquad (a \to 0).$$

$a \to \infty$:

$$I(a) \leq \text{const}\; a^{7/2} \int_{-1}^{+1} d\eta\, (1+\eta)^{3/2}\, [1 + a\,(1+\eta)]^{-P_1} \int_0^\infty dt\, t\, (t^2+1)^{3/2}\, [1 + a\, t^2]^{-P_2},$$

$$\leq \text{const} \int_0^\infty dr\, r^{3/2}\, (1+r)^{-P_1} \int_0^\infty ds\, (s+1)^{3/2-P_2} \qquad (a \geq 1),$$

where $p_1 + p_2 = p$. Taking $p_{1,2} > 5/2$, we have

$$I(a) \leq \text{const} \qquad (a \to \infty).$$

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