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Ergodicity of Quantum Many-Body Systems

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Synopsis. An important hypothesis in VAN HOVE's proof of the ergodicity of dissipative quantum systems is the generalized microscopic reversibility condition $X_{ll'}(\alpha\,\alpha') = X_{l'l}(\alpha'\,\alpha)$. In this paper we show how such an assumption can be avoided by deducing for a finite system a general symmetry relation which corresponds to the condition used by STUECKELBERG for deriving the H-theorem. This general symmetry relation considered in the limit of an infinite system ensures the correct statistical behaviour of the system for very large time. The calculation of the trace in the limiting case of an infinite system is briefly discussed at the end.

1. Introduction

The ergodic behaviour of large quantum systems was demonstrated by VAN HOVE[1]) in a series of papers (here quoted as S_1, S_2, S_3, and S_4) for an extensive class of physical systems and for initial states with incoherent phases. This remarkable property was established by supposing, in addition to the general properties of the system (considered on pp. 444 to 448 of S_2), the validity of a generalized microscopic reversibility and of the interconnexion of states having the same unperturbed energy (see pp. 470 to 472 of S_2).

The result has also been extended[2]) to the case of arbitrary initial states, but always by using the supplementary hypothesis introduced by VAN HOVE. The interconnexion of states having the same unperturbed energy implies the dissipative character of the unperturbed states and seems therefore to be a natural requirement for the type of system considered. However, the hypothesis of a generalized microscopic reversibility, as we remarked in our previous paper[2]), corresponds to a very strong assumption, and one knows simple examples of systems going towards equilibrium for which this property is not realized[3])[4]). The purpose of the present paper is to show that the ergodic behaviour is observed in dissipative many-body systems, without any assumption on generalized microscopic reversibility.

This paper was inspired by an article of STUECKELBERG[5]) based on a much weaker symmetry relation than the detailed balance condition. This symmetry relation was simply derived from the unitarity of the transition amplitude matrix S.

The present work follows essentially the lines indicated by VAN HOVE's "Les Houches" lectures (S_4), as the algebraic relations are best derived for the case of a very large but finite system. However, the analytic properties needed for the discussion of the asymptotic behaviour are considered only in the limit of an infinite system.

2. Definitions and General Properties

We consider a very large but finite quantum many-body system, described by the Hamiltonian:

$$H = H_0 + \lambda V \tag{2.1}$$

in terms of the normalized eigenstates $|\beta\rangle$ of the free system, which are supposed to be known and to form a complete set:

$$H_0 \mid \beta\rangle = \varepsilon(\beta) \mid \beta\rangle ; \quad \langle \beta \mid \beta' \rangle = \delta_{\beta\beta'} , \tag{2.2}$$

$\delta_{\beta\beta'}$ being 1 for $\beta = \beta'$ and 0 for $\beta \neq \beta'$, where β represents a collection of quantum numbers characterizing the state and $\varepsilon(\beta)$ the unperturbed energy of the system in the state $|\beta\rangle$.

We now define as a diagonal part of an operator 0, the diagonal operator which has the same diagonal matrix elements:

$$\langle \beta \mid 0^d \mid \beta\rangle = \langle \beta \mid 0 \mid \beta\rangle$$

and a non-diagonal part 0^{nd} as:

$$0^{nd} = 0 - 0^d ,$$

so that

$$\langle \beta \mid 0 \mid \beta' \rangle = 0(\beta\, \beta') = 0^d(\beta)\, \delta_{\beta\beta'} + 0^{nd}(\beta\, \beta') . \tag{2.3}$$

With this definition $0^{nd}(\beta\beta) = 0$. In the limit of an infinite system, however, it is convenient to include in $0^{nd}(\beta\,\beta')$ also its limiting value for $\beta \to \beta'$ and to subtract it from $0^d(\beta)$. The remaining "diagonal part" represents a singular part of 0 in the β-representation. Considered here is that class of operators 0 for which in the same limit, and in the corresponding continuous notation α defined belov, this singularity is a $\delta\,(\alpha - \alpha')$-one; weaker singularities are included in the "non-diagonal" part:

$$\left.\begin{aligned} \langle \alpha \mid 0 \mid \alpha' \rangle &= 0(\alpha\, \alpha') = 0^d(\alpha)\, \delta\,(\alpha - \alpha') + 0^{nd}(\alpha\, \alpha') , \\ \text{with} \qquad \langle \alpha \mid \alpha' \rangle &= \delta\,(\alpha - \alpha') . \end{aligned}\right\} \tag{2.3a}$$

For what follows this represents a restriction on the perturbation λV, and implies a certain number of properties. (See S_2 for further details).

It is assumed that the perturbation λV has a vanishing diagonal part in the β-representation, and that there are operators of the type $V A_1 V \ldots A_n V$ having non-vanishing diagonal parts. The A_j are diagonal operators with eigenvalues $A_j(\beta)$ which depend smoothly on β.

$$V^d \mid \beta\rangle \equiv 0 ; \quad \{V A_1 V \ldots A_n V\}^d \mid \beta\rangle \neq 0 . \tag{2.4}$$

As "irreducible diagonal part" $(V A_1 V \ldots A_n V)_{id}$ is defined the corresponding diagonal operator whose matrix elements are obtained by keeping all intermediate states different from one another, and from the initial or final states. In the same way we define the "irreducible non-diagonal part" $(V A_1 V \ldots A_n V)_{ind}$ for a non-diagonal operator.

We now consider the resolvent operator:

$$R_l = (H - l)^{-1} \tag{2.5}$$

where l is a complex number.

One can show that its diagonal part is given by:

$$R_l^d = D_l = (H_0 - l - \lambda^2 G_l)^{-1} \tag{2.6}$$

where

$$G_l = \{V D_l V - \lambda V D_l V D_l V + \ldots\}_{id} . \tag{2.7}$$

Defining

$$\langle \beta \mid R_l A R_{l'} \mid \beta' \rangle = \sum_{\beta_0} A(\beta_0)\, Z_{l l'}(\beta_0 \beta \beta') \tag{2.8}$$

the relation of $Z_{l l'}$ to $X_{l l'}$ and $Y_{l l'}$ introduced by VAN HOVE in S_2 is:

$$Z_{l l'}(\beta_0 \beta \beta') = X_{l l'}(\beta_0 \beta)\, \delta_{\beta \beta'} + Y_{l l'}(\beta_0 \beta \beta') . \tag{2.9}$$

As we have shown in I, $Y_{l l'}$ can be expressed by means of $V_{l l'}$ and $X_{l l'}$:

$$Y_{l l'}(\beta_0 \beta \beta') = \sum_{\beta_1} X_{l l'}(\beta_0 \beta_1)\, V_{l l'}(\beta_1 \beta \beta') , \tag{2.10}$$

where $V_{l l'}$ is defined by:

$$\left.\begin{aligned} &\langle \beta \mid \{(1 - \lambda D_l V + \lambda^2 D_l V D_l V - \ldots) \\ &\quad \times A\, (1 - \lambda V D_{l'} + \lambda^2 V D_{l'} V D_{l'} - \ldots)\}_{ind} \mid \beta' \rangle = \sum_{\beta_0} A(\beta_0)\, V_{l l'}(\beta_0 \beta \beta') . \end{aligned}\right\} \tag{2.11}$$

$Z_{l l'}$ obeys the following equation (see 7.6 in I):

$$\left.\begin{aligned} &(l - l')\, Z_{l l'}(\beta_0 \beta \beta') = [D_l(\beta_0) - D_{l'}(\beta_0)]\, [\delta_{\beta_0 \beta}\, \delta_{\beta \beta'} + V_{l l'}(\beta_0 \beta \beta')] - \\ &\quad - i \lambda^2 \sum_{\beta_1} \tilde{W}_{l l'}(\beta_0 \beta_1)\, Z_{l l'}(\beta_1 \beta \beta') + i \lambda^2 \sum_{\beta_1} \tilde{W}_{l l'}(\beta_1 \beta_0)\, Z_{l l'}(\beta_0 \beta \beta') , \end{aligned}\right\} \tag{2.12}$$

where

$$\tilde{W}_{l l'}(\beta_0 \beta) = i\, [D_l(\beta_0) - D_{l'}(\beta_0)]\, W_{l l'}(\beta_0 \beta) \tag{2.13}$$

and

$$\{(V - \lambda V D_l V + \ldots)\, A\, (V - \lambda V D_{l'} V + \ldots)\}_{id} \mid \beta \rangle = \mid \beta \rangle \sum_{\beta_0} A(\beta_0)\, W_{l l'}(\beta_0 \beta) \tag{2.14}$$

Actually (2.12) represents the Fourier-transform of a general master equation. As discussed in detail in S_4, (2.12) is only strictly valid in the limit of a very large system and for $A(\beta_0)$ being a continuous function of β_0. The latter conditions is due to the fact that, for it to be correct, (2.12) requires a coarse-graining over the intermediate state β_0. SWENSON[6]) was able to derive an exact general master equation corresponding to (2.12), valid for systems of arbitrary volume and which does not require a diagonal singularity condition, and therefore also not a coarse-graining. This is due to the fact that he uses non-reduced expressions, and does not discuss the approach to equilibrium, which is a property certainly not true for arbitrary systems or for arbitrary operators.

$Z_{l\,l'}(\beta_0\,\beta\,\beta')$ can equally well be expressed by:

$$Z_{l\,l'}(\beta_0\,\beta\,\beta') = \{\langle\beta \mid R_l \mid \beta_0\rangle \langle\beta_0 \mid R_{l'} \mid \beta'\rangle\}_{\beta_0} = \\ = \Big[\sum_{\beta_0 \in \Delta\beta_0} \langle\beta \mid R_l \mid \beta_0\rangle \langle\beta_0 \mid R_{l'} \mid \beta'\rangle\Big] \Big(\sum_{\beta_0 \in \Delta\beta_0} 1\Big)^{-1} \tag{2.15}$$

$\{\ldots\}_{\beta_0}$ meaning a coarse-graining over β_0.

From the relation:

$$X_{l\,l'}(\beta_0\,\beta) = D_l\,(\beta_0)\,D_{l'}(\beta_0)\,\Big\{\delta_{\beta_0\,\beta} + \lambda^2 \sum_{\beta_1} W_{l\,l'}(\beta_0\,\beta_1)\,X_{l\,l'}(\beta_1\,\beta)\Big\} \tag{2.16}$$

and using (2.10) we get:

$$Y_{l\,l'}(\beta_0\,\beta\,\beta') = D_l(\beta_0)\,D_{l'}(\beta_0)\,\Big\{V_{l\,l'}(\beta_0\,\beta\,\beta') + \lambda^2 \sum_{\beta_1} W_{l\,l'}(\beta_0\,\beta_1)\,Y_{l\,l'}(\beta_1\,\beta\,\beta')\Big\} \tag{2.17}$$

therefore we obtain a corresponding relation for $Z_{l\,l'}$:

$$Z_{l\,l'}(\beta_0\,\beta\,\beta') = D_l(\beta_0)\,D_{l'}(\beta_0)\,\Big\{\delta_{\beta_0\,\beta}\,\delta_{\beta\,\beta'} + V_{l\,l'}(\beta_0\,\beta\,\beta') + \\ + \lambda^2 \sum_{\beta_1} W_{l\,l'}(\beta_0\,\beta_1)\,Z_{l\,l'}(\beta_1\,\beta\,\beta')\Big\} \tag{2.18}$$

which of course is consistent with (2.12) and will also be naturally called "Master equation".

3. General Symmetry Relation

Before discussing the approach to equilibrium we derive a general symmetry relation replacing, as we shall show, the generalized microscopic reversibility, or detailed balance, assumed in S_2 and in I.

From the well-known identity:

$$R_l - R_{l'} = (l - l')\,R_l\,R_{l'} \tag{3.1}$$

we get:

$$\sum_{\beta_0} Z_{l l'}(\beta_0\,\beta\,\beta') = \sum_{\beta_0} Z_{l' l}(\beta_0\,\beta\,\beta') = (l - l')^{-1}\,\big(R_l(\beta\,\beta') - R_{l'}(\beta\,\beta')\big)\,. \tag{3.2}$$

A less trivial relation is obtained by summation over β of $Z_{l\,l'}(\beta_0\,\beta\,\beta)$ (notice $\beta' = \beta$).

$$\sum_{\beta} Z_{l\,l'}(\beta_0\,\beta\,\beta) = \Big\{\sum_{\beta} \langle\beta \mid R_l \mid \beta_0\rangle \langle\beta_0 \mid R_{l'} \mid \beta\rangle\Big\}_{\beta_0} = \Big\{\sum_{\beta} Z_{l' l}(\beta\,\beta_0\,\beta_0)\Big\}_{\beta_0}\,.$$

Using (3.2) we get the general symmetry relation:

$$\sum_{\beta} Z_{l\,l'}(\beta_0\,\beta\,\beta) = \Big\{\sum_{\beta} Z_{l\,l'}(\beta\,\beta_0\,\beta_0)\Big\}_{\beta_0}\,. \tag{3.3}$$

Of course a corresponding relation can be written down for the matrix elements, without coarse-graining, but we need a relation involving $Z_{l\,l'}$ which is only defined here in a coarse-grained sense.

We consider Z_t, which is the Fourier-transform of $Z_{l\,l'}$:

$$Z_t(\beta_0\,\beta\,\beta') = \frac{-1}{(2\,\pi)^2} \int\limits_{\gamma} dl \int\limits_{\gamma'} dl'\, e^{i(l-l')t}\, Z_{l\,l'}(\beta_0\,\beta\,\beta') \tag{3.4}$$

or

$$Z_t(\beta_0\,\beta\,\beta') = \{\langle\beta \mid U_{-t} \mid \beta_0\rangle \langle\beta_0 \mid U_t \mid \beta'\rangle\}_{\beta_0} \tag{3.5}$$

where $U_t = \exp[-i\,(H_0 + \lambda\,V)\,t]$, so that the Fourier transformed formulation of the general symmetry relation (3.3) corresponds to:

$$\Big\{\sum_{\beta} |\langle\beta_0 \mid U_t \mid \beta\rangle|^2\Big\}_{\beta_0} = \Big\{\sum_{\beta} |\langle\beta \mid U_t \mid \beta_0\rangle|^2\Big\}_{\beta_0} = 1 \tag{3.6}$$

which is the (coarse-grained) expression of the unitarity of the operator U_t. (3.6) corresponds to the relation used by STUECKELBERG[5]) for deriving the H-theorem:

$$\sum_i |S_{ik}|^2 = \sum_i |S_{ki}|^2 = 1 \tag{3.7}$$

also due to the unitarity of the transition matrix S. We remark however that (3.3) implies more than (3.6).

Let us introduce the operator $Q_{E,\eta}$ for $\eta > 0$:

$$Q_{E,\eta} = \frac{1}{2\,\pi\,i}\,(R_{E+i\eta} - R_{E-i\eta})\,. \tag{3.8}$$

Considering for the same η:

$$Z^{\pm}_{E,\eta}(\beta_0\,\beta\,\beta') = \frac{\eta}{\pi}\, Z_{E\mp i\eta,\,E\pm i\eta}(\beta_0\,\beta\,\beta') \tag{3.9}$$

we obtain from (3.2)

$$\sum_{\beta_0} Z^{\pm}_{E,\eta}(\beta_0\,\beta\,\beta') = Q_{E,\eta}(\beta\,\beta') \tag{3.10}$$

so that (3.3) yields:

$$\sum_{\beta} Z^{\pm}_{E,\eta}(\beta_0\,\beta\,\beta) = \Big\{\sum_{\beta} Z^{\pm}_{E,\eta}(\beta\,\beta_0\,\beta_0)\Big\}_{\beta_0} = \overline{Q}_{E,\eta}(\beta_0\,\beta_0)\,, \tag{3.11}$$

where we denote by $\overline{Q}_{E,\eta}(\beta_0\,\beta_0)$ the coarse-grained value of $Q_{E,\eta}(\beta_0\,\beta_0)$.

It is possible to avoid an explicit coarse-graining by summing up over the intermediate state β_0. In this case (3.3) takes the slightly different form:

$$\sum_{\beta_0\,\beta} A(\beta_0)\, Z_{l\,l'}(\beta_0\,\beta\,\beta) = \sum_{\beta_0\,\beta} Z_{l\,l'}(\beta\,\beta_0\,\beta_0)\, A(\beta_0) \tag{3.3a}$$

and (3.11) becomes:

$$\sum_{\beta_0\,\beta} A(\beta_0)\, Z^{\pm}_{E,\eta}(\beta_0\,\beta\,\beta) = \sum_{\beta_0} A(\beta_0)\, Q_{E,\eta}(\beta_0\,\beta_0)\,. \tag{3.11a}$$

In the same way, and taking

$$\{(V - \lambda\,V\,D_l\,V + \ldots)\,D_l\,A\,D_{l'}\,(V - \lambda\,V\,D_{l'}\,V + \ldots)\}_{id}$$

instead of A, we obtain for (3.3a)

$$\left.\begin{aligned}&\sum_{\beta_0\beta_1\beta} A(\beta_0)\, D_l(\beta_0)\, D_{l'}(\beta_0)\, W_{l\,l'}(\beta_0\,\beta_1)\, Z_{l\,l'}(\beta_1\,\beta\,\beta)\\ &\qquad = \sum_{\beta_0\beta_1\beta} A(\beta_0)\, D_l(\beta_0)\, D_{l'}(\beta_0)\, W_{l\,l'}(\beta_0\,\beta_1)\, Z_{l\,l'}(\beta\,\beta_1\,\beta_1)\end{aligned}\right\} \tag{3.3b}$$

and for (3.11a):

$$\left.\begin{aligned}&\sum_{\beta_0\beta_1\beta} A(\beta_0)\, D_{E+i\eta}(\beta_0)\, D_{E-i\eta}(\beta_0)\, W_{E\mp i\eta,\,E\pm i\eta}(\beta_0\,\beta_1)\, Z^{\pm}_{E,\eta}(\beta_1\,\beta\,\beta) =\\ &\qquad = \sum_{\beta_0\beta_1} A(\beta_0)\, D_{E+i\eta}(\beta_0)\, D_{E-i\eta}(\beta_0)\, W_{E\mp i\eta,\,E\pm i\eta}(\beta_0\,\beta_1)\, Q_{E,\eta}(\beta_1\,\beta_1)\,.\end{aligned}\right\} \tag{3.11b}$$

Summing up over β in the master equation (2.18) for $Z^{\pm}_{E,\eta}(\beta_0\,\beta\,\beta)$, one obtains:

$$\left.\begin{aligned}&\sum_{\beta_0} A(\beta_0)\, Q_{E,\eta}(\beta_0\,\beta_0) = \frac{\eta}{\pi} \sum_{\beta_0\beta} A(\beta_0)\, D_{E+i\eta}(\beta_0)\, D_{E-i\eta}(\beta_0)\\ &\times [\delta_{\beta_0\beta} + V_{E\mp i\eta,\,E\pm i\eta}(\beta_0\,\beta\,\beta)] + \lambda^2 \sum_{\beta_0\beta_1} A(\beta_0)\, D_{E+i\eta}(\beta_0)\, D_{E-i\eta}(\beta_0)\\ &\qquad\qquad \times W_{E\mp i\eta,\,E\pm i\eta}(\beta_0\,\beta_1)\, Q_{E,\eta}(\beta_1\,\beta_1)\,.\end{aligned}\right\} \tag{3.12}$$

It is in this form that we shall use the general symmetry relation, instead of assuming detailed balance.

4. Asymptotic Behaviour in Dissipative Systems

From now on we restrict our considerations to dissipative systems. Let us assume that for $t = 0$ the system is in the state $|\,\varphi_0\rangle$.

$$|\,\varphi_0\rangle = \sum_{\beta} |\,\beta\rangle\, C(\beta)\,; \qquad \langle\varphi_0\,|\,\varphi_0\rangle = 1\,. \tag{4.1}$$

The expectation value of the diagonal operator A at time t is given by:

$$\langle A\rangle_t = \langle\varphi_0\,|\,U_{-t}\,A\,U_t\,|\,\varphi_0\rangle = \sum_{\beta_0\beta\beta'} A(\beta_0)\, Z_t(\beta_0\,\beta\,\beta')\, C^*(\beta)\, C(\beta')\,. \tag{4.2}$$

To find out the asymptotic value of $\langle A\rangle_t$, i.e. for $t \to \pm\infty$, one has to take first the limit of an infinite system ($\Omega \to \infty$). This is however a very delicate limiting process, which requires not only another normalization for the eigenstates $|\,\beta\rangle$ but also a summation over these states before the limit $\Omega \to \infty$ is taken, as we show below.

Denoting by N_β the number of excitations present in the state β, we consider a new set of eigenfunctions $|\,\bar{\beta}\rangle$, differing from the previous ones only in the normalization, which is now given by:

$$|\,\bar{\beta}\rangle \equiv \left(\frac{\Omega}{8\,\pi^3}\right)^{N_\beta/2} |\,\beta\rangle\,. \tag{4.3}$$

Introducing the summation sign

$$\int_{\bar{\beta}} f(\bar{\beta}) \equiv \sum_{\bar{\beta}} \left(\frac{8\,\pi^3}{\Omega}\right)^{N_{\bar{\beta}}} f(\bar{\beta}) \tag{4.4}$$

it is easy to write down the formulae obtained so far in the new notation*), which is suitable for taking the limit $\Omega \to \infty$.

In this limit, however, some of the functions here considered become highly singular: this is the case for example of $Z_t(\bar\beta_0\, \bar\beta\, \bar\beta')$ for $\bar\beta = \bar\beta'$ if its diagonal part $P_t(\bar\beta_0\, \bar\beta)$ is different from zero, as it is for the systems dealt with here, whereas other functions like $\langle A \rangle_t$ remain perfectly regular.

For this reason it is better to look at the following master equation, which is simply obtained from (2.18) by using the definitions involved:

$$\left.\begin{aligned} &\int\limits_{\bar\beta_0 \bar\beta \bar\beta'} A(\bar\beta_0)\, Z^{\pm}_{E,\eta}(\bar\beta_0\, \bar\beta\, \bar\beta')\, C^*(\bar\beta)\, C(\bar\beta') = \lambda^2 \int\limits_{\bar\beta_0 \bar\beta_1 \bar\beta \bar\beta'} A(\bar\beta_0)\, D_{E+i\eta}(\bar\beta_0)\, D_{E-i\eta}(\bar\beta_0) \\ &\times W_{E \mp i\eta, E \pm i\eta}(\bar\beta_0\, \bar\beta_1)\, Z^{\pm}_{E,\eta}(\bar\beta_1\, \bar\beta\, \bar\beta')\, C^*(\bar\beta)\, C(\bar\beta') + \frac{\eta}{\pi} \int\limits_{\bar\beta_0 \bar\beta \bar\beta'} A(\bar\beta_0)\, D_{E+i\eta}(\bar\beta_0) \\ &\quad \times D_{E-i\eta}(\bar\beta_0) \left[\left(\frac{\Omega}{8\pi^3}\right)^{2N_{\bar\beta_0}} \delta_{\bar\beta_0 \bar\beta}\, \delta_{\bar\beta \bar\beta'} + V_{E \mp i\eta, E \pm i\eta}(\bar\beta_0\, \bar\beta\, \bar\beta')\right] C^*(\bar\beta)\, C(\bar\beta')\,. \end{aligned}\right\} \tag{4.5}$$

By means of the ansatz:

$$Z^{\pm}_{E,\eta}(\bar\beta_0\, \bar\beta\, \bar\beta') = \frac{Q_{E,\eta}(\bar\beta_0\, \bar\beta_0)\, Q_{E,\eta}(\bar\beta\, \bar\beta')}{\int\limits_{\bar\beta_1} Q_{E,\eta}(\bar\beta_1\, \bar\beta_1)} \tag{4.6}$$

one obtains for the left hand side of (4.5):

$$\int\limits_{\bar\beta_0 \bar\beta \bar\beta'} A(\bar\beta_0)\, Z^{\pm}_{E,\eta}(\bar\beta_0\, \bar\beta\, \bar\beta')\, C^*(\bar\beta)\, C(\bar\beta') = \frac{Tr(A\, Q_{E,\eta})}{Tr(Q_{E,\eta})}\, p_{E,\eta} \tag{4.7}$$

where Tr means trace, and $p_{E,\eta}$ is defined as:

$$p_{E,\eta} = \int\limits_{\bar\beta \bar\beta'} Q_{E,\eta}(\bar\beta\, \bar\beta')\, C^*(\bar\beta)\, C(\bar\beta')\,. \tag{4.8}$$

With this same ansatz (4.6) and using the general symmetry relation by means of (3.12), the right hand side of (4.5) becomes:

$$\begin{aligned} &\frac{Tr(A\, Q_{E,\eta})}{Tr(Q_{E,\eta})}\, p_{E,\eta} + \frac{\eta}{\pi} \int\limits_{\bar\beta_0} A(\bar\beta_0)\, D_{E+i\eta}(\bar\beta_0)\, D_{E-i\eta}(\bar\beta_0) \\ &\times \left\{ |\, C(\bar\beta_0)\,|^2 + \int\limits_{\bar\beta \bar\beta'} V_{E \mp i\eta, E \pm i\eta}(\bar\beta_0\, \bar\beta\, \bar\beta')\, C^*(\bar\beta)\, C(\bar\beta') \right\} - \frac{\eta}{\pi} \\ \times &\frac{Tr(D_{E+i\eta}\, A\, D_{E-i\eta} + \{(D_{E\mp i\eta} - \lambda D_{E\mp i\eta} V D_{E\mp i\eta} + \dots)\, A\, (D_{E\pm i\eta} - \lambda D_{E\pm i\eta} V D_{E\pm i\eta} + \dots)\}_{ind})}{Tr(Q_{E,\eta})}\, p_{E,\eta}\,. \end{aligned} \tag{4.9}$$

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*) One should remark that because of the changing in scale, we have in general:

$$0^{nd}(\bar\beta\, \bar\beta') \neq 0^{nd}(\beta\, \beta') \quad \text{whereas} \quad 0^{d}(\bar\beta) = 0^{d}(\beta)\,.$$

We now go over to the limit $\Omega \to \infty$ and for clarity we denote by $| \alpha\rangle$ the eigenfunctions of the unperturbed hamiltonian for this case; i.e.,

$$| \bar{\beta}\rangle \to | \alpha\rangle ; \quad \int_{\bar{\beta}} \to \int d\alpha \quad \text{for} \quad \Omega \to \infty \tag{4.10}$$

and

$$\langle \alpha \mid \alpha' \rangle = \delta (\alpha - \alpha') .$$

As discussed in I, for a dissipative infinite system, $D_{l'}$ $W_{l l}$ and $V_{l l'}$ are bounded operators in the complex plane and, except for a finite discontinuity across the real axis, the only singularity of

$$\int d\alpha_0 \, d\alpha \, d\alpha' \, A(\alpha_0) \, Z_{l l'}(\alpha_0 \, \alpha \, \alpha') \, C^*(\alpha) \, C(\alpha')$$

is a "pseudo-pole" of degree one for $l = E \pm i\, o$, $l' = E \mp i\, o$ and E real (for the definition of pseudo-pole see S_2 p. 465). It is this singularity which determine the asymptotic behaviour of $\langle A\rangle_t$ given by:

$$\lim_{t \to \pm \infty} \langle A\rangle_t = \int_{-\infty}^{+\infty} dE \lim_{0 \langle \eta \to 0} \int d\alpha_0 \, d\alpha \, d\alpha' \, A(\alpha_0) \, Z^{\pm}_{E,\eta}(\alpha_0 \alpha \alpha') \, C^*(\alpha) \, C(\alpha') . \tag{4.11}$$

We make now the reasonable assumption that the ratios of traces in (4.7) and (4.9) remain finite for $\Omega \to \infty$. Taking the limit $\eta \to 0$ only the first term in (4.9) is different from zero and represents therefore a solution of the master equation (4.5). We have:

$$\lim_{t \to \pm \infty} \langle A\rangle_t = \lim_{0 \langle \eta \to 0} \int_{-\infty}^{+\infty} dE \, \frac{T\, r(A \, Q_{E,\eta})}{T\, r(Q_{E,\eta})} \, p_{E,\eta} = \int_{-\infty}^{+\infty} dE \, \langle A\rangle_E \, p_E \tag{4.12}$$

$\langle A\rangle_E$ is the microcanonical average of the diagonal operator A on the energy shell, and p_E the probability distribution of the total energy in the initial state; we see that (4.12) ensures the ergodicity of the dissipative quantum many-body system considered.

5. Discussion

We add some remarks concerning the expressions for the trace adopted in previous papers (S_2, S_3 and I in particular) in the discussion of ergodic behaviour.

The correct expression for the trace of an operator 0 in the $\bar{\beta}$-representation in the case of a large but finite system is given by:

$$T\, r(0) = \int_{\bar{\beta}} \langle \bar{\beta} \,|\, 0 \,|\, \bar{\beta}\rangle = \int_{\bar{\beta}} \{\langle \bar{\beta} \,|\, \bar{\beta}\rangle \, 0^d(\bar{\beta}) + 0^{nd}(\bar{\beta}\, \bar{\beta})\} \simeq \int_{\bar{\beta}} 0^d(\bar{\beta}) \left(\frac{\Omega}{8\, \pi^3}\right)^{N_{\bar{\beta}}} \tag{5.1}$$

because for a large system and for a non-vanishing diagonal part, the non-diagonal one is negligible. In the limit of an infinite system the use (adopted in S_2, S_3 and I) of

$$\frac{Tr(A\,Q_E)}{Tr(Q_E)} = \frac{\int d\alpha_0\,A(\alpha_0)\,\Delta_E(\alpha_0)}{\int d\alpha\,\Delta_E(\alpha)} \tag{5.2}$$

$$\left(\text{where } Q_E = \lim_{0<\eta\to 0} Q_{E,\eta} \text{ and } \Delta_E \equiv Q_E^d\right)$$

instead of what we may indicate by:

$$\frac{Tr(A\,Q_E)}{Tr(Q_E)} = \lim_{\Omega\to\infty} \frac{\int d\alpha_0\,A(\alpha_0)\,\Delta_E(\alpha_0)\left(\frac{\Omega}{8\,\pi^3}\right)^{N_{\alpha_0}}}{\int d\alpha\,\Delta_E(\alpha)\left(\frac{\Omega}{8\,\pi^3}\right)^{N_\alpha}} \tag{5.3}$$

is consistent, in an approximative way, with the assumption underlying the hypothesis of generalized microscopic reversibility, i.e. that only states are considered "with a very large number of excitations present (of the order of the size of the system)" (see S_2, p. 471). It is then clear that for such states and in the limit of an infinite system, the normalization factor $(\Omega/8\,\pi^3)^{N_\alpha}$ becomes nearly the same for all the states, and the ratio of traces can be written as in (5.2).

Actually we clearly recognize that the limit $\Omega \to \infty$ should be taken only after a reduction of the formalism to extensive (proportional to Ω) and intensive (independent of Ω) operators, by means of some kind of linked-cluster expansion. The result is expected to be in accord with our assumption that ratios of traces remain finite.

Alternatively, if one considers the approach to equilibrium in finite systems, asymptotic behaviour means that the corresponding time is large compared to the relaxation times, but small with respect to the Poincaré cycles. In this case even the concept of dissipative system needs further investigation.

Finally, we underline the fact (always pointed out by VAN HOVE[1]), but not always clearly recognized) that the non-vanishing for $\Omega \to \infty$, of given diagonal operators (like $W_{ll'}$ for example) is an essential requirement for having ergodic behaviour and corresponds to a definite property of the perturbation in the unperturbed eigenfunction representation.

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References

1) VAN HOVE, L., Physica *21* (1955) 517; *23* (1957) 441; *25* (1959) 268; La théorie des gas neutres et ionisés, Hermann, Paris (1960) p. 151, Les Houches. (Here quoted as S_1, S_2, S_3, and S_4.)
2) JANNER, A., Helvetica Physica Acta *35* (1962) 47. (Here quoted as I.)
3) HAMILTON, J., and PENG, W. H., Proc. Roy. Ir. Acad. *49* (1944) 197.
4) HEITLER, W., Ann. Institut H. Poincaré *15* (1956) 67.
5) STUECKELBERG, E. C. G., Helvetica Physica Acta *25* (1952) 577.
6) SWENSON, R. J., J. of Math. Phys. *3* (1962) 1017.