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Multiple Coulomb Excitations of Deformed Nuclei

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Abstract. A formalism is developed for the purpose of investigating the multiple Coulomb excitation of deformed nuclei. The formalism yields an expression for the time development matrix describing the nuclear states which has a form similar to the solution given by the 'sudden approximation'. It is shown that this formalism may be applied to the problem of the excitation of rotational nuclei and an expression is obtained for corrections to the sudden approximation. This expression contains corrections of all orders in ξ and has the advantage that it is free of divergences. The corrections may be most readily calculated in the equal spacing approximation; an approximation in which the energy levels of the bombarded nucleus are assumed to be equally spaced. The formalism is also applied to the excitation of vibrational nuclei, taking both one-phonon and two-phonon processes into account. It is shown that the time development matrix may be expressed in closed form and earlier results concerning the effects of the two-phonon processes are extended.

1. Introduction

In recent years the Coulomb excitation process has been used extensively to gather information about low lying nuclear states. It has been possible to develop the theory of such excitation processes by using perturbation expansions. These methods are valid so long as the probability for nuclear excitation in a single encounter is small. When protons or α-particles are used as projectiles, and if the bombarding energy is kept so low that no nuclear reactions take place, this criterion is valid. However these conditions place a severe limit on the number of states which may be investigated by these methods. If heavier ions are used as projectiles the electric field exerted on the target nucleus is extremely strong, even at bombarding energies such that no nuclear reactions take place; this leads to multiple Coulomb excitations and to a method for investigating the higher lying nuclear states. One might still calculate excitation probabilities by the use of perturbation theory, but if many states are involved in the excitation process this is both complicated and unwieldly. Alternatively one might solve directly the set of coupled equations which describe the population of nuclear states during the collision. This latter method has previously been investigated by the present author[1]) but it has the inherent disadvantage that all calculations are necessarily numerical. In an attempt to give a theoretical description of

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these multiple excitation processes K. ALDER and A. WINTHER[2]) have considered an approximation method, 'the sudden approximation', which we will investigate in the present work more fully.

In order to understand the significance of the sudden approximation it is necessary to briefly discuss the parameters entering into the calculations. The methods are applicable to collisions in which the semi-classical orbit theory may be used[2])[3]); this presupposes that the projectile and target nucleus never approach very close to one another. In this framework there are two types of parameter involved in the theory. The first type of parameter is the strength parameter q_i which is a measure of the strength of the *i-th* multipole interaction. We will consider quadrupole excitations only, so that just one strength parameter enters. The second type of parameter is usually designated by ξ_i. A ξ-parameter occurs for each energy spacing of the target nuclei and the parameter can be roughly understood as the ratio of the interaction time to the characteristic time of the nuclear energy spacing. Of course the only ξ-parameters entering into a calculation are those associated with the energy spacings of levels between which transitions take place. If the relevant ξ_i are large the process is essentially adiabatic and if they are small the process is sudden. In the Coulomb excitation of deformed nuclei the ξ-parameters are small because the energy levels of the nuclei are closely spaced and the quadrupole operator causing the excitations places heavy restrictions on the allowed transitions. The sudden approximation assumes that all relevant ξ-parameters are zero, or equivalently that all nuclear energy levels are degenerate. This approximation yields a compact form for the matrix describing the time development of the nuclear states which may be evaluated without making a series expansion of the type encountered in perturbation theory. ALDER and WINTHER have also given a method for correcting the approximation for non-zero ξ. Their method of correction has, however, the disadvantage that divergences enter. In this paper we wish to show how a solution of the Schrödinger equation may be derived which has the same form as the sudden approximation solution and how this solution can be used for calculating corrections without encountering divergences. The methods used suggest an alternative approximation for the case of excitation of nuclei with a rotational spectrum. This approximation consists of assuming that all the energy levels of the nucleus are equally spaced and hence introduces just one ξ parameter into the problem.

Section 2 gives the development of the general formalism used in the ensuing investigations. Section 3 discusses the exitation of rotational nuclei and introduces the equal spacing approximation and Section 4 discusses vibrational nuclei.

2. General Formalism

The starting point for the development of our formalism is the Schrödinger equation for the nuclear state $\psi(t)$. This equation has the form

$$i\frac{d\psi}{dt} = [H + H_E(t)]\,\psi \tag{1}$$

where H is the Hamiltonian of the free nucleus, $H_E(t)$ is the time-dependent interaction Hamiltonian and we have taken $\hbar = 1$. We now transform to the interaction

representation and simultaneously introduce the time development operator of the nuclear states $U(t, t_0)$ i. e. we make the transformation

$$\psi(t) = e^{-iHt}\, U(t, t_0)\, e^{iHt_0}\, \psi(t_0)\,. \tag{2}$$

This transformation yields the equation

$$i\,\frac{dU(t, t_0)}{dt} = \tilde{H}(t)\, U(t, t_0) \tag{3}$$

where we have defined

$$\tilde{H}(t) = e^{iHt}\, H_E(t)\, e^{-iHt}\,. \tag{4}$$

As was explained in the introduction, the sudden approximation is equivalent to the assumption that all nuclear energy levels are degenerate. Therefore in this approximation

$$[H, H_E(t)] = 0$$

and thus

$$i\,\frac{dU(t, t_0)}{dt} = H_E(t)\, U(t, t_0)\,. \tag{5}$$

The interaction Hamiltonian $H_E(t)$ has the property that

$$[H_E(t_1), H_E(t_2)] = 0$$

so that (5) may be directly integrated to give the solution

$$U_S = U_S\,(\infty, -\infty) = \exp\{-\,i\,B\}$$

where

$$B = \int_{-\infty}^{\infty} H_E(t)\, dt\,. \tag{6}$$

The suffix S has been written to indicate that this solution is given by the sudden approximation. In order to examine the implications of this approximation we look for a solution of (3) having the same form as (6) i.e. we try to find a solution[7])

$$U(t, t_0) = \exp\{A(t, t_0)\}\,. \tag{7}$$

To find a solution of this form it is convenient to define operators

$$U(\lambda, t, t_0) = \exp\{\lambda\, A(t, t_0)\}$$

and

$$L(\lambda, t, t_0) = \frac{dU(\lambda, t, t_0)}{dt}\, U^{-1}(\lambda, t, t_0)$$

where λ is a real parameter.

Now

$$\frac{\partial}{\partial\lambda}\,\frac{\partial}{\partial t}\, U = \frac{\partial L}{\partial\lambda}\, U + L\, A\, U$$

and

$$\frac{\partial}{\partial t}\,\frac{\partial}{\partial\lambda}\, U = \frac{\partial A}{\partial t}\, U + A\, L\, U\,.$$

Equating these two derivatives we obtain the differential equation

$$\frac{\partial L}{\partial \lambda} = \frac{\partial A}{\partial t} + \Omega_A L \tag{8}$$

where Ω_A is an operator defined by the equation

$$\Omega_A L = [A, L] .$$

As (8) is a linear differential equation we may solve it to obtain

$$L(\lambda, t, t_0) = \varphi(\lambda \Omega_A) \frac{\partial A}{\partial t} \tag{9}$$

where

$$\varphi(x) = \frac{e^x - 1}{x} .$$

The solution of (3) has now been transformed to the problem of solving the equation

$$\tilde{H}(t) = i L(1, t, t_0) \tag{10}$$

where $L(1, t, t_0)$ is given by (9). Substituting (9) into (10) we obtain

$$\frac{\partial A}{\partial t} = - i \varphi^{-1} (\Omega_A) \tilde{H}(t) = - i \tilde{H}(t) + \frac{i}{2} [A, \tilde{H}(t)] - \frac{i}{12} [A [A, \tilde{H}(t)]] + \ldots . \tag{11}$$

Equation (11) may now be solved by iteration and we find that we may write the time development operator in the form $U = \exp \{i A\}$ where is given by

$$A = A (\infty, - \infty) = A_1 + A_2 + A_3 + \ldots$$

and

$$\left.\begin{aligned} A_1 &= - \int_{-\infty}^{\infty} \tilde{H}(t_1)\, dt , \quad A_2 = \frac{i}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{t_1} [\tilde{H}(t_1), \tilde{H}(t_2)]\, dt_1\, dt_2 , \\ A_3 &= \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} [\tilde{H}(t_1) [\tilde{H}(t_2), \tilde{H}(t_3)]]\, dt_1\, dt_2\, dt_3 \\ &\quad + \frac{1}{12} \int_{-\infty}^{\infty} \int_{-\infty}^{t_1} \int_{-\infty}^{t_1} [\tilde{H}(t_3) [\tilde{H}(t_2)\, \tilde{H}(t_1)]]\, dt_1\, dt_2\, dt_3 . \end{aligned}\right\} \tag{12}$$

In order to make comparisons of this solution with the sudden approximation solution it is useful to decompose U into the product

$$U = U_C\, U_S .$$

U_S is the sudden approximation solution (6) and U_C is the correction matrix

$$U_C = U\, U_S^{-1} = \exp \{i A\} \exp \{i B\}$$

where A and B are given by (12) and (6) respectively. The correction matrix may be re-expressed with the help of the BAKER-HAUSDORFF[4]) theorem and we find

$$U_C = \exp \{i C\} = \exp \left\{ i A + i B - \frac{1}{2} [A, B] - \frac{i}{12} [A [A, B]] + \ldots \right\} \tag{13}$$

where the exponent on the right hand side is an infinite sum of multiple commutators. Further details concerning this formula and methods of calculating the higher coefficients may be found in the cited references.

3. Rotational Nuclei

In this section we wish to apply the formalism developed in the previous section to the problem of Coulomb excitation of nuclei with a rotational spectrum. We will restrict our considerations to nuclei whose spectra form a pure rotational band. The free Hamiltonian for the rotation of the nuclear deformation is then given by

$$H = -\frac{1}{2I}\left\{\frac{\partial^2}{\partial\beta^2} + \operatorname{Cot}\beta\,\frac{\partial}{\partial\beta} + \frac{1}{\operatorname{Sin}^2\beta}\,\frac{\partial^2}{\partial\alpha^2}\right\} \tag{14}$$

where α and β are the Eulerian angles describing the orientation of the nuclear symmetry axis and I is the moment of inertia of the nucleus.

The Hamiltonian describing the electric quadrupole excitation of the nucleus due to its interaction with a charged projectile is given by

$$H_E(t) = \frac{4\pi}{5} Z_1 e \sum_{\mu} \frac{Y_{2\mu}(\theta_p, \varphi_p)}{r_p^3} M^*(E\,2, \mu)\,. \tag{15}$$

The time dependence and also the dependence of the Hamiltonian on the scattering angle θ enter through the coordinates of the projectile $(r_p, \theta_p, \varphi_p)$. The quadrupole operator can be expressed in terms of the intrinsic quadrupole moment Q_0 of the nucleus in the following manner

$$M(E\,2, \mu) = \frac{1}{2} Q_0\, \mathrm{Y}_{2\mu}(\beta, \alpha)\,. \tag{16}$$

In order to evaluate the time dependence of $H_E(t)$ we use the well known classical orbits method[2)3)]. We choose the coordinate system used by ALDER and WINTHER[2)] in which the projectile is scattered in the plane $\varphi_p = 0$ and where the z-axis is along the symmetry axis and we also make the following definitions

$$\left.\begin{aligned} q &= \frac{Z_1 e Q_0}{4 a^2 v}\,, \\ F_\mu(\theta, t) &= 2 a^2 v \sqrt{\frac{4\pi}{5}}\,\frac{Y_{2\mu}(\theta_p, \varphi_p)}{r_p^3}\,, \\ \text{and} \qquad M_\mu &= \sqrt{\frac{4\pi}{5}}\,\mathrm{Y}^*_{2\mu}(\beta, \alpha)\,. \end{aligned}\right\} \tag{18}$$

The parameter q was introduced in ref. 2, a is half the distance of closest approach of the projectile and the target nucleus, and v is the initial relative velocity of the projectile and target. The functions $F_\mu(\theta, t)$ have been defined in such a way that certain time integrals are related to the classical orbit integrals normally encountered in Coulomb excitation calculations. If

$$J_{2\mu}(\theta, \xi) = \int_{-\infty}^{\infty} F_\mu(\theta, t)\, e^{i\omega t}\, dt \tag{19}$$

where

$$\xi = \frac{a\,\omega}{v}$$

then

$$\left.\begin{aligned} J_{20}(\theta,\xi) &= \frac{1}{2} I_{20}(\theta,\xi) + \frac{3}{4} I_{22}(\theta,\xi) + \frac{3}{4} I_{2-2}(\theta,\xi)\,, \\ J_{2\pm1}(\theta,\xi) &= \frac{1}{2}\sqrt{\frac{3}{2}}\,[I_{2-2}(\theta,\xi) - I_{22}(\theta,\xi)]\,, \\ J_{2\pm2}(\theta,\xi) &= \frac{1}{2}\sqrt{\frac{3}{2}}\left[-I_{20}(\theta,\xi) + \frac{1}{2} I_{22}(\theta,\xi) + \frac{1}{2} I_{2-2}(\theta,\xi)\right] \end{aligned}\right\} \quad (20)$$

where the $I_{2\mu}(\theta,\xi)$ are the classical orbit integrals tabulated on page 478 ref. 3.

The advantage of the coordinate system chosen to evaluate the time dependence of the interaction Hamiltonian $H_E(t)$ is that for $\theta = \pi$

$$Y_{2\mu}(\theta_p, \varphi_p) = 0\,, \quad \text{for } \mu \neq 0\,.$$

This is also a very good approximation for all backward scattering angles if ξ is small. As the approximation (21) simplifies the formalism considerably we assume that it is generally valid. Thus the following considerations are only strictly true in the case of backward scattering. Corrections to this approximation have been discussed by ALDER and WINTHER [2]) and their discussion applies also to the following work. From the definitions (18), and with the approximation (21), the interaction Hamiltonian (15) now takes the simple form

$$H_E(t) = q\,F_0(\theta,t)\,M_0\,. \quad (22)$$

After this brief discussion of rotational nuclei we now return to the solution of the excitation problem given in Section 2. The time development operator describing the nuclear states is given by

$$U = \exp\{i\,A\}\,.$$

There are two essentially different ways of evaluating the matrix elements of an operator of this form. The first method is to expand the exponential and corresponds in some way to perturbation theory, the second method is to find a unitary matrix which diagonalizes i.e. to find a matrix S such that

$$S\,S^\dagger = S^\dagger\,S = 1$$

and

$$\langle n \mid S^\dagger\,A\,S \mid m\rangle = a_n\,\delta_{nm}\,.$$

Then the matrix elements of U are simply given by

$$\langle n \mid U \mid m\rangle = \sum_p \langle n \mid S \mid p\rangle\,\langle p \mid S^\dagger \mid m\rangle\,e^{-i a_p}\,. \quad (23)$$

However to apply either of these methods it is necessary to know the matrix elements of A and also to make certain approximations which are governed by the order of

magnitude of these matrix elements, we therefore now examine the structure of A. The matrix elements of A are determined in terms of the matrix elements of $\tilde{H}(t)$ these matrix elements are in turn determined by the matrix elements of the nuclear quadrupole operator which may be measured by nuclear spectroscopy. The matrix elements measured in this way often differ from those predicted by the simple rotational model but we will not consider these discrepancies. In the following we assume the values given by the rotational model although our methods could also be used in conjunction with the experimentally measured values. Now the matrix elements[5]) of the first term in (12) may be easily calculated and we find from (12), (18) and (19) that

$$\left.\begin{aligned}\langle I_1 M_1 \,|\, A_1 \,|\, I_2 M_2\rangle \\ = -\,\delta_{M_1 0}\,\delta_{M_2 0}\,q\,J_{20}(\theta,\xi_{12})\sqrt{(2I_1+1)\,(2I_2+1)}\begin{pmatrix} I_1 & I_2 & 2\\ 0 & 0 & 0\end{pmatrix}^2\end{aligned}\right\} \tag{24}$$

where the nuclear states $|I_i M_i\rangle$ are specified by means of the spin I_i and the magnetic quantum number M_i. The parameter ξ_{12} is defined by

$$\xi_{ij} = (E_i - E_j)\frac{a}{v} = \omega_{ij}\frac{a}{v}$$

where E_i and E_j are the energy eigenvalues of the states $|I_i M_i\rangle$ and $|I_j M_j\rangle$ respectively. The matrix elements of the higher terms occurring in the expression which defines A are not so easy to evaluate but we have

$$\left.\begin{aligned}\langle I_1 M_1 \,|\, A_2 \,|\, I_2 M_2\rangle = \delta_{M_1 0}\,\delta_{M_2 0}\frac{i q^2}{2}\sum_{I_3} K(\theta,\xi_{13},\xi_{23}) \\ \times (2I_3+1)\sqrt{(2I_1+1)\,(2I_2+1)}\begin{pmatrix} I_1 & I_3 & 2\\ 0 & 0 & 0\end{pmatrix}^2\begin{pmatrix} I_2 & I_3 & 2\\ 0 & 0 & 0\end{pmatrix}^2\end{aligned}\right\} \tag{25}$$

where

$$K(\theta,\xi_{13},\xi_{23})\int_{-\infty}^{\infty}\int_{-\infty}^{t_1}\{e^{i\omega_{13}t_1 - i\omega_{23}t_2} - e^{-i\omega_{23}t_1 + i\omega_{13}t_2}\}\,F_0(\theta,t_1)\,F_0(\theta,t_2)\,dt_1\,dt_2\,.$$

This integral may be put into a more convenient form by introducing the step function

$$\varepsilon(t_1 - t_2) = \lim_{\delta\to 0+}\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iz(t_1-t_2)}}{z + i\sigma}\,dz = \begin{cases}1 & t_1 > t_2\\ 0 & t_1 < t_2\end{cases}.$$

We find

$$\left.\begin{aligned}K(\theta,\xi_{13},\xi_{23}) = \frac{i}{2\pi}\,\mathcal{P}\int_{-\infty}^{\infty}\frac{d\xi}{\xi} \\ \times\,[J_{20}(\theta,\xi_{13}-\xi)\,J_{20}(\theta,\xi_{23}-\xi) - J_{20}(\theta,\xi_{23}+\xi)\,J_{20}(\theta,\xi_{13}+\xi)]\end{aligned}\right\} \tag{26}$$

where $\mathcal{P}$ indicates the Cauchy principal part.

The function $K(\theta, \xi_{13}, \xi_{23})$ has certain oddness and evenness properties which follow from the evenness of $J_{20}(\theta, \xi)$ in the variable ξ. We find

$$K(\theta, \xi_{13}, \xi_{23}) = -K(\theta, -\xi_{13}, -\xi_{23}) = K(\theta, \xi_{23}, \xi_{13}) . \tag{27}$$

These properties coupled with the selection rules of the quadrupole operator lead to cancellations in the sum occurring on the right hand side of (26). As $K(\theta, \xi_{13}, \xi_{23})$ is small for small values of ξ and as cancellations also occur in the sum the matrix elements of A_2 should be small in comparison with the matrix elements of A_1 for a large range of q. These properties are common to the matrix elements of all the higher terms occurring in the expression (12) and this suggests that it is a reasonable approximation to cut off the series at a low stage. A second approximation which aids greatly in evaluating the matrix elements of A_2 and higher terms is the equal spacing approximation. This approximation consists of assuming that the nuclear energy levels are equally spaced i.e. we assume

$$E_{i+1} - E_i = E .$$

With this approximation only one ξ-parameter enters the calculations of the matrix elements and we find from (25)

$$\left.\begin{aligned} \langle I_1 0 \,|\, A_2 \,|\, I_1 0\rangle &= i q^2 \frac{(2 I_1 + 1)}{2} K(\theta, \xi, \xi) \\ &\times \left[(2 I_1 - 3) \begin{pmatrix} I_1 & I_1 - 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}^4 - (2 I_1 + 5) \begin{pmatrix} I_1 & I_1 + 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}^4\right] \end{aligned}\right\} \tag{28}$$

and

$$\left.\begin{aligned} \langle I_1 0 \,|\, A_2 \,|\, I_1 + 2\, 0\rangle &= 3 i q^2 \frac{\sqrt{(2 I_1 + 1)(2 I_1 + 5)}}{(2 I_1 - 1)(2 I_1 + 3)(2 I_1 + 7)} \\ &\times K(\theta, 0, \xi) \begin{pmatrix} I_1 + 2 & I_1 & 2 \\ 0 & 0 & 0 \end{pmatrix}^2 = \langle I_1 + 2\, 0 \,|\, A_2 \,|\, I_1 0\rangle . \end{aligned}\right\} \tag{29}$$

It may be seen from these expressions that the matrix elements of A_2 decrease rapidly with respect to I_1 therefore ξ should be fitted to the value of the ξ-parameter corresponding to the lowest energy spacing to give the best results. The equal spacing approximation with ξ fitted in this manner tends to over estimate slightly the matrix elements corresponding to low spin and under estimate those corresponding to high spin. As the latter are very small this should have no noticeable effects.

The matrix elements of the higher terms in the expansion (12) of A may be treated in the same way as the second order term. We will however consider these no further in detail. The characteristic feature of each term is that a multiple integral over the classical orbital integrals occurs which is always of $O(\xi)$, an added factor q and an added numerical factor which is always rather small.

Having calculated the matrix elements of A in the above way it is now possible to calculate the matrix elements of U, either by expansion or by calculation of a

unitary matrix with the properties (23). ALDER and WINTHER have used both methods of calculation in examining the sudden approximation and have made comparisons of the results. The latter method is more exact and is instituted by approximating A by a finite matrix. The number of rows and columns which have to be included depends upon the number of states which are actively excited i.e. if for a certain q value only the lowest n states are appreciably excited then it suffices to approximate A by an $n \times n$ matrix.

The programme outlined above may also be used to calculate the correction matrix U_C defined by (13). This has the advantage that the exact results tabulated by ALDER and WINTHER for the sudden approximation solution U_S may be used to improve the accuracy of the calculation. The magnitude of the matrix elements of U_S determines the number of rows and columns necessary in the calculation of U_C.

The calculations outlined above are being carried out. Provisional results show the correct qualitative character for the corrections obtained by those methods. The full results of these calculations will be published in a separate work. We now turn our attention to the application of the formalism of section 2 to the problem of excitation of nuclei with a vibrational spectrum.

4. Vibrational Nuclei

The excitation of nuclei with a vibrational spectrum has been considered in a previous paper[6]) and the notation of this paper will be used in the following. The free Hamiltonian for the vibrations of the nuclear surface is given by

$$H = \sum_{\mu} \left[\frac{1}{2B} P_{\mu} P_{\mu}^{\dagger} + \frac{1}{2} C x_{\mu} x_{\mu}^{+} \right]. \tag{30}$$

Introducing phonon annihilation and creation operators by the definitions

$$\alpha_{\mu} = \frac{1}{\sqrt{2\omega}} \left[\sqrt{C}\, x_{\mu}^{\dagger} + i \frac{P_{\mu}}{\sqrt{B}} \right], \quad \alpha_{\mu}^{\dagger} = \frac{1}{\sqrt{2\omega}} \left[\sqrt{C}\, x_{\mu} - i \frac{P_{\mu}}{\sqrt{B}} \right], \tag{31}$$

we have

$$H = \omega \sum_{\mu} \left[\alpha_{\mu} \alpha_{\mu}^{\dagger} - \frac{1}{2} \right].$$

As in the previous section we approximate the excitation Hamiltonian by considering just the term with magnetic quantum number zero i.e. we take

$$H_E(t) = \frac{4\pi}{5} Z_1 e \frac{Y_{20}(\theta_p, \varphi_p)}{r_p^3} M^*(E\,2, 0).$$

The nuclear quadrupole operator is given by

$$M(E\,2, 0) = M_0^1 + M_0^2 + \ldots \tag{32}$$

where

$$M_0^1 = \frac{3}{4\pi\sqrt{5}} Z_2 e R_0^2 \beta [\alpha_0 + \alpha_0^\dagger] \tag{33}$$

and

$$M_0^2 = -\frac{3}{4\pi}\sqrt{\frac{2}{7\pi}} Z_2 e R_0^2 \beta^2 R \sum_{\lambda\mu} \begin{pmatrix} 2 & 2 & 2 \\ \lambda & \mu & 0 \end{pmatrix} [\alpha_\lambda^\dagger + (-1)^\lambda \alpha_\lambda] [\alpha_\mu^\dagger + (-1)^\mu \alpha_{-\mu}] .$$

Following the methods of the previous section we now define a characteristic q-parameter and a time dependent orbital function. The definitions differ from (18); we take

$$q = \frac{3}{5\sqrt{5}} \eta \frac{R_0^2}{a^2} \beta , \quad F_0(\theta, t) = a^2 v \frac{Y_{20}(\theta_p, \varphi_p)}{r_p^3} . \tag{34}$$

The parameter η is the well known parameter associated with Coulomb excitation calculations and is given by

$$\eta = \frac{Z_1 Z_2 e^2}{v} .$$

The function $F_0(\theta, t)$ has the property

$$J'_{20}(\theta, \xi) = \int_{-\infty}^{\infty} F_0(\theta, t) e^{i\omega t} dt .$$

The connection of $T_{20}(\theta, \xi)$ with the function defined by (19) is readily found to be

$$J'_{20}(\theta, \xi) = \frac{1}{2}\sqrt{\frac{5}{4\pi}} J_{20}(\theta, \xi) .$$

These definitions allow $H_E(t)$ to be written in the form

$$\left.\begin{aligned} H_E(t) = q F_0(\theta, t) [\alpha_0 + \alpha_0^\dagger] - \sqrt{\frac{10}{7\pi}} q k \beta F_0(\theta, t) \sum_{\lambda\mu} \\ \times \begin{pmatrix} 2 & 2 & 2 \\ \lambda & \mu & 0 \end{pmatrix} [\alpha_\lambda^\dagger + (-1)^\lambda \alpha_{-\lambda}] [\alpha_\mu^\dagger + (-1)^\mu \alpha_\mu] . \end{aligned}\right\} \tag{35}$$

This form of $H_E(t)$ now leads to certain simple properties for the matrix A defined by (12). $H_E(t)$ is a quadratic polynomial in the annihilation and creation operators and it may be easily seen that the algebra defined by quadratic polynomials in α_μ and $\alpha_\mu^\dagger$ is identical to its commutator algebra. Thus evaluation of (12) leads to the form

$$A = \sum_{\lambda\mu} \{I \alpha_0 + I^\dagger \alpha_0^\dagger + J_{\lambda\mu} \alpha_\lambda^\dagger \alpha_\mu^\dagger + J_{\lambda\mu}^\dagger \alpha_\lambda \alpha_\mu + K_{\lambda\mu} \alpha_\lambda \alpha_\mu^\dagger + K_{\lambda\mu} \alpha_\lambda^\dagger \alpha_\mu\} , \tag{36}$$

where the I, J, and K are determined as sums of integrals over the orbital functions and we have omitted a c-number which contributes only a phase factor to the time development matrix.

The first terms of I, J and K are given by

$$\left.\begin{aligned} I &= -q\, J'_{20}(\theta,\xi) + \frac{4}{7\sqrt{\pi}} q^2 k \beta \int_{-\infty}^{\infty}\int_{-\infty}^{t_1} \\ &\qquad \times F_0(\theta,t_1)\, F_0(\theta,t_2)\, \mathrm{Sin}\,\omega\,(t_1-t_2)\,[e^{-i\omega t_1}+e^{-i\omega t_2}]\, dt_1\, dt_2 + \ldots, \\ J_{\lambda\mu} &= \sqrt{\frac{10}{7\pi}}\, q k \beta \begin{pmatrix} 2 & 2 & 2 \\ \lambda & \mu & 0 \end{pmatrix} J'_{20}(\theta, 2\xi) + \ldots, \end{aligned}\right\} \tag{37}$$

and

$$K_{\lambda\mu} = \sqrt{\frac{10}{1\pi}}\, q k \beta \begin{pmatrix} 2 & 2 & 2 \\ -\lambda & \mu & 0 \end{pmatrix} (-1)^{\lambda} J'_{20}(\theta, 0) + \ldots .$$

The time development matrix U may be put into a convenient form by making a splitting

$$U = U_C\, U_S$$

where we define

$$U_C = \exp\left\{ i \sum_{\lambda\mu} [J_{\lambda\mu}\, \alpha_\lambda^\dagger\, \alpha_\mu^\dagger + J_{\lambda\mu}^\dagger\, \alpha_\lambda\, \alpha_\mu + K_{\lambda\mu}\, \alpha_\lambda\, \alpha_\mu^\dagger + K_{\lambda\mu}\, \alpha_\lambda^\dagger\, \alpha_\mu] \right\}. \tag{38}$$

U_S may be calculated with the help of the Baker Hausdorff theorem (13) and we find the simple form

$$U_S = \exp\{i\,[L\,\alpha_0 + L^\dagger\, \alpha_0^\dagger]\}$$

where L is given by a sum over products of the I, J and K. The first terms in this sum are

$$L = I - i\,(K_{00}\, I - J_{00}^\dagger\, I^\dagger) + \ldots .$$

To first order in k

$$\left.\begin{aligned} L &= -q\, J'_{20}(\theta,\xi) + \frac{4}{7\sqrt{\pi}} q^2 k \beta\, Q(\theta,\xi) \\ &\qquad - i\,\frac{2 q^2 k \beta}{7\sqrt{\pi}}\, J'_{20}(\theta,\xi)\,[J'_{20}(\theta,0) - J'_{20}(\theta,2\xi)] + \ldots \end{aligned}\right\} \tag{41}$$

where we have defined

$$Q(\theta,\xi) = \int_{-\infty}^{\infty}\int_{-\infty}^{t_1} F_0(\theta,t_1)\, F_0(\theta,t_2)\, \mathrm{Sin}\,\omega\,(t_1-t_2)\,[e^{-i\omega t_1}+e^{-i\omega t_2}]\, dt_1\, dt_2 .$$

This integral may be rearranged by introducing the step function and we find

$$L = -q\, J'_{20}(\theta,\xi) - \frac{1}{2} q^2 k \beta\, P(\theta,\xi) + \ldots \tag{42}$$

where

$$P(\theta,\xi) = \frac{4}{7\pi^{3/2}}\, \mathscr{P} \int_{-\infty}^{\infty} \frac{d\xi'}{\xi'}\, J'_{20}\,(\theta, \xi+\xi')\,[J'_{20}\,(\theta, 2\xi+\xi') + J'_{20}(\theta,\xi')] .$$

The function $P(\theta, \xi)$ has been encountered previously in an examination of the first order effects of the two-phonon part of the quadrupole operator (see Rf. 6. formula A.10). $P(\theta, \xi)$ is an odd function of ξ and for positive values of ξ it assumes negative values.

In order to evaluate the matrix elements of U we consider the matrix elements of U_C and U_S. The operator U_S may be rearranged using the Baker-Hausdorff theorem and we find

$$U_S = \exp\{i\, L^\dagger\, \alpha_0^\dagger\} \exp\{i\, L\, \alpha_0\} \exp\left\{-\frac{|L|^2}{2}\right\}.$$

We now have

$$\langle m \mid U_S \mid n\rangle = \sum_r \langle m \mid e^{i L^\dagger \alpha_0^\dagger} \mid r\rangle \langle r \mid e^{i L \alpha_0} \mid n\rangle \exp\left\{-\frac{|L|^2}{2}\right\}$$

$$= \sum_{r=0}^{p} \frac{(i\, L^\dagger)^{m-r}\, (i\, L)^{n-r} \sqrt{m!\, n!}}{(m-r)!\, (n-r)!\, r!} \exp\left\{-\frac{|L|^2}{2}\right\}$$

where

$$p = \text{Min}\{n, m\}.$$

These matrix elements assume a particularly simple form if $\mid n\rangle$ or $\mid m\rangle$ is the ground state of the nucleus. Then

$$\langle m \mid U_S \mid 0\rangle = \frac{(i\, L^\dagger)^m}{\sqrt{m!}} \exp\left\{-\frac{|L|^2}{2}\right\}$$

and further

$$P_m^S = |\langle m \mid U_S \mid 0\rangle|^2,$$

$$= \frac{|L|^{2m}}{m!} \exp\{-|L|^2\}.$$

At this stage it is instructive to make a comparison with the calculation[2)6)] of the excitation probabilities in the approximation that the quadrupole operator is given by the one-phonon part M_0^1. In this approximation the states of the excited nucleurs are also populated in a Poisson distribution and the probability of exciting the nucleus from its ground state to the state $\mid m\rangle$ is given by (46) with $k = 0$. The effect of setting $k = 0$ is equivalent of the replacement of L by $-q\, J'_{20}(\theta, \xi)$ as may be readily seen from (40). However, if instead of putting $k = 0$ we expand P_m^S in terms of k we find that P_m^S agrees, to first order in k with the result given previously (Ref. 6 formula (36)) for the excitation probability to all orders in the one-phonon processes and to first order in the two-phonon processes. Therefore to calculate the matrix elements of U to first order in the two-phonon processes it is perfectly consistent to take

$$U = U_S \quad \text{i.e.} \quad U_C = 1 \tag{47}$$

and to calculate the matrix elements of U_S to first order in k.

Although it is not consistent to make the approximation (47) and to retain all orders of k in the matrix elements of U_S this does shed some light upon the results obtained in Ref. 6. concerning the interference of the one-phonon processes and the

two-phonon processes. With the approximation (47), but to all orders in k the probability of exciting the nucleus from its ground state to the mth excited state is given by (46). Now from (42)

$$L = -\,q\,J'_{20}(\theta,\xi) - \frac{1}{2}q^2\,k\,\beta\,P(\theta,\xi) + O(k^2)\,.$$

Because $P(\theta,\xi)$ is negative for all positive values of ξ the absolute magnitude of L is smaller than its value for $k = 0$ (assuming that the terms of $O(k^2)$ are insignificant). Therefore in the approximation (47) the bombarding energy required to excite a state to its maximum is higher than that predicted by the one-phonon approximation.

We now turn our attention to the effects of the two-phonon interaction introduced through the correction matrix U_C. We consider only the matrix elements of U_C between the nuclear ground state and an excited state for reasons of simplicity. Calculation of these matrix elements is in general very complicated we therefore make two simplifying approximations. The first approximation is to restrict the sum in (38) to the terms with $\lambda = 0$ and $\mu = 0$. This approximation is inessential but considerably simplifies the following formalism without destroying the essential features of the result.

We now have

$$\langle 0 \mid U_C \mid m\rangle = \langle 0 \mid \exp\{i\,[J_{00}\,\alpha_0^\dagger\,\alpha_0^\dagger + J_{00}^\dagger\,\alpha_0\,\alpha_0 + K_{00}\,\alpha_0\,\alpha_0^\dagger + K_{00}\,\alpha_0^\dagger\,\alpha_0]\} \mid m\rangle\,. \tag{48}$$

There are various methods which may be used to proceed with the calculation. It is possible to expand the exponential in (48) or to diagonalize the exponent matrix as discussed in Section 3. However if we make the approximation $\xi = 0$ the matrix elements can then be calculated in a particularly simple form. We therefore consider this approximation and we find

where

$$\left.\begin{aligned} \langle 0 \mid U_C \mid m\rangle &= \left\langle 0 \,\middle|\, \exp\left\{-\frac{i\,X}{2}\,(\alpha_0 + \alpha_0^\dagger)^2\right\} \,\middle|\, m\right\rangle \\ X &= \frac{4}{7\sqrt{\pi}}\,q\,k\,\beta\,J'_{20}(\theta,0)\,. \end{aligned}\right\} \tag{49}$$

This follows directly from (37) and the fact that the higher order terms omitted in (37) are all of order ξ.

To proceed further we use the first identity of the Appendix to obtain

$$\langle 0 \mid U_C \mid m\rangle = \left\langle 0 \,\middle|\, \exp\left\{-\frac{1}{2}\ln(1 + i\,X)\,\alpha_0\,(\alpha_0 + \alpha_0^\dagger)\right\} \,\middle|\, m\right\rangle. \tag{50}$$

We now use the second identity given in the Appendix to simplify (50). Finally

$$\left.\begin{aligned} \langle 0 \mid U_C \mid m\rangle &= \left\langle 0 \,\middle|\, \exp\left\{-\frac{1}{2}\ln(1 + i\,X)\,\alpha_0\,\alpha_0^\dagger\right\} \exp\left\{\frac{-\,i\,X}{2\,(1 + i\,X)}\,\alpha_0^2\right\} \,\middle|\, m\right\rangle \\ &= \left\langle 0 \,\middle|\, \exp\left\{-\frac{i\,X}{2\,(1 + i\,X)}\,\alpha_0^2\right\} \,\middle|\, m\right\rangle \frac{1}{\sqrt{1 + i\,X}} \\ &= \frac{1}{\sqrt{1 + i\,X}}\left(\frac{-\,i\,X}{2\,(1 + i\,X)}\right)^{m/2} \frac{\sqrt{m\,!}}{(m/2)\,!} \qquad \text{if } m \text{ is even} \\ &= 0 \qquad \text{if } m \text{ is odd.} \end{aligned}\right\} \tag{51}$$

With the aid of (44) and (51) it is possible to calculate the matrix elements of U. After some rearrangement we find

$$\left.\begin{aligned}\langle 0 \mid U \mid n\rangle = e^{-(|L|^2/2)} \sqrt{\frac{n!}{1+iX}} \sum_{\substack{m=0 \\ r\geqslant 0}}^{\infty} \sum_{r=2m-n}^{2m} \\ \times \left(\frac{-iX}{2(1+iX)}\right)^m \frac{(iL^+)^r (iL)^{n-2m+r} 2m!}{(2m-r)!\,(n-2m+r)!\,m!\,r!}.\end{aligned}\right\} \quad (52)$$

The excitation amplitudes may be expressed in the more convenient form

$$\langle n \mid U \mid 0\rangle = \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{1+iX}} g_n(L, X) \exp\left\{-\frac{LL^+}{2} + \frac{iXL^{+2}}{2(1+iX)}\right\}$$

where the polynomials $g_n(L, X)$ are defined by means of the generating function

$$\exp\left\{\frac{-iXs^2}{2(1+iX)} + \frac{iL^\dagger s}{1+iX}\right\} = \sum_n g\,n(L, X) \frac{s^n}{n!}.$$

The excitation probabilities may now be directly calculated. Some simplification takes place if we assume that L is real. We then find

$$\begin{aligned} P_n &= |\langle 0 \mid U \mid n\rangle|^2, \\ &= \frac{1}{n!} \frac{1}{\sqrt{1+X^2}} |g_n(L, X)|^2 \exp\left\{-\frac{L^2}{1+X^2}\right\} \end{aligned}$$

and explicitly

$$\left.\begin{aligned} P_0 &= \frac{1}{\sqrt{1+X^2}} \exp\left\{-\frac{L^2}{1+X^2}\right\}, \quad P_1 = \frac{1}{\sqrt{1+X^2}} \frac{L^2}{1+X^2} \exp\left\{-\frac{L^2}{1+X^2}\right\}, \\ P_2 &= \frac{1}{\sqrt{1+X^2}} \frac{1}{2!} \frac{(L^2-X^2)^2+X^2}{(1+X^2)^2} \exp\left\{-\frac{L^2}{1+X^2}\right\}, \\ P_3 &= \frac{1}{\sqrt{1+X^2}} \frac{1}{3!} \frac{L^2}{(1+X^2)^3} [(L^2-3X^2)^2+qX^2] \exp\left\{-\frac{L^2}{1+X^2}\right\}. \end{aligned}\right\} \quad (53)$$

As we have assumed that L is real L^2 in (53) should only be evaluated to first order in k

Thus the inclusion of the corrections due to the two-phonon processes destroys the simplicity of the results obtained for the excitation probabilities and we do not obtain the simple prediction that the nuclear states are populated in a Poisson distribution. However for small X the probabilities given by (53) do not differ very greatly from those given by the Poisson distribution

$$P_n = \frac{1}{n!} \frac{L^{2n}}{(1+X^2)^n} \exp\left\{-\frac{L^2}{1+X^2}\right\}. \quad (54)$$

We may again compare this distribution with the result obtained in the approximation of considering one-phonon processes only and we find the same conclusion that was discussed in connection with the approximation (47). i.e. bombarding energy required to excite a state to its maximum is higher than that predicted by the one-phonon approximation.

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Appendix

The first identity used to simplify formula (49) in the text is stated as follows;

$$\exp\left\{-\frac{i\,X}{2}\,(\alpha_0+\alpha_0^\dagger)^2\right\}=\exp\left\{-\frac{1}{2}\ln(1+i\,X)\,\alpha_0^\dagger\,(\alpha_0+\alpha_0^\dagger)\right\}$$
$$\times\exp\left\{-\frac{1}{2}\ln(1+i\,X)\,\alpha_0\,(\alpha_0+\alpha_0^\dagger)\right\}.$$

To prove this identity we make the ansatz

$$\exp\{t\,\alpha_0^\dagger\,(\alpha_0+\alpha_0^\dagger)\}\exp\{t\,\alpha_0\,(\alpha_0+\alpha_0^\dagger)\}=\exp\{f(t)\,(\alpha_0+\alpha_0^\dagger)^2\}.$$

That this ansatz is reasonable follows from applying the Baker-Hausdorff theorem to the left hand side. We now differentiate the ansatz with respect to t and after some elementary rearrangement we obtain

$$f'(t)=e^{-2t}.$$

We may now integrate with the boundary condition $f(0)=0$ to obtain the desired result.

Similarly we may prove the second identity;

$$\exp\{t\,\alpha_0\,(\alpha_0+\alpha_0^+)\}=\exp\{t\,\alpha_0\,\alpha_0^+\}\exp\left\{\frac{1}{2}\,(e^{2t}-1)\,\alpha_0^2\right\}.$$

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Note added in proof: Prof. K. ALDER has informed the author, in a private communication, that he had earlier obtained results similar to those contained in the present paper in collaboration with A. WINTHER. Their work remained however unpublished.